Pseudoentropy for descendant operators in two-dimensional conformal field theories

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We study the late-time behaviors of pseudo-(Rényi) entropy of locally excited states in rational conformal field theories. To construct the transition matrix, we utilize two nonorthogonal locally excited states that are created by the application of different descendant operators to vacuum. We show that when two descendant operators are generated by a single Virasoro generator acting on the same primary operator, the late-time excess of pseudoentropy and pseudo-Rényi entropy corresponds to the logarithm of the quantum dimension of the associated primary operator, in agreement with the case of entanglement entropy. However, for linear combination operators generated by the generic summation of Virasoro generators, we obtain a distinct late-time excess formula for the pseudo-(Rényi) entropy compared to that for (Rényi) entanglement entropy. As the mixing of holomorphic and antiholomorphic generators enhances the entanglement, in this case, the pseudo-(Rényi) entropy can receive an additional contribution. The additional contribution can be expressed as the pseudo-(Rényi) entropy of an effective transition matrix in a finite-dimensional Hilbert space.

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I. INTRODUCTION

The discovery of the AdS/CFT correspondence [1–3] has motivated much research related to quantum information theory in the high-energy physics community in recent years. Among them, quantum entanglement, as a carrier of quantum information, plays an increasingly significant role in probing the structure of quantum field theories (QFTs) [4–10], the emergence of geometry [11–13], and the black hole information paradox [14–18].

Recently, a new entanglement measure called pseudoentropy was proposed in [19] as a generalization of entanglement entropy. Specifically, pseudoentropy is a two-state vector version of entanglement entropy defined as follows. Given two nonorthogonal states $|\psi\rangle$ and $|\varphi\rangle$ in the Hilbert space $\mathcal{H}_{\mathcal{S}}$ of a composed quantum system $\mathcal{S} = A \cup B$, we first construct an operator called the transition matrix acting on $\mathcal{H}_{\mathcal{S}}$ [19,20],

$$\mathcal{T}^{\psi|\varphi} \equiv \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle} = \frac{\rho_\psi \rho_\varphi}{\mathrm{tr}[\rho_\psi \rho_\varphi]}. \tag{1}$$

The pseudoentropy of subsystem A is then obtained by calculating the von Neumann entropy of the reduced transition matrix $\mathcal{T}_A^{\psi|\varphi} \equiv \mathrm{tr}_B[\mathcal{T}^{\psi|\varphi}]$,

$$S(\mathcal{T}_A^{\psi|\varphi}) = -\mathrm{tr}[\mathcal{T}_A^{\psi|\varphi} \log \mathcal{T}_A^{\psi|\varphi}]. \tag{2}$$

Generally, the reduced transition matrix is non-Hermitian, requiring careful consideration when discussing pseudoentropy in systems with infinite-dimensional Hilbert spaces (such as in QFTs) since taking the logarithm of a generic operator requires choosing a radial line in the complex plane that does not intersect the spectrum [21]. To avoid dealing directly with the logarithm of the non-Hermitian matrix, in practice, one usually computes a quantity called pseudo-Rényi entropy,

$$S_A^{(n)} \equiv S^{(n)}(\mathcal{T}_A^{\psi|\varphi}) \coloneqq \frac{1}{1-n} \log \mathrm{tr}[(\mathcal{T}_A^{\psi|\varphi})^n], \tag{3}$$

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instead of pseudoentropy, and the branch of the logarithm function is chosen to be $-\pi < \mathrm{Im}[\log(z)] < \pi$. For $n \in N^+$, $n \geq 2$, (3) admits an alternative expression:

$$S^{(n)}(\mathcal{T}_A^{\psi|\varphi}) := \frac{1}{1-n} \log\left[\sum_j \lambda_j(\mathcal{T}_A^{\psi|\varphi})^n\right], \quad \left(\sum_j \lambda_j(\mathcal{T}_A^{\psi|\varphi}) = 1\right), \tag{4}$$

where $\lambda_j(\mathcal{T}_A^{\psi|\varphi})$ are the eigenvalues of $\mathcal{T}_A^{\psi|\varphi}$ directly following from a Jordan decomposition of $\mathcal{T}_A^{\psi|\varphi}$. In this paper, we mainly focus on the pseudo-Rényi entropy for general $n (n \geq 2)$. When discussing pseudoentropy, we refer to the Shannon entropy defined by the eigenvalues of $\mathcal{T}_A^{\psi|\varphi}$,

$$-\sum_j \lambda_j(\mathcal{T}_A^{\psi|\varphi}) \log[\lambda_j(\mathcal{T}_A^{\psi|\varphi})]. \tag{5}$$

For finite-dimensional systems, it is clear that (2) equals (5), and the latter can be obtained from taking an analytic continuation of $n \to 1$ on (4). However, as previously mentioned, for infinite-dimensional systems, the definition of pseudoentropy in (2) may not be well defined in general.

Pseudoentropy was originally proposed from the study of the generalization of holography entanglement entropy [19]. In the AdS/CFT context, the pseudoentropy of a boundary subsystem is proposed to be dual to the area of a minimal surface in a Euclidean time-dependent AdS space [19]. In addition, it is found that pseudoentropy is closely related to postselection experiments in quantum information [19,22] (i.e., in addition to the initial state, the system's final state is also specified [23]). There are also many research interests and prospects driving the study of pseudoentropy in QFTs [24–30]. See [20,31–38] for other related developments of pseudoentropy.

Nonequilibrium dynamics in quantum many-body systems is a subject of intensive research [39,40]. One of the recurring themes is how quantum entanglement arises and propagates in nonequilibrium processes known as entanglement dynamics. Research shows that chaotic quantum many-body systems can nonlocally disrupt quantum information. The scrambling of quantum information will at least lead to the loss of local initial state information and lead to thermalization [41–43]. A typical nonequilibrium process in quantum many-body systems is quantum quench [44,45]. The process usually involves two steps: First, prepare an initial state $|\psi\rangle$, which can be the ground state of a certain Hamiltonian H, and then evolve it with a different Hamiltonian H'. One can also quench the system by a local perturbation (generally called a local quench [46,47]), for instance, acting on a local operator (generally called a local operator quench [48,49]). Then, the entanglement dynamics are diagnostic about the nature of this excitation.

The present paper aims to study the properties of pseudo-(Rényi) entropy of states obtained by acting on vacuum with a descendant of a local primary operator (also referred to as descendant states in this paper) in two-dimensional conformal field theories (2D CFTs). Our study can be traced back to the research on entanglement entropy in local operator quantum quenches in 2D CFTs [49–65]. The local operator quench exhibits broad applicability in measuring scrambling and thermalization effects in CFTs with large central charge [53,54,58,66,67], which can be regarded as a manifestation of quantum chaos, as well as in probing the bulk geometry [68] and characterizing bulk dynamics [63,69–71] in the context of AdS/CFT correspondence—an essential avenue for comprehending quantum gravity. It is found that the excess of Rényi entropy of the local primary or descendant excited states in rational conformal field theories (RCFTs) saturates to a constant equal to the logarithm of the quantum dimension [72] of the local operator's conformal family [51,56,57]. Such a saturation is well explained by the picture of quasiparticle pair propagation [49]. The related research has been extended to the pseudoentropy in parallel [30]. Specifically, when studying the real-time evolution of the pseudo-Rényi entropy, such as the second pseudo-Rényi entropy, for locally primary excited states in RCFTs, the conformal block at early times relies on the spatial positions of two identical primary operators, leading to a model-dependent pseudo-Rényi entropy. Nevertheless, the pseudo-Rényi entropy shows a universal behavior at late times, which only depends on the quantum dimension of the primary operator, just like the entanglement entropy. The result suggests that the picture of quasiparticle pair propagation is preserved in the pseudoentropy. We generalize the previous study [30] of the pseudo-(Rényi) entropy to descendant operators in this paper to understand the intricate connections between fusion rules and entanglement properties [73], where fusion rules play a fundamental role in characterizing algebraic and structural properties of a CFT [74,75]. The algebraic and structural properties would be encoded in the dynamics of entanglement. Specifically, we would like to explore the late-time behavior of the pseudo-Rényi entropy of two descendant operators in RCFTs. We construct the transition matrix using two locally excited states created by the operator

$$V_\alpha(x) = \sum_{\{n_i\},\{\bar{n}_j\}} \alpha_{\{n_i\}\{\bar{n}_j\}} \cdot \prod_{i,j} L_{-n_i} \bar{L}_{-\bar{n}_j} \mathcal{O}(x) \tag{6}$$

and evaluate the pseudo-Rényi entropy using the replica method [5] and conformal mapping. In (6), $\mathcal{O}(x)$ is a primary operator in the Schrödinger picture with chiral and antichiral conformal dimension Δ, L_{-n} ($\bar{L}_{-n}$) are holomorphic (antiholomorphic) Virasoro generators, and $\alpha_{\{n_i\},\{\bar{n}_j\}} \in \mathbb{C}$ are superposition coefficients. Since the two-point function between descendant operators of different levels does not vanish, the transition matrices we are permitted to construct have more degrees of freedom than the cases of the primary operator [76]. It is interesting to see whether the late-time behavior of the pseudo-(Rényi) entropy of subsystems corresponding to these transition matrices has contributions other than the quantum dimension.

The rest of this paper is organized as follows. In Sec. II, we briefly review the replica method for locally excited states in 2D CFTs and provide our convention and some useful formulas for the later calculations. In Sec. III, we mainly focus on the late-time behavior of the second pseudo-Rényi entropy of locally descendant excited states. For simplicity, we study the cases in that a single holomorphic Virasoro generator generates the descendants. More general and complicated situations are discussed in Sec. IV, where we derive the late-time behavior of the kth pseudo-Rényi entropy for the generic descendant states. We end with conclusions and prospects in Sec. V. Some calculation details are presented in the appendixes.

II. SETUP IN 2D CFTs

A. Replica method with local operators

Our focus is on the pseudo-Rényi entropy of locally excited states created by acting with the operator V_α (6) on the ground state in RCFTs, and the subsystem A under consideration in this paper is always taken to be the interval $[0,\infty)$. In this scenario, the pseudo-Rényi entropy can be formulated in the path integral formalism using the replica method. Given an RCFT that lives on a plane and has a vacuum state $|\Omega\rangle$, we first prepare two locally excited states using V_α to construct a real-time evolved transition matrix $\mathcal{T}^{1|2}(t)$,

$$|\psi_1\rangle \equiv e^{-\epsilon H} V_\alpha(x_1)|\Omega\rangle, \qquad |\psi_2\rangle \equiv e^{-\epsilon H} V_\beta(x_2)|\Omega\rangle,$$
$$\mathcal{T}^{1|2}(t) \equiv e^{-iHt}\frac{|\psi_1\rangle\langle\psi_2|}{\langle\psi_2|\psi_1\rangle}e^{iHt}. \tag{7}$$

Notice that an infinitesimally small parameter ϵ has been introduced to suppress the high-energy modes [44]. We can obtain the reduced transition matrix of subsystem A at time t by tracing out the degrees of freedom of A^c (the complement of A), $\mathcal{T}_A^{1|2}(t) = \mathrm{tr}_{A^c}[\mathcal{T}^{1|2}(t)]$. It turns out that the excess of the nth pseudo-Rényi entropy of A with respect to the ground state, defined as $\Delta S^{(n)}(\mathcal{T}_A^{1|2}(t)) \coloneqq S^{(n)}(\mathcal{T}_A^{1|2}(t)) - S^{(n)}(\mathrm{tr}_{A^c}[|\Omega\rangle\langle\Omega|])$, is of the form [30]

$$\Delta S^{(n)}(\mathcal{T}_A^{1|2}(t)) = \frac{1}{1-n}\left[\log\left\langle\prod_{k=1}^{n} V_\alpha(w_{2k-1},\bar{w}_{2k-1})V_\beta^\dagger(w_{2k},\bar{w}_{2k})\right\rangle_{\Sigma_n} - n\log\langle V_\alpha(w_1,\bar{w}_1)V_\beta^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1}\right] \tag{8}$$

using the replica method. In (8), Σ_n denotes an n-sheeted Riemann surface with cuts on each copy corresponding to A, and $(w_{2k-1},\bar{w}_{2k-1})$ and $(w_{2k},\bar{w}_{2k})$ are coordinates on the kth-sheet surface. The first term in Eq. (8) is given by a 2n-point correlation function on Σ_n, while a two-point function on Σ_1 gives the second term. We have

$$w_{2k-1} = x_1 + t - i\epsilon, \qquad w_{2k} = x_2 + t + i\epsilon, \qquad \bar{w}_{2k-1} = x_1 - t + i\epsilon, \qquad \bar{w}_{2k} = x_2 - t - i\epsilon, \qquad (k = 1,2,\ldots,n). \tag{9}$$

B. Convention and useful formulas

The $2n$-point correlation function on Σ_n in Eq. (8) can be evaluated with the help of a conformal mapping of Σ_n to the complex plane Σ_1. We can then map Σ_n to Σ_1 using the simple conformal mapping

$$w = z^n. \tag{10}$$

Let us first focus on the case of $n = 2$. The calculation of $\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t))$ is related to the four-point function known quite well for exactly solvable CFTs. In our convention, using Eq. (10), the four points z_1, z_2, z_3, z_4 in the complex plane are given by

$$z_1 = -z_3 = i\sqrt{-x_1 - t + i\epsilon}, \quad \bar{z}_1 = -\bar{z}_3 = -i\sqrt{-x_1 + t - i\epsilon},$$
$$z_2 = -z_4 = i\sqrt{-x_2 - t - i\epsilon}, \quad \bar{z}_2 = -\bar{z}_4 = -i\sqrt{-x_2 + t + i\epsilon}. \tag{11}$$

