# Uniqueness of photon sphere for Reissner-Nordström electric-magnetic system

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The uniqueness of the static, asymptotically flat, nonextremal *photon sphere* in Einstein-Maxwell spacetime with electric and magnetic charges has been proved. Using the conformal positive energy theorem, as well as the positive mass theorem and adequate conformal transformations, we envisage the two alternative ways of proving that the exterior region of a certain radius of the studied static photon sphere is characterized by Arnowitt-Deser-Misner mass, electric, and magnetic charges.

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## I. INTRODUCTION

In view of the first-ever obtained images of M87 and Milky Way supermassive black holes [1,2] and the measure of polarization (a signature of magnetic fields close to the edge of a black hole) performed by the Event Horizon Telescope (EHT) Collaboration [3–5], the growth of the hope for the future possible verifications of the other black hole characteristics and observations of new physics effects in the vicinity of them can be observed.

From both theoretical and observational points of views the regions of spacetime, where the photon orbits are closed, forming timelike hypersurfaces on which the bending angle of light is unrestrictedly high, attract much attention. In essence, one can suppose that compact objects such as black holes, wormholes, and neutron stars are surrounded by *photon spheres*.

What is more, general relativity and its generalizations foresee the existence of such kinds of regions. The concept of *photon sphere* and *photon surface* are of great importance in studies of black hole shadows [6,7], triggered by the recent achievements concerning black hole images at the center of Milky Way and M87 galaxies [1,2]. They are also used in the search for the traces of new physics beyond the Standard Model.

In four-dimensional spacetime, photon sphere special properties, which recall features of a black hole event horizon, allow us to classify their spacetimes in terms of their asymptotical charges. It constitutes the alternative for the black hole uniqueness theorem [8–20]. Consequently, the generalizations of the uniqueness theorem for $n$-dimensional gravity were also investigated. In [21] the higher-dimensional problem of the photon sphere and the uniqueness of higher-dimensional Schwarzschild spacetime was elaborated, while the photon sphere uniqueness for electrovacuum $n$-dimensional spacetime was given in [22]. On the other hand, in [23] the studies of the so-called *trapped photons*, i.e., photons that never pass the event horizon or escape toward spatial infinity, in the spacetime of the higher-dimensional Schwarzschild-Tangherlini black hole, have been elaborated.

The aforementioned concepts of the photon sphere and surface play the key role in studies of Penrose inequalities [24,25]. It turns out that they are timelike totally umbilic hypersurfaces with the proportionality between their first and second fundamental forms. Other mathematical and geometrical aspects of these objects were scrutinized for both static and stationary axisymmetric spacetimes [26–37].

On the other hand, the generalization of the photon sphere concept to the case of the massive charged particle surface, was presented. They describe the case of timelike hypersurfaces to which any wordline of particles initially touching them remains in the hypersurface in question [38,39].

As was mentioned above, the studies of the photon sphere properties reveal that it is a totally umbilical hypersurface (i.e., its second fundamental form is a pure trace) with a constant mean curvature and surface gravity, strongly resembling the black hole event horizon. On the other hand, from black hole theory one knows that the presence of the black hole event horizon enables one to classify asymptotically flat spacetimes in terms of their asymptotic charges (authorizes the uniqueness theorems for various kind of black hole solutions).

Therefore the tantalizing question arises, whether the presence of the photon sphere delivers a new tool for the classification of spacetimes with asymptotical charges or other physical quantities.

Our paper is concerned with the problem of classification of static asymptotically flat spacetimes being the solution

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of Einstein-Maxwell gravity with electric $Q_{(F)}$ and magnetic $Q_{(F)}$ charges, having the line element of the form

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2_{(F)} + Q^2_{(B)}}{r^2}\right)dt^2 + \frac{dr^2}{(1 - \frac{2M}{r} + \frac{Q^2_{(F)}+Q^2_{(B)}}{r^2})} + r^2 d\Omega^2, \tag{1}$$

where $d\Omega^2$ is the metric of the unit sphere, containing a photon sphere. This paper is a continuation of our previous work [14], devoted to the uniqueness of the photon sphere for Einstein-Maxwell-dilaton black hole solutions with an arbitrary coupling constant. In addition, the influence of the magnetic field on the photon sphere region is very interesting due to the measurements and observations of the black hole magnetic field by EHT Collaboration and in the context of future planned experiments [3–5,40].

The organization of the paper is as follows. In Sec. II we describe the basic features of the Maxwell gauge field in spacetime with the presence of an asymptotically timelike Killing vector field orthogonal to the hypersurface of constant time. Section III will be devoted to the basic characteristics of the photon sphere with electric and magnetic charges. We found the functional dependence among the lapse function and aforementioned charges, which will be of key importance for revealing that the photon sphere has scalar constant curvature. Section IV is connected with the basic steps in the proof of the uniqueness theorem, using the conformal positive energy theorem. The alternative way of obtaining the classification (uniqueness) of the nonextremal static asymptotically flat solution in Einstein-Maxwell gravity with electric and magnetic charges will be presented in Sec. V. The proof is based on the method that uses the appropriate conformal transformation and positive energy theorem. In the last section we conclude our investigations.

## II. EQUATIONS OF MOTION WITH THE PRESENCE OF STATIONARY KILLING VECTOR FIELD

In this section we recall the basic features of electric and magnetic Maxwell fields in the presence of an asymptotically timelike Killing vector field. In what follows we shall consider the ordinary Einstein-Maxwell system given by the action

$$S_{\rm EM} = \int d^4x \sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu}), \tag{2}$$

where $g$ is the determinant of the four-dimensional metric tensor and $F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]}$ stands for the $U(1)$-gauge field strength. Variation of the action (2) with respect to metric tensor $g_{\mu\nu}$ and $A_\mu$ reveals the standard form of Einstein-Maxwell equations of motion

$$\nabla_\mu F^{\mu\nu} = 0, \qquad R_{\mu\nu} = T_{\mu\nu}(F), \tag{3}$$

where the energy momentum tensors, defined as $T_{\mu\nu} = -\delta S/\sqrt{-g}\delta g^{\mu\nu}$, are provided by

$$T_{\mu\nu}(F) = 2F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}F^2. \tag{4}$$

