Motion of a rotating black hole in a homogeneous scalar field

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In the present paper, we consider a rotating black hole moving in a homogeneous massless scalar field. We assume that the field is weak and neglect its backreaction so that the metric at far distance from the black hole is practically flat. In this domain, one can introduce two reference frames, K and $\tilde{K}$. The frame $\tilde{K}$ is associated with the homogeneous scalar field, in which its constant gradient has only time component. The other frame, K, is the frame in which the black hole is at rest. To describe the Kerr metric of the black hole, we use its Kerr-Schild form $g_{\mu\nu} = \eta_{\mu\nu} + \Phi l_\mu l_\mu$, where $\eta_{\mu\nu}$ is the (asymptotic) flat metric in K frame. We find an explicit solution of the scalar field equation, which is regular at the horizon, and properly reproduce the asymptotic form of the scalar field at the infinity. Using this solution, we calculate the fluxes of the energy, momentum and the angular momentum of the scalar field into the black hole. This allows us to derive the equation of motion of the rotating black hole. We discuss main general properties of solutions of these equations and obtain explicit solutions for special type of the motion of the black hole.

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I. INTRODUCTION

Scalar field plays an immense role in the modern physics and cosmology. In the high energy physics, a scalar Higgs field is used to provide the particles their mass as a result of the spontaneous symmetry breaking. In cosmology, the inflation can be driven by the potential part of the scalar field ("inflaton"). The scalar field and Higgs mechanism are important parts of the models describing possible symmetry breaking and phase transitions in the cosmology. It is believed that in the early Universe, there might be several of such phase transitions that played an important role in its evolution. During these transitions, formation of primordial black holes and cosmic strings might become possible. More recently, another mechanism of symmetry breaking known as the ghost condensation was proposed [1,2]. The corresponding phase of the ghost condensate is formed due to the special form of the kinetic part of the scalar field action. In such a model, the ground state is the scalar field with a nonvanishing vector of its gradient. The presence of such a field breaks the Lorentz invariance.

The ghost condensation model belongs to a wide class of scalar field models that are invariant with respect to the scalar field shift $\Psi \to \Psi + \mathrm{const}$. At the lowest order in derivatives, the Poincaré invariant Lagrangian for such a theory takes the form

$$L = L(X), \qquad X = \eta^{\mu\nu}\Psi_{,\mu}\Psi_{,\nu}. \tag{1.1}$$

Such a model can also be used as the low-energy effective field theory for zero- and finite-temperature relativistic superfluids [3,4]. For the superfluid state with finite charge density and vanishing spatial current, the corresponding solution is $\Psi = \mu t$, where μ is the chemical potential [3]. Similar solutions with a constant spacelike gradient of the scalar field were considered in application to the cosmology in the framework of a so-called solid inflation model [5]. Let us also mention an interesting approach in which the scalar field is used to describe a phenomenon emergence of time and dynamics in originally Euclidean spacetime (see, e.g., [6–9]).

A natural and interesting question is how a black hole interacts with a scalar field in different models and under different conditions. Let us note that black holes cannot have their own scalar field. Discussion of the no-hair theorem for the scalar field and further references can be found in [10]. At the same time, a black hole can exist in the presence of an external scalar field. Accretion of the ghost condensate by black holes in the expanding Universe was discussed in [11–14]. Primordial black holes and Higgs field vacuum decay were considered in [15–18]. Scalar field accretion by black holes was also discussed in [19].

In this paper, we consider a motion of a rotating black hole in an external homogeneous massless minimally coupled scalar field. This model allows a rather complete analysis. In the absence of the black hole, the scalar field equation in the flat spacetime

$$\Box\Psi = 0 \tag{1.2}$$

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has a simple solution

$$\Psi = \Psi_0 t. \tag{1.3}$$

Such a field has constant gradient and is homogeneous in space. This form of the solution is valid in a specially chosen inertial reference frame, and, in this sense, it breaks the Lorentz invariance. This choice of the solution is motivated by the ghost condensate model. We consider a rotating black hole moving in such a scalar field. In the presence of the black hole, the scalar field Ψ is distorted. We assume that the scalar field is weak, and its back-reaction on the metric can be neglected.

Our first goal is to find a solution for the scalar field that is regular at the horizon of the moving black hole and has the asymptotic form (1.3) at far distance from it. We shall demonstrate that this problem allows an exact solution. To find this solution, we proceed as follows.

A remarkable property of the Kerr metric is that it can be written in the Kerr-Schild form [20]

$$g_{\mu\nu} = \eta_{\mu\nu} + \Phi l_\mu l_\nu, \tag{1.4}$$

where $\eta_{\mu\nu}$ is a flat metric, Φ is a scalar field, and $\boldsymbol{l}$ is a tangent vector to a shear-free geodesic null congruence. It has been shown that these solutions of the Einstein equations can be obtained by complex coordinate transformations from the Schwarzschild metric [21,22]. In particular, the potential Φ for the Kerr metric can be obtained as a solution of the Laplace equation in flat coordinates (X, Y, Z)

$$\triangle\Phi = 4\pi j, \tag{1.5}$$

with a pointlike source j located at the complex coordinate $Z + ia$, where a is the rotation parameter of the Kerr black hole [23,24]. A comprehensive review of the Kerr-Schild metrics and complex space approaches can be found in [25]. More recently, the Kerr-Newman representation of the spacetime geometry attracted a lot of attention in the so-called double copy formalism. This formalism is based on the following result: Einstein equations for the metrics that allow the Kerr-Schild representation can be reduced to the linear equations for Maxwell and scalar fields. At the moment, there exist dozens of publications on this subject. Related references can be found, e.g., in the following review articles [26–28].

