Constraint characterization and degree of freedom counting in Lagrangian field theory

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We present a Lagrangian approach to counting degrees of freedom in first-order field theories. The emphasis is on the systematic attainment of a complete set of constraints. In particular, we provide the first comprehensive procedure to ensure the functional independence of all constraints and discuss in detail the possible closures of the constraint algorithm. We argue degrees of freedom can but need not correspond to physical modes. The appendix comprises fully worked out, physically relevant examples of varying complexity.

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I. INTRODUCTION

In most Lagrangian field theories, there exists a mismatch between the number of *a priori* independent field variables and the number of degrees of freedom N_{DoF} being propagated. The determination of N_{DoF} is of crucial importance, as it directly affects the physics. More often than not, this calls for nontrivial analyses.

Within the realm of modified gravity theories, the counting of N_{DoF} has been the subject of vivid interest in recent years. Indeed, there exists a plethora of studies in this regard around the Dvali-Gabadadze-Porrati (DGP) model [1], Galileons [2], the de Rham-Gabadadze-Tolley (dRGT) massive gravity [3], bimetric gravity [4,5], beyond Horndeski theories [6,7], generalized Proca [8,9], and degenerate-higher-order-scalar-tensor (DHOST) theories [10], to mention but a few popular settings. The recently proposed $f(Q)$ gravity [11] provides an example wherein settlement of N_{DoF} remains elusive; see Ref. [12] and references therein.

All approaches to the degree of freedom count developed thus far revolve around constraint algorithms, whose origin dates back to the work of Dirac first and Bergmann shortly afterwards. These authors considered a coordinate-dependent approach for autonomous systems within the Hamiltonian formalism. A nice review of the initial proposal can be found in [13]. It was only sensibly later that the geometrization of the procedure was carried out, yielding the celebrated presymplectic constraint algorithm (PCA) [14]. A different yet equivalent geometric algorithm was put forward in [15].

The PCA was adapted into the Lagrangian formalism stepwise [16–18]. The relation between constraints emerging in the Lagrangian and Hamiltonian algorithms was studied in a coordinate-dependent manner in [19]. The said relation is based on the so-called (temporal) evolution operator, whose geometric definition and properties were clarified in [20].

As a remark, we note that even more general geometric constraint algorithms exist. Prominent examples include an algorithm for nonautonomous systems in the Lagrangian formalism [21] and algorithms for dynamical systems described implicitly via differential equations [22–24]. Unfortunately, there does not seem to exist a complete review of constraint algorithms and relations between them.

In spite of the vast and sophisticated bodywork on geometric constraint algorithms, such approaches are intrinsically covariant and thus demanding for application in physical theories, where time plays a distinct role. The need to construct multiple auxiliary objects prior to

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implementation adds to the challenge. This is especially true for elaborate settings, including modified gravity proposals. The said and kindred hindrances presumably explain (at least in part) the extensive use of the Dirac-Bergmann algorithm, often supplemented by the Arnowitt-Deser-Misner (ADM) foliation [25], by the gravitational physics community.

In synergy with such fruitful background, few but noteworthy Lagrangian algorithms have been put forward, either providing or developed as a coordinate-dependent prescription [26–29]. Invariably, they aim to facilitate implementation in exigent theories and abridge the obtention of physically relevant quantities, such as the number of degrees of freedom N_{DoF}. The present work delves into this line of research.

A. Organization

In the main section II, we consider first-order Lagrangian field theories and present a coordinate-dependent constraint algorithm for them. Supplemented with information on local (gauge-like) symmetries, the algorithm yields the significant number N_{DoF} in the theory. Most remarkably, a thorough procedure for the verification of functional independence among constraints is given in sections II B and II C. This essential feature had only received modest attention thus far. Distinct closures of the algorithm are expounded in section II D. For clarity, appendix consists of minute implementations of the method in diverse physical theories. In section III, we reflect on the relation between N_{DoF} and physics. We draw our conclusions in the final section IV.

B. Conventions

We work on a $(d \geq 2)$-dimensional Minkowski manifold $\mathcal{M}$. Spacetime indices are denoted by the Greek letters $(\mu, \nu, \ldots)$ and raised/lowered with the metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \ldots, 1)$ and its inverse $\eta^{\mu\nu}$. Spacetime coordinates are indicated by $x^\mu = (x^0, x^1, \ldots, x^{d-1}) \equiv (x^0, x^i)$, with Latin indices $(i, j, \ldots)$ labeling spacelike directions. Dot stands for derivation with respect to time x^0. We employ the shorthand $(\partial_\mu, \partial_i)$ for derivation with respect to (x^μ, x^i), respectively. Summation over repeated indices applies throughout.

II. METHOD

Let $Q^A = Q^A(x^\mu)$ be a finite set of real field variables. $A = 1, 2, \ldots, N$ is a collective index, running over all *a priori* independent components of possibly multiple fields of different types. Consider a first-order classical field theory, defined by the action

$$S = \int_{\mathcal{M}} d^d x \mathcal{L}, \qquad \mathcal{L} = \mathcal{L}(Q^A, \partial_\mu Q^A). \tag{2.1}$$

Notice we do not consider Lagrangians with explicit spacetime dependence. The action (2.1) may but need not be invariant under local field transformations of the form

$$Q^A \to Q^A + \delta_\theta Q^A,$$
$$\delta_\theta Q^A = \sum_{p=0}^{P} (-1)^p (\partial_{\mu_1} \partial_{\mu_2} \ldots \partial_{\mu_p} \theta^M)(\mathcal{R}_M{}^A)^{\mu_1 \mu_2 \ldots \mu_p}, \tag{2.2}$$

with $P \in \mathbb{N}_0$. Here, θ^M denotes arbitrary functions of spacetime labeled by a collective index M, while $\mathcal{R}_M{}^A$ refers to fixed functions of $(Q^A, \partial_\mu Q^A)$. All $(\theta^M, \mathcal{R}_M{}^A)$ are taken to be smooth.

Gauge, Lorentz, and diffeomorphism transformations comprise the physically most relevant examples of (2.2). Discrete, global, and conformal (including Weyl) symmetries of the action do not affect the degree of freedom count and hence are not discussed. Typically, Lagrangians are postulated on the basis of a specific field content Q^A entertaining certain symmetries, if any. Accordingly and when pertinent, we assume *a priori* knowledge of (2.2). Given a Lagrangian whose symmetries of the relevant form (2.2) are unknown *a priori*, there exist systematic procedures to their disclosure [27,30,31]. See also the earlier studies [32,33].

The number of degrees of freedom propagated in the theory can be calculated as [26,27]

$$N_{\mathrm{DoF}} = N - \frac{1}{2}(g + e + l), \tag{2.3}$$

with $l, g, e \in \mathbb{N}_0$. g is the number of distinct θ^M functions in (2.2), while $e \geq g$ is the number of distinct θ^M functions plus their successive time derivatives $(\dot{\theta}^M, \ddot{\theta}^M, \ldots)$ in (2.2). The focus henceforth is on the number l of functionally independent Lagrangian constraints, iteratively determined as

$$l = l_1 + l_2 + \ldots, \tag{2.4}$$

where $l_1 \geq l_2 \geq \ldots$ count primary, secondary, etc. functionally independent Lagrangian constraints.

We stress that (2.3) counts degrees of freedom exclusively in terms of Lagrangian parameters. In particular, it does not require the classification of constraints into first and second class. Nonetheless, such information is not lost, as (2.3) follows from the map between Hamiltonian and Lagragian parameters [26,34,35]

$$N_1 = e, \qquad N_2 = l + g - e, \tag{2.5}$$

where (N_1, N_2) denote the number of first and second class constraints in Dirac's canonical formalism, respectively. Accordingly, (2.3) applies to both point particle systems and classical field theories, with the latter counting

degrees of freedom per point in spacetime. An intrinsically Lagrangian count of degrees of freedom is an important result that was obtained geometrically in [27]—see Eqs. (2), (3), and (16) therein. We remark that (2.3) was derived for first-order theories but has also been employed in higher-order settings [36,37].

A. Initialization

Consider the (primary) equations of motion of the theory, in the form

$$E_A \coloneqq \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu Q^A)}\right)-\frac{\partial\mathcal{L}}{\partial Q^A}=W_{AB}\ddot{Q}^B+U_A=0. \tag{2.6}$$

The (primary) Hessian

$$W_{AB}=\partial_{\dot{A}}\partial_{\dot{B}}\mathcal{L} \tag{2.7}$$

captures the linear dependence on the generalized accelerations $\ddot{Q}^A$, while

$$\begin{aligned}U_A=&(\partial_{\dot{A}}\partial^i_B\mathcal{L}+\partial_{\dot{B}}\partial^i_A\mathcal{L})\partial_i\dot{Q}^B\\&+(\partial^i_A\partial^j_B\mathcal{L})\partial_i\partial_jQ^B+(\partial^\mu_A\partial_B\mathcal{L})\partial_\mu Q^B-\partial_A\mathcal{L}.\end{aligned} \tag{2.8}$$

In the above, we have introduced the shorthand

$$\partial_{\dot{A}}=\frac{\partial}{\partial\dot{Q}^A},\quad \partial^i_A=\frac{\partial}{\partial(\partial_iQ^A)},\quad \partial^\mu_A=\frac{\partial}{\partial(\partial_\mu Q^A)},\quad \partial_A=\frac{\partial}{\partial Q^A}. \tag{2.9}$$

We further rewrite the (primary) equations of motion as

$$\mathbf{E}^{(1)}\coloneqq\mathbf{W}^{(1)}\ddot{\mathbf{Q}}+\mathbf{U}^{(1)}=0, \tag{2.10}$$

where $(\mathbf{E}^{(1)},\ddot{\mathbf{Q}},\mathbf{U}^{(1)})$ are N-dimensional column vectors and $\mathbf{W}^{(1)}$ is an $N\times N$ square matrix.

B. First iteration

The (primary) equations of motion (2.10) may but need not encode second order differential equations (SODEs) in time for all Q^A's. The number of functionally independent such SODEs is given by the (row) rank of the (primary) Hessian $\mathbf{W}^{(1)}$. The theory may have up to $(N-\mathrm{rank}\mathbf{W}^{(1)})$ primary Lagrangian constraints.