We introduce an asymptotically timelike Killing vector field $k_\delta$ and assume that the field strength in the considered theory will be stationary, i.e., $\mathcal{L}_k F_{\alpha\beta} = 0$. The exact form of the energy-momentum tensor $T_{\alpha\beta}(F)$ envisages that it also fulfills the stationarity assumption, $\mathcal{L}_k T_{\alpha\beta}(F) = 0$. The existence of stationary Killing vector field $k_a$ enables one to introduce into consideration the meaning of the twist vector $\omega_a$, defined as

$$\omega_a = \frac{1}{2}\epsilon_{abcd}k^b\nabla^c k^d. \tag{5}$$

Furthermore, for any Killing vector field one has $\nabla_\alpha\nabla_\beta\chi_\gamma = -R_{\beta\gamma\alpha}{}^\delta\chi_\delta$, which implies in turn the relation of the form as

$$\nabla_\beta\omega_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}k^\gamma R^{\delta\chi}k_\chi. \tag{6}$$

It can also be found that $\nabla_\alpha(\frac{\omega^\alpha}{N^4}) = 0$, where we set $N^2 = -k_\gamma k^\gamma$.

The introduction of the Killing vector field in question allows one to define electric and magnetic components for gauge field strengths $F_{\alpha\beta}$ as follows:

$$E_\alpha = -F_{\alpha\beta}k^\beta, \qquad B_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}k^\beta F^{\gamma\delta}, \tag{7}$$

and consequently the field strength $F_{\alpha\delta}$ can be rewritten in terms of $E_\alpha$ and $B_\alpha$, i.e., $N^2F_{\alpha\beta} = -2k_{[\alpha}E_{\beta]} + \epsilon_{\alpha\beta\gamma\delta}k^\gamma B^\delta$. On the other hand, the equations of motion for magnetic and electric parts gauge field strength are provided by

$$\nabla_\alpha\left(\frac{E^\alpha}{N^2}\right) = 2\frac{B^\gamma}{N^4}\omega_\gamma, \tag{8}$$

$$\nabla_\alpha\left(\frac{B^\alpha}{N^2}\right) = -2\frac{E^\gamma}{N^4}\omega_\gamma, \tag{9}$$

while the field invariance conditions $\mathcal{L}_kF_{\alpha\beta} = 0$, as well as the relations $\nabla_{[\gamma}F_{\alpha\beta]} = 0$, establish the generalized Maxwell source-free equations in the form

$$\nabla_{[\alpha}E_{\beta]} = 0, \qquad \nabla_{[\alpha}B_{\beta]} = 0. \tag{10}$$

We shall consider the simply connected spacetime. It permits us to implement the electric and magnetic potentials in the forms as

$$E_\alpha = \nabla_\alpha \psi_F, \qquad B_\alpha = \nabla_\alpha \psi_B. \tag{11}$$

Keeping in mind the relations (6) and the explicit form of the Ricci tensor, one finds the Poynting flux in Einstein-Maxwell gravity with electric and magnetic charges. It yields

$$\nabla_{[\alpha}\omega_{\beta]} = 4E_{[\alpha}B_{\beta]}. \tag{12}$$

In what follows we shall pay attention to static spacetime; i.e., one supposes that there exists a smooth Riemannian manifold and a smooth lapse function $N\colon M^3 \to R^+$, such that $M^4 = R \times M^3$. The assumptions provide that the line element in the studied spacetime can be written in the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + g_{ab}dx^a dx^b, \tag{13}$$

where $N$ and $g_{ab}$ are time independent, as they are determined on the hypersurface of constant time.

The spacetime under consideration is asymptotically flat containing a dataset $(\Sigma_{\rm end}, g_{ij}, K_{ij})$ with gauge fields $A_\mu$ such that $\Sigma_{\rm end}$ constitutes a manifold diffeomorphic to $R^{(3)}$ minus a closed unit ball at the origin of $R^{(3)}$. Besides, it is subject to the following asymptotic behaviors of $g_{ij}, F_{\mu\nu}$:

$$|g_{ij} - \delta_{ij}| + r|\partial_a g_{ij}| + \cdots + r^k|\partial_{a_1\cdots a_k} g_{ij}| + r|K_{ij}| + \cdots + r^k|\partial_{a_1\cdots a_k} K_{ij}| \leq \mathcal{O}\left(\frac{1}{r}\right), \tag{14}$$

$$F_{\alpha\beta} + r|\partial_a F_{\alpha\beta}| + \cdots + r^k|\partial_{a_1\cdots a_k} F_{\alpha\beta}| \leq \mathcal{O}\left(\frac{1}{r^2}\right). \tag{15}$$

The Einstein-Maxwell equations are provided by the following:

$${}^{(g)}\nabla_i{}^{(g)}\nabla^i N = \frac{1}{N}\left({}^{(g)}\nabla_i\psi_F{}^{(g)}\nabla^i\psi_F + {}^{(g)}\nabla_i\psi_B{}^{(g)}\nabla^i\psi_B\right), \tag{16}$$

$$N{}^{(g)}\nabla_i{}^{(g)}\nabla^i\psi_F = {}^{(g)}\nabla_i N{}^{(g)}\nabla^i\psi_F, \tag{17}$$

$$N{}^{(g)}\nabla_i{}^{(g)}\nabla^i\psi_B = {}^{(g)}\nabla_i N{}^{(g)}\nabla^i\psi_B, \tag{18}$$

$${}^{(g)}R = \frac{1}{N^2}\left({}^{(g)}\nabla_i\psi_F{}^{(g)}\nabla^i\psi_F + {}^{(g)}\nabla_i\psi_B{}^{(g)}\nabla^i\psi_B\right), \tag{19}$$

$$\begin{aligned}{}^{(g)}R_{ij} = {}&\frac{1}{N}{}^{(g)}\nabla_i{}^{(g)}\nabla_j N + \frac{1}{N^2}[g_{ij}({}^{(g)}\nabla_k\psi_F{}^{(g)}\nabla^k\psi_F \\ &+ {}^{(g)}\nabla_k\psi_B{}^{(g)}\nabla^k\psi_B) \\ &- 2({}^{(g)}\nabla_i\psi_F{}^{(g)}\nabla_j\psi_F + {}^{(g)}\nabla_i\psi_B{}^{(g)}\nabla_j\psi_B)],\end{aligned} \tag{20}$$

where ${}^{(g)}\nabla_i$ is the covariant derivative with respect to metric tensor $g_{ij}$. ${}^{(g)}R_{ij}$ denotes the three-dimensional Ricci tensor, while ${}^{(g)}R$ stands for the Ricci scalar curvature.