One can interpret this result as follows. The form (1.4) of the metric allows one to treat the metric $\boldsymbol{\eta}$ as the metric of the background flat spacetime, while the term $\Phi l_\mu l_\nu$ describes its "perturbation" due to the black hole located at the origin of the background space. Denote by M the mass of the black hole. The gravitational field of the black hole is strong in its vicinity. For an observer located at the distance $L \gg M$ from the black hole, this field is weak. One can say that such an observer "lives" in the space with the background metric $\boldsymbol{\eta}$. Such an observer can describe the black hole as a small compact object and use for the description of its motion a "point particle" approximation.

If the scalar field is present and has the form (1.3), the interaction of this field results in its accretion by the black hole. In this paper, we consider a black hole moving in such a homogenous scalar field. In this case, their exist two natural reference frames. One of them, which we denote by $\tilde{K}$, is the frame in which at far distance from the black hole, the scalar field has the form (1.3). The other frame, moving with respect to $\tilde{K}$ with the velocity $\vec{V}$, is the frame in which the black hole is at rest. We denote it by K. In this frame, the form of the solution for the scalar field differs from (1.3), and it can be obtained by making the correspondent Lorentz transformation. We shall use both frames. Namely, we use the Kerr-Schild form of the Kerr metric associated with K frame to solve the scalar field equations in the presence of the black hole. Using this result, we calculate the force acting on the black hole due to its motion in the scalar field and obtain the equation of motion of the black hole in $\tilde{K}$ frame.

The paper is organized as follows. In Sec. II, we collect useful formulas connected with the Kerr-Schild form of the Kerr metric. Section III describes a solution for the scalar field in the presence of the moving rotating black hole. Fluxes of the scalar field in K frame are obtained in Sec. IV. Section V contains calculation of the 4D force acting on the black hole in the frame $\tilde{K}$. It also discusses the equations of motion of the black hole, general properties of their solutions, and special cases. Useful information concerning complex null tetrads is collected in Appendix A. Appendix B contains details of the calculations of the fluxes of the energy and angular momentum of the scalar field through the horizon of the black hole. In the paper, we use units in which $G = c = 1$ and sign conventions of the book [29].

II. KERR METRIC AND ITS KERR-SCHILD FORM

A. The Kerr metric

The Kerr metric, describing a vacuum stationary rotating black hole, written in the Boyer-Lindquist coordinates is

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\varphi \\ &\quad + \left(r^2 + a^2 + \frac{2Ma^2r}{\Sigma}\sin^2\theta\right)\sin^2\theta d\varphi^2 \\ &\quad + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \\ \Sigma &= r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2. \end{aligned} \tag{2.1}$$

Here M is the black hole mass, and a is its rotation parameter. This metric has two commuting Killing vectors $\boldsymbol{\xi} = \partial_t$ and $\boldsymbol{\zeta} = \partial_\phi$. Let us denote

$$r_\pm = M \pm b, \qquad b = \sqrt{M^2 - a^2}. \tag{2.2}$$

Equation $r = r_+$, where $\Delta = 0$, describes the event horizon. The surface area of the horizon is

$$\mathcal{A} = 4\pi(r_+^2 + a^2) = 8\pi M r_+. \tag{2.3}$$

Coordinates (t, r, θ, φ) are singular at this surface. To describe both the exterior and interior of a rotating black hole, one can use so-called Kerr incoming coordinates $(v, r, \theta, \tilde{\varphi})$, which are regular at the future event horizon [30]

$$dv = dt + dr_*, \qquad dr_* = (r^2 + a^2)\frac{dr}{\Delta}, \qquad d\tilde{\varphi} = d\varphi + a\frac{dr}{\Delta}. \tag{2.4}$$

In these coordinates, the metric [30] takes the form

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dv - \frac{1}{a}\Delta_y^{(0)} d\tilde{\varphi}\right)^2 + \frac{\Delta_y^{(0)}}{\Sigma}\left(dv - \frac{1}{a}\Delta_r^{(0)} d\tilde{\varphi}\right)^2 + \frac{\Sigma}{\Delta_y^{(0)}} dy^2 + 2dr\left(dv - \frac{1}{a}\Delta_y^{(0)} d\tilde{\varphi}\right).$$

Similarly, one can introduce Kerr outgoing coordinates $(u, r, \theta, \tilde{\varphi})$

$$dv = dt - dr_*, \qquad d\tilde{\varphi} = d\varphi - a\frac{dr}{\Delta}, \tag{2.5}$$

which are regular at the past horizon and cover the white hole domain.