The key point is that one should treat $t \pm i\epsilon$ as a pure imaginary number in all algebraic calculations and take t to be real only in the final expression of the pseudo-Rényi entropy. To evaluate the four-point correlation function, it is useful to focus on the cross ratios [30]

$$\eta := \frac{z_{12}z_{34}}{z_{13}z_{24}} = \frac{(x_1+x_2+2t)+2\sqrt{(x_1+t)(x_2+t)+\epsilon^2+i\epsilon(x_1-x_2)}}{4\sqrt{(x_1+t)(x_2+t)+\epsilon^2+i\epsilon(x_1-x_2)}},$$
$$\bar{\eta} := \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}} = \frac{(x_1+x_2-2t)+2\sqrt{(x_1-t)(x_2-t)+\epsilon^2-i\epsilon(x_1-x_2)}}{4\sqrt{(x_1-t)(x_2-t)+\epsilon^2-i\epsilon(x_1-x_2)}}, \tag{12}$$

where $z_{ij} = z_i - z_j$, and a useful relation is

$$1-\eta = \frac{z_{14}z_{23}}{z_{13}z_{24}}. \tag{13}$$

Since we are mainly interested in the late-time ($t \to \infty$) behavior of pseudo-Rényi entropy, one can find some useful late-time formulas from (11)

$$\lim_{t\to\infty} z_1 \sim \lim_{t\to\infty} z_4 \sim -\sqrt{t}, \qquad \lim_{t\to\infty} z_2 \sim \lim_{t\to\infty} z_3 \sim \sqrt{t},$$
$$\lim_{t\to\infty} z_{12} \sim \lim_{t\to\infty} z_{13} \sim -\sqrt{t}, \qquad \lim_{t\to\infty} z_{24} \sim \lim_{t\to\infty} z_{34} \sim \sqrt{t},$$
$$\lim_{t\to\infty} z_{14} \sim \lim_{t\to\infty} z_{23} \sim \sqrt{\frac{1}{t}}. \tag{14}$$

For the cross ratios $(\eta, \bar{\eta})$, as shown in [30], we have

$$\lim_{t\to\infty}(\eta,\bar{\eta}) = \left(1+\frac{(x_2-x_1+2i\epsilon)^2}{16t^2}, -\frac{(x_2-x_1-2i\epsilon)^2}{16t^2}\right) \simeq (1,0),$$
$$\partial_i\eta \sim \frac{1}{t^{\frac{3}{2}}}, \qquad \partial_i\partial_j\eta \sim \frac{1}{t}, \qquad \partial_i\partial_j\partial_k\eta \sim \frac{1}{t^{\frac{5}{2}}}, \qquad \partial_i\partial_j\partial_k\partial_l\eta \sim \frac{1}{t^2}(i \neq j \neq k \neq l). \tag{15}$$

For general nth pseudo-Rényi entropy, the $2n$ points $z_1, z_2, \ldots, z_{2n}$ in the z coordinates are given by

$$z_{2k+1} = \mathrm{e}^{2\pi i\frac{k+1/2}{n}}(-x_1-t+i\epsilon)^{\frac{1}{n}}, \qquad \bar{z}_{2k+1} = \mathrm{e}^{-2\pi i\frac{k+1/2}{n}}(-x_1+t-i\epsilon)^{\frac{1}{n}},$$
$$z_{2k+2} = \mathrm{e}^{2\pi i\frac{k+1/2}{n}}(-x_2-t-i\epsilon)^{\frac{1}{n}}, \qquad \bar{z}_{2k+2} = \mathrm{e}^{-2\pi i\frac{k+1/2}{n}}(-x_2+t+i\epsilon)^{\frac{1}{n}}, \qquad (k=0,\ldots,n-1). \tag{16}$$

III. SECOND PSEUDO-RÉNYI ENTROPY $\Delta S_A^{(2)}$ FOR DESCENDANT OPERATORS

In RCFTs, it is known that the excess of the Rényi entropy for the primary/descendant operator saturates to a constant equal to the logarithm of the quantum dimension of the inserted primary operator [51,56,57]. To study the entanglement entropy of local operators, one needs to use two identical operators with the same spatial coordinates to generate the density matrix. However, as mentioned in the Introduction, pseudoentropy provides us with greater flexibility; we can use descendant operators of different levels and with different spatial coordinates to construct the transition matrix. This section will explore the second pseudo-Rényi entropy for some specific descendant operators.

A. $\Delta S_A^{(2)}$ for $V_\alpha = L_{-1}\mathcal{O}$, $V_\beta = \mathcal{O}$

Let us initially examine the simplest scenario that deviates from the previous studies [30]: $V_\alpha(x_1) = L_{-1}\mathcal{O}(x_1), V_\beta(x_2) = \mathcal{O}(x_2)$. The second pseudo-Rényi entropy, according to (8), is related to a four-point function on Σ_2,

$$e^{-\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t))} = \frac{\langle L_{-1}\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)L_{-1}\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2}}{\langle L_{-1}\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1}^2}. \tag{17}$$

For the first descendant operators, the transformation law under the conformal mapping $w = z^2$ is given by

$$\partial\mathcal{O}(w_i,\bar{w}_i)=(w_i')^{-\Delta}(\bar{w}_i')^{-\Delta}\left((w_i')^{-1}\partial\mathcal{O}(z_i,\bar{z}_i)-\Delta\frac{w_i''}{(w_i')^2}\mathcal{O}(z_i,\bar{z}_i)\right), \tag{18}$$

where the prime denotes the derivative with respect to z or $\bar{z}$. Then the four-point function in (17) can be written in terms of correlators on the plane as

$$\begin{aligned}&\langle L_{-1}\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)L_{-1}\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2}\\&=\left(\prod_{i=1}^4|w_i'|^{-2\Delta}\right)\cdot\left(\frac{\partial z_1}{\partial w_1}\frac{\partial z_3}{\partial w_3}\langle\partial\mathcal{O}(1)\mathcal{O}^\dagger(2)\partial\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}+\Delta^2\left(\frac{\partial z_1}{\partial w_1}\right)^2\frac{\partial^2w_1}{\partial z_1^2}\left(\frac{\partial z_3}{\partial w_3}\right)^2\frac{\partial^2w_3}{\partial z_3^2}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}\right.\\&\quad\left.-\Delta\frac{\partial z_1}{\partial w_1}\left(\frac{\partial z_3}{\partial w_3}\right)^2\frac{\partial^2w_3}{\partial z_3^2}\langle\partial\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}-\Delta\frac{\partial z_3}{\partial w_3}\left(\frac{\partial z_1}{\partial w_1}\right)^2\frac{\partial^2w_1}{\partial z_1^2}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\partial\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}\right),\end{aligned} \tag{19}$$

where we use the notation $\mathcal{O}(i)\equiv\mathcal{O}(z_i,\bar{z}_i)$. Because of the conformal symmetry, we can express the four-point functions involved in (19) as follows:

$$\begin{aligned}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}&=|z_{13}z_{24}|^{-4\Delta}G(\eta,\bar{\eta}),\\\langle\partial\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}&=|z_{13}z_{24}|^{-4\Delta}\partial_{z_1}G(\eta,\bar{\eta})-\frac{2\Delta}{z_{13}}|z_{13}z_{24}|^{-4\Delta}G(\eta,\bar{\eta}),\\\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\partial\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}&=|z_{13}z_{24}|^{-4\Delta}\partial_{z_3}G(\eta,\bar{\eta})+\frac{2\Delta}{z_{13}}|z_{13}z_{24}|^{-4\Delta}G(\eta,\bar{\eta}),\\\langle\partial\mathcal{O}(1)\mathcal{O}^\dagger(2)\partial\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}&=|z_{13}z_{24}|^{-4\Delta}\partial_{z_1}\partial_{z_3}G(\eta,\bar{\eta})+\frac{2\Delta}{z_{13}}|z_{13}z_{24}|^{-4\Delta}(\partial_{z_1}-\partial_{z_3})G(\eta,\bar{\eta})\\&\quad+\frac{-2\Delta(2\Delta+1)}{z_{13}^2}|z_{13}z_{24}|^{-4\Delta}G(\eta,\bar{\eta}),\end{aligned} \tag{20}$$

where

$$G(\eta,\bar{\eta})\coloneqq\lim_{z\to\infty}|z|^{4\Delta}\langle\mathcal{O}(z,\bar{z})\mathcal{O}(1,1)\mathcal{O}(\eta,\bar{\eta})\mathcal{O}(0,0)\rangle_{\Sigma_1}. \tag{21}$$

Under the conformal mapping between Σ_2 and Σ_1, we have

$$\begin{aligned}&\langle L_{-1}\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)L_{-1}\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2}\\&=2^{-8\Delta}|z_1z_2z_3z_4|^{-2\Delta}|z_{13}z_{24}|^{-4\Delta}\cdot\left\{\frac{1}{4z_1z_3}\left[\partial_{z_1}\partial_{z_3}+\frac{2\Delta}{z_{13}}(\partial_{z_1}-\partial_{z_3})-\frac{2\Delta(2\Delta+1)}{z_{13}^2}\right]G(\eta,\bar{\eta})\right.\\&\quad\left.+\frac{\Delta^2}{4z_1^2z_3^2}G(\eta,\bar{\eta})-\frac{\Delta}{4z_1z_3^2}\left[\partial_{z_1}-\frac{2\Delta}{z_{13}}\right]G(\eta,\bar{\eta})-\frac{\Delta}{4z_1^2z_3}\left[\partial_{z_3}+\frac{2\Delta}{z_{13}}\right]G(\eta,\bar{\eta})\right\}.\end{aligned} \tag{22}$$

At late times ($t\to\infty$), as shown in [30], η and $\bar{\eta}$ approach 1 and 0, respectively, which leads to the following late-time behavior of $G(\eta,\bar{\eta})$ for RCFTs:

$$\lim_{t\to\infty}G(\eta,\bar{\eta})\sim d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta}, \tag{23}$$

where $d_{\mathcal{O}}$ is so called the quantum dimension, and by using modular S matrix $\mathcal{S}_{ab}$ this is given by $d_{\mathcal{O}_a}=\mathcal{S}_{0a}/\mathcal{S}_{00}$ [75,77]. Hence, we can obtain

$$\begin{aligned}&\lim_{t\to\infty}\partial_{z_1}G(\eta,\bar{\eta})\sim\frac{2\Delta\partial_{z_1}\eta}{1-\eta}d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta},\qquad\lim_{t\to\infty}\partial_{z_3}G(\eta,\bar{\eta})\sim\frac{2\Delta\partial_{z_3}\eta}{1-\eta}d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta},\\&\lim_{t\to\infty}\partial_{z_1}\partial_{z_3}G(\eta,\bar{\eta})\sim\frac{2\Delta\partial_{z_1}\partial_{z_3}\eta}{1-\eta}d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta}+\frac{2\Delta(2\Delta+1)\partial_{z_1}\eta\partial_{z_3}\eta}{(1-\eta)^2}d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta}.\end{aligned} \tag{24}$$

On the other hand, the two-point function in (17) is

$$\langle L_{-1}\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1} = \partial_{w_1}\frac{1}{|w_{12}|^{4\Delta}} = \frac{-2\Delta}{w_{12}}\cdot\frac{1}{|w_{12}|^{4\Delta}}. \tag{25}$$

Substituting (22), (24), and (25) into (17) and setting $z_3 = -z_1$, $z_4 = -z_2$, we obtain, at late times,

$$\begin{aligned} e^{-\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t))} &\sim \frac{w_{12}^2}{4\Delta^2}\eta^{2\Delta}(1-\bar{\eta})^{2\Delta}\left\{\frac{-1}{4z_1^2}\left[\frac{2\Delta\frac{z_1^2+z_2^2}{8z_1^3z_2}}{(1-\eta)d_{\mathcal{O}}} - \frac{2\Delta(2\Delta+1)\frac{(z_1^2-z_2^2)^2}{64z_1^4z_2^2}}{(1-\eta)^2 d_{\mathcal{O}}} + \frac{2\Delta^2\frac{z_2^2-z_1^2}{4z_1^2z_2}}{z_1(1-\eta)d_{\mathcal{O}}} - \frac{\Delta(2\Delta+1)}{2z_1^2 d_{\mathcal{O}}}\right]\right. \\ &\quad\left. + \frac{\Delta^2}{4z_1^4 d_{\mathcal{O}}} - \frac{\Delta}{4z_1^3}\left[\frac{2\Delta\frac{z_2^2-z_1^2}{8z_1^2z_2}}{(1-\eta)d_{\mathcal{O}}} - \frac{\Delta}{z_1 d_{\mathcal{O}}}\right] + \frac{\Delta}{4z_1^3}\left[\frac{2\Delta\frac{z_1^2-z_2^2}{8z_1^2z_2}}{(1-\eta)d_{\mathcal{O}}} + \frac{\Delta}{z_1 d_{\mathcal{O}}}\right]\right\} \\ &\sim d_{\mathcal{O}}^{-1}. \end{aligned} \tag{26}$$

In going from the second to the third line, we use Eq. (11) and perform the Laurent expansion at infinity. The late-time limit of the second pseudo-Rényi entropy is thus given by

$$\lim_{t\to\infty}\Delta S^{(2)}(\mathcal{T}^{1|2}(t)) = \log d_{\mathcal{O}}. \tag{27}$$

In this simplest case, the late-time behavior of the second pseudo-Rényi entropy of $L_{-1}\mathcal{O}$ with $\mathcal{O}$ is the same as that of the primary operator $\mathcal{O}$. Note that the four-point functions in the plane in Eq. (19) are also encountered when studying the entanglement entropy of $L_{-1}\mathcal{O}$ [57]. However, they are discarded as subleading terms. Our finding shows that these subleading correlators can also reproduce the result of $\log d_{\mathcal{O}}$, as long as we consider the pseudo-Rényi entropy instead of the Rényi entropy.