## III. GEOMETRY OF PHOTON SPHERE IN STATIC ASYMPTOTICALLY FLAT SPACETIME WITH ELECTRIC AND MAGNETIC POTENTIALS

This section will be devoted to the description of the photon sphere with one single component [11]. Namely, a photon surface is an embedded timelike hypersurface for which any null geodesics initially tangent to it remains tangent during the passage of time of its existence. On the other hand, by photon sphere one defines a photon surface with constant lapse function $N$, and the additional conditions imposed on the electric and magnetic charges emerging in the studied theory.

We suppose that the lapse function regularly foliates the manifold outside the photon sphere. Thus it effects that all level sets with $N = \text{const}$ are topological spheres. It yields that outside the photon sphere one has $1/\rho^2 = {}^{(g)}\nabla_i N{}^{(g)}\nabla^i N \neq 0$.

Further, we define the electric and magnetic static system as a time slice of the static spacetime $(R \times M^3, -N^2dt^2 + g_{ij}dx^i dx^j)$. Then, we define the notion of the photon surface. Namely, let $(M^3, g_{ij}, N, \psi_F, \psi_B)$ be a Maxwell electric-magnetic system bounded with spacetime defined above, with the metric (13). By the photon sphere we shall understand a timelike hypersurface embedded $(P^3, h_{ij}) \hookrightarrow (R \times M^3, -N^2dt^2 + g_{ij}dx^i dx^j)$. If the embedding is umbilic and the gradient of the lapse function, the electric one-form is normal to $P^3$.

### A. Mean curvature of photon sphere

It has been revealed in [7] that the second fundamental form of $P^3$ may be written as $K_{ij} = \frac{1}{3}\Theta h_{ij}$, where $\Theta$ stands for the expansion of the unit normal to the photon sphere. Moreover, the condition for a timelike hypersurface to be a photon sphere is its total umbilicity (its second fundamental form is a pure trace).

To commence, we use the Codazzi equations to analyze the properties of the photon sphere in question. We denote by $n_a$ the unit normal to the photon sphere, $Y_\beta$ will represent the element of tangent space $TP^3$, and using these quantities one obtains that for all vectors $Y_c$, belonging to $TP^3$, the following relation is satisfied:

$$\frac{1}{3}\Theta_{,b}(1-3)Y^b = {}^{(g)}R_{cd}n^c Y^d, \tag{21}$$

where the right-hand side of Eq. (21) is given by

$$
\begin{aligned}
{}^{(g)}R_{cd}n^cY^d = &\, 2\frac{1}{N^4}k_aY^ak_bn^b(E_mE^m + B_mB^m) \\
&- 2\frac{1}{N^2}(E_aE_b + B_aB_b)Y^an^b \\
&+ \frac{n_kY^k}{N^2}(E_mE^m + B_mB^m).
\end{aligned}
\tag{22}
$$

Keeping in mind the fact that electric fields $E_a$, $B_a$ ($E_a$ is normal to $P^3$, and by its definition and the results of Sec. III C show that this is the case for $B_a$ in static spacetime) are normal to $P^3$ and $k_\alpha Y^\alpha = 0, k_\beta n^\beta = 0$, one concludes that ${}^{(g)}R_{cd}n^cY^d$ is equal to zero, and we arrive at

$$
0 = (1-3)Y^\zeta \frac{\Theta_{,\zeta}}{3}. \tag{23}
$$

Thus for an arbitrary vector $Y^\beta$, the mean curvature of the considered photon sphere is constant.

It can also be shown [11] that $\mathcal{L}_X(n^{j(g)}\nabla_j N) = 0$, where $X$ is an arbitrary tangent vector to $\Sigma^2$, envisaging that $n^{j(g)}\nabla_j N$ is constant on $\Sigma^2$.

### B. Scalar curvature of electric-magnetic photon sphere

The scalar curvature of the photon sphere in question will be found by means of the contracted Gauss equation. It implies

$$
{}^{(g)}R - 2{}^{(g)}R_{ij}n^in^j = {}^{(p)}R - \frac{2}{3}\Theta^2, \tag{24}
$$

where in our case ${}^{(g)}R_{ij}n^in^j$ yields

$$
{}^{(g)}R_{ij}n^in^j = -\frac{1}{N^2}(E_aE_b + B_aB_b)n^an^b. \tag{25}
$$

As a result, one achieves the relation for the scalar curvature of the photon sphere

$$
{}^{(p)}R = \frac{2}{3}\Theta^2 + 2\frac{1}{N^2}(E_aE_b + B_aB_b)n^an^b. \tag{26}
$$

To show that the photon sphere has a constant scalar curvature one needs to prove that $E_an^a$ and $B_kn^k$ are constant on $P^3$. Above we mentioned that $n^{j(g)}\nabla_jN$ is constant on $P^3$ (see for the proof [11]), and in the next subsection we envisage that electric and magnetic potentials are functions of $N$; see relation (39). It leads to the conclusion that $E_an^a$ nad $B_kn^k$ are constant on $P^3$, implying that $P^3$ has the constant scalar curvature.

### C. Functional dependence–lapse function electric and magnetic potentials

In static spacetime with Killing vector field $k_\mu$, one has that the twist vector $\omega_\alpha$ (12) is equal to zero. It implies proportionality between magnetic and electric fields [8]. Because of the fact that the electric one-form is spacelike ($k_\mu$ is timelike), every one-form parallel and orthogonal to it vanishes, Eqs. (8) and (9). Moreover, keeping in mind the asymptotic conditions $\psi_F \to 0$ and $\psi_B \to 0$, when $r \to \infty$, we get that

$$
\psi_B = \mu\psi_F, \tag{27}
$$

where $\mu$ is constant.