B. Useful coordinates

For $M = 0$, the Riemann curvature of the Kerr metric vanishes, and the metric (2.1) becomes flat. We write it in the form[1]

$$\begin{aligned} dS^2 &= -dT^2 + dh^2, \\ dh^2 &= \frac{\Sigma}{r^2 + a^2} dr^2 + \Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2. \end{aligned} \tag{2.6}$$

The coordinates (r, θ, ϕ) are oblate spheroidal coordinates taking the following values $r \geq 0$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$.

[1] Let us note that we use notations T and ϕ for the time and angle variables in the flat spacetime. These coordinates are used in the Kerr-Schild form of the Kerr metric, while the standard Bouer-Lindquist coordinates t and φ are related to T and ϕ by means of relations (2.18).

These coordinates are related to the Cartesian coordinates as follows:

$$\begin{aligned} X &= \sqrt{r^2 + a^2}\sin\theta\cos\phi, \\ Y &= \sqrt{r^2 + a^2}\sin\theta\sin\phi, \\ Z &= r\cos\theta. \end{aligned} \tag{2.7}$$

In these coordinates, the flat metric dS^2 takes the standard form

$$dS^2 = \eta_{\mu\nu} dX^2\, dX^\nu = -dT^2 + dX^2 + dY^2 + dZ^2. \tag{2.8}$$

For $r > 0$, the surfaces $r = \text{const}$ are oblate ellipsoids. Figure 1 shows the coordinate lines of the oblate spheroidal coordinates (r, θ) in the plane $Y = 0$ $(\phi = 0)$. For $r = 0$ and $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, one has a disc of radius a located in the $Z = 0$ plane. The coordinate θ is discontinuous on the disc. For $(0, \pi/2)$, the coordinate θ covers the upper part of the disc, while for $(\pi/2, \pi)$, it covers the lower part of it. The boundary of this disc is a ring of radius a. Equations $\theta = 0$ and $\theta = \pi$ describe the axis of symmetry $X = Y = 0$. For $\theta = 0$ and $Z = r$, it is positive, while for $\theta = \pi$ $Z = -r$, it is negative.

In what follows, we shall also use another coordinate, y, related to the angle θ as follows:

$$y = a\cos\theta. \tag{2.9}$$

The flat metric dS^2 in the spheroidal coordinates (T, r, y, ϕ) is

$$\begin{aligned} dS^2 &= -dT^2 + \Sigma\left(\frac{dr^2}{\Delta_r^0} + \frac{dy^2}{\Delta_y^0}\right) + \frac{\Delta_r^0 \Delta_y^0}{a^2} d\phi^2, \\ \Sigma &= r^2 + y^2, \quad \Delta_r^0 = r^2 + a^2, \quad \Delta_y^0 = a^2 - y^2. \end{aligned} \tag{2.10}$$

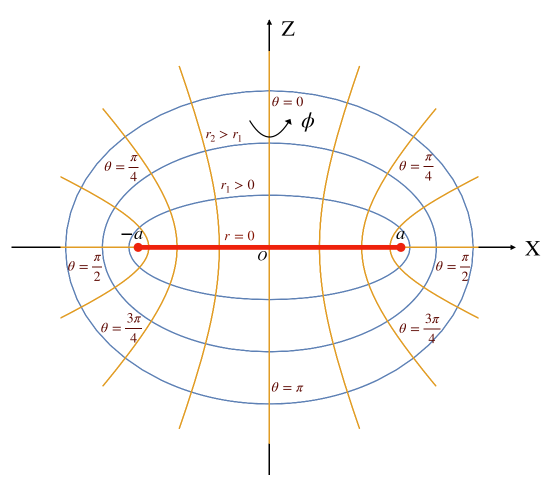

FIG. 1. Coordinate lines of the oblate spheroidal coordinates (r, θ) in the plane $Y = 0$.

In these coordinates, the Cartesian coordinates take the form

$$
\begin{aligned}
X &= \frac{1}{a}\sqrt{\Delta_r^{(0)}\Delta_y^{(0)}}\cos\phi,\\
Y &= \frac{1}{a}\sqrt{\Delta_r^{(0)}\Delta_y^{(0)}}\sin\phi,\\
Z &= \frac{1}{a}ry.
\end{aligned}
\tag{2.11}
$$

We denote

$$
e_{(X)\mu} = X_{,\mu}, \quad e_{(Y)\mu} = Y_{,\mu}, \quad e_{(Z)\mu} = Z_{,\mu}. \tag{2.12}
$$

We also denote by $\boldsymbol{e}_i$, $i = 1, 2, 3$ a set of 4D unit vectors e_i^μ along X, Y and Z axes.