1. Step I. Rank of the Hessian

In full generality, the determination of the (row) rank of the (primary) Hessian $\mathbf{W}^{(1)}$ is a challenging task. A conceptually neat and algebraically convenient manner to do so is as follows. Assume $\mathbf{W}^{(1)}$ admits left null vectors

$$\mathbf{V}^{(1)}\cdot\mathbf{W}^{(1)}=0. \tag{2.11}$$

If no nontrivial solution to (2.11) exists, then the (row) rank of the (primary) Hessian is N and the theory is said to be regular. In this case, the theory possesses no (primary) Lagrangian constraints $l=l_1=0$. The constraint determining algorithm thus terminates.

Else, let $\mathbf{V}^{(1)}$ itself denote a maximal set of $M_1\in[1,N)$ linearly independent solutions to (2.11), normalized as per convenience. In this case, the (row) rank of the (primary) Hessian is $N-M_1$, the theory is said to be singular and up to M_1 primary Lagrangian constraints may exist, which we proceed to unveil.

2. Step II. Lagrangian constraints

Consider the left contraction of the M_1 null vectors $\mathbf{V}^{(1)}$ with the (primary) equations of motion (2.10)

$$\phi^{(1)}\coloneqq\mathbf{V}^{(1)}\cdot\mathbf{E}^{(1)}=\mathbf{V}^{(1)}\cdot\mathbf{U}^{(1)}=0. \tag{2.12}$$

By definition, $\phi^{(1)}$ is a set of M_1 relations that do not depend on the generalized accelerations $\ddot{Q}^A$. Any maximal (sub)set of functionally independent relations among (2.12) can be regarded as the primary Lagrangian constraints in the theory. Hence, $l_1\in[0,M_1]$.

It may happen that all relations in (2.12) identically vanish. If so, there exist no (primary) Lagrangian constraints $l=l_1=0$ and the constraint algorithm thus terminates.

In the absence of such trivialization, the distillation of a maximal (sub)set of functionally independent relations from (2.12) is generally demanding. A systematic way around the hurdle is similar to the determination of the (row) rank of the (primary) Hessian before. Assume the set of relations in (2.12), written as an M_1-dimensional column vector $\boldsymbol{\phi}^{(1)}$, admits solutions to

$$\mathbf{\Gamma}^{(1)}\cdot\boldsymbol{\phi}^{(1)}=0. \tag{2.13}$$

A generic ansatz for such a vector is

$$\mathbf{\Gamma}^{(1)}=(\Gamma^1,\Gamma^2,\ldots,\Gamma^{M_1}),\quad \Gamma^I=\Gamma^I_0+(\Gamma^I_1)^i\partial_i+(\Gamma^I_2)^{ij}\partial_i\partial_j. \tag{2.14}$$

Notice that every component of $\mathbf{\Gamma}^{(1)}$ includes spatial derivative operators up to second order. In this manner, both algebraic and spatial derivative dependences among the relations in (2.12) can be identified.

If no nontrivial solution to (2.13) exists, then all relations in (2.12) are functionally independent, implying $l_1=M_1$. Moreover, the relations (2.12) themselves can be interpreted as the primary Lagrangian constraints in the theory.

Else, let $\mathbf{\Gamma}^{(1)}$ itself denote a maximal set of $m_1\in[1,M_1)$ linearly independent solutions to (2.13), normalized as per convenience. In this case, there exist $l_1=M_1-m_1$ primary

Lagrangian constraints in the theory, which can be parametrized as

$$\phi_*^{(1)} \coloneqq (\mathbf{\Gamma}_0^{(1)})^{\perp} \cdot \boldsymbol{\phi}^{(1)} = 0, \tag{2.15}$$

where $(\mathbf{\Gamma}_0^{(1)})^{\perp}$ stands for a maximal set of linearly independent row vectors orthogonal to

$$\mathbf{\Gamma}_0^{(1)} = (\Gamma_0^1, \Gamma_0^2, \ldots, \Gamma_0^{M_1}) \subseteq \mathbf{\Gamma}^{(1)}, \tag{2.16}$$

normalized as per convenience.[1]

To sum up, for $l_1 \neq 0$, let

$$\mathbf{\Phi}^{(1)} = \begin{cases} \phi^{(1)} \text{ in } (2.12) & \text{if } l_1 = M_1, \\ \phi_*^{(1)} \text{ in } (2.15) & \text{if } l_1 = M_1 - m_1 \end{cases} \tag{2.17}$$

parametrize the primary Lagrangian constraints in the theory. For later convenience, let $\mathbf{\Phi}^{(1)}$ denote those same primary Lagrangian constraints, written as an l_1-dimensional column vector.

3. Step III. Stability of the Lagrangian constraints

Let $R = 1, 2, \ldots, l_1$ label the primary Lagrangian constraints (2.17). Self-consistency of the theory under time evolution implies

$$E_R \equiv \dot{\Phi}_R^{(1)} \coloneqq W_{RA}\ddot{Q}^A + U_R = 0 \tag{2.18}$$

must hold true, where

$$\begin{aligned} W_{RA} &= \partial_{\dot{A}}\Phi_R^{(1)} + (\partial_{\dot{A}}^i \Phi_R^{(1)})\partial_i, \\ U_R &= (\partial_{\dot{A}}^{ij}\Phi_R^{(1)})\partial_i\partial_j\dot{Q}^A + (\partial_{\dot{A}}^i\Phi_R^{(1)})\partial_i\dot{Q}^A + (\partial_A\Phi_R^{(1)})\dot{Q}^A. \end{aligned} \tag{2.19}$$

It is convenient to rewrite the above as

$$\mathbf{E}^{(2)} \equiv \dot{\mathbf{\Phi}}^{(1)} \coloneqq \mathbf{W}^{(2)}\ddot{\mathbf{Q}} + \mathbf{U}^{(2)} = 0, \tag{2.20}$$

where $(\mathbf{E}^{(2)}, \mathbf{U}^{(2)})$ are l_1-dimensional column vectors and $\mathbf{W}^{(2)}$ is an $l_1 \times N$ rectangular matrix.

The l_1 conditions (2.18), viewed as the secondary equations of motion for the system (2.20), may give rise to up to l_1 secondary Lagrangian constraints, which are to be unveiled in a second iteration of the constraint algorithm. Indeed, a conceptually simple repetition of the just described procedure yields the secondary Lagrangian constraints in the theory, if any. The only formal subtlety amounts to ensuring that only functionally independent secondary Lagrangian constraints are considered in the degree of freedom count (2.3). To this aim, let

$$\mathbf{E}^{(2)\downarrow} \coloneqq \mathbf{W}^{(2)\downarrow}\ddot{\mathbf{Q}} + \mathbf{U}^{(2)\downarrow} = 0 \tag{2.21}$$

encompass the primary (2.10) and secondary (2.20) equations of motion, in the form

$$\mathbf{E}^{(2)\downarrow} = \begin{pmatrix} \mathbf{E}^{(1)} \\ \mathbf{E}^{(2)} \end{pmatrix}, \quad \mathbf{W}^{(2)\downarrow} = \begin{pmatrix} \mathbf{W}^{(1)} \\ \mathbf{W}^{(2)} \end{pmatrix}, \quad \mathbf{U}^{(2)\downarrow} = \begin{pmatrix} \mathbf{U}^{(1)} \\ \mathbf{U}^{(2)} \end{pmatrix}. \tag{2.22}$$

Here, $\mathbf{E}^{(2)\downarrow}$ and $\mathbf{U}^{(2)\downarrow}$ are $(N + l_1)$-dimensional column vectors, while $\mathbf{W}^{(2)\downarrow}$ is an $(N + l_1) \times N$ rectangular matrix. The second iteration in the constraint algorithm takes $\mathbf{E}^{(2)\downarrow}$ as a starting point, as opposed to merely $\mathbf{E}^{(2)}$.

C. Generic iteration $n \geq 2$

For $m < n$, let $\mathbf{\Phi}^{(m)}$, denote the m-stage Lagrangian constraints, written as an l_m-dimensional column vector. Then, let

$$\mathbf{\Phi}^{(n-1)\downarrow} \coloneqq \begin{pmatrix} \mathbf{\Phi}^{(1)} \\ \mathbf{\Phi}^{(2)} \\ \vdots \\ \mathbf{\Phi}^{(n-1)} \end{pmatrix} \tag{2.23}$$

denote the ordered collection of all Lagrangian constraints unveiled thus far. We stress that $\mathbf{\Phi}^{(n-1)\downarrow}$ is an $(N^+ - N)$-dimensional column vector whose components have already been proven functionally independent, where

$$N^+ = N + \sum_{p=1}^{n-1} l_p. \tag{2.24}$$

On the other hand, let

$$\mathbf{E}^{(n)} \equiv \dot{\mathbf{\Phi}}^{(n-1)} \coloneqq \mathbf{W}^{(n)} \cdot \ddot{\mathbf{Q}} + \mathbf{U}^{(n)} = 0 \tag{2.25}$$

denote the n-stage equations of motion, written as an (l_{n-1})-dimensional column vector. Notice that the n-stage Hessian $\mathbf{W}^{(n)}$ is an $(l_{n-1} \times N)$ rectangular matrix. Further, let

$$\mathbf{E}^{(n)\downarrow} \coloneqq \mathbf{W}^{(n)\downarrow} \cdot \ddot{\mathbf{Q}} + \mathbf{U}^{(n)\downarrow} = 0 \tag{2.26}$$

denote the ordered collection of all equations of motion up to and including the n-stage, in the form

$$\mathbf{E}^{(n)\downarrow} = \begin{pmatrix} \mathbf{E}^{(1)} \\ \mathbf{E}^{(2)} \\ \vdots \\ \mathbf{E}^{(n)} \end{pmatrix}, \quad \mathbf{W}^{(n)\downarrow} = \begin{pmatrix} \mathbf{W}^{(1)} \\ \mathbf{W}^{(2)} \\ \vdots \\ \mathbf{W}^{(n)} \end{pmatrix}, \quad \mathbf{U}^{(n)\downarrow} = \begin{pmatrix} \mathbf{U}^{(1)} \\ \mathbf{U}^{(2)} \\ \vdots \\ \mathbf{U}^{(n)} \end{pmatrix}. \tag{2.27}$$

[1]Toy models for functional dependence detection and characterization by the above procedure are given in Appendix A 1.

Here, $(\mathbf{E}^{(n)\downarrow}, \mathbf{U}^{(n)\downarrow})$ are N^+-dimensional column vectors, while $\mathbf{W}^{(n)\downarrow}$ is an $N^+ \times N$ rectangular matrix.