As in [41], one can introduce coordinates on the $N = \text{const}, t = \text{const}$ manifold provided by

$$
g_{ab}dx^adx^b = {}^{(2)}g_{ab}dy^ady^b + \rho^2dN^2. \tag{28}
$$

Keeping in mind equations of motion for electric and magnetic potentials and the relation (27), we obtain

$$
\frac{1}{\sqrt{{}^{(2)}g}}\left[\sqrt{{}^{(2)}g}\frac{\phi_F}{N}\right] = -\frac{(\rho\psi_F^{;a})_{;a}}{N}, \tag{29}
$$

where we have denoted

$$
\frac{\partial\psi_F}{\partial N} = \rho\phi_F, \tag{30}
$$

and the gravitational relation of the form

$$
\frac{1}{\rho^2}\frac{\partial\rho}{\partial N} = K + \frac{2\rho(1+\mu^2)}{N}(\phi_F^2 + \psi_{F;a}\psi_F^{;a}), \tag{31}
$$

where $K = K_m{}^m$ is the extrinsic scalar curvature of $N = \text{const}$ spacetime. Based on equations of motion (29)–(31) for the theory in question, one can arrive at the integral identity given by

$$
\begin{aligned}
&\frac{1}{\sqrt{{}^{(2)}g}}\frac{\partial}{\partial N}\left[\sqrt{{}^{(2)}g}\left(\frac{1}{N}F(N,\tilde\psi)\tilde\phi + \frac{G(N,\tilde\psi)}{\rho}\right)\right] \\
&= A\rho(\tilde\phi^2 + \tilde\psi_{;a}\tilde\psi^{;a}) + C\tilde\psi + \frac{1}{\rho}\frac{\partial G}{\partial N} - \frac{1}{N}(F\rho\tilde\psi^{;a})_{;a},
\end{aligned}
\tag{32}
$$

for differentiable arbitrary (for the time being) functions $F$, $G$ and the new potential $\tilde\psi = \sqrt{1+\mu^2}\psi_F$ (where we have used the dependence of electric and magnetic potentials in the static spacetime), for the same reason we get that $\tilde\phi = \sqrt{1+\mu^2}\phi_F$. The functions $A$ and $B$ are provided by

$$
A = \frac{1}{N}\left(G + \frac{\partial F}{\partial\tilde\psi}\right), \tag{33}
$$

$$
B = \frac{1}{N}\frac{\partial F}{\partial N} + \frac{\partial G}{\partial\tilde\psi}. \tag{34}
$$

To achieve the integral conservation laws from (32), we have to restrict our consideration to the case where

$A = B = \frac{\partial G}{\partial N} = 0$. The general solutions of the above overdetermined linear system of differential equation for $F$ and $G$ constitute a linear combination of the following particular solutions:

$$F = 1, \quad G = 0, \quad F = 2\tilde{\psi}, \qquad G = 1,$$
$$F = 2\tilde{\psi}^2 - N^2, \quad G = 2\tilde{\psi}. \tag{35}$$

One can integrate the relation (32), with respect to all the aforementioned values of functions $F$ and $G$, keeping in mind that the integral of two-dimensional divergence over a closed $N = \text{const}$ space disappears. The two boundary surfaces $\Sigma_0$ and $\Sigma_\infty$ were taken into account with the appropriate asymptotic conditions imposed on fields and characteristic features of them. Namely, for approaching $\Sigma_\infty$ one has that $r\psi_F \to Q_{(F)}, r^2\phi_F \to -Q_{(F)}, \frac{\rho}{r^2} \to \frac{1}{M}$. For $\Sigma_0$ we have that $\phi_F = \mathcal{O}(N), \psi_{F;a} = \mathcal{O}(N)$. The $\Sigma_0$ $\psi_F$ and $1/\rho$ are constant [41].

Finally, one arrives at the following:

$$\int_{\Sigma_0} dS\left(\frac{\phi_F}{N}\right) = -Q_{(F)}, \tag{36}$$

$$2(1+\mu^2)\psi_{(0)F}\int_{\Sigma_0} dS\left(\frac{\phi_F}{N}\right) + \frac{S_0}{\rho_0} = M, \tag{37}$$

$$2(1+\mu^2)\psi^2_{(0)F}\int_{\Sigma_0} dS\left(\frac{\phi_F}{N}\right) + 2\frac{S_0}{\rho_0}\psi_{(0)F} = Q_{(F)}, \tag{38}$$

where $S_0$ is the area of two-space $\Sigma_0$.

It can be seen that the addition of magnetic charge does not change the basic features of the photon sphere (as obtained in the Maxwell case in [11]). Qualitative features such as the constancy of its mean curvature and scalar curvature are the same; however, quantitative ones are different. Namely, they are valid for the modified potential $\tilde{\psi} = \sqrt{1+\mu^2}\psi_F$, on which the magnetic potential imprints its influence.

All the above reveal that one arrives at the following functional dependence among the $N_0$ lapse function on $\Sigma_0$, the $\psi_{(0)F}$ electric potential at $\Sigma_0$, and the constant $\mu$ bounded magnetic and electric potentials

$$2(1+\mu^2)\psi^2_{(0)F} + 2\psi_{(0)F}\frac{M}{Q_{(F)}} - 1 = N_0^2; \tag{39}$$

as was mentioned above $\psi_{(0)F}$ and $N_0$ are constant on the considered hypersurface and $\psi_F \to 0$, as $r \to \infty$.