C. Kerr metric in the Kerr-Schild form

Let us consider the following 1-form

$$
l_\mu dx^\mu = -dT - \frac{\Sigma}{\Delta_r^0}dr + \frac{\Delta_y^0}{a}d\phi. \tag{2.13}
$$

We define a metric

$$
ds^2 = dS^2 + \Phi(l_\mu dx^\mu)^2, \tag{2.14}
$$

where $\Phi = \Phi(r, y)$ is some function. The metric coefficients of the metrics ds^2 and dS^2 are related as follows:

$$
g_{\mu\nu} = \eta_{\mu\nu} + \Phi l_\mu l_\nu. \tag{2.15}
$$

The following statements are valid for each of the metrics ds^2 and dS^2. In other words, these statements are valid for an arbitrary function Φ, including $\Phi = 0$. The vector field $\boldsymbol{l}$ has the following properties:

(i) The contravariant components of the vector $\boldsymbol{l}$ in (T, r, θ, ϕ) coordinates are

$$
l^\mu = \left(1, -1, 0, \frac{a}{r^2 + a^2}\right). \tag{2.16}
$$

(ii) $\boldsymbol{l}$ is a null vector $\boldsymbol{l}^2 = l_\mu l^\mu = 0$.
(iii) Vectors $\boldsymbol{l}$ are tangent vectors to incoming null geodesics in the affine parametrization, $l^\nu l^\mu{}_{;\nu} = 0$.
(iv) $l^\mu{}_{;\mu} = -\frac{2r}{\Sigma}$.
(v) $l_{(\mu;\nu)}l^{(\mu;\nu)} - \frac{1}{2}(l^\mu_{;\mu})^2 = 0$.

The last property implies that the congruence of null vectors $\boldsymbol{l}$ is shear free (for more details see, e.g., [31,32]). Such a null geodesic congruence is related to the light cones with apex on the worldline in the complex space. The twist is a measure of how far the complex worldline is from the real slice [33].

It is easy to check that for a special choice of the function Φ

$$
\Phi_0 = \frac{2Mr}{\Sigma}, \tag{2.17}
$$

the metric ds^2 given by (2.14) is Ricci flat, and in fact, it coincides with the Kerr metric. In order to prove this, it is sufficient to make the following coordinate transformation:

$$
\begin{aligned}
T &= t + t_0(r), \qquad \phi = \varphi + \varphi_0(r),\\
t_0(r) &= \int \frac{2Mr}{\Delta}dr\\
&= \frac{M}{\sqrt{M^2 - a^2}}[r_+\ln(r - r_+) - r_-\ln(r - r_-)],\\
\varphi_0(r) &= \int \frac{2Mar}{(r^2 + a^2)\Delta}dr\\
&= \frac{a}{2\sqrt{M^2 - a^2}}\ln\left(\frac{r - r+}{r - r_-}\right) - \arctan(r/a) + \frac{1}{2}\pi.
\end{aligned}
\tag{2.18}
$$

Here Δ is defined in ([30]). These coordinates (t, r, θ, φ) are chosen so that the nondiagonal components g_{rt} and $g_{r\varphi}$ of the metric ds^2 vanish. One can check that the metric ds^2 written in the (t, r, θ, φ) coincides with the Kerr metric dS^2, provided one identifies the coordinates t and φ in ds^2 with the standard Boyer-Lindquist coordinates t and φ in the metric (2.1). The integration constant in the expression for φ_0 is chosen so that this quantity vanishes when $r \to \infty$. Hence, in this limit, the angle variables φ and ϕ coincide.

It is easy to check that the coordinates T and ϕ are related to the Kerr incoming coordinates v and $\tilde{\varphi}$ as follows:

$$
T = v - r, \qquad \phi = \tilde{\varphi} - \arctan(r/a). \tag{2.19}
$$

Coordinates (T, r, y, ϕ) are regular at the future event horizon and cover the exterior region of the black hole as well as a part of its interior. Similarly, by a simple change of the sign of the coefficient of dr term in the expression (2.13), one can obtain outgoing null vector and use it to construct the Kerr-Schild metric, which is regular at the past horizon.

III. SCALAR FIELD

A. Flat spacetime

Let us consider a minimally coupled massless scalar field Ψ that obeys the equation

$$
\Box\Psi = 0. \tag{3.1}
$$

Its stress-energy tensor is

$$
T_{\mu\nu} = \Psi_{,\mu}\Psi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Psi_{,\alpha}\Psi^{,\alpha}. \tag{3.2}
$$

Let us consider first the flat spacetime. We choose some inertial frame in it. We denote it by $\tilde{K}$ and denote by $\tilde{X}^\mu = (\tilde{T}, X, Y, \tilde{Z})$ the Cartesian coordinates associated with this frame. In this frame, there exists a simple solution of the (3.1)

$$\Psi = \Psi_0 \tilde{T}, \tag{3.3}$$

describing a homogeneous scalar field. The stress-energy tensor for this solution is diagonal and has the following nonvanishing components:

$$T_{\tilde{T}\tilde{T}} = T_{\tilde{X}\tilde{X}} = T_{\tilde{Y}\tilde{Y}} = T_{\tilde{Z}\tilde{Z}} = \frac{1}{2}\Psi_0^2. \tag{3.4}$$

The gradient of the field Ψ has only a time component. In this sense, this solution breaks the Lorentz invariance and singles out an inertial frame in which the field Ψ does not depend on spatial coordinates. As we already mentioned, such solutions with a constant gradient of the field play an important role in the shift-invariant scalar field theories (see, e.g., [1,2,4,5]).