B. $\Delta S_A^{(2)}$ for $V_\alpha = L_{-n}\mathcal{O}$, $V_\beta = \mathcal{O}$

We next consider a more complicated case in which V_α is a general n-level descendant associated with the Virasoro generator L_{-n}, and V_β is still a primary. The two-point function of V_α and V_β reads [78]

$$\langle L_{-n}\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}(w_2,\bar{w}_2)\rangle_{\Sigma_1} = \frac{(n+1)\Delta}{w_{21}^n}|w_{12}|^{-4\Delta}. \tag{28}$$

We then compute the four-point function on Σ_2. Under the conformal transformation, the n-level descendant transforms as

$$L_{-n}O(w_i,\bar{w}_i) = (w_i')^{-(\Delta+n)}(\bar{w}_i')^{-\Delta}L_{-n}\mathcal{O}(z_i,\bar{z}_i) + \ldots \tag{29}$$

The ellipsis stands for operators with lower conformal dimensions contributing to lower-order singularities in the correlation functions; that is, we have

$$\langle\mathcal{O}^{(-n)}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}^{(-n)}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2} \sim \left(\prod_{i=1}^4|w_i'|^{-2\Delta}\right)(w_1')^{-n}(w_3')^{-n}\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} \tag{30}$$

at late times. We next pick out the most singular terms of the four-point function on the z plane in (30). According to (14) and (A1) in Appendix A, the leading contribution at late times in $\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1}$ should be

$$\begin{aligned} \langle\mathcal{O}^{(-n)}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} &= \frac{(n-1)\Delta}{z_{41}^n}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} + \frac{-\partial_{z_4}}{z_{41}^{n-1}}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} + \cdots \\ &= \left(\frac{(n-1)\Delta}{z_{41}^n} - \frac{\partial_{z_4}}{z_{41}^{n-1}}\right)\left(\frac{(n-1)\Delta}{z_{23}^n} - \frac{\partial_{z_2}}{z_{23}^{n-1}}\right)\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\mathcal{O}(3)\mathcal{O}^\dagger(4)\rangle_{\Sigma_1} + \ldots \\ &= |z_{13}z_{24}|^{-4\Delta}d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta}\left(\frac{(n-1)^2\Delta^2}{z_{41}^n z_{23}^n} - \frac{(n-1)\Delta}{z_{41}^n z_{23}^{n-1}}\cdot\frac{2\Delta\partial_{z_2}\eta}{1-\eta} - \frac{(n-1)\Delta}{z_{41}^{n-1}z_{23}^n}\cdot\frac{2\Delta\partial_{z_4}\eta}{1-\eta}\right. \\ &\quad\left. + \frac{1}{z_{41}^{n-1}z_{23}^{n-1}}\cdot\left(\frac{2\Delta(2\Delta+1)\partial_{z_2}\eta\cdot\partial_{z_4}\eta}{(1-\eta)^2} + \frac{2\Delta\partial_{z_2}\partial_{z_4}\eta}{1-\eta}\right)\right) + \ldots. \end{aligned} \tag{31}$$

Again, the ellipsis represents the terms that give rise to lower-order singularities in the correlation functions.

Combining (28)–(31) and taking the limit $t \to \infty$, the leading-order behavior of $\exp\{-\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t))\}$ is given by

$$\begin{aligned}\lim_{t\to\infty} e^{-\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t))} &= \frac{w_{12}^{2n}}{(n+1)^2\Delta^2} \times \frac{1}{4^n z_1^n z_3^n} d_{\mathcal{O}}^{-1}\Bigg(\frac{(n-1)^2\Delta^2}{z_{41}^n z_{23}^n} - \frac{(n-1)\Delta}{z_{41}^n z_{23}^{n-1}}\cdot\frac{2\Delta\partial_{z_2}\eta}{1-\eta} - \frac{(n-1)\Delta}{z_{41}^{n-1} z_{23}^n}\cdot\frac{2\Delta\partial_{z_4}\eta}{1-\eta} \\ &\quad + \frac{1}{z_{41}^{n-1} z_{23}^{n-1}}\cdot\left(\frac{2\Delta(2\Delta+1)\partial_{z_2}\eta\cdot\partial_{z_4}\eta}{(1-\eta)^2} + \frac{2\Delta\partial_{z_2}\partial_{z_4}\eta}{1-\eta}\right)\Bigg) + \ldots \\ &= \frac{1}{d_{\mathcal{O}}} + \ldots . \end{aligned} \tag{32}$$

The ellipsis here denotes the subleading terms that vanish as t goes to infinity. Hence, the late-time limit of the second pseudo-Rényi entropy of the transition matrix constructed by a primary $\mathcal{O}$ and its n-level descendant $L_{-n}\mathcal{O}$ is still $\log d_{\mathcal{O}}$.

C. $\Delta S_A^{(2)}$ for $V_\alpha = L_{-n}\mathcal{O}$, $V_\beta = L_{-m}\mathcal{O}$

In this subsection, we use the conformal block and operator product expansion (OPE) to show that the phenomenon discovered in previous subsections is true for a general case: $V_\alpha = L_{-n}\mathcal{O}$, $V_\beta = L_{-m}\mathcal{O}$.

In terms of [78], the two-point function of V_α and V_β reads [79]

$$\begin{aligned}\langle L_{-n}\mathcal{O}(w_1,\bar{w}_1)L_{-m}\mathcal{O}(w_2,\bar{w}_2)\rangle_{\Sigma_1} &= \frac{1}{12}(-1)^n(w_1-w_2)^{-m-n}\frac{1}{|w_{12}|^{4\Delta}} \\ &\quad\times\left(\frac{\Gamma(m+n)(cm(m^2-1)n(n^2-1)+24\Delta(m+n)(m+n+1)(mn-1))}{\Gamma(m+2)\Gamma(n+2)}\right. \\ &\quad\left. + 12\Delta(\Delta(m+1)(n+1)+2)\right). \end{aligned} \tag{33}$$

The late-time behavior of the four-point function on Σ_2 of (8) can be derived according to (29)

$$\begin{aligned}&\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_2} \\ &\sim\left(\prod_{i=1}^4|w_i'|^{-2\Delta}\right)(w_1')^{-n}(w_2')^{-m}(w_3')^{-n}(w_4')^{-m}\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}. \end{aligned} \tag{34}$$

We can next pick out the most singular terms of the four-point function on the z plane in (34). According to (14), the leading contribution at late times in $\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}$ comes from the OPE of $\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(4)$ and $\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)$, and its complete result is given by (B8) in Appendix B.

Combining (33) and (B8) and taking the limit $t \to \infty$, the leading-order behavior of $e^{-\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t))}$ is

$$\lim_{t\to\infty} e^{-\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t))} = \frac{1}{d_{\mathcal{O}}} + \ldots . \tag{35}$$

Again, the ellipsis denotes the subleading terms that vanish as $t \to \infty$. The late-time limit of the second pseudo-Rényi entropy of the transition matrix constructed by an m-level descendant operator $L_{-m}\mathcal{O}$ and an n-level descendant operator $L_{-n}\mathcal{O}$ is $\log d_{\mathcal{O}}$, being consistent with the studies in previous sections.

IV. *k*TH PSEUDO-RÉNYI ENTROPY FOR GENERIC DESCENDANT STATES

In the previous section, we found that the second pseudo-Rényi entropy corresponding to $L_{-n}\mathcal{O}$ and $L_{-m}\mathcal{O}$ is the same as the second pseudo-Rényi entropy of the corresponding primary operator $\mathcal{O}$ at late times, i.e., the logarithm of the quantum dimension of the primary operator $\mathcal{O}$. In this section, we shall investigate the kth pseudo-Rényi entropy for general descendant states and take $k \to 1$ to obtain the corresponding pseudoentropy.

A. $\Delta S_A^{(k)}$ for $V_\alpha = L_{-n}\mathcal{O}$, $V_\beta = L_{-m}\mathcal{O}$

We begin with studying the case discussed above: $V_\alpha = L_{-n}\mathcal{O}$, $V_\beta = L_{-m}\mathcal{O}$. According to (29), the $2k$-point function on Σ_k at late times can be reformulated as the $2k$-point function on Σ_1 as follows:

$$\begin{aligned}&\langle \mathcal{O}^{(-n)}(w_1,\bar{w}_1)\mathcal{O}^{(-m)\dagger}(w_2,\bar{w}_2)\ldots\mathcal{O}^{(-n)}(w_{2k-1},\bar{w}_{2k-1})\mathcal{O}^{(-m)\dagger}(w_{2k},\bar{w}_{2k})\rangle_{\Sigma_k},\\ &\sim \mathcal{F}(w_1,w_2,\ldots,w_{2k},m,n,\Delta)\langle \mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\ldots\mathcal{O}^{(-n)}(2k-1)\mathcal{O}^{(-m)\dagger}(2k)\rangle_{\Sigma_1}+\ldots,\end{aligned} \tag{36}$$

where

$$\mathcal{F}(w_1,w_2,\ldots,w_{2k},m,n,\Delta)=\left(\prod_{i=1}^{2k}|w_i'|^{-2\Delta}\right)(w_1')^{-n}(w_2')^{-m}\ldots(w_{2k-1}')^{-n}(w_{2k}')^{-m} \tag{37}$$

is the leading factor coming from the conformal transformation between correlation functions on Σ_k and correlation functions on Σ_1, and the ellipsis denotes terms contributing to lower-order singularity in the correlation functions.

Based on (16), it can be found that during the late time, $2k$ holomorphic coordinates and $2k$ antiholomorphic coordinates approach each other in distinct pairings [30],

$$\begin{aligned}&\lim_{t\to\infty}(z_{2j+1}-z_{2j+4})\sim e^{2\pi i\frac{j+1}{k}}\frac{w_1-w_2}{kt^{1-\frac{1}{k}}}\sim 0,\\ &\lim_{t\to\infty}(\bar{z}_{2j+1}-\bar{z}_{2j+2})\sim e^{-2\pi i\frac{j+\frac{1}{2}}{k}}\frac{\bar{w}_2-\bar{w}_1}{kt^{1-\frac{1}{k}}}\sim 0,\qquad (j=0,1,\ldots,k-1;z_{2k+2}\equiv z_2).\end{aligned} \tag{38}$$

Hence, at late times, the most divergent part of the $2k$-point correlation function on the plane in (36) arises from the OPE of $\mathcal{O}^{(-n)}(2j+1)\mathcal{O}^{(-m)\dagger}(2j+4)$, i.e.,

$$\langle \mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\ldots\mathcal{O}^{(-n)}(2k-1)\mathcal{O}^{(-m)\dagger}(2k)\rangle_{\Sigma_1}\sim\mathcal{D}_{1,4}\mathcal{D}_{3,6}\ldots\mathcal{D}_{2k-3,2k}\mathcal{D}_{2k-1,2}\langle\mathcal{O}(1)\mathcal{O}^\dagger(2)\ldots\mathcal{O}(2k-1)\mathcal{O}^\dagger(2k)\rangle_{\Sigma_1}, \tag{39}$$

where $\mathcal{D}_{2i+1,2i+4}$ is a derivative operator that only contains constants related to the information of two descendant operators and derivatives coming from the most singular part of the OPE of $\mathcal{O}^{(-n)}(2i+1)\mathcal{O}^{(-m)\dagger}(2i+4)$, i.e., $\mathcal{D}_{2i+1,2i+4}=\mathcal{D}(\partial_{2i+1},\partial_{2i+4};m,n,c,\Delta)$. See Appendix B for a concrete example of the $\mathcal{D}$ operator. We need to pick up the proper channel to expand the $2k$-point function into the holomorphic and the antiholomorphic part, as graphically shown in Fig. 1. In each channel, only the identity operator contributes to the final result. Hence, the $2k$-point function breaks up into k two-point functions for the holomorphic part (and k for the antiholomorphic part),

$$\begin{aligned}&\langle \mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\ldots\mathcal{O}^{(-n)}(2k-1)\mathcal{O}^{(-m)\dagger}(2k)\rangle_{\Sigma_1}\\ &\quad\sim (F_{00}[\mathcal{O}])^{k-1}\mathcal{D}_{1,4}\ldots\mathcal{D}_{2k-3,2k}\mathcal{D}_{2k-1,2}\langle\mathcal{O}(z_1)\mathcal{O}^\dagger(z_4)\rangle_{\Sigma_1}\ldots\langle\mathcal{O}(z_{2k-3})\mathcal{O}^\dagger(z_{2k})\rangle_{\Sigma_1}\langle\mathcal{O}(z_{2k-1})\mathcal{O}^\dagger(z_2)\rangle_{\Sigma_1}\\ &\qquad\times\langle\mathcal{O}(\bar{z}_1)\mathcal{O}^\dagger(\bar{z}_2)\rangle_{\Sigma_1}\ldots\langle\mathcal{O}(\bar{z}_{2k-3})\mathcal{O}^\dagger(\bar{z}_{2k-2})\rangle_{\Sigma_1}\langle\mathcal{O}(\bar{z}_{2k-1})\mathcal{O}^\dagger(\bar{z}_{2k})\rangle_{\Sigma_1}\\ &\quad\sim (F_{00}[\mathcal{O}])^{k-1}\langle\mathcal{O}^{(-n)}(z_1)\mathcal{O}^{(-m)\dagger}(z_4)\rangle_{\Sigma_1}\ldots\langle\mathcal{O}^{(-n)}(z_{2k-3})\mathcal{O}^{(-m)\dagger}(z_{2k})\rangle_{\Sigma_1}\langle\mathcal{O}^{(-n)}(z_{2k-1})\mathcal{O}^{(-m)\dagger}(z_2)\rangle_{\Sigma_1}\\ &\qquad\times\langle\mathcal{O}(\bar{z}_1)\mathcal{O}^\dagger(\bar{z}_2)\rangle_{\Sigma_1}\ldots\langle\mathcal{O}(\bar{z}_{2k-3})\mathcal{O}^\dagger(\bar{z}_{2k-2})\rangle_{\Sigma_1}\langle\mathcal{O}(\bar{z}_{2k-1})\mathcal{O}^\dagger(\bar{z}_{2k})\rangle_{\Sigma_1},\end{aligned} \tag{40}$$

where we formally decompose the operator $\mathcal{O}(z,\bar{z})$ into a product of a holomorphic operator $\mathcal{O}(z)$ and an antiholomorphic operator $\mathcal{O}(\bar{z})$, in the sense of the two-point function $\langle\mathcal{O}(1)\mathcal{O}(2)\rangle = z_{12}^{-2\Delta}\bar{z}_{12}^{-2\Delta} = \langle\mathcal{O}(z_1)\mathcal{O}(z_2)\rangle\langle\mathcal{O}(\bar{z}_1)\mathcal{O}(\bar{z}_2)\rangle$. In the last line, the fact that $\mathcal{D}_{2i+1,2i+4}$ is a linear operator, and coordinates z_i and z_j are independent for $i\neq j$, has been applied.