Equation (39) is valid not only on the surface in question but also in all its exterior region. Namely, let us compose the divergence identity based on the above equations

$$\frac{1}{2}{}^{(g)}\nabla^m\left[\left(-N^2 + 2(1+\mu^2)\psi_F^2 + \frac{2\psi_F M}{Q_{(F)}} - 1\right)\theta_m\right] = N\theta_m\theta^m, \tag{40}$$

where $\theta^m$ yields

$$\theta^m = -{}^{(g)}\nabla^m N + \frac{1}{N}\left(2(1+\mu^2)\psi_F{}^{(g)}\nabla^m\psi_F + \frac{M}{Q_{(F)}}{}^{(g)}\nabla^m\psi_F\right). \tag{41}$$

In the next step one applies the Gauss theorem to the relation (40), and taking into account the asymptotic behaviors of $N, \psi_F$, and the fact that $N > 0$ in the exterior region of the photon sphere, one can draw a conclusion that $\theta^m = 0$. Fixing in this relation the integration constant as equal to 1, we arrive at the equation expressing a functional dependence among electric/magnetic potentials and $N$.

It proves the constancy of $E^a n_a$ and $B_c n^c$ on $P^3$, implying that ${}^{(g)}R$ is a constant scalar curvature.

### D. Auxiliary formulas

Some additional formulas envisaging the influence of the magnetic charge on the photon sphere can be obtained by the equation of motion (16), for the isometric embedding $(\Sigma^2, \sigma_{ij}) \hookrightarrow (M^3, g_{ij})$. Namely, if one considers the contracted Gauss relation, it yields

$$N{}^{(\sigma)}R = \frac{2}{N}(E_aE_b + B_aB_b)n^an^b + 2Hn^{k(g)}\nabla_k N + \frac{H^2}{2}N, \tag{42}$$

where we have denoted $H = \frac{2}{3}\Theta$. The integration of (42) over the hypersurface $\Sigma$ results in

$$\int_\Sigma d\Sigma N^{(\sigma)}R = \int_\Sigma d\Sigma\frac{2}{N}(E_aE_b + B_aB_b)n^an^b + 2\int_\Sigma d\Sigma Hn^{k(g)}\nabla_k N + \int_\Sigma d\Sigma\frac{H^2}{2}N. \tag{43}$$

Let us examine the area of the hypersurface $\Sigma$ denoted by $A_\Sigma$ and apply the Gauss-Bonnet theorem. Consequently, we arrive at

$$N_0 = \frac{1}{4\pi N_0}(E_aE_b + B_aB_b)n^an^bA_\Sigma + HM_{Phs} + \frac{1}{16\pi}H^2A_\Sigma N_0, \tag{44}$$

where the mass of the photon sphere implies

$$M_{\rm phs} = \frac{1}{4\pi}n^{k(g)}\nabla_k NA_\Sigma. \tag{45}$$

Next, we take into account the contracted Gauss equation ${}^{(\sigma)}R = {}^{(p)}R - 2{}^{(p)}R_{ij}\eta^i\eta^j$, for $(\Sigma^2, \sigma_{ij}) \hookrightarrow (P^3, h_{ij})$ isometric embedding, with a unit normal $\eta_i$.

The same procedure as above leads us to the equation provided by

$$1 = \frac{3}{16\pi} H^2 A_\Sigma + \frac{1}{4\pi N_0^2} (E_a E_b + B_a B_b) n^a n^b A_\Sigma, \quad (46)$$

which can be rewritten using the definition of the electric and magnetic charges

$$Q_{(F)} = -\frac{A_\Sigma E_k n^k}{4\pi N_0}, \qquad Q_{(B)} = -\frac{A_\Sigma B_k n^k}{4\pi N_0}, \quad (47)$$

and Eqs. (44) and (46), in the form as

$$1 = \frac{4\pi(Q_{(F)}^2 + Q_{(B)}^2)}{A_\Sigma} + \frac{3}{2}\frac{H}{N_0} M_{\mathrm{phs}}. \quad (48)$$

On the other hand, the relation among $n^a$, $N_0$, and $H$ yields

$$2n^{a(g)}\nabla_a N = H N_0. \quad (49)$$

Then, using (49) we get the expression envisaging how magnetic charge influences $A_\Sigma$:

$$\frac{A_\Sigma}{4\pi} = (Q_{(F)}^2 + Q_{(B)}^2) + 3\frac{M_{\mathrm{phs}}^2}{N_0^2} = (1+\mu^2) Q_{(F)}^2 + 3\frac{M_{\mathrm{phs}}^2}{N_0^2}. \quad (50)$$

## IV. UNIQUENESS OF PHOTON SPHERE WITH ELECTRIC AND MAGNETIC CHARGES

The uniqueness proof of the photon sphere in Einstein-Maxwell dark photon gravity will be conducted in several steps, keeping in mind the attitude presented in Refs. [9–14].

In this section the photon sphere emerges as the inner boundary of the studied spacetime [12]. Namely one has that

$$(P^3, h_{ij}) = \cup_{i=1}^{I} \, (R \times \Sigma_i^2, -N_i^2 dt^2 + \sigma_{ij}^{(i)} dx^i dx^j), \quad (51)$$

where $P_i^3$ denotes each connected component of $P^3$.

To begin with, we define the electric-magnetic Einstein-Maxwell dark photon system as $(M^3, g_{ij}, N, \psi_F, \psi_B)$, being asymptotic to the static spherically symmetric solution in the considered theory, and possessing the Killing horizon boundary. It can be done by glueing spatial pieces of the aforementioned solution, having the adequate mass and Maxwell electric and magnetic charges. Namely, each of the photon spheres $\Sigma_i^2$ will be glued at the neck piece of the Einstein-Maxwell manifold with a mass greater than zero and having charges $Q_i^{(F)}, Q_i^{(B)}$.

The manifold $M^3$ will be smooth, and the metric tensor, the lapse function, and the potentials $\psi_F$ and $\psi_B$ will be smooth away from the glueing surface. The considered manifold will be characterized by non-negative scalar curvature (away from the glueing surface) and will be geodesically complete.

Next the adequate conformal transformations will be applied, and we use the conformal positive energy theorem to show that the photon sphere is isometric to Einstein-Maxwell spacetime characterized by the Arnowitt-Deser-Misner (ADM) mass, electric, and magnetic charges. The nondegenerate case of the Einstein-Maxwell static, spherically symmetric system will be considered.