Consider a rotating black hole moving with respect to $\tilde{K}$ with a constant velocity $\vec{V}$ (see Fig. 2). We denote by K a reference frame associated with the black hole and denote by $X^\mu = (T, X, Y, Z)$ the Cartesian coordinates associated with this frame. We choose the coordinate axes in both frames to be in the same directions. We also choose the Z axis to be parallel to the spin of the black hole. The vector $\vec{V}$ has the following components:

$$\vec{V} = (V_X, V_Y, V_Z),$$
$$V^2 = (\vec{V})^2 = V_X^2 + V_Y^2 + V_Z^2. \tag{3.5}$$

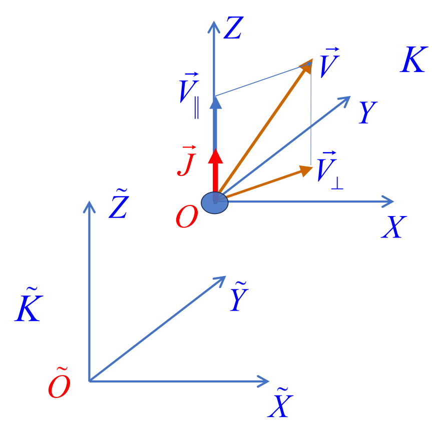

FIG. 2. Two inertial frames, K and $\tilde{K}$, are schematically shown at this figure. The Cartesian coordinates in these frames are (T, X, Y, Z) and $(\tilde{T}, \tilde{X}, \tilde{Y}, \tilde{Z})$, respectively. The frame $\tilde{K}$ is the "rest frame" of the homogeneous scalar field, in which it has the form $\Psi = \Psi_0 \tilde{T}$. The frame K is the rest frame of the rotating black hole. The origin of this frame O is at the position of the black hole. The frame K moves with respect to $\tilde{K}$ with the velocity $\vec{V}$. The coordinate axes in both frames are parallel. The Z axis in the K frame coincides with the direction of the spin $\vec{J}$ of the black hole. The velocity $\vec{V}$ in the frame K can be decomposed as follows: $\vec{V} = \vec{V}_\parallel + \vec{V}_\perp$, where $\vec{V}_\parallel$ is parallel to the spin of the black hole, and $\vec{V}_\perp$ is orthogonal to it.

Denote by $\vec{R} = (X, Y, Z)$ a 3D vector connecting the origin $\tilde{O}$ of the frame $\tilde{K}$ with the origin O of the moving frame K. Then the Lorentz transformation implies

$$\tilde{T} = \gamma(T + (\vec{V}, \vec{R})),$$
$$\gamma = 1/\sqrt{1 - V^2}. \tag{3.6}$$

The solution (3.3) written in the frame K comoving with the black hole takes the form

$$\Psi = \bar{\Psi}_0 (T + V_X X + V_Y Y + V_Z Z),$$
$$\bar{\Psi}_0 = \frac{\Psi_0}{\sqrt{1 - V^2}}. \tag{3.7}$$

B. Scalar field solution in the presence of a moving rotating black hole

In the previous subsection, we ignore the gravitational field of the black hole. To obtain a solution for the scalar field in the presence of a moving rotating black hole, we proceed as follows. We use the Kerr-Schild form of the metric (2.14) associated with K frame, in which the black hole is at rest. In this form, the Einstein equations are linearized. One can identify the metric dS^2 with the flat background geometry in the K frame, while the $\Phi l_\mu l_\nu$ describes its "perturbation" due to the presence of the black hole. We are looking for a solution of the scalar field equation (3.1), which is regular at the horizon of the black hole and at far distance has the asymptotic form (3.7).

The required solution satisfying the imposed boundary conditions in the coordinates (T, r, y, ϕ) is

$$\Psi = \bar{\Psi}_0 (T + T_0(r) + V_X \mathcal{V}_X + V_Y \mathcal{V}_Y + V_Z \mathcal{V}_Z),$$
$$T_0(r) = -2M \ln(r - r_-),$$
$$\mathcal{V}_X = \frac{1}{a}\sqrt{\frac{\Delta_y^{(0)}}{\Delta_r^{(0)}}}((\Delta_r^{(0)} - Mr)\cos\phi - Ma\sin\phi),$$
$$\mathcal{V}_Y = \frac{1}{a}\sqrt{\frac{\Delta_y^{(0)}}{\Delta_r^{(0)}}}((\Delta_r^{(0)} - Mr)\sin\phi + Ma\cos\phi),$$
$$\mathcal{V}_Z = \frac{y}{a}(r - M). \tag{3.8}$$

To check the validity of the boundary conditions at the infinity, it is sufficient to use following asymptotics of the functions $\mathcal{V}$:

$$\mathcal{V}_X \approx r \sin\theta \cos\phi = X,$$
$$\mathcal{V}_Y \approx r \sin\theta \sin\phi = Y,$$
$$\mathcal{V}_Z \approx r \cos\theta = Z. \tag{3.9}$$

Since the coordinates (T, r, y, ϕ) are regular on the future horizon, and the components of its gradient $\Phi_{;\mu}$ are regular functions of these coordinates, one can conclude that the presented solution (3.8) does satisfy the required condition of the regularity at the horizon.