1. Step I. Rank of the Hessian

First, the row rank of $\mathbf{W}^{(n)\downarrow}$ is to be determined. To this aim, assume it admits left null vectors

$$\mathbf{V}^{(n)\downarrow} \cdot \mathbf{W}^{(n)\downarrow} = 0. \tag{2.28}$$

A generic ansatz for such a null vector is[2]

$$\mathbf{V}^{(n)\downarrow} \equiv (\mathbf{V}^{(1)}, \mathbf{V}^{(2)}, \ldots, \mathbf{V}^{(n)}),$$
$$\mathbf{V}^{(m)} = \mathbf{V}_0 + \sum_{p=1}^{n-m} (\mathbf{V}_p)^{i_1 \ldots i_p} \partial_{i_1} \ldots \partial_{i_p}. \tag{2.29}$$

By construction, there indeed exist solutions to (2.28): they trivially extend the left null vector(s) $\mathbf{V}^{(n-1)\downarrow}$ found in the immediately previous stage through $\mathbf{V}^{(n)} = 0$. Such solutions do not carry new information. Consequently, they are to be dismissed.

If no left null vector to $\mathbf{W}^{(n)\downarrow}$ exists such that $\mathbf{V}^{(n)} \neq 0$, then

$$\mathrm{rrank}(\mathbf{W}^{(n)\downarrow}) = \mathrm{rrank}(\mathbf{W}^{(n-1)\downarrow}) + l_{n-1}, \tag{2.30}$$

where rrank stands for row rank. It follows that $l_n = 0$ and the algorithm thus terminates. In this case, $l = N^+ - N$.

Else, let $\mathbf{V}^{(n)\downarrow}$ itself denote a maximal set of $M_n \in [1, l_{n-1}]$ linearly independent left null vectors to $\mathbf{W}^{(n)\downarrow}$ such that $\mathbf{V}^{(n)} \neq 0$, normalized as per convenience. In this case,

$$\mathrm{rrank}(\mathbf{W}^{(n)\downarrow}) = \mathrm{rrank}(\mathbf{W}^{(n-1)\downarrow}) + l_{n-1} - M_n, \tag{2.31}$$

implying that up to M_n n-stage Lagrangian constraints may exist.

Algebraic ease dictates that the disclosure of the n-stage Lagrangian constraints, if any, is carried out in two steps. The first step guarantees functional independence within the nth iteration. The second step guarantees functional independence with respect to previous iterations.

2. Step II. Lagrangian constraints

Substep IIA. Functional independence within the stage. Consider the set of M_n relations

$$\boldsymbol{\phi}^{(n)} \coloneqq \mathbf{V}^{(n)\downarrow} \cdot \mathbf{E}^{(n)\downarrow} = \mathbf{V}^{(n)\downarrow} \cdot \mathbf{U}^{(n)\downarrow} = 0. \tag{2.32}$$

If the above relations trivially vanish, there exist no n-stage Lagrangian constraints $l_n = 0$. The constraint algorithm thus terminates, yielding $l = N^+ - N$.

Else, a maximal (sub)set of functionally independent relations among (2.32) is to be extracted. To this aim, let $\boldsymbol{\phi}^{(n)}$ denote the relations (2.32), written as an M_n-dimensional column vector. Assume $\boldsymbol{\phi}^{(n)}$ admits solutions to

$$\boldsymbol{\Gamma}^{(n)} \cdot \boldsymbol{\phi}^{(n)} = 0. \tag{2.33}$$

A generic ansatz for such a vector is

$$\boldsymbol{\Gamma}^{(n)} = (\Gamma^1, \Gamma^2, \ldots, \Gamma^{M_n}), \quad \Gamma^I = \Gamma^I_0 + \sum_{p=1}^{n-1} (\Gamma^I_p)^{i_1 \ldots i_p} \partial_{i_1} \ldots \partial_{i_p}. \tag{2.34}$$

If no nontrivial solution to (2.33) exists, then all relations in (2.32) are functionally independent among themselves.

Else, let $\boldsymbol{\Gamma}^{(n)}$ itself denote a maximal set of $m_n \in [1, M_n)$ linearly independent solutions to (2.33), normalized as per convenience. In this case, a maximal subset of $M_n - m_n$ functionally independent relations among (2.32) is

$$\boldsymbol{\phi}^{(n)}_* \coloneqq (\boldsymbol{\Gamma}^{(n)}_0)^{\perp} \cdot \boldsymbol{\phi}^{(n)} = 0, \tag{2.35}$$

where $(\boldsymbol{\Gamma}^{(n)}_0)^{\perp}$ stands for a maximal set of linearly independent row vectors orthogonal to

$$\boldsymbol{\Gamma}^{(n)}_0 = (\Gamma^1_0, \Gamma^2_0, \ldots, \Gamma^{M_n}_0) \subseteq \boldsymbol{\Gamma}^{(n)}, \tag{2.36}$$

normalized as per convenience.[1]

In conclusion, let

$$\boldsymbol{\varphi}^{(n)} = \begin{cases} \boldsymbol{\phi}^{(n)} \text{ in (2.32)} & \text{if (2.33) does not admit nontrivial solutions,} \\ \boldsymbol{\phi}^{(n)}_* \text{ in (2.35)} & \text{otherwise} \end{cases} \tag{2.37}$$

denote a maximal (sub)set of functionally independent relations among (2.32). Let $\mathfrak{M}_n$ denote their number, where $\mathfrak{M}_n = M_n$ or $M_n - m_n$, as per (2.37). For subsequent convenience, let $\boldsymbol{\varphi}^{(n)}$ denote those same relations, arranged in a column vector of dimension $\mathfrak{M}_n$.

Substep IIB. Functional independence with respect to previous stages. In an $n \geq 2$ iteration of the constraint algorithm, the disclosed maximal (sub)set of functionally independent relations (2.37) cannot be immediately

[2]Examples requiring the postulation and calculation of a nonobvious vector $\mathbf{V}^{(2)\downarrow}$ are given in Appendixes A 5 and A 6.

regarded as parametrizing the n-stage Lagrangian constraints. This is because (2.37) is not necessarily functionally independent from the Lagrangian constraints unveiled in previous iterations. We proceed to ensure such retroactive functional independence.

Let

$$\boldsymbol{\Psi}^{(n)\downarrow} \coloneqq \begin{pmatrix} \boldsymbol{\Phi}^{(n-1)\downarrow} \\ \boldsymbol{\varphi}^{(n)} \end{pmatrix} \tag{2.38}$$

denote the ordered collection of all previous stages' Lagrangian constraints $\boldsymbol{\Phi}^{(n-1)\downarrow}$ in (2.23) and the relations $\boldsymbol{\varphi}^{(n)}$ in (2.37). Recall that, by construction, both distinct sets $\boldsymbol{\Phi}^{(n-1)\downarrow}$ and $\boldsymbol{\varphi}^{(n)}$ comprise only functionally independent relations. As a result, upon joint consideration,

$$l_n = \operatorname{rrank}(\boldsymbol{\Psi}^{(n)\downarrow}) - \operatorname{rrank}(\boldsymbol{\Phi}^{(n-1)\downarrow}) \in [0, \mathfrak{M}_n]. \tag{2.39}$$

In order to determine l_n, assume $\boldsymbol{\Psi}^{(n)\downarrow}$ admits solutions to

$$\boldsymbol{\Upsilon}^{(n)\downarrow} \cdot \boldsymbol{\Psi}^{(n)\downarrow} = 0. \tag{2.40}$$

A generic ansatz for such a vector is[3]

$$\begin{aligned} \boldsymbol{\Upsilon}^{(n)\downarrow} &= (\Upsilon^{(1)}, \Upsilon^{(2)}, \ldots, \Upsilon^{(n)}), \\ \Upsilon^{(m)} &= \Upsilon_0^{(m)} + \sum_{p=1}^{n-m} (\Upsilon_p^{(m)})^{i_1 \ldots i_p} \partial_{i_1} \ldots \partial_{i_p}. \end{aligned} \tag{2.41}$$

If a maximal set of $\mathfrak{M}_n$ linearly independent solutions to (2.40) exists, then all relations (2.37) are functionally dependent with respect to $\boldsymbol{\Phi}^{(n-1)\downarrow}$. In this case, (2.38) has the minimal row rank

$$\operatorname{rrank}(\boldsymbol{\Psi}^{(n)\downarrow}) = \operatorname{rrank}(\boldsymbol{\Phi}^{(n-1)\downarrow}) = N^+ - N \tag{2.42}$$

and therefore $l_n = 0$. The constraint algorithm thus terminates, yielding $l = N^+ - N$.

Else, start by considering the diametrically opposite instance. If no nontrivial solution to (2.40) exists, then the relations (2.37) are functionally independent with respect to $\boldsymbol{\Phi}^{(n-1)\downarrow}$. In this case, $l_n = \mathfrak{M}_n$. Moreover, the relations (2.37) can be regarded as the n-stage Lagrangian constraints in the theory.

Next, consider all intermediate instances. Let $\boldsymbol{\Upsilon}^{(n)\downarrow}$ itself denote a maximal set of $\mathfrak{m}_n \in [1, \mathfrak{M}_n)$ linearly independent solutions to (2.40), normalized as per convenience. In this case, $l_n = \mathfrak{M}_n - \mathfrak{m}_n$ and a maximal subset of functionally independent relations among (2.37) is

$$\varphi_*^{(n)} \coloneqq (\boldsymbol{\Upsilon}_0^{(n)\downarrow})^{\perp} \cdot \boldsymbol{\Psi}^{(n)\downarrow} = 0, \tag{2.43}$$

where $(\boldsymbol{\Upsilon}_0^{(n)\downarrow})^{\perp}$ stands for a maximal set of linearly independent row vectors orthogonal to

$$\boldsymbol{\Upsilon}_0^{(n)\downarrow} = (\Upsilon_0^{(1)}, \Upsilon_0^{(2)}, \ldots, \Upsilon_0^{(n)}) \subseteq \boldsymbol{\Upsilon}^{(n)\downarrow}, \tag{2.44}$$

normalized as per convenience.

In short, for $l_n \neq 0$, let

$$\Phi^{(n)} = \begin{cases} \varphi^{(n)} \text{ in (2.37)} & \text{if (2.40) does not admit nontrivial solutions,} \\ \varphi_*^{(n)} \text{ in (2.43)} & \text{otherwise} \end{cases} \tag{2.45}$$

denote the n-stage Lagrangian constraints.

3. Step III. Stability of the Lagrangian constraints

Self-consistency demands that the n-stage Lagrangian constraints are preserved under time evolution: $\dot{\Phi}^{(n)} = 0$, for all l_n relations in (2.45). This condition may yield up to l_n Lagrangian constraints in a subsequent iteration of the constraint algorithm.