### A. Conformal positive theorem and uniqueness of electric-magnetic photon sphere

In our attitude to the problem, the conformal positive energy theorem, derived in Refs. [42,43], will account for the key role in the uniqueness proof. To implement it to our considerations one ought to fulfill its assumptions. Namely, we have to consider two asymptotically flat Riemannian $(n-1)$-dimensional manifolds, $(\Sigma^{(\Phi)}, {}^{(\Phi)}g_{ij})$ and $(\Sigma^{(\Psi)}, {}^{(\Psi)}g_{ij})$ for which metric tensors are bounded with the conformal transformation given by

$${}^{(\Psi)}g_{ij} = \Omega^{2(\Phi)} g_{ij}, \quad (52)$$

where $\Omega$ stands for a conformal factor. On the other hand, the masses of the above manifolds satisfy the relation of the form ${}^{(\Phi)}m + \beta^{(\Psi)}m \geq 0$, under the auxiliary requirement putting on the Ricci curvature scalar tensor ${}^{(\Phi)}R + \beta\Omega^{2(\Psi)}R \geq 0$, where ${}^{(\Phi)}R$ and ${}^{(\Psi)}R$ are the Ricci scalars with respect to the adequate metric tensors, defined on the two manifolds, while $\beta$ is a positive constant. The inequalities are satisfied if $(n-1)$-dimensional manifolds are flat [42].

The conformal positive energy theorem was widely applied in proving the uniqueness of four- and higher-dimensional black objects [43–51] and wormhole solutions [52].

In our considerations we implement the conformal transformation of the form as follows:

$$\tilde{g}_{ij} = N^2 g_{ij}, \quad (53)$$

leading to the conformally rescaled Ricci tensor given by

$$\tilde{R}_{ij}(\tilde{g}) = \frac{2}{N^2} {}^{(g)}\nabla_i N {}^{(g)}\nabla_j N - \frac{2}{N^2} ({}^{(g)}\nabla_i \psi_F {}^{(g)}\nabla_j \psi_F + {}^{(g)}\nabla_i \psi_B {}^{(g)}\nabla_j \psi_B). \quad (54)$$

Next, we define the quantities provided by the relations, for electric potential $\psi_F$,

$$\Phi_1 = \frac{1}{2}\left(N + \frac{1}{N} - \frac{2}{N}\psi_F^2\right), \quad (55)$$

$$\Phi_0 = \frac{\sqrt{2}}{N}\psi_F, \quad (56)$$

$$\Phi_{-1} = \frac{1}{2}\left(N - \frac{1}{N} - \frac{2}{N}\psi_F^2\right), \tag{57}$$

and the quantities including the $\psi_B$ potential are given by

$$\Psi_1 = \frac{1}{2}\left(N + \frac{1}{N} - \frac{2}{N}\psi_B^2\right), \tag{58}$$

$$\Psi_0 = \frac{\sqrt{2}}{N}\psi_B, \tag{59}$$

$$\Psi_{-1} = \frac{1}{2}\left(N - \frac{1}{N} - \frac{2}{N}\psi_B^2\right). \tag{60}$$

It can be observed that the auxiliary constraint relation can be found when one defines the metric tensor $\eta_{AB} = \mathrm{diag}(1, -1, -1)$. They are provided by

$$\Phi_A \Phi^A = \Psi_A \Psi^A = -1, \tag{61}$$

where we set $A = -1,\ 0,\ 1$. Consequently, the other symmetric tensors can be constructed for the potential $\Phi_A$,

$$\tilde{G}_{ij} = \widetilde{\nabla}_i \Phi_{-1} \widetilde{\nabla}_j \Phi_{-1} - \widetilde{\nabla}_i \Phi_0 \widetilde{\nabla}_j \Phi_0 - \widetilde{\nabla}_i \Phi_1 \widetilde{\nabla}_j \Phi_1, \tag{62}$$

and similarly for the potential $\Psi_A$,

$$\tilde{H}_{ij} = \widetilde{\nabla}_i \Psi_{-1} \widetilde{\nabla}_j \Psi_{-1} - \widetilde{\nabla}_i \Psi_0 \widetilde{\nabla}_j \Psi_0 - \widetilde{\nabla}_i \Psi_1 \widetilde{\nabla}_j \Psi_1, \tag{63}$$

where $\widetilde{\nabla}_i$ is denoted for the covariant derivative with respect to the conformally rescaled metric $\tilde{g}_{ij}$.

Because of relations (61) one arrives at

$$\tilde{\nabla}^2 \Phi_A = \tilde{G}_i{}^i \Phi_A, \qquad \tilde{\nabla}^2 \Psi_A = \tilde{H}_i{}^i \Psi_A. \tag{64}$$

Moreover, the Ricci curvature tensor $\tilde{R}_{ij}$ connected with conformally rescaled metric $\tilde{g}_{ij}$ may be rewritten in terms of $\tilde{G}_{ij}$ and $\tilde{H}_{ij}$, i.e.,

$$\tilde{R}_{ij} = \tilde{G}_{ij} + \tilde{H}_{ij}. \tag{65}$$

Relations (64) and (65) can be derived by varying the Lagrangian density [44,53,54]

$$\mathcal{L} = \sqrt{-\tilde{g}}\left(\tilde{G}_i{}^i + \tilde{H}_i{}^i + \frac{\widetilde{\nabla}^i \Phi_A \widetilde{\nabla}_i \Phi^A}{\Phi_A \Phi^A} + \frac{\widetilde{\nabla}^i \Psi_A \widetilde{\nabla}_i \Psi^A}{\Psi_A \Psi^A}\right), \tag{66}$$

with respect to $\tilde{g}_{ij}, \Phi_A, \Psi_A$, and taking into account the constraint relations (61).