The same solution written in the Boyer-Lindquist coordinates (t, r, θ, φ) is

$$\Psi = \bar{\Psi}_0(t + \tilde{t}_0 + V_X\mathcal{U}_X + V_Y\mathcal{U}_Y + V_Z\mathcal{U}_Z),$$
$$\mathcal{U}_X = \frac{\sin\theta}{\sqrt{\Delta_r^{(0)}}}[(\Delta_r^{(0)} - Mr)\cos\psi - Ma\sin\psi],$$
$$\mathcal{U}_Y = \frac{\sin\theta}{\sqrt{\Delta_r^{(0)}}}[(\Delta_r^{(0)} - Mr)\sin\psi + Ma\cos\psi],$$
$$\mathcal{U}_Z = V(r - M)\cos\theta,$$
$$\tilde{t}_0 = t_0(r) + T_0(r) = \frac{M}{\sqrt{M^2 - a^2}}[r_+\ln(r - r_+) - r_-\ln(r - r_-)] - 2M\ln(r - r_-),$$
$$\varphi_0(r) = \frac{a}{2\sqrt{M^2 - a^2}}\ln\left(\frac{r - r_+}{r - r_-}\right) - \arctan(r/a),$$
$$\psi = \varphi + \varphi_0(r). \tag{3.10}$$

IV. FLUXES

A. Energy and angular momentum fluxes through $r = \mathrm{const}$ surface

Let us calculate the fluxes of the energy, angular momentum, and the momentum through a 2D surface of constant radius $r = r_0$ surrounding the black hole. For these calculations, it is convenient to use the Boyer-Lindquist coordinates (t, r, θ, φ). Denote by Σ_0 a 3D timelike surface describing the "evolution" of S for the time interval (t_-, t_+).

Denote by $\boldsymbol{q}$ a 3D metric on Σ induced by its embedding in the 4D space. Then

$$q \equiv \sqrt{-\det(\boldsymbol{q})} = \sqrt{\Delta\Sigma}\sin\theta. \tag{4.1}$$

A unit vector $\boldsymbol{n}$ orthogonal to Σ_0 and inward directed is

$$n^\mu = -\sqrt{\Delta/\Sigma}\,\delta_r^\mu. \tag{4.2}$$

The vector of the volume element of the surface Σ is

$$d\sigma^\mu = n^\mu dt\, d\theta\, d\varphi = -\delta_r^\mu \Delta \sin\theta\, dt\, d\theta\, d\varphi. \tag{4.3}$$

We denote the fluxes of the energy and angular momentum through 3D surface Σ_0 into its interior per a unit time t by $\mathcal{E}$ and $\mathcal{J}$, respectively. Then one has

$$\mathcal{E} = -\frac{1}{t_+ - t_-}\int_\Sigma \xi^\mu T_{\mu\nu} d\sigma^\mu = \Delta\int_S T_{tr} d\omega,$$
$$\mathcal{J} = \frac{1}{t_+ - t_-}\int_\Sigma \zeta^\mu T_{\mu\nu} d\sigma^\mu = -\Delta\int_S T_{r\varphi} d\omega,$$
$$d\omega = \sin\theta d\theta d\varphi. \tag{4.4}$$

The signs in these expressions are chosen so that these quantities describe the flux into the surface S from its exterior.

Since g_{tr} and $g_{r\varphi}$ components of the Kerr metric in the Boyer-Lindquist coordinates vanish, one has

$$T_{tr} = \Psi_{,t}\Psi_{,r}, \qquad T_{t\varphi} = \Psi_{,r}\Psi_{,\varphi}. \tag{4.5}$$

Calculating these expressions and taking integrals in (4.4), one obtains

$$\mathcal{E} = 8\pi\bar{\Psi}_0^2 M r_+,$$
$$\mathcal{J} = -\frac{4}{3}\pi\bar{\Psi}_0^2 a M^2 (V_X^2 + V_Y^2). \tag{4.6}$$

Let us emphasize that these quantities do not depend on the radius $r = r_0$ of the surface S, which was used for their calculation. (For explanation of this property and more details, see Appendix B.)

B. Momentum fluxes

We use the vectors e_i^μ, defined in (2.12) to define the following objects[2]:

$$\mathcal{P}_X = \frac{1}{t_+ - t_-}\int_\Sigma T_{\mu\nu} e_{(X)}^\mu d\sigma^\nu = -\Delta\int_S T_{Xr} d\omega,$$
$$\mathcal{P}_Y = \frac{1}{t_+ - t_-}\int_\Sigma T_{\mu\nu} e_{(Y)}^\mu d\sigma^\nu = -\Delta\int_S T_{Yr} d\omega,$$
$$\mathcal{P}_Z = \frac{1}{t_+ - t_-}\int_\Sigma T_{\mu\nu} e_{(Z)}^\mu d\sigma^\nu = -\Delta\int_S T_{Zr} d\omega,$$
$$T_{Xr} = T_{\mu r} e_{(X)}^\mu, \quad T_{Yr} = T_{\mu r} e_{(Y)}^\mu, \quad T_{Zr} = T_{\mu r} e_{(Z)}^\mu. \tag{4.7}$$

[2] Let us note that the norm of the vectors e_i^μ, calculated in the metric $g_{\mu\nu}$, at far distance slightly differs from one. However, it is possible to check that using the normalized versions of these vectors in the expressions given below does not change the results when the limit $r = r_0 \to \infty$ is taken.