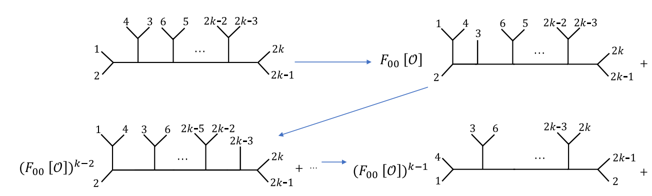

FIG. 1. $k-1$ fusion transformations to obtain $\Delta S_A^{(k)}$.

Changing back into the w coordinate, with the leading divergent term being transformed homogeneously and keeping the most divergent term, we find that

$$\begin{aligned}&\langle\mathcal{O}^{(-n)}(w_1,\bar{w}_1)\mathcal{O}^{(-m)\dagger}(w_2,\bar{w}_2)\ldots\mathcal{O}^{(-n)}(w_{2k-1,\bar{w}_{2k-1}})\mathcal{O}^{(-m)\dagger}(w_{2k},\bar{w}_{2k})\rangle_{\Sigma_k}\sim(F_{00}[\mathcal{O}])^{k-1}\mathcal{F}(w_1,w_2,\ldots,w_{2k},m,n,\Delta)\\&\langle\mathcal{O}^{(-n)}(z_1)\mathcal{O}^{(-m)\dagger}(z_4)\rangle_{\Sigma_1}\ldots\langle\mathcal{O}^{(-n)}(z_{2k-3})\mathcal{O}^{(-m)\dagger}(z_{2k})\rangle_{\Sigma_1}\langle\mathcal{O}^{(-n)}(z_{2k-1})\mathcal{O}^{(-m)\dagger}(z_2)\rangle_{\Sigma_1}\\&\quad\times\langle\mathcal{O}(\bar{z}_1)\mathcal{O}^{\dagger}(\bar{z}_2)\rangle_{\Sigma_1}\ldots\langle\mathcal{O}(\bar{z}_{2k-3})\mathcal{O}^{\dagger}(\bar{z}_{2k-2})\rangle_{\Sigma_1}\langle\mathcal{O}(\bar{z}_{2k-1})\mathcal{O}^{\dagger}(\bar{z}_{2k})\rangle_{\Sigma_1}\\&\sim(F_{00}[\mathcal{O}])^{k-1}\langle\mathcal{O}^{(-n)}(w_1)\mathcal{O}^{(-m)\dagger}(w_4)\rangle_{\Sigma_k}\ldots\langle\mathcal{O}^{(-n)}(w_{2k-3})\mathcal{O}^{(-m)\dagger}(w_{2k})\rangle_{\Sigma_k}\langle\mathcal{O}^{(-n)}(w_{2k-1})\mathcal{O}^{(-m)\dagger}(w_2)\rangle_{\Sigma_k}\\&\quad\times\langle\mathcal{O}(\bar{w}_1)\mathcal{O}^{\dagger}(\bar{w}_2)\rangle_{\Sigma_k}\ldots\langle\mathcal{O}(\bar{w}_{2k-3})\mathcal{O}^{\dagger}(\bar{w}_{2k-2})\rangle_{\Sigma_k}\langle\mathcal{O}(\bar{w}_{2k-1})\mathcal{O}^{\dagger}(\bar{w}_{2k})\rangle_{\Sigma_k}.\end{aligned}\tag{41}$$

By utilizing Eqs. (16), (29), and (38), we find that in the late-time limit, the correlation functions of both holomorphic and antiholomorphic two-point functions on Σ_k are equal to those on Σ_1, up to a unitary factor,

$$\begin{aligned}&\langle\mathcal{O}^{(-n)}(w_{2j+1})\mathcal{O}^{(-m)\dagger}(w_{2j+4})\rangle_{\Sigma_k}\sim e^{-2\pi i(1+j)(2\Delta+m+n)}\langle\mathcal{O}^{(-n)}(w_1)\mathcal{O}^{(-m)\dagger}(w_2)\rangle_{\Sigma_1},\\&\qquad\langle\mathcal{O}(\bar{w}_{2j+1})\mathcal{O}^{\dagger}(\bar{w}_{2j+2})\rangle_{\Sigma_k}\sim e^{2\pi i(1+j)2\Delta}\langle\mathcal{O}(\bar{w}_1)\mathcal{O}^{\dagger}(\bar{w}_2)\rangle_{\Sigma_1},\qquad(j=1,2,\ldots,k-1;w_{2k+2}\equiv w_2).\end{aligned}\tag{42}$$

Substituting (42) into (41), the $2k$-point function on Σ_k is reduced to

$$\langle\mathcal{O}^{(-n)}(w_1,\bar{w}_1)\mathcal{O}^{(-m)\dagger}(w_2,\bar{w}_2)\ldots\mathcal{O}^{(-n)}(w_{2k-1,\bar{w}_{2k-1}})\mathcal{O}^{(-m)\dagger}(w_{2k},\bar{w}_{2k})\rangle_{\Sigma_k}\sim d_{\mathcal{O}}^{1-k}\langle\mathcal{O}^{(-n)}(w_1,\bar{w}_1)\mathcal{O}^{(-m)\dagger}(w_2,\bar{w}_2)\rangle_{\Sigma_1}^{k},\tag{43}$$

where we use the relation between the quantum dimension and the fusion matrix: $d_{\mathcal{O}}=1/F_{00}[\mathcal{O}]$.

Finally, in accordance with Eq. (43), the excess of the kth pseudo-Rényi entropy of $L_{-n}\mathcal{O}$ and $L_{-m}\mathcal{O}$ at late times can be deduced as equal to

$$\begin{aligned}\lim_{t\to\infty}\Delta S^{(k)}(\mathcal{T}_A^{1|2}(t))&=\lim_{t\to\infty}\frac{1}{1-k}\log\frac{\langle\mathcal{O}^{(-n)}(w_1,\bar{w}_1)\ldots\mathcal{O}^{(-m)\dagger}(w_{2k},\bar{w}_{2k})\rangle_{\Sigma_k}}{\langle\mathcal{O}^{(-n)}(w_1,\bar{w}_1)\mathcal{O}^{(-m)\dagger}(w_2,\bar{w}_2)\rangle_{\Sigma_1}^{k}}\\&=\log d_{\mathcal{O}},\end{aligned}\tag{44}$$

which is independent of the level k and consistent with the results of the second pseudo-Rényi entropy in the previous sections. Based on the above results, we can conclude that the late-time excess of the pseudoentropy of $L_{-n}\mathcal{O}$ and $L_{-m}\mathcal{O}$ is consistent with the entanglement entropy of $L_{-n}\mathcal{O}$ and also equals $\log d_{\mathcal{O}}$.

B. $\Delta S_A^{(k)}$ for linear combination of descendant operators

Let us consider two linear combination operators constructed by operators in $\mathcal{O}$'s conformal family,

$$\begin{aligned}V_\alpha(w,\bar{w})&=\sum_{i=1}^{M}C_iV_i(w,\bar{w}),\qquad V_i(w,\bar{w})=L_{-\{K_i\}}\bar{L}_{-\{\bar{K}_i\}}\mathcal{O}(w,\bar{w}),\\V_\beta(w,\bar{w})&=\sum_{j=1}^{M'}C'_jV'_j(w,\bar{w}),\qquad V'_j(w,\bar{w})=L_{-\{K'_j\}}\bar{L}_{-\{\bar{K}'_j\}}\mathcal{O}(w,\bar{w}),\end{aligned}\tag{45}$$

where $L_{-\{K_i\}}\equiv L_{-k_{i1}}L_{-k_{i2}}\ldots L_{-k_{in_i}}$, $(0\le k_{i1}\le k_{i2}\le\ldots\le k_{in_i})$, and $L_{-\{\bar{K}_i\}}\equiv L_{-\bar{k}_{i1}}L_{-\bar{k}_{i2}}\ldots L_{-\bar{k}_{i\bar{n}_i}}$, $(0\le\bar{k}_{i1}\le\bar{k}_{i2}\le\ldots\le\bar{k}_{i\bar{n}_i})$, and likewise for $L_{-\{K'_j\}}$ and $L_{-\{\bar{K}'_j\}}$. If the combination coefficients C_i (C'_i) are required to be dimensionless, all $V_i(w,\bar{w})$ $(V'_i(w,\bar{w}))$ should have the same mass dimension denoted as N (N'). This indicates that $\{K_i\}$ and $\{K'_i\}$ satisfy

$$|K_i|+|\bar{K}_i|=N,\qquad|K'_i|+|\bar{K}'_i|=N'\qquad\left(|K_i|\equiv\sum_{j=1}^{n_i}k_{ij},|\bar{K}_i|\equiv\sum_{j=1}^{\bar{n}_i}\bar{k}_{ij}\right).\tag{46}$$

First, the two-point function of V_α and $V_\beta^\dagger$ on Σ_1 is given by

$$\begin{aligned}\langle V_\alpha(w_1,\bar{w}_1)V_\beta^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1} &= \sum_{i=1}^{M}\sum_{j=1}^{M'} C_i C_j'^{*}\langle L_{-\{K_i\}}\mathcal{O}(w_1)L_{-\{K_j'\}}\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}\langle \bar{L}_{-\{\bar{K}_i\}}\mathcal{O}(\bar{w}_1)\bar{L}_{-\{\bar{K}_j'\}}\mathcal{O}^\dagger(\bar{w}_2)\rangle_{\Sigma_1}\\ &= \sum_{i=1}^{M}\sum_{j=1}^{M'} C_i C_j'^{*}\frac{c_0(\{K_i\},\{K_j'\})}{(w_1-w_2)^{2\Delta+|K_i|+|K_j'|}}\frac{\bar{c}_0(\{\bar{K}_i\},\{\bar{K}_j'\})}{(\bar{w}_1-\bar{w}_2)^{2\Delta+|\bar{K}_i|+|\bar{K}_j'|}}. \end{aligned} \tag{47}$$

Similar to the previous subsection, in the above, we formally decompose the operator $L_{-\{K_i\}}\bar{L}_{-\{\bar{K}_i\}}O(w,\bar{w})$ into a holomorphic operator $L_{-\{K_i\}}O(w)$ and an antiholomorphic operator $\bar{L}_{-\{\bar{K}_i\}}O(\bar{w})$ in the sense of the two-point function. c_0 and $\bar{c}_0$ in Eq. (47) are, respectively, the coefficients of the holomorphic and antiholomorphic two-point correlation function.

We then deal with the $2k$-point function on Σ_k. At late times, the $2k$-point function is given by

$$\begin{aligned}&\langle V_\alpha(w_1,\bar{w}_1)V_\beta^\dagger(w_2,\bar{w}_2)\ldots V_\alpha(w_{2k-1},\bar{w}_{2k-1})V_\beta^\dagger(w_{2k},\bar{w}_{2k})\rangle_{\Sigma_k}\\ &\quad= \sum_{i_1,i_3,\ldots,i_{2k-1}}\sum_{j_2,j_4,\ldots j_{2k}} C_{i_1}C_{j_2}'^{*}\ldots C_{i_{2k-1}}C_{j_{2k}}'^{*}\langle V_{i_1}(w_1,\bar{w}_1)V_{j_2}'^{\dagger}(w_2,\bar{w}_2)\ldots V_{i_{2k-1}}(w_{2k-1},\bar{w}_{2k-1})V_{j_{2k}}'^{\dagger}(w_{2k},\bar{w}_{2k})\rangle_{\Sigma_k}\\ &\quad\sim d_{\mathcal{O}}^{1-k}\sum_{i_1,i_3,\ldots,i_{2k-1}}\sum_{j_2,j_4,\ldots j_{2k}} C_{i_1}C_{j_2}'^{*}\ldots C_{i_{2k-1}}C_{j_{2k}}'^{*}\\ &\qquad\times\langle L_{-\{K_{i_1}\}}\mathcal{O}(w_1)L_{-\{K_{j_4}'\}}\mathcal{O}^\dagger(w_4)\rangle_{\Sigma_k}\ldots\langle L_{-\{K_{i_{2k-1}}\}}\mathcal{O}(w_{2k-1})L_{-\{K_{j_{2k+2}}'\}}\mathcal{O}^\dagger(w_{2k+2})\rangle_{\Sigma_k}(2k+2\equiv 2)\\ &\qquad\times\langle \bar{L}_{-\{\bar{K}_{i_1}\}}\mathcal{O}(\bar{w}_1)\bar{L}_{-\{\bar{K}_{j_2}'\}}\mathcal{O}^\dagger(\bar{w}_2)\rangle_{\Sigma_k}\ldots\langle \bar{L}_{-\{\bar{K}_{i_{2k-1}}\}}\mathcal{O}(\bar{w}_{2k-1})\bar{L}_{-\{\bar{K}_{j_{2k}}'\}}\mathcal{O}^\dagger(\bar{w}_{2k})\rangle_{\Sigma_k}\\ &\quad\sim d_{\mathcal{O}}^{1-k}\sum_{i_1,i_3,\ldots,i_{2k-1}}\sum_{j_2,j_4,\ldots j_{2k}} C_{i_1}C_{j_2}'^{*}\ldots C_{i_{2k-1}}C_{j_{2k}}'^{*}\frac{c_0(\{K_{i_1}\},\{K_{j_4}'\})}{(w_1-w_2)^{2\Delta+|K_{i_1}|+|K_{j_4}'|}}\cdots\frac{c_0(\{K_{i_{2k-3}}\},\{K_{j_{2k}}'\})}{(w_1-w_2)^{2\Delta+|K_{i_{2k-3}}|+|K_{j_{2k}}'|}}\frac{c_0(\{K_{i_{2k-1}}\},\{K_{j_2}'\})}{(w_1-w_2)^{2\Delta+|K_{i_{2k-1}}|+|K_{j_2}'|}}\\ &\qquad\times\frac{\bar{c}_0(\{\bar{K}_{i_1}\},\{\bar{K}_{j_2}'\})}{(\bar{w}_1-\bar{w}_2)^{2\Delta+|\bar{K}_{i_1}|+|\bar{K}_{j_2}'|}}\cdots\frac{\bar{c}_0(\{\bar{K}_{i_{2k-1}}\},\{\bar{K}_{j_{2k}}'\})}{(\bar{w}_1-\bar{w}_2)^{2\Delta+|\bar{K}_{i_{2k-1}}|+|\bar{K}_{j_{2k}}'|}}. \end{aligned} \tag{48}$$