The conformal positive energy theorem accounts for the main point in the uniqueness theorem. Thus, we suppose that one has two asymptotically flat Riemannian three-dimensional manifolds $(\Sigma^\Phi, {}^{(\Phi)}g_{ij})$ and $(\Sigma^\Psi, {}^{(\Psi)}g_{ij})$. The conformal transformation between two manifolds will be of the form as ${}^{(\Psi)}g_{ij} = \Omega^{2(\Phi)} g_{ij}$. It provides that the corresponding masses satisfy the relation ${}^{\Phi}m + \beta {}^{\Psi}m \geq 0$ if ${}^{(\Phi)}R + \beta \Omega^{2(\Psi)} R \geq 0$, for some positive constant $\beta$. The inequalities in question ensure that the three-dimensional Riemannian manifolds are flat.

To satisfy the requirement of the conformal positive energy theorem, we introduce into consideration conformal transformations provided by

$${}^{(\Phi)}g^{\pm}_{ij} = {}^{(\Phi)}\omega^2_{\pm} \tilde{g}_{ij}, \qquad {}^{(\Psi)}g^{\pm}_{ij} = {}^{(\Psi)}\omega^2_{\pm} \tilde{g}_{ij}, \tag{67}$$

where the conformal factors imply

$${}^{(\Phi)}\omega_{\pm} = \frac{\Phi_1 \pm 1}{2}, \qquad {}^{(\Psi)}\omega_{\pm} = \frac{\Psi_1 \pm 1}{2}. \tag{68}$$

Then, the standard procedure of pasting three-dimensional manifolds $(\Sigma^\Phi_\pm, {}^{(\Phi)}g^{\pm}_{ij})$ and $(\Sigma^\Psi_\pm, {}^{(\Psi)}g^{\pm}_{ij})$ across their shared minimal boundary can be put into application. As a result, we obtain four manifolds $(\Sigma^\Phi_+, {}^{(\Phi)}g^{+}_{ij})$, $(\Sigma^\Phi_-, {}^{(\Phi)}g^{-}_{ij})$, $(\Sigma^\Psi_+, {}^{(\Psi)}g^{+}_{ij})$, and $(\Sigma^\Psi_-, {}^{(\Psi)}g^{+}_{ij})$, which will be pasted across shared minimal boundaries $B^\Psi$ and $B^\Phi$. Thus, complete regular hypersurfaces $\Sigma^\Phi = \Sigma^\Phi_+ \cup \Sigma^\Phi_-$ and $\Sigma^\Psi = \Sigma^\Psi_+ \cup \Sigma^\Psi_-$ can be constructed.

It can be checked that the total gravitational mass ${}^{\Phi}m$ on hypersurface $\Sigma^\Phi$ and ${}^{\Psi}m$ on $\Sigma^\Psi$ vanish; i.e., it can be shown that the metric tensors connected with the adequate hypersurface are proportional to the Kronecker delta [43].

On this account, it is customary to define another conformal transformation provided by the relation

$$\hat{g}^{\pm}_{ij} = [({}^{(\Phi)}\omega_{\pm})^2 ({}^{(\Psi)}\omega_{\pm})^2]^{\frac{1}{2}} \tilde{g}_{ij}, \tag{69}$$

leading to the following form of the Ricci curvature tensor on the defined space:

$$\begin{aligned} \hat{R}_{\pm} = {} & [{}^{(\Phi)}\omega^2_{\pm}\, {}^{(\Psi)}\omega^2_{\pm}]^{-\frac{1}{2}} ({}^{(\Phi)}\omega^2_{\pm}\, {}^{(\Phi)}R_{\pm} + {}^{(\Psi)}\omega^2_{\pm}\, {}^{(\Psi)}R_{\pm}) \\ & + (\hat{\nabla}_i \ln {}^{(\Phi)}\omega_{\pm} - \hat{\nabla}_i \ln {}^{(\Psi)}\omega_{\pm}) \\ & \times (\hat{\nabla}^i \ln {}^{(\Phi)}\omega_{\pm} - \hat{\nabla}^i \ln {}^{(\Psi)}\omega_{\pm}). \end{aligned} \tag{70}$$

The direct calculations revealed that the first term on the right-hand side of the above equation can be rewritten as

$$\begin{aligned} {}^{(\Phi)}\omega^2_{\pm}\, {}^{(\Phi)}R_{\pm} + {}^{(\Psi)}\omega^2_{\pm}\, {}^{(\Psi)}R_{\pm} = {} & 2\left|\frac{\Phi_0 \tilde{\nabla}_i \Phi_{-1} - \Phi_{-1} \tilde{\nabla}_i \Phi_0}{\Phi_1 \pm 1}\right|^2 \\ & + 2\left|\frac{\Psi_0 \tilde{\nabla}_i \Psi_{-1} - \Psi_{-1} \tilde{\nabla}_i \Psi_0}{\Psi_1 \pm 1}\right|^2. \end{aligned} \tag{71}$$

It leads to the conclusion that in terms of Eqs. (70) and (71), the Ricci scalar $\hat{R}_{\pm}$ is greater than or equal to zero.

Consequently, on account of the conformal positive energy theorem, it is revealed that the manifolds $(\Sigma^{\Phi}, {}^{\Phi}g_{ij})$, $(\Sigma^{\Psi}, {}^{\Psi}g_{ij})$, and $(\hat{\Sigma}, \hat{g}_{ij})$ are flat, which in turn enables us to claim that ${}^{(\Phi)}\omega = \text{const}{}^{(\Psi)}\omega$, $\Phi_0 = \text{const}\Phi_{-1}$, and $\Psi_0 = \text{const}\Psi_{-1}$.

We conclude that the manifold $(\Sigma, g_{ij})$ is conformally flat. Moreover, its metric tensor $\hat{g}_{ij}$ can be rearranged in conformally flat form $\hat{g}_{ij} = \mathcal{U}^{4(\Phi)} g_{ij}$, where the conformal factor is given by $\mathcal{U} = ({}^{\Phi}\omega_{\pm} N)^{-1/2}$.

Keeping in mind relation (72), calculating Ricci scalar $\hat{R}$, we obtain ${}^{(\Phi)}R$ plus a term proportional to $\nabla^2 \mathcal{U}$ [43]. Because $\hat{R} = {}^{(\Phi)}R = 0$, thus $\mathcal{U}$ is a harmonic function on the three-dimensional Euclidean manifold $\nabla_i \nabla^i \mathcal{U} = 0$, where $\nabla$ is the covariant derivative on a flat manifold.