Calculating these integrals and taking the limit $r_0 \to \infty$, one obtains the following results:

$$\mathcal{P}_X = -8\pi\bar{\Psi}_0^2\left[Mr_+V_X + \frac{2}{3}aMV_Y\right],$$
$$\mathcal{P}_Y = -8\pi\bar{\Psi}_0^2\left[Mr_+V_Y - \frac{2}{3}aMV_X\right],$$
$$\mathcal{P}_Z = -8\pi\bar{\Psi}_0^2 Mr_+V_Z. \tag{4.8}$$

The obtained results can be presented in the following 3D vector form. Let us denote

$$\beta = 8\pi\Psi_0^2, \qquad \vec{V} = (V_X, V_Y, V_Z),$$
$$\vec{\mathcal{P}} = (\mathcal{P}_X, \mathcal{P}_Y, \mathcal{P}_Z), \quad \vec{J} = M\vec{a}, \quad \vec{\mathcal{J}} = \frac{\vec{a}}{a}\mathcal{J}. \tag{4.9}$$

Here $\vec{a}$ is a 3D vector with the norm a and directed along Z-axis. Consider an observer that is located at a very far distance from the black and assume that the black hole is initially at rest in his/her frame. As a result of the interaction with "moving" scalar field, the black hole absorbs energy and momentum. To describe its further evolution as a result of this effect, a far-distant observer can neglect the black hole size and approximate it by the massive point with spin that has energy and momentum. In this interpretation, the calculated quantities $\mathcal{E}$ and $\vec{\mathcal{P}}$ are nothing but the components of the 4D force $\boldsymbol{F}$ acting on such a massive point.

In the asymptotic flat coordinates (T, X, Y, Z), the components of this 4D force are

$$F^\mu = (\mathcal{E}, \mathcal{P}_X, \mathcal{P}_Y, \mathcal{P}_Z),$$
$$F^T = \frac{\beta M r_+}{1 - V^2},$$
$$\vec{F} = -\frac{\beta}{1 - V^2}\left(Mr_+\vec{V} + \frac{2}{3}\vec{J}\times\vec{V}\right). \tag{4.10}$$

In addition to these expressions for the 4D force acting on the black hole, there exists one more equation that demonstrates that the spin of the black changes when the black hole has a nonvanishing transverse component of the velocity $V_\perp$

$$\vec{\mathcal{J}} = -\frac{1}{6}\beta M V_\perp^2 \vec{J},$$
$$V_\perp^2 = \vec{V}^2 - \frac{1}{J^2}(\vec{J}\cdot\vec{V})^2. \tag{4.11}$$

V. MOTION OF A ROTATING BLACK HOLE THROUGH THE SCALAR FIELD

A. Friction force in $\tilde{K}$ frame

The 4D force F^μ is calculated in the frame where the black hole is initially at rest. Under the action of this force, the black hole has a nonvanishing acceleration. Since the velocity of the black hole interacting with the scalar fields changes in time, the frame in which the black hole is at rest is not inertial. For this reason, it is more convenient to write the equations of motion in the frame $\tilde{K}$ associated with the scalar field. Let us denote by $f^\mu \equiv \tilde{F}^\mu$ the components of the 4D force in $\tilde{K}$ frame. The corresponding Lorentz transformation implies

$$f^0 = \gamma(F^T + \vec{V}\cdot\vec{F}),$$
$$\vec{f} = \vec{F} + \gamma F^T\vec{V} + (\gamma - 1)\frac{\vec{V}\cdot\vec{F}}{V^2}\vec{V}. \tag{5.1}$$

Here $\gamma = 1/\sqrt{1 - V^2}$. Using (4.10), one obtains

$$f^0 = \frac{\beta M r_+}{\sqrt{1 - V^2}},$$
$$\vec{f} = -\frac{2}{3}\frac{\beta}{\sqrt{1 - V^2}}[\vec{J}\times\vec{V}]. \tag{5.2}$$

Let us notice that the following two useful relations are valid:

$$\vec{V}\cdot\vec{f} = 0, \qquad \vec{J}\cdot\vec{f} = 0. \tag{5.3}$$

B. Equations of motion

We denote by $\boldsymbol{P}$ the 4D momentum of the black hole in $\tilde{K}$ frame. In Cartesian coordinates, it has the following form:

$$P^\mu = (E, \vec{P}), \qquad \vec{P} = (P_X, P_Y, P_Z). \tag{5.4}$$

The energy E and momentum $\vec{P}$ of the black hole with mass M and velocity $\vec{V}$ are

$$E = \frac{M}{\sqrt{1 - V^2}}, \qquad \vec{P} = \frac{M\vec{V}}{\sqrt{1 - V^2}}. \tag{5.5}$$

Such a black hole is similar to a massive relativistic particle, with two important differences: (i) The mass of the black hole changes as a result of the absorption of the scalar field, and (ii) the black hole has spin $\vec{J}$, which also changes with time.