In the above derivation, from the first equation to the first tilde, we follow the approach outlined in the preceding subsection: First, we map the $2k$-point function on Σ_k to the plane through conformal transformation $w=z^k$ and extract its leading behavior; subsequently, using Eq. (38) and fusion transformation $k-1$ times, we decompose the leading $2k$-point function on the plane into k holomorphic two-point functions and k antiholomorphic two-point functions, and finally, map the $2k$ two-point functions back to Σ_k. From the first tilde to the second tilde, we utilize a late-time relation similar to (42) as follows:

$$\begin{aligned}\langle L_{-\{K\}}\mathcal{O}(w_{2j+1})L_{-\{K'\}}\mathcal{O}^\dagger(w_{2j+4})\rangle_{\Sigma_k}&\sim e^{-2\pi i(1+j)(2\Delta+|K|+|K'|)}\langle L_{-\{K\}}\mathcal{O}(w_1)L_{-\{K'\}}\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1},\\ \langle L_{-\{\bar{K}\}}\mathcal{O}(\bar{w}_{2j+1})L_{-\{\bar{K}'\}}\mathcal{O}^\dagger(\bar{w}_{2j+2})\rangle_{\Sigma_k}&\sim e^{2\pi i(1+j)(2\Delta+|\bar{K}|+|\bar{K}'|)}\langle L_{-\{\bar{K}\}}\mathcal{O}(\bar{w}_1)L_{-\{\bar{K}'\}}\mathcal{O}^\dagger(\bar{w}_2)\rangle_{\Sigma_1},\\ &(j=1,2,\ldots,k-1;w_{2k+2}\equiv w_2), \end{aligned} \tag{49}$$

which can be readily derived using Eqs. (16), (29), and (38).

Upon substituting Eqs. (47) and (48) into the kth pseudo-Rényi entropy expression (8) and attempting to eliminate $w_{1,2}$, we encounter some subtleties. Specifically, after analytic continuation, the expressions for $w_{1,2}$ and $\bar{w}_{1,2}$ (9) imply that when $x_1\neq x_2$, we have $w_1-w_2=\bar{w}_1-\bar{w}_2=x_1-x_2$ in the limit $\epsilon\to 0$. Consequently, based on the initial constraint (46), we can extract the power of x_1-x_2 in (48) from the summation, which is equal to $(x_1-x_2)^{-k(4\Delta+N+N')}$ [for Eq. (47), it is $(x_1-x_2)^{-(4\Delta+N+N')}$]. The late-time excess formula of the kth pseudo-Rényi entropy is thus given by

$$\begin{aligned}\lim_{t\to\infty}\Delta S^{(k)}(\mathcal{T}_A^{1|2}(t)) &= \log d_{\mathcal{O}}\\ &\quad+\frac{1}{1-k}\log\left(\frac{\sum_{i_1,i_3,\ldots,i_{2k-1}=1}^{M}\sum_{j_2,j_4,\ldots,j_{2k}=1}^{M'}\prod_{u=1}^{k}C_{i_{2u-1}}C_{j_{2u}}'^{*}c_0(\{K_{i_{2u-1}}\},\{K_{j_{2u+2}}'\})\bar{c}_0(\{\bar{K}_{i_{2u-1}}\},\{\bar{K}_{j_{2u}}'\})}{(\sum_i\sum_j C_iC_j'^{*}c_0(\{K_i\},\{K_j'\})\bar{c}_0(\{\bar{K}_i\},\{\bar{K}_j'\}))^k}\right)\\ &\quad(\text{for } x_1\neq x_2; 2k+2\equiv 2). \end{aligned} \tag{50}$$

However, things become slightly different when $x_1=x_2$. This is because, in this case, $w_1-w_2=-(\bar{w}_1-\bar{w}_2)=-2i\epsilon$. When attempting to eliminate the normalization parameter ϵ by dividing Eq. (48) by the kth power of Eq. (47), we will be left with a negative power in both the numerator and denominator summations, leading to another late-time excess formula for the kth pseudo-Rényi entropy,

$$\lim_{t\to\infty}\Delta S^{(k)}(\mathcal{T}_A^{1|2}(t)) = \log d_{\mathcal{O}} + \frac{1}{1-k}\log\left(\frac{\sum_{i_1,i_3,\ldots,i_{2k-1}=1}^{M}\sum_{j_2,j_4,\ldots,j_{2k}=1}^{M'}\prod_{u=1}^{k}(-1)^{|K_{i_{2u-1}}|+|K'_{j_{2u}}|}C_{i_{2u-1}}C'^*_{j_{2u}}c_0(\{K_{i_{2u-1}}\},\{K'_{j_{2u+2}}\})\bar{c}_0(\{\bar{K}_{i_{2u-1}}\},\{\bar{K}'_{j_{2u}}\})}{(\sum_{i,j}(-1)^{|K_i|+|K'_j|}C_iC'^*_j c_0([K_i],[K'_j])\bar{c}_0([\bar{K}_i],[\bar{K}'_j]))^k}\right)$$
$$(\text{for } x_1 = x_2; 2k+2\equiv 2). \tag{51}$$

Indeed, one can absorb such a negative power in Eq. (51) into the coefficient of the holomorphic two-point function to obtain a formula similar to Eq. (50),

$$\lim_{t\to\infty}\Delta S^{(k)}(\mathcal{T}_A^{1|2}(t)) = \log d_{\mathcal{O}} + \frac{1}{1-k}\log\left(\frac{\sum_{i_1,i_3,\ldots,i_{2k-1}=1}^{M}\sum_{j_2,j_4,\ldots,j_{2k}=1}^{M'}\prod_{u=1}^{k}C_{i_{2u-1}}C'^*_{j_{2u}}c_0(\{K'_{j_{2u+2}}\},\{K_{i_{2u-1}}\})\bar{c}_0(\{\bar{K}_{i_{2u-1}}\},\{\bar{K}'_{j_{2u}}\})}{(\sum_{i,j}C_iC'^*_j c_0(\{K'_j\},\{K_i\})\bar{c}_0(\{\bar{K}_i\},\{\bar{K}'_j\}))^k}\right)$$
$$(\text{for } x_1 = x_2; 2k+2\equiv 2), \tag{52}$$

where $c_0(\{K'_j\},\{K_i\}) \coloneqq (-1)^{2\Delta+|K_i|+|K'_j|}c_0(\{K_i\},\{K'_j\})$. When we delve into a detailed analysis of these two formulas, we may find that Eq. (51) [or Eq. (52)] is compatible with the late-time excess formula for entanglement entropy given in [57]. This can be verified by simply removing the prime from Eq. (51) [or Eq. (52)]. However, generally speaking, since Eq. (50) is not equal to Eq. (51), Eq. (50) cannot be reduced to the formula for entanglement entropy. We verify the discontinuity of the pseudo-Rényi entropy in these two cases (i.e., $x_1 = x_2$ and $x_1 \neq x_2$) through numerical calculations in the critical Ising model; see Fig. 2. Mathematically, we can attribute this discontinuity of pseudo-Rényi entropy to the noncommutativity of the limits as $\epsilon \to 0$ and $x_1 \to x_2$. It would be interesting to comprehend this point from a physical perspective.

More importantly, regardless of the case (whether $x_1 = x_2$ or $x_1 \neq x_2$), the late-time excess of the pseudo-Rényi entropy of two linear combination operators is composed of two parts. The first part is the logarithm of the quantum dimension of the corresponding primary operator, which reflects the entanglement properties of the primary/descendant operators used to construct the linear combination operators. The second part involves the coefficients of the superposition C_i, the coefficients of the holomorphic and antiholomorphic two-point functions, which reflect the additional entanglement generated by the process of linear combination. We can express this part of the additional entanglement contribution as the entanglement entropy (pseudoentropy) of an effective density (transition) matrix in a finite-dimensional Hilbert space. Taking Eq. (50) as an example [Eq. (51) shares a similar treatment], we use the superposition coefficients, the coefficients of holomorphic and antiholomorphic two-point functions to define the following $M \times M$ matrix $\mathcal{T}_{\text{eff}}$:

$$X_{ij} \coloneqq \sqrt{C_iC'^*_j}\cdot c_0(\{K_i\},\{K'_j\}), \qquad \bar{X}_{ij} \coloneqq \sqrt{C_iC'^*_j}\cdot \bar{c}_0(\{\bar{K}_i\},\{\bar{K}'_j\}),$$
$$\mathcal{T}_{\text{eff}} \coloneqq \frac{X\bar{X}^{\mathrm{T}}}{\mathrm{tr}[X\bar{X}^{\mathrm{T}}]}, \qquad (i = 1,2,\ldots,M; j = 1,2,\ldots,M'). \tag{53}$$

We refer to $\mathcal{T}_{\text{eff}}$ as an effective transition matrix because $\mathcal{T}_{\text{eff}}$ is usually non-Hermitian, and we will see that it characterizes the additional pseudo-Rényi entropy at the late time. With the help of $\mathcal{T}_{\text{eff}}$, Eq. (50) can be equivalently written as

$$\lim_{t\to\infty}\Delta S^{(k)}(\mathcal{T}_A^{1|2}(t)) = \log d_{\mathcal{O}} + \frac{1}{1-k}\log \mathrm{tr}[(\mathcal{T}_{\text{eff}})^k]. \tag{54}$$

From the above equation, it is clear that the additional pseudo-Rényi entropy generated by the linear combination process equals the pseudo-Rényi entropy of an effective transition matrix in an M-dimensional Hilbert space, and the additional pseudoentropy thus is equal to $-\mathrm{tr}[\mathcal{T}_{\text{eff}}\log\mathcal{T}_{\text{eff}}]$.

It is evident that the late-time additional contributions for all levels of pseudo-Rényi entropy resulting from a linear combination are zero only when $\mathcal{T}_{\text{eff}}$ possesses a single

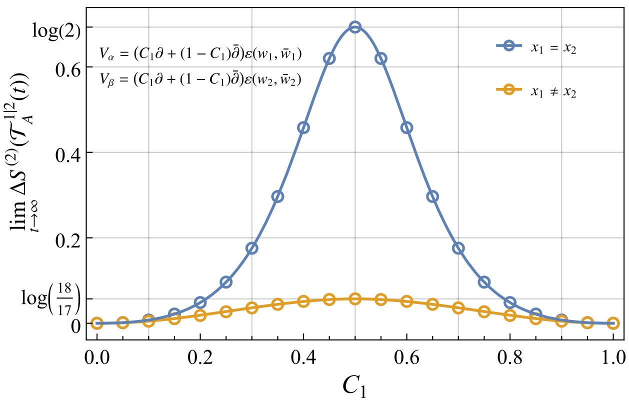

FIG. 2. The late-time excess of the second Rényi entropy (in blue) or the second pseudo-Rényi entropy (in orange) of the linear combination operator $(C_1\partial + (1-C_1)\bar{\partial})\varepsilon$, where ε is the energy density operator in the critical Ising model. We have $d_\varepsilon = 1$. The hollow circles represent the numerical data obtained by using the known four-point function of ε, while the solid lines represent the theoretical result obtained by using Eq. (51) [(or (52)] (corresponding to the blue line) and Eq. (50) (corresponding to the orange line). It should be noted that when the linear combination operator is the equally weighted sum of $L_{-1}\varepsilon$ and $\bar{L}_{-1}\varepsilon$, i.e., $C_1 = 1/2$, the late-time excess of the Rényi entropy ($x_1 = x_2$) is $\log 2$, while the late-time excess of the pseudo-Rényi entropy ($x_1 \neq x_2$) is $\log\frac{18}{17} \approx 0.057$.