One can define a local coordinate for the base space in the form

$$ {}^{(\Phi)} g_{ij} dx^i dx^j = \tilde{\rho}^2 d\mathcal{U}^2 + \tilde{h}_{AB} dx^A dx^B. \tag{72} $$

The photon sphere will be located at some constant value of $\mathcal{U}$, and the radius of the photon sphere can be given at the fixed value of the $\rho$-coordinate [9]. All these enable that on the hypersurface $\Sigma$ the metric tensor can be given in the form of

$$ \hat{g}_{ij} dx^i dx^j = \rho^2 dN^2 + h_{AB} dx^A dx^B, $$

and a connected component of the photon surface can be identified at a fixed value of the $\rho$-coordinate.

Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ comprise two solutions of the boundary value problem of the Einstein-Maxwell system with electric and magnetic charges. Keeping in mind the Green identity, integrating over the volume element, one gets

$$ \left( \int_{r\to\infty} - \int_{\mathcal{H}} \right) (\mathcal{U}_1 - \mathcal{U}_2) \frac{\partial}{\partial r} (\mathcal{U}_1 - \mathcal{U}_2) dS = \int_{\Omega} |\nabla(\mathcal{U}_1 - \mathcal{U}_2)|^2 d\Omega. \tag{73} $$

The surface integrals on the left-hand side of Eq. (73) vanish because of the imposed boundary conditions and provide that the volume integrals have to be identically equal to zero. It all leads to the conclusion that the considered two solutions of the Laplace equation with the Dirichlet boundary conditions are identical.

### B. Positive mass theorem and uniqueness

For the completeness of the presented results, we propose the alternative way of conducting the uniqueness proof of electric and magnetic charged photon spheres, based on another conformal transformation and on the use of the positive energy theorem [55–58]. Namely, consider the conformal transformation on $(\Sigma, \Omega^2 g_{ij})$; then one pastes two copies of $\Sigma_{\pm}$ along the boundary and takes into account the conformal transformations on each copy of $\Sigma$, i.e., $\Omega_{\pm}^2 g_{ij}$. The conformal factors yield [8,56]

$$ \Omega_{\pm} = \frac{1}{4} [(1 \pm N)^2 - ZZ^*]. \tag{74} $$

The Ricci curvature for the metric $\Omega^2 g_{ij}$, where for brevity we denote $\Omega = \Omega_{\pm}$, has the form

$$ \frac{1}{2} \Omega^4 N^2 R(\Omega^2 g_{ij}) = \left| \left( \Omega - N \frac{\partial \Omega}{\partial N} \right) {}^{(g)}\nabla_i Z - 2N \frac{\partial \Omega}{\partial Z^*} {}^{(g)}\nabla_i N \right|^2 - \frac{1}{16} N^2 |Z^{(g)}\nabla_i Z^* - Z^{*(g)}\nabla_i Z|^2. \tag{75} $$

It turns out that the relation between electric and magnetic potentials in the static spacetime causes the last term in (75) to disappear, and one can conclude that $(\Sigma, \Omega^2 g_{ij})$ is an asymptotically flat complete three-dimensional manifold with non-negative scalar curvature and vanishing mass. Next, the implementation of the positive energy theorem implies that the manifold in question is isometric to $(R^3, \delta_{ij})$.

The requirements for the positive energy theorem point out that it cannot be implemented for $(\Sigma_+, \Omega_+^2 g_{ij})$ [57]. However, they are satisfied for

$$ (\Sigma, g_{ij}) = (\Sigma_+, \Omega_+^2 g_{ij}) \cup (\Sigma_- \cup \{p\}, \Omega_-^2 g_{ij}), $$

where $\{p\}$ is a point at infinity at $\Sigma_-$ [57,58]. On the other hand, the conformal flatness of $(\Sigma, g_{ij})$ entails its spherical symmetry [8,57].

The arguments, presented, for instance, in [8,56–58], lead to the final conclusion that the metric $g_{ij}$ is spherically symmetric, and we arrive at the uniqueness of the photon sphere characterized by ADM mass $M$ and electric and magnetic charges as the only static spherically symmetric spacetime possessing the photon sphere in Einstein-Maxwell gravity with electric and magnetic potentials.

Summing it all up, we achieve the main result, the uniqueness of the photon sphere for the nonextremal Reissner-Nordström electric-magnetic system.

*Theorem.* Suppose that the set $(M^3, g_{ij}, N, \psi_F, \psi_B)$ is the asymptotic to the static nonextremal Einstein-Maxwell black hole spacetime with electric and magnetic charges. Moreover, the spacetime in question has the photon sphere $(P^3, h_{ij}) \hookrightarrow (R \times M^3, -N^2 dt^2 + g_{ij} dx^i dx^j)$, which can be regarded as the inner boundary of $R \times M^3$. Suppose further that $M$, $Q_{(F)}$, and $Q_{(B)}$ are the ADM mass and the total charges connected with the Maxwell electric and magnetic fields of $(R \times M^3, -N^2 dt^2 + g_{ij} dx^i dx^j)$. Then, $(R \times M^3, -N^2 dt^2 + g_{ij} dx^i dx^j)$ is isometric to the region exterior to the photon sphere in the electrically and magnetically charged nonextremal Einstein-Maxwell black hole spacetime.

## V. CONCLUSIONS

Our paper is devoted to the problem of uniqueness of the black hole photon sphere in Einstein-Maxwell gravity with electric and magnetic charges. Keeping in mind the special features of electric and magnetic fields in the spacetime with an asymptotically timelike Killing vector field, which is orthogonal to the hypersurface of constant time, we find the functional dependence among lapse function and electric and magnetic potentials. It authorizes that the Ricci curvature scalar of the photon sphere is a constant scalar curvature one.

The conformal positive energy and positive energy theorems allow us to find the two alternative proofs of the uniqueness of a static, nonextremal asymptotically flat black hole photon sphere in Einstein-Maxwell gravity with electric and magnetic charges (the Reissner-Nordström electric-magnetic black hole photon sphere).

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