D. Closure

In order to unequivocally establish the number of degrees of freedom N_{DoF} in a given theory, it is imperative to pursue any constraint algorithm to its closure. Unfortunately and especially within coordinate-dependent Lagrangian approaches, persistence to termination is not always the case, as alerted against in [29,38]. In the method just advocated, there exist three distinct manners in which the constraint algorithm may close.

Let n_{f} denote the (finite) final iteration, wherein $l_{n_{\mathrm{f}}} = 0$. As per (2.25), let $\mathbf{E}^{(n_{\mathrm{f}})}$ and $\mathbf{W}^{(n_{\mathrm{f}})}$ denote the associated n_{f}-stage equations of motions and Hessian, respectively. $l_{n_{\mathrm{f}}} = 0$ is a direct consequence of one of the following instances:

(i) $\mathbf{W}^{(n_{\mathrm{f}})}$ has maximal row rank (2.30). In this case, consistency under time evolution of the $(n_{\mathrm{f}} - 1)$-stage Lagrangian constraints is ensured dynamically, through second-order (in time) differential equations of the variables Q^A. Examples of this closure can be found in Appendixes A 2 and A 4–A 6.

(ii) $\mathbf{W}^{(n_{\mathrm{f}})}$ does not have maximal row rank, but all contractions of its chosen left null vectors with $\mathbf{E}^{(n_{\mathrm{f}})}$

[3]An example requiring the postulation and calculation of a nonobvious vector $\boldsymbol{\Upsilon}^{(2)\downarrow}$ is given in Appendix A 7.

identically vanish. Namely, (2.32) is identically satisfied. This closure is the trivial expression of the functional dependence of the would-be Lagrangian constraints arising at the n_{f} stage on the previous stages' Lagrangian constraints. An example of this closure is provided in Appendix A 3.

(iii) Equation (2.32) is not identically satisfied, but it exclusively comprises relations that are functionally dependent on the previous stages' Lagrangian constraints. Namely, (2.42) is fulfilled. This constitutes the nontrivial counterpart to the previously described closure. An example is given in Appendix A 7.

We are not aware of any physical example within the scope of this work where constraint algorithms fail to close at a finite number of iterations.

E. Remarks

Lagrangian constraints are not uniquely defined, only the space they span is. At every iteration, we have advocated for the most convenient choice. Such choice is model dependent.

For all iterations in the constraint algorithm, it has been implicitly assumed that the row rank of the relevant Hessian remains constant. Presumably, the distinct dynamical behavior of the field configuration(s) for which there is a change in the rank of one or more of the Hessians remains encoded in the final stage's stack of equations of motion $\mathbf{E}^{(n_{\mathrm{f}})\downarrow}$—and hence in the later discussed dynamical problem (3.2).

The method readily applies to higher-order field theories whose equations of motion are linear in the generalized accelerations $\ddot{Q}^A$, as in (2.10). In this case, the (primary) Hessian and remaining terms have a more complicated, order-dependent relation to the Lagrangian than (2.7) and (2.8), but the algorithm *per se* remains unaltered.

III. DYNAMICAL VERSUS PHYSICAL MODES

Degrees of freedom are a foundational subject in physics. As such, they drive sustained investigations on and around themselves. Prominent questions under survey include their very definition, the well-posedness and solvability of their associated dynamical equations, and their relation to physical observables. In this section, we briefly ponder on such open-ended problems.

It is customary to view N_{DoF} as counting the (pairs of) initial conditions needed to define the dynamical problem of a given theory. We proceed to elucidate the previous assertion within the scope of Sec. II. This allows us to confront dynamical and physical modes.

Consider a theory of the form (2.1) for which the constraint algorithm has been successfully pursued until closure $l_{n_{\mathrm{f}}} = 0$. As per (2.26) and (2.27), let $\mathbf{E}^{(n_{\mathrm{f}})\downarrow}$ and $\mathbf{W}^{(n_{\mathrm{f}})\downarrow}$ denote the exhaustive stack of associated equations of motions and Hessians, respectively.

By construction and for singular theories, $\mathbf{W}^{(n_{\mathrm{f}})\downarrow}$ is an $N_{\mathrm{f}}^{+} \times N$ matrix with nonmaximal row rank

$$\varrho \equiv \mathrm{rrank}(\mathbf{W}^{(n_{\mathrm{f}})\downarrow}) = N_{\mathrm{f}}^{+} - M_{\mathrm{tot}} < N_{\mathrm{f}}^{+}, \qquad N_{\mathrm{f}}^{+} = N + l,$$
$$M_{\mathrm{tot}} = \sum_{p=1}^{n_{\mathrm{f}}} M_p. \tag{3.1}$$

(Recall that M_n denotes the maximal number of linearly independent left null vectors to $\mathbf{W}^{(n_{\mathrm{f}})\downarrow}$ that necessarily and at most involve the n-stage Hessian $\mathbf{W}^{(n)}$.) It follows that $\mathbf{W}^{(n_{\mathrm{f}})\downarrow}$ admits a maximal set of M_{tot} linearly independent left null vectors $\mathbf{V}^{(n_{\mathrm{f}})\downarrow}$, normalized as per convenience. Let $(\mathbf{V}^{(n_{\mathrm{f}})\downarrow})^{\perp}$ denote a maximal set of ϱ linearly independent row vectors orthogonal to $\mathbf{V}^{(n_{\mathrm{f}})\downarrow}$, normalized as per convenience.

Consider

$$\mathbb{E} \coloneqq (\mathbf{V}^{(n_{\mathrm{f}})\downarrow})^{\perp} \cdot \mathbf{E}^{(n_{\mathrm{f}})\downarrow} = 0. \tag{3.2}$$

The above comprises ϱ functionally independent second order differential equations in time for (some of) the variables Q^A. Supplemented by $2N_{\mathrm{DoF}}$ initial values for $(Q^A, \dot{Q}^A)$, (3.2) defines the dynamic problem of the theory (2.1). Conversely, N_{DoF} counts the pairs $(Q^A, \dot{Q}^A)$ whose time dependence is encoded in the just described dynamical problem. In other words, N_{DoF} counts dynamical modes. If any, variables Q^A present in the Lagrangian (2.1) but not determined by the dynamical problem comprise pure gauge modes. In this regard, the interested reader is gladly referred to [39].

The dynamical problem may but need not fall within the scope of the Cauchy-Kovalevskaya (CK) theorem. When it does, a unique analytical solution is guaranteed to exist. When it does not, existence (let alone uniqueness) of analytical solutions cannot be generically ascertained. Physics-driven extensions to the CK theorem thus comprise an enticing line of mathematical research.

Consider a generic Lagrangian field theory, possibly beyond the subclass in Sec. II. Presume a fortunate case in which one or more analytical solutions to its dynamical problem can be found. Even then, the obtained dynamical modes should not be immediately identified with physical modes, in the sense that dynamical modes may exhibit a behavior that is incompatible with well-established physical principles and/or observation. Reasons are plentiful.

First, consider stability criteria. For instance, solutions could be perturbatively unstable, in the sense of lacking robustness against small deviations in the initial data and/or free parameters. Whenever in conflict with observation, such solutions are to be disregarded. A lucid introduction to the most frequent perturbative instabilities in gravitational settings is [40]. Numerical examples of critical values for free parameters which dramatically destabilize a theory can be found in [41]. Moreover, it has been known for a while that instabilities could also appear only at

the nonperturbative level [42]. For an enlightening recent review apropos, see Ref. [43].

Causality is yet another essential requirement for a dynamical mode to be deemed physical. As a remarkable example, we note [44], where causality was employed to constrain dynamical modes for certain massive gravity theories. Overall, the quest remains for necessary and sufficient conditions that guarantee physicality of dynamical modes, even within the theoretical realm.

More generally, it is worth considering physicality of a (classical) field theory as a whole. From a philosophical perspective, degrees of freedom might help to address the question of physical equivalence between theories, which enjoys a long tradition in the philosophy of science, see e.g. [45–47]. While several notions of equivalence are discussed in the recent literature [48,49], there exists widespread consensus that dynamical equivalence of two theories is a necessary condition for their empirical equivalence [50]. In this context, we regard a match in the number of degrees of freedom as a prefatory necessary (but not sufficient) condition for physical equivalence between two theories.

IV. CONCLUSIONS

We have presented a Lagrangian method to count degrees of freedom in first-order classical field theories. The emphasis is two-fold. First, a systematic and algebraically convenient procedure to establish the functional independence of the constraints that may be present in such theories. Second, a detailed discussion on the possible closures of the associated constraint algorithm. Both are consequential aspects that are rarely explicitly addressed in akin Lagrangian approaches. Nonexhaustive counterexamples to the former omission can be found in [27,29]. The latter was painstakingly discussed in [29] and, for a certain family of massive electrodynamics theories, in [38].

We stress that functional independence among constraints is essential to the postulation of self-consistent theories. This is particularly relevant when a given constraint structure is being pursued, i.e. fixed values for the number of primary, secondary, etc. (Lagrangian) constraints. Against this background, we note that a suitable (row) rank reduction of an n-stage Hessian does not ensure the desired number of n-stage constraints are generated. While such (row) rank reduction is a necessary condition for the sought constraint structure, it is premature to regard it as sufficient on its own [38].

Failure to close a constraint algorithm may yield an incorrect number of degrees of freedom N_{DoF}. It could happen that N_{DoF} is overestimated, via the overlooking of overconstrained systems. Contrastively, N_{DoF} may be underestimated, via the misidentification of functionally dependent relations of the type in (2.32) as (Lagrangian) constraints.

Last but not least, we have discussed several nontrivial conditions that propagating degrees of freedom must fulfill before they can be regarded as physical modes.

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APPENDIX: EXAMPLES

We begin in Sec. A 1 by providing simple yet illuminating examples for the obtention of functionally independent constraints from a given out-of-context set. The remainder of the appendix is devoted to the explicit application of the method presented in Sec. II to count degrees of freedom in various examples of physical relevance. The appendix thus serves to amply illustrate the use of the method, at various levels of algebraic intricacy.

Notation. Brackets denoting symmetrization and antisymmetrization of indices are defined as $T_{(\mu\nu)} = (T_{\mu\nu} + T_{\nu\mu})/2$ and $T_{[\mu\nu]} = (T_{\mu\nu} - T_{\nu\mu})/2$, respectively. For the two-dimensional examples in Secs. A 1, A 2, and A 6, we use the shorthand $T' = \partial_1 T$. Natural units are employed throughout.