nonzero eigenvalue of 1. We show that the physical origin of these additional corrections is attributed to the mixing of holomorphic Virasoro generators and antiholomorphic Virasoro generators. Considering

$$V_\alpha = \sum_i C_i L_{-\{K_i\}}\bar{L}_{-\{\bar{K}\}}\mathcal{O}, \quad V_\beta = \sum_j C'_j L_{-\{K'_j\}}\bar{L}_{-\{\bar{K}'\}}\mathcal{O}, \tag{55}$$

the holomorphic and antiholomorphic Virasoro generators in $V_{\alpha(\beta)}$ appear in the form of product (not mixed). Based on Eq. (53), we can write down the matrix element of $\mathcal{T}_{\rm eff}$,

$$(\mathcal{T}_{\rm eff})_{ij} = \frac{(\vec{a})_i(\vec{b})_j}{\vec{a}\cdot\vec{b}},$$
$$(\vec{a})_i \coloneqq \sqrt{C_i}\sum_l C'^*_l c_0(\{K_i\},\{K'_l\}),$$
$$(\vec{b})_j \coloneqq \bar{c}_0(\{\bar{K}\},\{\bar{K}'\})\sqrt{C_j}. \tag{56}$$

Evidently, $\mathcal{T}_{\rm eff}$ under this scenario takes the form of the pure state transition matrix (1), resulting in a single nonzero eigenvalue of 1 for $\mathcal{T}_{\rm eff}$. Therefore, we prove that the linear combination process does not generate any additional correction in this case. To provide a heuristic understanding of this result, we may draw an analogy to [49]: When $V_{\alpha(\beta)}(w,\bar{w})$ takes the form of $\sum_i C_i L_{-\{K_i\}}\bar{L}_{-\{\bar{K}\}}\mathcal{O}(w,\bar{w})$, $V_{\alpha(\beta)}$ can be decomposed into a holomorphic operator $\sum_i C_i L_{-\{K_i\}}\mathcal{O}(w)$ and an antiholomorphic operator $\bar{L}_{-\{\bar{K}\}}\mathcal{O}(\bar{w})$ in the sense of the two-point function $\langle V_\alpha V_\beta^\dagger\rangle$, producing a product state $\sum_i C_i|L_{-\{K_i\}}\mathcal{O}\rangle_{\mathcal{H}} \otimes |\bar{L}_{-\{\bar{K}\}}\mathcal{O}\rangle_{\bar{\mathcal{H}}}$ in the Verma module $\mathcal{H}\otimes\bar{\mathcal{H}}$ when acting on the vacuum state. However, when $V_{\alpha(\beta)}(w,\bar{w}) = \sum_i C_i L_{-\{K_i\}}\bar{L}_{-\{\bar{K}_i\}}\mathcal{O}(w,\bar{w})$, where $L_{-\{K_i\}}\bar{L}_{-\{\bar{K}_i\}}\mathcal{O}(w,\bar{w})$ can each be decomposed into a holomorphic operator $L_{-\{K_i\}}\mathcal{O}(w)$ and an antiholomorphic operator $\bar{L}_{-\{\bar{K}_i\}}\mathcal{O}(\bar{w})$ in the sense of two-point functions, V_α acting on the vacuum state produces an entangled state $\sum_i C_i|L_{-\{K_i\}}\mathcal{O}\rangle_{\mathcal{H}} \otimes |\bar{L}_{-\{\bar{K}_i\}}\mathcal{O}\rangle_{\bar{\mathcal{H}}}$, enhancing the entanglement.

Finally, we consider the late-time excess formula (50) for the second pseudo-Rényi entropy

$$\lim_{t\to\infty}\Delta S^{(2)}(\mathcal{T}_A^{1|2}(t)) = \log d_{\mathcal{O}} - \log\left(\frac{\sum_{i_1,i_3}\sum_{j_2,j_4} C_{i_1}C_{i_3}C'^*_{j_2}C'^*_{j_4} c_0(\{K_{i_1}\},\{K'_{j_4}\})c_0(\{K_{i_3}\},\{K'_{j_2}\})\bar{c}_0(\{\bar{K}_{i_1}\},\{\bar{K}'_{j_2}\})\bar{c}_0(\{\bar{K}_{i_3}\},\{\bar{K}'_{j_4}\})}{(\sum_i\sum_j C_i C'^*_j c_0(\{K_i\},\{K'_j\})\bar{c}_0(\{\bar{K}_i\},\{\bar{K}'_j\}))^2}\right) \tag{57}$$

to show the phenomenon of "mixing enhancing the entanglement."

(i) Example 1 with $V_\alpha(w_1,\bar{w}_1) = (L_{-1}+\bar{L}_{-1})\mathcal{O}(w_1,\bar{w}_1)$, $V_\beta(w_2,\bar{w}_2) = (L_{-1}+\bar{L}_{-1})\mathcal{O}(w_2,\bar{w}_2)$. The two-point function is

$$\langle V_\alpha(w_1,\bar{w}_1)V_\beta^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1} = \frac{-4\Delta(4\Delta+1)}{(x_1-x_2)^{4\Delta+2}}. \tag{58}$$

Formula (58) is easy to check. Here, we replace x_1+t and x_2+t with w_1 and w_2 in the final result. The four-point function at the late time is

$$\langle V_\alpha(w_1,\bar{w}_1)V_\beta^\dagger(w_2,\bar{w}_2)V_\alpha(w_3,\bar{w}_3)V_\beta^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2} \sim \frac{1}{d_{\mathcal{O}}}\frac{8\Delta^2(1+16\Delta(1+2\Delta)}{(x_1-x_2)^{8\Delta+4}}. \tag{59}$$

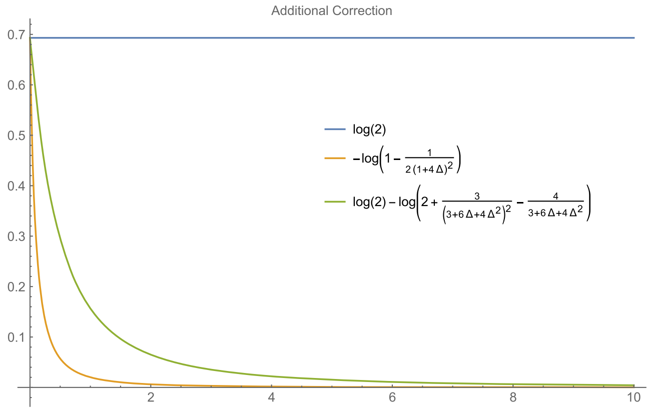

FIG. 3. Additional correction of the late-time $\Delta S_A^{(2)}$ due to the mixing of holomorphic and antiholomorphic Virasoro generators. The horizontal axis is the conformal dimension of the primary operator $\mathcal{O}$.

From (58) and (59), we have

$$\lim_{t\to\infty}\Delta S_A^{(2)}(t)=\log d_{\mathcal{O}}-\log\left(1-\frac{1}{2(1+4\Delta)^2}\right). \quad (60)$$

In this case, the correlation function of V_α and V_β cannot be divided into the product of the holomorphic part and antiholomorphic part, and $\Delta S^{(2)}$ contains an extra correction $\log\left(1-\frac{1}{2(1+4\Delta)^2}\right)$ besides $\log d_{\mathcal{O}}$. The relation between extra correction and the conformal weight is shown in Fig. 3 (the orange curve). Note that the extra correction will be $\log 2$ when we consider another late-time excess formula (51), reproducing the result of entanglement entropy in [57].

(ii) Example 2 with $V_\alpha(w_1,\bar{w}_1)=(L_{-1}L_{-1}+\bar{L}_{-1}\bar{L}_{-1})\mathcal{O}(w_1,\bar{w}_1)$, $V_\beta(w_2,\bar{w}_2)=(L_{-1}L_{-1}+\bar{L}_{-1}\bar{L}_{-1})\mathcal{O}(w_2,\bar{w}_2)$. The two-point function is

$$\langle V_\alpha(w_1,\bar{w}_1)V_\beta(w_2,\bar{w}_2)\rangle_{\Sigma_1}=\frac{8\Delta(1+2\Delta)(3+6\Delta+4\Delta^2)}{(x_1-x_2)^{-4(1+\Delta)}}. \quad (61)$$

The four-point function at the late time is

$$\langle V_\alpha(w_1,\bar{w}_1)V_\beta(w_2,\bar{w}_2)V_\alpha(w_3,\bar{w}_3)V_\beta(w_4,\bar{w}_4)\rangle_{\Sigma_2}\sim\frac{32\Delta^2(1+2\Delta)^2(9+8\Delta(3+2\Delta)(2+\Delta(3+2\Delta)))}{(x_1-x_2)^{-8(1+\Delta)}}. \quad (62)$$

Combining (61) and (62), the second pseudo-Rényi entropy is given by

$$\lim_{t\to\infty}\Delta S_A^{(2)}(t)=\log d_{\mathcal{O}}+\log 2-\log\left[2+\frac{3}{(3+6\Delta+4\Delta^2)^2}-\frac{4}{3+6\Delta+4\Delta^2}\right]. \quad (63)$$

Notice that there is an additional correction depending on the conformal weight of the corresponding primary operator, and its relation with the conformal weight can be seen in Fig. 3 (the green curve). For two general linear combination operators, its pseudo-Rényi entropy may also acquire extra correction depending on the central charge and conformal weight of the theory at the late time, and one can calculate the extra correction in general cases.

V. CONCLUSION AND PROSPECTS

In this paper, we investigate the pseudo-Rényi entropy of local descendant operators in RCFTs, extending the previous studies in [30,51,57]. In [30,57], it has been found that the late-time excess of the pseudo-Rényi entropy of two primary states and the Rényi entropy of a descendant state equal the logarithmic quantum dimension of the primary operator in RCFTs. It is a natural question to consider the pseudo-Rényi entropy of the descendant states.

First, we show that in some special cases: $V_\alpha=L_{-1}\mathcal{O}$, $V_\beta=\mathcal{O}$ and $V_\alpha=L_{-n}\mathcal{O}$, $V_\beta=\mathcal{O}$ with $\mathcal{O}$ being primary, the late-time excess of the second pseudo-Rényi entropy (8) is still logarithmic of the quantum dimension of the primary operator. Using the conformal block and operator product expansion, we compute the second pseudo-Rényi entropy constructed by two descendant operators with different Virasoro generators. We show that their second pseudo-Rényi entropy is the same as their primaries for such states. Although the calculation looks quite complicated, the leading divergent terms in the late-time limit are simple, behaving as the one for primary operators.

Further, we compute kth pseudo-Rényi entropy with two descendant operators $L_{-n}\mathcal{O}$ and $L_{-m}\mathcal{O}$. We extract the most divergent term of the $2k$-point function on Σ_k with an overall factor $\mathcal{F}$ (36), and then associate the $2k$-point function of descendant operators with the $2k$-point function of primary operators (39) with some derivative operators of the form

$$\mathcal{D}_{2i+1,2i+4}=\mathcal{D}(\partial_{2i+1},\partial_{2i+4};m,n,c,\Delta). \quad (64)$$

We find the $2k$-point function breaks up into k two-point functions for the holomorphic part (and k for the antiholomorphic part). The two-point function only depends on the conformal weight and some constant (42). As a result, in this case, the pseudoentropy of the descendant operators is the same as the primaries.

Finally, we discuss the most generic descendant operators, which are two linear combination operators constructed by operators in $\mathcal{O}$'s conformal family

$$V_\alpha(w_1,\bar{w}_1)=\sum_i C_i V_i(w_1,\bar{w}_1),\qquad V_\beta(w_2,\bar{w}_2)=\sum_j C'_j V_j(w_2,\bar{w}_2). \tag{65}$$

We derive the formula for the kth pseudo-Rényi entropy of linear combination operators at the late time

$$\lim_{t\to\infty}\Delta S^{(k)}(\mathcal{T}^{1|2}(t)) = \log d_{\mathcal{O}}+\frac{1}{1-k}\log\left(\frac{\sum_{i_1,i_3,\ldots,i_{2k-1}=1}^{M}\sum_{j_2,j_4,\ldots,j_{2k}=1}^{M'}\prod_{u=1}^{k}C_{i_{2u-1}}C'^{*}_{j_{2u}}c_0(\{K_{i_{2u-1}}\},\{K'_{j_{2u+2}}\})\bar{c}_0(\{\bar{K}_{i_{2u-1}}\},\{\bar{K}'_{j_{2u}}\})}{(\sum_i\sum_j C_iC'^{*}_j c_0(\{K_i\},\{K'_j\})\bar{c}_0(\{\bar{K}_i\},\{\bar{K}'_j\}))^k}\right)$$
$$(\text{for } x_1\neq x_2;\, 2k+2\equiv 2), \tag{66}$$

which is quite different from the formula derived when $x_1\neq x_2$,

$$\lim_{t\to\infty}\Delta S^{(k)}(\mathcal{T}^{1|2}(t)) = \log d_{\mathcal{O}}+\frac{1}{1-k}\log\left(\frac{\sum_{i_1,i_3,\ldots,i_{2k-1}=1}^{M}\sum_{j_2,j_4,\ldots,j_{2k}=1}^{M'}\prod_{u=1}^{k}(-1)^{|K_{i_{2u-1}}|+|K'_{j_{2u}}|}C_{i_{2u-1}}C'^{*}_{j_{2u}}c_0(\{K_{i_{2u-1}}\},\{K'_{j_{2u+2}}\})\bar{c}_0(\{\bar{K}_{i_{2u-1}}\},\{\bar{K}'_{j_{2u}}\})}{(\sum_{i,j}(-1)^{|K_i|+|K'_j|}C_iC'^{*}_j c_0([K_i],[K'_j])\bar{c}_0([\bar{K}_i],[\bar{K}'_j]))^k}\right)$$
$$(\text{for } x_1=x_2;\, 2k+2\equiv 2). \tag{67}$$

For convenience, we also introduce an effective transition matrix $\mathcal{T}_{\text{eff}}$,

$$X_{ij}\coloneqq\sqrt{C_iC'^{*}_j}c_0(\{K_i\},\{K'_j\}),\qquad \bar{X}_{ij}\coloneqq\sqrt{C_iC'^{*}_j}\bar{c}_0(\{\bar{K}_i\},\{\bar{K}'_j\}),$$
$$\mathcal{T}_{\text{eff}}\coloneqq\frac{X\bar{X}^{\mathrm{T}}}{\mathrm{tr}[X\bar{X}^{\mathrm{T}}]},\qquad (i=1,2,\ldots,M;\ j=1,2,\ldots,M') \tag{68}$$

to simplify the formula (66) [Eq. (67) shares a similar treatment]. Using the formula (50), we find that the pseudo-Rényi entropy of linear combination operators is generally different from that of the primary operator $\mathcal{O}$. The pseudo-Rényi entropies are the same as the ones of the primary when the correlation function of V_α and V_β can be divided into the product of the holomorphic part and the anti-holomorphic part. A typical example is

$$V_\alpha(w_1,\bar{w}_1)=\sum_i C_iL_{-\{K_i\}}\bar{L}_{-\{\bar{K}_1\}}\mathcal{O}(w_1,\bar{w}_1),$$
$$V_\beta(w_2,\bar{w}_2)=\sum_j C'_jL_{-\{K'_j\}}\bar{L}_{-\{\bar{K}'_2\}}\mathcal{O}(w_2,\bar{w}_2). \tag{69}$$

Otherwise, there is an extra contribution due to the mixing of the holomorphic and antiholomorphic Virasoro generators. A typical example of extra contribution is

$$V_\alpha(w_1,\bar{w}_1)=(L_{-1}+\bar{L}_{-1})\mathcal{O}(w_1,\bar{w}_1),$$
$$V_\beta(w_2,\bar{w}_2)=(L_{-1}+\bar{L}_{-1})\mathcal{O}(w_2,\bar{w}_2). \tag{70}$$

In general, the kth pseudo-Rényi entropy for two linear combination operators at the late time only depends on the quantum dimension and the contribution from a finite-dimensional Hilbert space,

$$\lim_{t\to\infty}\Delta S^{(k)}(\mathcal{T}^{1|2}(t))=\log d_{\mathcal{O}}+\frac{1}{1-k}\log\mathrm{tr}[(\mathcal{T}_{\text{eff}})^k]. \tag{71}$$

Noticing the current results in RCFTs, one can directly calculate the pseudoentropy of generic local operators in Liouville CFT, holographic CFTs, nondiagonal CFTs, etc. Since the spectra in such theories have different structures, the associated pseudoentropy will be highly different from those in RCFTs. In particular, since holomorphic and antiholomorphic conformal blocks have different structures in nondiagonal CFTs, the late-time behavior of the entanglement entropy and pseudoentropy associated with locally excited states will not be the same as the ones demonstrated in the current paper. We would like to leave them to future work.

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APPENDIX A: REDUCTION OF $\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}$

Following the standard way [78], we compute the four-point function $\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}$ in this section.

$$\begin{aligned}
&\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\\
&=-\frac{1}{2\pi i}\sum_{i=2}^{4}\oint_{\mathcal{C}(z_i)}\mathrm{d}z(z-z_1)^{-n+1}\langle T(z)\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\\
&=\frac{-1}{2\pi i}\oint_{\mathcal{C}(z_2)}\frac{\mathrm{d}z}{(z-z_1)^{n-1}}\left\{\frac{\Delta\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}}{(z-z_2)^2}+\frac{\partial_{z_2}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}}{z-z_2}+\mathrm{reg}(z-z_2)\right\}\\
&\quad+\frac{-1}{2\pi i}\oint_{\mathcal{C}(z_3)}\frac{\mathrm{d}z}{(z-z_1)^{n-1}}\left\{\frac{n(n^2-1)c/12+2n\Delta}{(z-z_3)^{n+2}}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\right.\\
&\quad+\sum_{k=1}^{n-1}\frac{(n+k)\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-(n-k))}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}}{(z-z_3)^{k+2}}+\frac{(\Delta+n)\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}}{(z-z_3)^2}\\
&\quad\left.+\frac{\partial_{z_3}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}}{z-z_3}+\mathrm{reg}(z-z_3)\right\}\\
&\quad+\frac{-1}{2\pi i}\oint_{\mathcal{C}(z_4)}\frac{\mathrm{d}z}{(z-z_1)^{n-1}}\left\{\frac{\Delta\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}}{(z-z_4)^2}+\frac{\partial_{z_4}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}}{z-z_4}+\mathrm{reg}(z-z_4)\right\}\\
&=\frac{(n-1)\Delta}{z_{21}^n}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}+\frac{-\partial_{z_2}}{z_{21}^{n-1}}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\\
&\quad+\frac{(n-1)\Delta}{z_{41}^n}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}+\frac{-\partial_{z_4}}{z_{41}^{n-1}}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\\
&\quad+(-1)^n\frac{(n(n^2-1)c/12+2n\Delta)(2n-1)!}{(n+1)!(n-2)!}\frac{\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}}{z_{13}^{2n}}\\
&\quad+(-1)^n\sum_{k=1}^{n-1}\frac{(n+k)!}{(k+1)!(n-2)!}\frac{\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-(n-k))}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}}{z_{13}^{n+k}}\\
&\quad+\frac{(n-1)(\Delta+n)}{z_{31}^n}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}+\frac{-\partial_{z_3}}{z_{31}^{n-1}}\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}.
\end{aligned}\tag{A1}$$

APPENDIX B: REDUCTION OF $\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_2}$

In terms of (14), the most divergent term of $\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}$ should only contain z_{14} and z_{23}, as any terms containing z_{13}, z_{24}, z_{12}, and z_{34} are subleading. So, we can first expand $\mathcal{O}(1)$'s Virasoro generator,

$$\begin{aligned}
&\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}\\
&\sim-\frac{1}{2\pi i}\oint_{\mathcal{C}(z_4)}\frac{\mathrm{d}z}{(z-z_1)^{n-1}}\langle T(z)\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}\\
&\sim-\frac{1}{2\pi i}\oint_{\mathcal{C}(z_4)}\frac{\mathrm{d}z}{(z-z_1)^{n-1}}\left\langle\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\left(\frac{m(m^2-1)c/12+2m\Delta}{(z-z_4)^{m+2}}\mathcal{O}^{\dagger}(4)\right.\right.\\
&\quad\left.\left.+\sum_{k=1}^{m-1}\frac{(m+k)}{(z-z_4)^{k+2}}\mathcal{O}^{-(m-k)\dagger}(4)+\frac{(\Delta+m)}{(z-z_4)^2}\mathcal{O}^{(-m)\dagger}(4)+\frac{\partial_4\mathcal{O}^{(-m)\dagger}(4)}{z-z_4}\right)\right\rangle_{\Sigma_1}
\end{aligned}$$

$$
\begin{aligned}
&\sim (-1)^m \frac{(n+m-1)!}{(m+1)!(n-2)!}\frac{m(m^2-1)c/12+2m\Delta}{z_{41}^{n+m}}\langle \mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1} \\
&\quad + (-1)^n \sum_{k=1}^{m-1}\frac{(n+k-1)!}{(k+1)!(n-2)!}\frac{(m+k)}{z_{14}^{n+k}}\langle \mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-(m-k))\dagger}(4)\rangle_{\Sigma_1} \\
&\quad + \frac{(n-1)(\Delta+m)}{z_{41}^{n}}\langle \mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1} \\
&\quad - \frac{\partial_4}{z_{41}^{n-1}}\langle \mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}.
\end{aligned} \tag{B1}
$$

The correlation function with four Virasoro generators is deformed into correlation functions containing no more than three Virasoro generators. We can then expand $\mathcal{O}(4)$'s Virasoro generator,

$$
\begin{aligned}
&\langle \mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1} \\
&\quad \sim -\frac{1}{2\pi i}\oint_{c(z_1)}\frac{\mathrm{d}z}{(z-z_4)^{m-1}}\langle T(z)\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1} \\
&\quad \sim -\oint_{c(z_1)}\frac{\mathrm{d}z}{(z-z_4)^{m-1}}\left\langle \left(\frac{\Delta}{(z-z_1)^2}\mathcal{O}(1)+\frac{\partial_1 O(1)}{Z-Z_1}\right)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\right\rangle_{\Sigma_1} \\
&\quad \sim \frac{(m-1)\Delta}{z_{14}^{m}}\langle (\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1} - \frac{\partial_1}{z_{14}^{m-1}}\langle (\mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}.
\end{aligned} \tag{B2}
$$

From (B1) and (B2), we can read the exact form of $\mathcal{D}_{1,4}$ introduced in (39)

$$
\begin{aligned}
\mathcal{D}_{1,4} &= (-1)^m \frac{(n+m-1)!}{(m+1)!(n-2)!}\frac{m(m^2-1)c/12+2m\Delta}{z_{41}^{n+m}} + (-1)^n\sum_{k=1}^{m-1}\frac{(n+k-1)!}{(k+1)!(n-2)!}\frac{(m+k)}{z_{14}^{n+k}}\left(\frac{(m-k-1)\Delta}{z_{14}^{m-k}}-\frac{\partial_1}{z_{14}^{m-k-1}}\right) \\
&\quad + \frac{(n-1)(\Delta+m)}{z_{41}^{n}}\left(\frac{(m-1)\Delta}{z_{14}^{m}}-\frac{\partial_1}{z_{14}^{m-1}}\right) - \frac{\partial_4}{z_{41}^{n-1}}\left(\frac{(m-1)\Delta}{z_{14}^{m}}-\frac{\partial_1}{z_{14}^{m-1}}\right).
\end{aligned} \tag{B3}
$$

We can expand $\mathcal{O}(2)$'s Virasoro generator and $\mathcal{O}(3)$'s Virasoro generator in a similar way,

$$
\begin{aligned}
\langle \mathcal{O}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1} &\sim -\oint_{c(z_3)}\frac{\mathrm{d}z}{(z-z_2)^{m-1}}\langle \mathcal{O}(1)T(z)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1} \\
&\sim -\oint_{c(z_3)}\frac{\mathrm{d}z}{(z-z_2)^{m-1}}\Bigg\langle \mathcal{O}(1)\mathcal{O}^{\dagger}(2)\Bigg(\frac{n(n^2-1)c/12+2n\Delta}{(z-z_3)^{n+2}}\mathcal{O}(3) \\
&\quad + \sum_{l=1}^{n-1}\frac{(n+l)}{(z-z_3)^{l+2}}\mathcal{O}^{-(n-l)}(3) + \frac{(\Delta+n)}{(z-z_3)^2}\mathcal{O}^{-n}(3) + \frac{\partial_3}{z-z_3}\mathcal{O}^{(-n)}(3)\Bigg)\mathcal{O}^{\dagger}(4)\Bigg\rangle_{\Sigma_1} \\
&\sim (-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}\langle \mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1} \\
&\quad + (-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}\langle \mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-(n-l)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1} \\
&\quad + \frac{(m-1)(\Delta+n)}{z_{32}^{m}}\langle \mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1} - \frac{\partial_3}{z_{32}^{m-1}}\langle \mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}.
\end{aligned} \tag{B4}
$$

Finally, we have

$$\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}\sim\frac{(n-1)\Delta}{z_{23}^{n}}G-\frac{\partial_2}{z_{23}^{n-1}}G, \tag{B5}$$

where G is $\langle\mathcal{O}(1)\mathcal{O}^{\dagger}(2)\mathcal{O}(3)\mathcal{O}^{\dagger}(4)\rangle_{\Sigma_1}$, and in the late-time limit, it is $d_{\mathcal{O}}^{-1}(1-\eta)^{-2\Delta}\bar{\eta}^{-2\Delta}$.