1. Detection and avoidance of functional dependence among *ad hoc* constraints

a. Toy model I

Consider a theory of the form (2.1) in two-dimensional Minkowski spacetime. Further consider the set of M_n relations in (2.12) for $n = 1$ or (2.32) for $n \geq 2$. Suppose $M_n = 2$ has been obtained, with the relations arranged into a column vector of the form

$$\boldsymbol{\phi} = \begin{pmatrix} F \\ F' \end{pmatrix}, \qquad F = F(Q^A, \partial_\mu Q^A). \tag{A1}$$

(For simplicity, we omit indices indicating the iteration.) In view of the spatial derivatives' order difference between the

two relations, a generic ansatz to (2.13) for $n = 1$ or to (2.33) for $n \geq 2$ is particularly simple in this case:

$$\mathbf{\Gamma} = (\Gamma_0 + \Gamma_1\partial_1, \tilde{\Gamma}_0), \tag{A2}$$

which readily yields a single linearly independent solution $m_n = 1$ parametrized by $\Gamma_0 = 0$ and $\Gamma_1 = -\tilde{\Gamma}_0$. We choose a convenient normalization for the solution, look into its algebraic subspace and choose a convenient normalization for the $M_n - m_n = 1$ linearly independent orthogonal vector:

$$\mathbf{\Gamma} = (-\partial_1, 1), \qquad \mathbf{\Gamma}_0 = (0, 1), \qquad \mathbf{\Gamma}_0^{\perp} = (1, 0). \tag{A3}$$

The left contraction of the latter with (A1) yields a functionally independent relation: $\phi_* = \mathbf{\Gamma}_0^{\perp} \cdot \boldsymbol{\phi} = F$. For $n = 1$, ϕ_* can be regarded as the primary Lagrangian constraint. For $n \geq 2$, functional independence of ϕ_* with respect to Lagrangian constraints unveiled in previous iterations must be ensured before regarding ϕ_* as the n-stage Lagrangian constraint.

b. Toy model II

Consider a theory of the form (2.1) in three-dimensional Minkowski spacetime. Further, consider the following set of $M_n = 3$ relations in (2.12) for $n = 1$ or (2.32) for $n \geq 2$, arranged into a column vector

$$\boldsymbol{\phi} = \begin{pmatrix} F_x \\ F + G_y \\ G \end{pmatrix}, \qquad F_x \equiv \partial_1 F, \qquad G_y \equiv \partial_2 G, \tag{A4}$$

where (F, G) denote obviously functionally independent relations; for instance $F = F(Q^A, \hat{Q}^B, \partial_\mu Q^A)$ and $G = G(Q^A, \partial_\mu Q^A)$, with the hat denoting a specific coordinate within the set Q^A that is not present. Observation of the relative difference in the order of spatial derivatives between the relations leads us to postulate a generic ansatz to (2.13) for $n = 1$ or to (2.33) for $n \geq 2$ of the form

$$\mathbf{\Gamma} = (\Gamma_0 + \Gamma_1^x\partial_1 + \Gamma_1^y\partial_2, \quad \tilde{\Gamma}_0 + \tilde{\Gamma}_1^x\partial_1 + \tilde{\Gamma}_1^y\partial_2, \quad \hat{\Gamma}_0 + \hat{\Gamma}_1^x\partial_1 + \hat{\Gamma}_1^y\partial_2 + \hat{\Gamma}_2^{xx}\partial_1\partial_1 + 2\hat{\Gamma}_2^{xy}\partial_1\partial_2 + \hat{\Gamma}_2^{yy}\partial_2\partial_2). \tag{A5}$$

The above readily yields a single linearly independent solution $m_n = 1$ parametrized by $\tilde{\Gamma}_1^x = -\Gamma_0$ and $\hat{\Gamma}_2^{xy} = \Gamma_0/2$, with all other free functions set to zero. We choose a convenient normalization for this solution, look into its algebraic subspace and choose a convenient normalization for the $M_n - m_n = 2$ linearly independent orthogonal vectors:

$$\mathbf{\Gamma} = (1, -\partial_1, \partial_{(1}\partial_{2)}), \qquad \mathbf{\Gamma}_0 = (1, 0, 0), \qquad \mathbf{\Gamma}_0^{\perp} = (0, 1, 0), \qquad \tilde{\mathbf{\Gamma}}_0^{\perp} = (0, 0, 1). \tag{A6}$$

The left contraction of the last two with (A4) yields two functionally independent relations

$$\phi_* = \mathbf{\Gamma}_0^{\perp} \cdot \boldsymbol{\phi} = F + G_y, \qquad \tilde{\phi}_* = \tilde{\mathbf{\Gamma}}_0^{\perp} \cdot \boldsymbol{\phi} = G. \tag{A7}$$

For $n = 1$, (A7) can be readily regarded as the primary Lagrangian constraints. For $n \geq 2$, functional independence of (A7) with respect to Lagrangian constraints unveiled in previous iterations must be ensured before reaching such a conclusion.

Notice that the simplest choice of orthogonal vectors $(\mathbf{\Gamma}_0^{\perp}, \tilde{\mathbf{\Gamma}}_0^{\perp})$ does not yield the obviously simplest span of the constraint space, given by $\{F, G\}$. Toy model II thus illustrates our first remark in Sec. II E.

2. Floreanini-Jackiw chiral boson

The Lagrangian for the two-dimensional theory of a chiral boson due to Floreanini and Jackiw [51] is

$$\mathcal{L}_{\mathrm{FJ}} = \frac{1}{2}\phi'(\dot{\phi} - \phi'). \tag{A8}$$

Here, the scalar field $\phi = \phi(x^0, x^1)$ is the only *a priori* independent field variable Q^A and so $N = 1$. As is well known, this theory possesses no local symmetries—neither of the relevant form (2.2) nor otherwise—in its original formulation (A8). Therefore, $g, e = 0$. It is worth mentioning that a manifestly Lorentz invariant action for the Floreanini-Jackiw chiral boson exists, which has been further generalized into the so-called $2k$-form electrodynamics family of higher-dimensional theories [52].

The (primary) equations of motion following from (A8) are of the form (2.10), with

$$\mathbf{W}^{(1)} = 0, \qquad \mathbf{U}^{(1)} = \dot{\phi}' - \phi''. \tag{A9}$$

Obviously, the (row) rank of the (primary) Hessian is zero. A convenient left null vector for it is simply $\mathbf{V}^{(1)} = 1$. As per (2.12) and since no identical vanishing happens,

$$\phi^{(1)} \coloneqq \dot{\phi}' - \phi'' = 0 \tag{A10}$$

can be readily regarded as the only primary Lagrangian constraint in the theory $l_1 = 1$.

The (primary) equations of motion, together with the demand for stability under time evolution of the primary Lagrangian constraint, conform the starting point of the second iteration (2.21), where

$$\mathbf{W}^{(2)\downarrow} = \begin{pmatrix} 0 \\ \partial_1 \end{pmatrix}, \qquad \mathbf{U}^{(2)\downarrow} = \begin{pmatrix} \mathbf{U}^{(1)} \\ -\dot{\phi}'' \end{pmatrix}. \tag{A11}$$

Clearly, $\mathbf{W}^{(2)\downarrow}$ only admits left null vectors of the form

$$\mathbf{V}^{(2)\downarrow} = (\mathbf{V}^{(1)}, 0). \tag{A12}$$

According to the discussion below (2.29), there exists no secondary Lagrangian constraint in the theory $l_2 = 0$. The constraint algorithm thus terminates, by means of closure 1.

Using (2.3) and (2.4), we reproduce the renowned result that the theory propagates $N_{\mathrm{DoF}} = 1/2$ degrees of freedom.

3. Maxwell electrodynamics

The Lagrangian for standard electrodynamics in $d \geq 2$ dimensions is

$$\mathcal{L}_{\mathrm{M}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}, \tag{A13}$$

where the components of the vector field $A_\mu = A_\mu(x^0, x^i)$ conform the *a priori* independent field variables Q^A and thus $N = d$. The theory enjoys a manifest $U(1)$ gauge invariance, under the transformation

$$A_\mu \to A_\mu + \partial_\mu \theta, \tag{A14}$$

which is of the relevant form (2.2). It follows that $g = 1$ and $e = 2$.

The (primary) equations of motion for A_μ following from (A13) are of the form (2.10), with

$$\begin{aligned} W^{(1)\mu\nu} &= \eta^{\mu\nu} + \eta^{\mu 0}\eta^{\nu 0} \equiv \mathcal{W}^{\mu\nu}, \\ U^{(1)\mu} &= 2(\eta^{0(\mu}\eta^{\nu)i}\partial_i \dot{A}_\nu + \eta^{i[\nu}\eta^{j]\mu}\partial_i\partial_j A_\nu) \equiv \mathcal{A}^\mu. \end{aligned} \tag{A15}$$

It is easy to see that the (row) rank of the (primary) Hessian is $d - 1$. A convenient left null vector for it is $\mathbf{V}^{(1)} = \delta^0_\mu$. As per (2.12) and since no identical vanishing happens,

$$\phi^{(1)} \coloneqq \partial^i F_{i0} = 0 \tag{A16}$$

can be identified with the primary Lagrangian constraint in the theory $l_1 = 1$. In fact, this is Gauss's law for the electric field.

The (primary) equations of motion, together with the demand for stability under time evolution of the primary Lagrangian constraint, conform the starting point of the second iteration. They can be written as (2.21), where

$$\mathbf{W}^{(2)\downarrow} = \begin{pmatrix} \mathcal{W}^{\mu\nu} \\ -\eta^{i\nu}\partial_i \end{pmatrix}, \qquad \mathbf{U}^{(2)\downarrow} = \begin{pmatrix} \mathcal{A}^\mu \\ \nabla^2 \dot{A}_0 \end{pmatrix}. \tag{A17}$$

It is easy to see that, up to normalization, there exists only one linearly independent left null vector to $\mathbf{W}^{(2)\downarrow}$ that does not trivially extend $\mathbf{V}^{(1)}$. We choose it as $\mathbf{V}^{(2)\downarrow} = (\delta^i_\mu \partial_i, 1)$. We remark that the above is a particular instance of the general form prescribed in (2.29). As per (2.32),

$$\phi^{(2)} \coloneqq \partial^i \partial^j F_{ij} \equiv 0. \tag{A18}$$

Hence, no secondary Lagrangian constraint exists $l_2 = 0$. The constraint algorithm thus terminates, by means of closure 2.

Using (2.3) and (2.4), we obtain the familiar result $N_{\mathrm{DoF}} = d - 2$.