Combining (B1), (B2), (B4), and (B5), we have

$$\begin{aligned}
&\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_1}\\
&\sim(-1)^m\frac{(n+m-1)!}{(m+1)!(n-2)!}\frac{m(m^2-1)c/12+2m\Delta}{z_{41}^{n+m}}\left\{(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}G\right.\\
&\quad+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}\left(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}G-\frac{\partial_2}{z_{23}^{n-l-1}}G\right)\\
&\quad\left.+\frac{(m-1)(\Delta+n)}{z_{32}^{m}}\left(\frac{(n-1)\Delta}{z_{23}^{n}}G-\frac{\partial_2}{z_{23}^{n-1}}G\right)-\frac{1}{z_{32}^{m-1}}\left(\frac{n(n-1)\Delta}{z_{23}^{n+1}}G+\frac{(n-1)\Delta\partial_3}{z_{23}^{n}}G-\frac{(n-1)\partial_2}{z_{23}^{n}}G-\frac{\partial_3\partial_2}{z_{23}^{n-1}}G\right)\right\}\\
&\quad+(-1)^n\sum_{k=1}^{m-1}\frac{(n+k-1)!}{(k+1)!(n-2)!}\frac{(m+k)}{z_{14}^{n+k}}\left\{\frac{(m-k-1)\Delta}{z_{14}^{m-k}}\left[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}G\right.\right.\\
&\quad+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}\left(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}G-\frac{\partial_2}{z_{23}^{n-l-1}}G\right)\\
&\quad\left.+\frac{(m-1)(\Delta+n)}{z_{32}^{m}}\left(\frac{(n-1)\Delta}{z_{23}^{n}}G-\frac{\partial_2}{z_{23}^{n-1}}G\right)-\frac{1}{z_{32}^{m-1}}\left(\frac{n(n-1)\Delta}{z_{23}^{n+1}}G+\frac{(n-1)\Delta\partial_3}{z_{23}^{n}}G-\frac{(n-1)\partial_2}{z_{23}^{n}}G-\frac{\partial_3\partial_2}{z_{23}^{n-1}}G\right)\right]\\
&\quad-\frac{1}{z_{14}^{m-k-1}}\left[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}\partial_1G\right.\\
&\quad+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}\left(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}\partial_1G-\frac{\partial_1\partial_2}{z_{23}^{n-l-1}}G\right)+\frac{(m-1)(\Delta+n)}{z_{32}^{m}}\left(\frac{(n-1)\Delta}{z_{23}^{n}}\partial_1G-\frac{\partial_1\partial_2}{z_{23}^{n-1}}G\right)\\
&\quad\left.\left.-\frac{1}{z_{32}^{m-1}}\left(\frac{n(n-1)\Delta}{z_{23}^{n+1}}\partial_1G+\frac{(n-1)\Delta\partial_1\partial_3}{z_{23}^{n}}G-\frac{(n-1)\partial_1\partial_2}{z_{23}^{n}}G-\frac{\partial_1\partial_3\partial_2}{z_{23}^{n-1}}G\right)\right]\right\}\\
&\quad+\frac{(n-1)(\Delta+m)}{z_{41}^{n}}\left\{\frac{(m-1)\Delta}{z_{14}^{m}}\left[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}G\right.\right.\\
&\quad+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}\left(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}G-\frac{\partial_2}{z_{23}^{n-l-1}}G\right)\\
&\quad\left.+\frac{(m-1)(\Delta+n)}{z_{32}^{m}}\left(\frac{(n-1)\Delta}{z_{23}^{n}}G-\frac{\partial_2}{z_{23}^{n-1}}G\right)-\frac{1}{z_{32}^{m-1}}\left(\frac{n(n-1)\Delta}{z_{23}^{n+1}}G+\frac{(n-1)\Delta\partial_3}{z_{23}^{n}}G-\frac{(n-1)\partial_2}{z_{23}^{n}}G-\frac{\partial_3\partial_2}{z_{23}^{n-1}}G\right)\right]\\
&\quad-\frac{1}{z_{14}^{m-1}}\left[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}\partial_1G\right.\\
&\quad+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}\left(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}\partial_1G-\frac{\partial_1\partial_2}{z_{23}^{n-l-1}}G\right)+\frac{(m-1)(\Delta+n)}{z_{32}^{m}}\left(\frac{(n-1)\Delta}{z_{23}^{n}}\partial_1G-\frac{\partial_1\partial_2}{z_{23}^{n-1}}G\right)\\
&\quad\left.\left.-\frac{1}{z_{32}^{m-1}}\left(\frac{n(n-1)\Delta}{z_{23}^{n+1}}\partial_1G+\frac{(n-1)\Delta\partial_1\partial_3}{z_{23}^{n}}G-\frac{(n-1)\partial_1\partial_2}{z_{23}^{n}}G-\frac{\partial_1\partial_3\partial_2}{z_{23}^{n-1}}G\right)\right]\right\}\\
&\quad-\frac{1}{z_{41}^{n-1}}\left\{\frac{m(m-1)\Delta}{z_{14}^{m+1}}\left[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}G\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}\left(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}G-\frac{\partial_2}{z_{23}^{n-l-1}}G\right)+\frac{(m-1)(\Delta+n)}{z_{32}^m}\left(\frac{(n-1)\Delta}{z_{23}^n}G-\frac{\partial_2}{z_{23}^{n-1}}G\right)\\
&-\frac{1}{z_{32}^{m-1}}\left(\frac{n(n-1)\Delta}{z_{23}^{n+1}}G+\frac{(n-1)\Delta\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_2}{z_{23}^n}G-\frac{\partial_3\partial_2}{z_{23}^{n-1}}G\right)\Bigg]\\
&+\frac{(m-1)\Delta}{z_{14}^m}\Bigg[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}\partial_4G\\
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}\left(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}\partial_4G-\frac{\partial_4\partial_2}{z_{23}^{n-l-1}}G\right)\\
&+\frac{(m-1)(\Delta+n)}{z_{32}^m}\left(\frac{(n-1)\Delta}{z_{23}^n}\partial_4G-\frac{\partial_4\partial_2}{z_{23}^{n-1}}G\right)\\
&-\frac{1}{z_{32}^{m-1}}\left(\frac{n(n-1)\Delta}{z_{23}^{n+1}}\partial_4G+\frac{(n-1)\Delta\partial_4\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_4\partial_2}{z_{23}^n}G-\frac{\partial_4\partial_3\partial_2}{z_{23}^{n-1}}G\right)\Bigg]\\
&-\frac{(m-1)}{z_{14}^m}\Bigg[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}\partial_1G\\
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}\left(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}\partial_1G-\frac{\partial_1\partial_2}{z_{23}^{n-l-1}}G\right)+\frac{(m-1)(\Delta+n)}{z_{32}^m}\left(\frac{(n-1)\Delta}{z_{23}^n}\partial_1G-\frac{\partial_1\partial_2}{z_{23}^{n-1}}G\right)\\
&-\frac{1}{z_{32}^{m-1}}\left(\frac{n(n-1)\Delta}{z_{23}^{n+1}}\partial_1G+\frac{(n-1)\Delta\partial_1\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_1\partial_2}{z_{23}^n}G-\frac{\partial_1\partial_3\partial_2}{z_{23}^{n-1}}G\right)\Bigg]\\
&-\frac{1}{z_{14}^{m-1}}\Bigg[(-1)^n\frac{(n+m-1)!}{(n+1)!(m-2)!}\frac{n(n^2-1)c/12+2n\Delta}{z_{32}^{m+n}}\partial_4\partial_1G\\
&+(-1)^m\sum_{l=1}^{n-1}\frac{(m+l-1)!}{(l+1)!(m-2)!}\frac{(n+l)}{z_{23}^{m+l}}\left(\frac{(n-l-1)\Delta}{z_{23}^{n-l}}\partial_4\partial_1G-\frac{\partial_4\partial_1\partial_2}{z_{23}^{n-l-1}}G\right)\\
&+\frac{(m-1)(\Delta+n)}{z_{32}^m}\left(\frac{(n-1)\Delta}{z_{23}^n}\partial_4\partial_1G-\frac{\partial_4\partial_1\partial_2}{z_{23}^{n-1}}G\right)\\
&-\frac{1}{z_{32}^{m-1}}\left(\frac{n(n-1)\Delta}{z_{23}^{n+1}}\partial_4\partial_1G+\frac{(n-1)\Delta\partial_4\partial_1\partial_3}{z_{23}^n}G-\frac{(n-1)\partial_4\partial_1\partial_2}{z_{23}^n}G-\frac{\partial_4\partial_1\partial_3\partial_2}{z_{23}^{n-1}}G\right)\Bigg]\Bigg\}. \qquad \text{(B6)}
\end{aligned}
$$

The correlation function of four descendant operators becomes the correlation functions of their corresponding primary operators with some constants and derivatives. For $i\neq j\neq k\neq l$, we have

$$
\begin{aligned}
\partial_iG&=\frac{2\Delta\partial_i\eta}{1-\eta}G,\\
\partial_j\partial_iG&=\frac{2\Delta\partial_j\partial_i\eta}{1-\eta}G+\frac{2\Delta(2\Delta+1)\partial_j\eta\partial_i\eta}{(1-\eta)^2}G,\\
\partial_k\partial_j\partial_iG&=\frac{2\Delta\partial_k\partial_j\partial_i\eta}{1-\eta}G+2\Delta(2\Delta+1)\left[\frac{\partial_j\partial_i\eta\partial_k\eta+\partial_j\partial_k\eta\partial_i\eta+\partial_k\partial_i\eta\partial_j\eta}{(1-\eta)^2}G+\frac{(2\Delta+2)\partial_j\eta\partial_i\eta\partial_k\eta}{(1-\eta)^3}G\right]\\
&\sim 2\Delta(2\Delta+1)\left[\frac{\partial_j\partial_i\eta\partial_k\eta+\partial_j\partial_k\eta\partial_i\eta+\partial_k\partial_i\eta\partial_j\eta}{(1-\eta)^2}G+\frac{(2\Delta+2)\partial_j\eta\partial_i\eta\partial_k\eta}{(1-\eta)^3}G\right],\\
\partial_l\partial_k\partial_j\partial_iG&=\frac{2\Delta\partial_l\partial_k\partial_j\partial_i\eta}{1-\eta}G+\frac{(2\Delta(2\Delta+1))}{(1-\eta)^2}(\partial_k\partial_j\partial_i\eta\partial_l\eta+\partial_k\partial_j\partial_l\eta\partial_i\eta+\partial_l\partial_j\partial_i\eta\partial_k\eta+\partial_k\partial_l\partial_i\eta\partial_j\eta\\
&\quad+\partial_i\partial_j\eta\partial_l\partial_k\eta+\partial_i\partial_k\eta\partial_j\partial_l\eta+\partial_i\partial_l\eta\partial_k\partial_j\eta)G+\frac{2\Delta(2\Delta+1)(2\Delta+2)}{(1-\eta)^3}(\partial_j\partial_i\eta\partial_k\eta\partial_l\eta+\partial_j\partial_k\eta\partial_i\eta\partial_l\eta\\
&\quad+\partial_k\partial_i\eta\partial_j\eta\partial_l\eta+\partial_l\partial_i\eta\partial_k\eta\partial_j\eta+\partial_j\partial_l\eta\partial_k\eta\partial_i\eta+\partial_l\partial_k\eta\partial_i\eta\partial_j\eta)G
\end{aligned}
$$

$$+\frac{2\Delta(2\Delta+1)(2\Delta+1)(2\Delta+3)}{(1-\eta)^4}\partial_i\eta\partial_j\eta\partial_k\eta\partial_l\eta G$$

$$\sim\frac{(2\Delta(2\Delta+1))}{(1-\eta)^2}(\partial_i\partial_j\eta\partial_l\partial_k\eta+\partial_i\partial_k\eta\partial_j\partial_l\eta+\partial_i\partial_l\eta\partial_k\partial_j\eta)G+\frac{2\Delta(2\Delta+1)(2\Delta+1)}{(1-\eta)^3}(\partial_j\partial_i\eta\partial_k\eta\partial_l\eta+\partial_j\partial_k\eta\partial_i\eta\partial_l\eta$$

$$+\partial_k\partial_i\eta\partial_j\eta\partial_l\eta+\partial_l\partial_i\eta\partial_k\eta\partial_j\eta+\partial_j\partial_l\eta\partial_k\eta\partial_i\eta+\partial_l\partial_k\eta\partial_i\eta\partial_j\eta)G+\frac{2\Delta(2\Delta+1)(2\Delta+2)(2\Delta+3)}{(1-\eta)^4}\partial_i\eta\partial_j\eta\partial_k\eta\partial_l\eta G. \tag{B7}$$

From (B6), (B7), (19), (15), and (34) we derive the leading behavior of $\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_2}$ at the late-time limit,

$$\begin{aligned}
&\langle\mathcal{O}^{(-n)}(1)\mathcal{O}^{(-m)\dagger}(2)\mathcal{O}^{(-n)}(3)\mathcal{O}^{(-m)\dagger}(4)\rangle_{\Sigma_2}\\
&\sim\left(\prod_{i=1}^{4}|w_i'|^{-2\Delta}\right)(w_1-w_2)^{-m-n}(w_3-w_4)^{-m-n}G(\eta,\bar\eta)(-1)^{m+n}\\
&\times\Bigg(-\frac{1}{12\Gamma(m+2)^2\Gamma(n+2)^2}\Delta(-1)^{-m}((m(m+1)n-2)\Gamma(m+2)\Gamma(n+2)\\
&-2(m+1)(m^2n+m(2n^2-1)-n(n+1))\Gamma(m+n))((-1)^{-m-n}\Gamma(m+n)\\
&\times(m(m^2-1)n(-1)^m(c(n^2-1)+24\Delta)+24\Delta(-1)^m(n+1)(m(2mn-m+n^2-1)-n))\\
&+12\Delta(-1)^{-n}\Gamma(m+2)\Gamma(n+2)((n+1)(\Delta+m(\Delta+n))+(2-mn(n+1))))\\
&+\frac{1}{12}\Delta(m+1)(\Delta+m)(-1)^{-2m-n}\Bigg(\frac{1}{n(n+1)\Gamma(m+2)\Gamma(n-1)}\Gamma(m+n)\\
&\times(m(m^2-1)(-1)^n n(-1)^{m+n}(c(n^2-1)+24\Delta)+24\Delta(-1)^m(n+1)(m(m(2n-1)+n^2-1)-n)\Bigg)\\
&+12\Delta(n-1)((-1)^m(n+1)(\Delta+m(\Delta+n))+(-1)^{m+1}(mn(n+1)-2)))\\
&\times\frac{1}{12}\Delta(2\Delta+m)(-1)^{-2m-n}\Bigg(\frac{1}{m\Gamma(m)\Gamma(n+2)}\Gamma(m+n)\\
&\times(m(m^2-1)(-1)^n n(-1)^{m+n}(c(n^2-1)+24\Delta)+24\Delta(-1)^m(n+1)(m(m(2n-1)+n^2-1)-n))\\
&+12\Delta(m+1)((-1)^m(n+1)(\Delta+m(\Delta+n))+(-1)^{m+1}(mn(n+1)-2))\Bigg)\\
&+\frac{1}{144(m+1)\Gamma(m-1)\Gamma(m)\Gamma(m+2)^2\Gamma(n-1)\Gamma(n+2)}\\
&\times((-1)^{m+1}m(-1)^{-m-n}(c(m^2-1)+24\Delta)\Gamma(m+n)((m+1)(-1)^{n+1}n(-1)^{m+n}\Gamma(m)\Gamma(m+2)\\
&\times(c(n^2-1)+24\Delta)\Gamma(m+n)+12\Delta\Gamma(m-1)((m+1)\Gamma(m)((-1)^m(-n-1)\Gamma(m+2)\Gamma(n+2)\\
&\times(\Delta+m(\Delta+n))+(-1)^m((mn(n+1)-2)\Gamma(m+2)\Gamma(n+2)+2n(n+1)\Gamma(m+n)))\\
&-2(-1)^m(n+1)(2mn-m+n^2-1)\Gamma(m+2)\Gamma(m+n))))\Bigg)+\ldots
\end{aligned}\tag{B8}$$

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