4. Proca electrodynamics

Consider the simplest massive electrodynamics theory in $d \geq 2$ dimensions

$$\mathcal{L}_{\mathrm{P}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu, \qquad m \in \mathbb{R}_{>0}. \tag{A19}$$

As for Maxwell electrodynamics before, the components of the vector field $A_\mu = A_\mu(x^0, x^i)$ conform the *a priori* independent field variables Q^A and thus $N = d$. Contrastively, the mass term explicitly breaks the $U(1)$ gauge invariance of (A13), leaving no residual symmetry. Hence, $g = e = 0$.

The (primary) equations of motion for A_μ following from (A19) are of the form (2.10), with

$$W^{(1)\mu\nu} = \mathcal{W}^{\mu\nu}, \qquad U^{(1)\mu} = \mathcal{A}^\mu + m^2 A^\mu, \tag{A20}$$

where $(\mathcal{W}^{\mu\nu}, \mathcal{A}^\mu)$ were defined in (A15). The (primary) Hessian is the same as for Maxwell electrodynamics (A15), with rank $d - 1$. We again chose $\mathbf{V}^{(1)} = \delta^0_\mu$ as a convenient left null vector for it. Using (2.12) and since no identical vanishing happens,

$$\phi^{(1)} \coloneqq \partial^i F_{i0} - m^2 A_0 = 0 \tag{A21}$$

can be identified with the primary Lagrangian constraint in the theory $l_1 = 1$.

Next, consider the (primary) equations of motion, together with the demand for stability under time evolution of the primary Lagrangian constraint, in the form (2.21), where

$$\mathbf{W}^{(2)\downarrow} = \begin{pmatrix} \mathcal{W}^{\mu\nu} \\ -\eta^{i\nu}\partial_i \end{pmatrix}, \quad \mathbf{U}^{(2)\downarrow} = \begin{pmatrix} \mathcal{A}^\mu + m^2 A^\mu \\ (\nabla^2 - m^2)\dot{A}_0 \end{pmatrix}. \tag{A22}$$

The above conforms the starting point of the second iteration in the constraint algorithm. We note that $\mathbf{W}^{(2)\downarrow}$ for Proca electrodynamics matches that of Maxwell's theory (A17) and we repeat our choice $\mathbf{V}^{(2)\downarrow} = (\delta^i_\mu \partial_i, 1)$ for a conveniently normalized left null vector that is linearly independent from a trivial extension of $\mathbf{V}^{(1)}$. Using (2.32), we obtain the relation

$$\phi^{(2)} \coloneqq m^2 \partial^\mu A_\mu = 0. \tag{A23}$$

Clearly, the above does not identically vanish. It is also easy to see that (A21) and (A23) are functionally independent. Therefore, $l_2 = 1$ and (A23) can be taken as the secondary Lagrangian constraint.

In a third iteration of the constraint algorithm, we consider the (primary) equations of motion, along with the demand for stability under time evolution of both the primary and the secondary Lagrangian constraints, in the form (2.21), where the tertiary equations of motion are given by

$$\mathbf{W}^{(3)} = (m^2 \eta^{\mu 0}), \qquad \mathbf{U}^{(3)} = (m^2 \nabla \dot{A}). \tag{A24}$$

It is obvious that $\mathbf{W}^{(3)\downarrow}$ does not admit left null vectors beyond trivial extensions of $(\mathbf{V}^{(1)}, \mathbf{V}^{(2)})$ before. Consequently, there exists no tertiary Lagrangian constraints $l_3 = 0$ and the algorithm terminates according to closure 1.

Using (2.3) and (2.4), we reproduce the well-known result that Proca electrodynamics propagates $N_{\mathrm{DoF}} = d - 1$ degrees of freedom.

5. Podolsky electrodynamics

Podolsky's proposal for a generalized electrodynamics theory [53]

$$\mathcal{L}_{\mathrm{Po}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{a^2}{2} \partial_\mu F^{\mu\nu} \partial^\rho F_{\rho\nu} \tag{A25}$$

is arguably the best-known higher-order field theory. So as to remain within the scope of Sec. II, we consider its first-order formulation [54] in $d \geq 2$ dimensions

$$\mathcal{L}_{\mathrm{Po1}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} B_\mu B^\mu - \frac{a^2}{2} G_{\mu\nu} F^{\mu\nu}, \quad G_{\mu\nu} = 2\partial_{[\mu} B_{\nu]}, \quad a \in \mathbb{R}_{>0}. \tag{A26}$$

The components of the vector fields $A_\mu = A_\mu(x^0, x^i)$, $B_\mu = B_\mu(x^0, x^i)$ are the *a priori* independent field variables Q^A and hence $N = 2d$. The Lagrangian (A26) inherits the symmetry of Maxwell electrodynamics and is gauge invariant under the field transformations

$$A_\mu \to A_\mu + \partial_\mu \theta, \qquad B_\mu \to B_\mu, \tag{A27}$$

which are of the relevant form (2.2). It follows that $g = 1$ and $e = 2$, as in the Maxwell case earlier on.

The (primary) equations of motion following from (A26) are of the form (2.10), with

$$\mathbf{W}^{(1)} = \begin{pmatrix} 1 & a^2 \\ a^2 & 0 \end{pmatrix} \mathcal{W}^{\mu\nu}, \quad \mathbf{U}^{(1)} = \begin{pmatrix} \mathcal{A}^\mu + a^2 \mathcal{B}^\mu \\ a^2 \mathcal{A}^\mu - a^2 B^\mu \end{pmatrix}, \tag{A28}$$

where $(\mathcal{W}^{\mu\nu}, \mathcal{A}^\mu)$ were introduced in (A15) and $\mathcal{B}^\mu$ stands for the same quantity as $\mathcal{A}^\mu$, but in terms of B^μ instead of A^μ. It is rather obvious that the (row) rank of the (primary) Hessian is $2(d-1)$. A convenient choice for the two linearly independent left null vectors is $\mathbf{V}_1^{(1)} = (\delta_\mu^0, 0)$ and $\mathbf{V}_2^{(1)} = (0, \delta_\mu^0)$. Using (2.12), since no identical vanishing happens and taking into consideration the manifest functional independence,

$$\mathbf{\Phi}^{(1)} \coloneqq \begin{pmatrix} \partial^i F_{i0} + a^2 \partial^i G_{i0} \\ a^2 \partial^i F_{i0} + a^2 B_0 \end{pmatrix} = 0 \tag{A29}$$

can be identified with the two primary Lagrangian constraints in the theory $l_1 = 2$.

In order to ensure the stability of the primary Lagrangian constraints, we calculate the theory's secondary equations of motions (2.20). We find

$$\mathbf{W}^{(2)} = \begin{pmatrix} -1 & -a^2 \\ -a^2 & 0 \end{pmatrix} \eta^{\mu i} \partial_i, \quad \mathbf{U}^{(2)} = \begin{pmatrix} \nabla^2 \dot{A}_0 + a^2 \nabla^2 \dot{B}_0 \\ a^2 \nabla^2 \dot{A}_0 + a^2 \dot{B}_0 \end{pmatrix} \tag{A30}$$

and consider them together with the (primary) equations of motion, as described in (2.21) and (2.22). Beyond trivial extensions of $\mathbf{V}_1^{(1)}$ and $\mathbf{V}_2^{(1)}$ before and up to normalization, there exists another linearly independent left null vector $\mathbf{V}^{(2)\downarrow}$ to $\mathbf{W}^{(2)\downarrow}$. It can be found as prescribed around (2.28). Explicitly, we postulate

$$\mathbf{V}^{(2)\downarrow} = (V_\mu^i \partial_i, \quad \tilde{V}_\mu^i \partial_i, \quad \hat{V}, \quad \bar{V}) \quad \text{such that} \quad \begin{cases} (V_\mu^i + a^2 \tilde{V}_\mu^i) \mathcal{W}^{\mu\nu} - (\hat{V} + a^2 \bar{V}) \eta^{\nu i} = 0, \\ V_\mu^i \mathcal{W}^{\mu\nu} - \hat{V} \eta^{\nu i} = 0. \end{cases} \tag{A31}$$

Our simplified ansatz is a direct consequence of the overall spatial derivatives' order difference between $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$. We choose as representative of the one-parameter family of linearly independent solutions

$$\mathbf{V}^{(2)\downarrow} = (a^2 \delta_\mu^i \partial_i, \quad -\delta_\mu^i \partial_i, \quad a^2, \quad -1). \tag{A32}$$

Using the above and (2.32), we obtain the relation

$$\phi^{(2)} \coloneqq a^2 \partial^\mu B_\mu = 0. \tag{A33}$$

It is easy to see that (A33) does neither identically vanish nor functionally depend on the primary Lagrangian constraints (A29). Therefore, $l_2 = 1$ and (A33) can be regarded as the secondary Lagrangian constraint in the theory. Note it matches the secondary Lagrangian constraint (A23) in Proca electrodynamics.

In a third iteration of the constraint algorithm, we consider the (primary) equations of motion, together with the demand for stability of the primary and secondary Lagrangian constraints, in the form (2.27). For the tertiary equations of motion, we obtain

$$\mathbf{W}^{(3)} = (\,0, \quad a^2\eta^{\mu 0}\,), \qquad \mathbf{U}^{(3)} = (a^2 \nabla \dot{B}). \tag{A34}$$

It is a matter of conjunct inspection of $(\mathbf{W}^{(1)}, \mathbf{W}^{(2)}, \mathbf{W}^{(3)})$ to deduce that $\mathbf{W}^{(3)\downarrow}$ admits no other left null vectors beyond trivial extensions of the left null vectors obtained in earlier iterations. Consequently, no tertiary Lagrangian constraints exist $l_3 = 0$ and the algorithm terminates according to closure 1.

Using (2.3) and (2.4), we reproduce the well-known result $N_{\mathrm{DoF}} = 2d - 3$. This count supports the interpretation of Podolsky electrodynamics as the theory of an interacting pair of vector fields, one of which is massive e.g. [55,56]. It is also instructive to compare the constraint structures in this and the two previous sections A 3 and A 4.

6. Minimal model in extended Proca-Nuevo

We consider the simplest, two-dimensional case in the so-called extended Proca-Nuevo (EPN) class of massive vector field theories, dubbed minimal model. EPN was originally proposed in [57] as an extension of the Proca-Nuevo (PN) construction in [58]. A study of the constraint structure of both PN and EPN, including an explicit discussion of the minimal model here revisited, can be found in [59].

The Lagrangian for the minimal model is

$$\mathcal{L}_{\mathrm{MM}} = \Lambda^2(\alpha + 2N - 4), \tag{A35}$$

where Λ is a constant of length dimension (-1), α is a dimensionless and at least twice differentiable function of the square of the vector

$$\alpha = \alpha(X), \qquad X = -A_0^2 + A_1^2, \tag{A36}$$

and $N \neq 0$ is given by

$$N = \sqrt{x^2 - y^2}, \quad x = 2 + \frac{A_1' - \dot{A}_0}{\Lambda}, \quad y = \frac{\dot{A}_1 - A_0'}{\Lambda}. \tag{A37}$$

In (A35), the components of the vector field $A_\mu = A_\mu(x^0, x^1)$ are the *a priori* independent field variables Q^A and so $N = 2$. By definition, theories within the (E)PN class(es) do not possess any local symmetry. Therefore, $g,\ e = 0$.

The (primary) equations of motion following from (A35) are of the form (2.10), with

$$\mathbf{W}^{(1)} = -\frac{2}{N^3}\begin{pmatrix} y^2 & xy \\ xy & x^2 \end{pmatrix},$$
$$\mathbf{U}^{(1)} = \begin{pmatrix} \frac{2}{N^3}[xy(2\dot{A}_0' - A_1'') - x^2 A_0'' + \bar{N}^2 \dot{A}_1'] + 2\Lambda^2 \alpha_X A_0 \\ \frac{2}{N^3}[xy(2\dot{A}_1' - A_0'') - y^2 A_1'' + \bar{N}^2 \dot{A}_0'] - 2\Lambda^2 \alpha_X A_1 \end{pmatrix}, \tag{A38}$$

where we have introduced the shorthand

$$\bar{N} = \sqrt{x^2 + y^2}, \quad \alpha_X = \frac{d\alpha(X)}{dX}, \quad \alpha_{XX} = \frac{d^2\alpha(X)}{dX^2}, \tag{A39}$$

and the second (total) derivative of α has been defined for later convenience. It is easy to see that the (row) rank of the (primary) Hessian is 1. Following [58,59], we choose a left null vector to $\mathbf{W}^{(1)}$ as

$$\mathbf{V}^{(1)} = \frac{1}{N}(x, -y). \tag{A40}$$

As per (2.12) and since no identical vanishing happens,

$$\phi^{(1)} \coloneqq \frac{2\Lambda}{N^2}(xy' - yx') + \frac{2\Lambda^2}{N}\alpha_X(xA_0 + yA_1) = 0 \tag{A41}$$

can be readily regarded as the only primary Lagrangian constraint in the theory $l_1 = 1$.

Following (2.20), at the second iteration we find

$$\mathbf{W}^{(2)} = (\,\Omega_1 \quad \Omega_2\,),$$
$$\mathbf{U}^{(2)} = -\frac{2}{N^2}(x\dot{A}_0'' + y\dot{A}_1'') + \frac{2\bar{N}^2}{N^4}(x'\dot{A}_0' - y'\dot{A}_1') - \frac{4xy}{N^4}(y'\dot{A}_0' - x'\dot{A}_1')$$
$$-\frac{2\Lambda}{N^3}\alpha_X(yA_0 + xA_1)(x\dot{A}_0' + y\dot{A}_1') + \frac{2\Lambda^2}{N}\alpha_X(x\dot{A}_0 + y\dot{A}_1) + \frac{2\Lambda^2}{N}\alpha_{XX}(xA_0 + yA_1)\dot{X}, \tag{A42}$$

where we have introduced

$$\Omega_1 = \omega_1 + \bar{\omega}_1\partial_1 = \frac{4x}{N^4}(xy' - yx') - \frac{2y'}{N^2} + \frac{2\Lambda}{N^3}\alpha_X y(xA_1 + yA_0) + \frac{2y}{N^2}\partial_1,$$
$$\Omega_2 = \omega_2 + \bar{\omega}_2\partial_1 = \frac{4y}{N^4}(xy' - yx') - \frac{2x'}{N^2} + \frac{2\Lambda}{N^3}\alpha_X x(xA_1 + yA_0) + \frac{2x}{N^2}\partial_1. \tag{A43}$$

As per (2.21) and (2.22), we proceed to the joint consideration of the primary and secondary equations of motion. In particular, we inspect the row rank of $\mathbf{W}^{(2)\downarrow}$ via a conveniently normalized generic left null vector of the form (2.29)

$$\mathbf{V}^{(2)\downarrow} = (V_0 + V_1\partial_1, \tilde{V}_0 + \tilde{V}_1\partial_1, 1). \tag{A44}$$

For the above to fulfill (2.28), four *a priori* independent equations must be fulfilled. These are

$$\begin{cases} -\frac{2y}{N^3}(V_1 y + \tilde{V}_1 x) + \bar{\omega}_1 = 0, \\ -\frac{2x}{N^3}(V_1 y + \tilde{V}_1 x) + \bar{\omega}_2 = 0, \\ -\frac{2y}{N^3}(V_0 y + \tilde{V}_0 x) - \frac{2y}{N^3}(V_1 y' + \tilde{V}_1 x') - 2\left(\partial_1 \frac{y}{N^3}\right)(V_1 y + \tilde{V}_1 x) + \omega_1 = 0, \\ -\frac{2x}{N^3}(V_0 y + \tilde{V}_0 x) - \frac{2x}{N^3}(V_1 y' + \tilde{V}_1 x') - 2\left(\partial_1 \frac{x}{N^3}\right)(V_1 y + \tilde{V}_1 x) + \omega_2 = 0. \end{cases} \tag{A45}$$

The first (second) equation can be easily seen to be redundant with respect to the second (first) equation. Assuming $x,\ y \neq 0$, either of these equations poses the same null vector condition on (A44):

$$\frac{1}{N}(V_1 y + \tilde{V}_1 x) - 1 = 0. \tag{A46}$$

Additionally and using (A46), the third and fourth equations can be seen to boil down to the same null vector condition on (A44):

$$V_0 y + \tilde{V}_0 x + \frac{xy' - yx'}{y}\left(\frac{x}{N} - \tilde{V}_1\right) - \Lambda\alpha_X(xA_1 + yA_0) = 0. \tag{A47}$$

On the whole, a (unique, up to normalization) linearly independent solution is given by

$$V_0 = \Lambda\alpha_X A_0, \quad \tilde{V}_0 = -\frac{y}{N}, \quad V_1 = \Lambda\alpha_X A_1, \quad \tilde{V}_1 = \frac{x}{N}. \tag{A48}$$

Pursuing the constraint algorithm by means of (2.32), we find that the left contraction of the above null vector with $\mathbf{E}^{(2)\downarrow}$ gives rise to the relation

$$\phi^{(2)} := \frac{2\Lambda}{N^4}(xy' - yx')^2 + \frac{2\Lambda^3}{N}\alpha_X(2x - N^2) + \frac{2\Lambda^2}{N^3}\alpha_X(xy' - yx')(xA_0 + yA_1) - 2\Lambda^3\alpha_X^2 X + \frac{2\Lambda^2}{N}\alpha_{XX}[\dot{X}(xA_0 + yA_1) - X'(yA_0 + xA_1)] = 0. \tag{A49}$$

Close inspection readily reveals that (A41) and (A49) are functionally independent. Complementarily, it can be checked that there exists no nontrivial solution to (2.40) for the minimal model. As a result, there exists a secondary constraint in the theory $l_2 = 1$, which can be parametrized by (A49) itself. However, it is more convenient to use (A41) so as to simplify (A49) to

$$\phi^{(2)} = \frac{2\Lambda^3}{N}\alpha_X(2x - N^2) - 2\Lambda^3\alpha_X^2 X + \frac{2\Lambda^2}{N}\alpha_{XX}[\dot{X}(xA_0 + yA_1) - X'(yA_0 + xA_1)] = 0. \tag{A50}$$

A posteriori, we streamline the discussion on the third iteration for the minimal model. In particular, we limit ourselves to proving that $\mathbf{W}^{(3)\downarrow}$, as defined around (2.27), has maximal row rank equal to 2. As a direct consequence, there exist no tertiary Lagrangian constraints $l_3 = 0$ and the constraint algorithm thus terminates, by means of closure 1.

Consider the first and fourth row in $\mathbf{W}^{(3)\downarrow}$. All entries are scalars and thus it is a matter of easy algebra to verify that their determinant does not vanish:

$$\det\begin{pmatrix} -\frac{2y^2}{N^3} & -\frac{2xy}{N^3} \\ \frac{\partial\phi^{(2)}}{\partial A_0} & \frac{\partial\phi^{(2)}}{\partial A_1} \end{pmatrix} = \frac{4\Lambda^2}{N^4} y[N^2\alpha_X - 2\alpha_{XX}(xA_0 + yA_1)^2] \neq 0. \tag{A51}$$

Moreover, straightforward examination unequivocally indicates that the above determinant is functionally independent from both the primary (A41) and the secondary (A50) Lagrangian constraints. This proves there exists a maximal row rank minor within $\mathbf{W}^{(3)\downarrow}$. Hence, $\mathbf{W}^{(3)\downarrow}$ itself has maximal row rank and does not admit left null vectors that nontrivially extend $\mathbf{V}^{(1)}$ and $\mathbf{V}^{(2)\downarrow}$ above.

Using (2.3) and (2.4), we independently reproduce the recent result in [59] that the minimal model propagates $N_{\mathrm{DoF}} = 1$ degree of freedom, as a massive electrodynamics theory in two dimensions must do.

7. Two-dimensional Palatini

Consider Einstein-Hilbert theory of gravity in two dimensions. A popular first-order reformulation attributed to Palatini [60] is

$$\mathcal{L}_{\mathrm{Pa}} = -(\partial_\rho h^{\mu\nu})G^\rho_{\mu\nu} + h^{\mu\nu}(G^\rho_{\rho\mu}G^\sigma_{\sigma\nu} - G^\rho_{\sigma\mu}G^\sigma_{\rho\nu}), \tag{A52}$$

where $h^{\mu\nu} = h^{\nu\mu}$ and $G^\lambda_{\mu\nu} = G^\lambda_{\nu\mu}$ denote tensors proportional to the metric and the connection, respectively. Their independent components conform the *a priori* independent field variables

$$Q^A = \{h \equiv h^{00}, h^1 \equiv h^{01}, h^{11}, G \equiv G^0_{00}, G_1 \equiv G^0_{10}, G_{11} \equiv G^0_{11}, \mathcal{G}^1 \equiv G^1_{00}, \mathcal{G}^1_1 \equiv G^1_{10}, \mathcal{G}^1_{11} \equiv G^1_{11}\}, \tag{A53}$$

where we have introduced multiple renamings for notational simplicity. It readily follows that $N = 9$. The theory is invariant under the field transformations [61,62]

$$h^{\mu\nu} \to h^{\mu\nu} + 2\epsilon^{\rho(\mu}h^{\nu)\sigma}\theta_{\rho\sigma}, \qquad G^\rho_{\mu\nu} \to G^\rho_{\mu\nu} + \epsilon^{\rho\sigma}\partial_\sigma\theta_{\mu\nu} + 2\epsilon^{\sigma\lambda}G^\rho_{\sigma(\mu}\eta_{\nu)\lambda}, \tag{A54}$$

which are of the relevant form (2.2). Here, $\theta_{\mu\nu} = \theta_{\nu\mu}$, so it has $g = 3$ independent components. It is easy to see that $e = 6$.

As a direct consequence of the exclusively linear dependence of the Lagrangian (A52) on the generalized velocities $\dot{Q}^A$, the (primary) Hessian vanishes $\boldsymbol{W}^{(1)} = 0$ and its linearly independent left null vectors can be chosen as

$$(V^{(1)}_I)^A = \delta^A_I, \qquad I = 1, 2, \ldots, M_1 = 9. \tag{A55}$$

Then, the components of the $\boldsymbol{U}^{(1)}$ vector coincide with the relations (2.12): $\boldsymbol{U}^{(1)} = \boldsymbol{\phi}^{(1)}$, where

$$\begin{aligned}
&\phi_1 \coloneqq -[\dot{G} + \partial_1\mathcal{G}^1 + 2(G\mathcal{G}^1_1 - G_1\mathcal{G}^1)] = 0, && \phi_2 \coloneqq -2(\dot{G}_1 + \partial_1\mathcal{G}^1_1 + G\mathcal{G}^1_{11} - G_{11}\mathcal{G}^1) = 0,\\
&\phi_3 \coloneqq -[\dot{G}_{11} + \partial_1\mathcal{G}^1_{11} + 2(G_1\mathcal{G}^1_{11} - G_{11}\mathcal{G}^1_1)] = 0, && \phi_4 \coloneqq \dot{h} - 2(h\mathcal{G}^1_1 + h^1\mathcal{G}^1_{11}) = 0,\\
&\phi_5 \coloneqq 2(\dot{h}^1 + h\mathcal{G}^1 - h^{11}\mathcal{G}^1_{11}) = 0, && \phi_6 \coloneqq \dot{h}^{11} + 2(h^1\mathcal{G}^1 + h^{11}\mathcal{G}^1_1) = 0,\\
&\phi_7 \coloneqq \partial_1 h + 2(hG_1 + h^1G_{11}) = 0, && \phi_8 \coloneqq 2(\partial_1 h^1 - hG + h^{11}G_{11}) = 0,\\
&\phi_9 \coloneqq \partial_1 h^{11} - 2(h^1G + h^{11}G_1) = 0.
\end{aligned} \tag{A56}$$

(For simplicity, we omit indices indicating the iteration.) Observe that the first six relations depend on a distinct generalized velocity each, while the latter three do not depend on generalized velocities. Namely, functional independence is manifest for $\{\phi_1, \ldots, \phi_6\}$. Functional (in)dependence among $\boldsymbol{\phi}_{\mathrm{red}} = \{\phi_7, \phi_8, \phi_9\}$ is nonobvious and so we proceed to test it via (2.13). Postulating

$$\boldsymbol{\Gamma}_{\mathrm{red}} = (\Gamma_0 + \Gamma_1\partial_1, \quad \tilde{\Gamma}_0 + \tilde{\Gamma}_1\partial_1, \quad \hat{\Gamma}_0 + \hat{\Gamma}_1\partial_1), \tag{A57}$$

the reduced left null vector condition $\boldsymbol{\Gamma}_{\mathrm{red}} \cdot \boldsymbol{\phi}_{\mathrm{red}} = 0$ yields no nontrivial solution, without much algebraic effort. It follows that $\boldsymbol{\phi}_{\mathrm{red}}$ indeed comprises functionally independent relations. Consequently, $l_1 = 9$ and the relations (A56) themselves can be regarded as the primary Lagrangian constraints in the theory.

We proceed to the second iteration. As per (2.18), the secondary equations of motion are given by

$$\boldsymbol{W}^{(2)} = \begin{pmatrix} 0 & -\boldsymbol{w} & 0 \\ \boldsymbol{w} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{w} = \mathrm{diag}(1,2,1), \qquad \boldsymbol{U}^{(2)} = \begin{pmatrix} \boldsymbol{U} \\ \dot{\phi}_{\mathrm{red}} \end{pmatrix} \tag{A58}$$

where the components of $\boldsymbol{U}$ are

$$\begin{aligned} U_1 &= -\partial_1 \dot{\mathcal{G}}^1 - 2\partial_0(G\mathcal{G}^1_1 - G_1\mathcal{G}^1), & U_2 &= -2\partial_1 \dot{\mathcal{G}}^1_1 - 2\partial_0(G\mathcal{G}^1_{11} - G_{11}\mathcal{G}^1), \\ U_3 &= -\partial_1 \dot{\mathcal{G}}^1_{11} - 2\partial_0(G_1\mathcal{G}^1_{11} - G_{11}\mathcal{G}^1_1), & U_4 &= -2\partial_0(h\mathcal{G}^1_1 + h^1\mathcal{G}^1_{11}), \\ U_5 &= 2\partial_0(h\mathcal{G}^1 - h^{11}\mathcal{G}^1_{11}), & U_6 &= 2\partial_0(h^{11}\mathcal{G}^1 + h^1\mathcal{G}^1_1). \end{aligned} \tag{A59}$$

It is evident that $\mathrm{rrank}(\boldsymbol{W}^{(2)}) = 6$. Following (2.28) and leaving aside trivial extensions of the primary left null vectors (A52), we find $M_2 = 3$ additional linearly independent left null vectors, up to normalization. We conveniently choose them as

$$\mathbf{V}^{(2)\downarrow}_R = (\underbrace{0,0,\ldots,0}_{15\text{ components}},\hat{R}), \qquad R = 1,2,3, \tag{A60}$$

where $\hat{R}$ denotes the standard Cartesian unit vector in Euclidean 3 dimensions. In accordance with (2.32), we find the relations

$$\begin{aligned} \phi^{(2)}_1 &\coloneqq \partial_1 \dot{h} + 2(\dot{h}G_1 + \dot{h}^1 G_{11} + h\dot{G}_1 + h^1 \dot{G}_{11}) = 0, \\ \phi^{(2)}_2 &\coloneqq 2(\partial_1 \dot{h}^1 - \dot{h}G + \dot{h}^{11} G_{11} - h\dot{G} + h^{11}\dot{G}_{11}) = 0, \\ \phi^{(2)}_3 &\coloneqq \partial_1 \dot{h}^{11} - 2(\dot{h}^1 G + \dot{h}^{11} G_1 + h^1 \dot{G} + h^{11}\dot{G}_1) = 0. \end{aligned} \tag{A61}$$

Following the prescription in (2.33), the search for functional (in)dependence within the secondary stage involves an ansatz for a vector $\boldsymbol{\Gamma}^{(2)}$ with up to two spatial derivatives. Nonetheless, due to the relative structure of the first term in each relation, it is easy to see that $\boldsymbol{\Gamma}^{(2)} \cdot \boldsymbol{\phi}^{(2)} = 0$ only admits the trivial solution. As a result, all relations in (A61) are functionally independent among themselves.

Establishing the functional (in)dependence of (A61) with respect to the primary Lagrangian constraints (A56) is a more delicate endeavour. Consider the row vector (2.38). In this case, the first nine components are (A56), while the latter three components are (A61). We search for solutions to (2.40). A generic ansatz Υ has components given by

$$(\Upsilon)^{T\leq 9} = \Upsilon^T_0 + \Upsilon^T_1 \partial_1, \quad (\Upsilon)^{T>9} = \Upsilon^T_0, \quad T = 1,\ldots,12. \tag{A62}$$

The simplest implementation of (2.41) consists in imposing three independent conditions, each involving a single $\Upsilon^{T>9}_0$ at a time:

$$\sum_{T=1}^{9} (\Upsilon^T_0 + \Upsilon^T_1 \partial_1)\phi_T - \phi^{(2)}_R = 0 \quad \forall \ R = 1,2,3, \tag{A63}$$

where, without loss of generality, we have set $\Upsilon^{T>9}_0 = -1$. The advantage of fixing the terms accompanying the secondary constraints is that we can factor the conditions $\Upsilon \cdot \boldsymbol{\Psi} = 0$ in a more efficient way. For instance, evaluating (A63) for $\phi^{(2)}_1$ yields the nontrivial solution

$$\Upsilon_1 = (0, -h, -2h^1, 2G_1 + \partial_1, G_{11}, 0, 2\mathcal{G}^1_1, \mathcal{G}^1_{11}, 0, -1, 0, 0). \tag{A64}$$

The above implies $\phi^{(2)}_1$ is functionally dependent on the primary Lagrangian constraints. The analogous strategy for $\phi^{(2)}_2$ and $\phi^{(2)}_3$ reveals these too are functionally dependent on the primary Lagrangian constraints. Therefore there are no secondary Lagrangian constraints $l_2 = 0$ and the algorithm terminates according to closure 3.

For completeness, we present the relations (A61) in terms of exclusively the primary Lagrangian constraints:

$$\begin{aligned} \phi^{(2)}_1 &= (2G_1 + \partial_1)\phi_4 - h\phi_2 - 2h^1\phi_3 + G_{11}\phi_5 + 2\mathcal{G}^1_1\phi_7 + \mathcal{G}^1_{11}\phi_8, \\ \phi^{(2)}_2 &= \partial_1\phi_5 + 2(h\phi_1 - h^{11}\phi_3 - G\phi_4 + G_{11}\phi_6 - \mathcal{G}^1\phi_7 + \mathcal{G}^1_{11}\phi_9), \\ \phi^{(2)}_3 &= (-2G_1 + \partial_1)\phi_6 + 2h^1\phi_1 + h^{11}\phi_2 - (G\phi_5 - \mathcal{G}^1\phi_8 + 2\mathcal{G}^1_1\phi_9). \end{aligned} \tag{A65}$$

Written in this manner, it becomes obvious that vanishing of the primary Lagrangian constraints implies vanishing of (A61) without further restrictions on the field variables Q^A.

Using (2.3) and (2.4), we obtain the familiar result $N_{\mathrm{DoF}} = 0$.

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