The equations of motion are

$$\frac{dE}{d\tau} = f^0, \qquad \frac{d\vec{P}}{d\tau} = \vec{f}, \tag{5.6}$$

where f^0 and $\vec{f}$ are given in (5.2), and τ is the proper time along the worldline of the black hole. There exists an additional equation for the spin evolution

$$\frac{dJ}{d\tau} = -\mathcal{J}, \tag{5.7}$$

where $\mathcal{J}$ is given by (4.6).

Since vector $\vec{P}$ is collinear with $\vec{V}$, one has

$$\frac{1}{2}\frac{d\vec{P}^2}{d\tau} = \vec{P}\cdot\frac{d\vec{P}}{d\tau} = \frac{M}{\sqrt{1-V^2}}\vec{V}\cdot\vec{f} = 0. \tag{5.8}$$

Hence, $\vec{P}^2 = P_0^2 = \text{const}$.

Let us write the momentum vector in the form

$$\vec{P} = \vec{P}_{\parallel} + \vec{P}_{\perp}, \tag{5.9}$$

where $\vec{P}_{\parallel}$ is parallel to the spin vector $\vec{J}$, and $\vec{P}_{\perp}$ is perpendicular to it. Then one has

$$\vec{P}_{\parallel}\cdot\frac{d\vec{P}_{\parallel}}{d\tau} = 0. \tag{5.10}$$

This means that the norms of the both vectors $P_{\parallel} = \sqrt{\vec{P}_{\parallel}^2}$ and $P_{\perp} = \sqrt{\vec{P}_{\perp}^2}$ are conserved quantities.

Using relations

$$E = \sqrt{M^2+P_0^2},$$
$$\frac{1}{\sqrt{1-V^2}} = \frac{1}{M}\sqrt{M^2+P_0^2}, \tag{5.11}$$

one can rewrite the first equation in (5.6) as follows:

$$\frac{dM}{d\tau} = \beta\frac{r_+}{M}(M^2+P_0^2),$$
$$r_+ = M + \sqrt{M^2 - J^2/M^2}. \tag{5.12}$$

Using relation

$$V_{\perp}^2 = \frac{P_{\perp}^2}{M^2+P_0^2}, \tag{5.13}$$

one can write (5.7) in the form

$$\frac{dJ}{d\tau} = -\frac{1}{6}\beta\frac{MP_{\perp}^2}{M^2+P_0^2}J. \tag{5.14}$$

The second equation in (5.6) implies

$$\frac{d\vec{P}_{\perp}}{d\tau} = -\frac{2}{3}\beta\frac{1}{M}[\vec{J}\times\vec{P}_{\perp}]. \tag{5.15}$$

Let us denote

$$P_X + iP_Y = P_{\perp}\exp(i\alpha). \tag{5.16}$$

Then the equation (5.15) implies

$$\frac{d\alpha}{d\tau} = -\frac{2}{3}\frac{\beta J}{\sqrt{M^2+P_0^2}}. \tag{5.17}$$

The obtained equations (5.12) and (5.14) allow one to find the time dependence of the black hole mass M and spin J for a given initial values M_0 and J_0 of these parameters. The equation (5.17) determines the time evolution of the transverse momentum $\vec{P}_{\perp}$. In order to find a unique solution, one needs also specify the conserved parameters P_0 and $P_{\perp}$.

Let us make a following remark. In the above consideration, we assumed that a distant observer is located at far distance L from the black hole, $L \gg M$. This allows one to describe a black hole as a "pointlike particle," which has mass M and spin J. We also assume that these parameters change slowly in time so that

$$\dot{M}/M \ll 1/M, \qquad \dot{J}/J \ll 1/M. \tag{5.18}$$

In this adiabatic approximation, one can describe the black hole metric by the Kerr solution with a slowly changing parameters $M(t)$ and $J(t)$. Note that such an adiabatic approximation is broken in the vicinity of the moment of time, at which the mass of the black hole formally becomes infinite.

C. General properties of solutions

1. Mass evolution

Let us discuss general properties of the solutions of the equations of motion for the rotating black hole moving in the homogeneous scalar field. First of all, let us notice that equations (5.12) and (5.14) show that the black hole mass M is a monotonically increasing function of time τ, while its spin J monotonically decreases in time.

One can obtain a more detailed information about the time dependence of these parameters by using the following trick. Let us denote $r_+ = bM$, where $1 \le b \le 2$. The quantity b takes the value 1 for the extremely rotating black hole when $J = M^2$, and $b = 2$ for a nonrotating black hole. Solving (5.12) for $b = \text{const}$, one gets

$$\arctan(M_b/P_0) - \arctan(M_0/P_0) = b\beta P_0\tau, \tag{5.19}$$

where M_0 is the mass at the initial moment of time $\tau = 0$. If $M(\tau)$ is the exact solution of the equation (5.12), then for the same initial mass M_0, one has

$$M_1(\tau) \le M(\tau) \le M_2(\tau). \tag{5.20}$$

Denote

$$\tau_0 = \frac{1}{\beta P_0}\left(\frac{\pi}{2} - \arctan(M_0/P_0)\right). \tag{5.21}$$

Then the black hole mass M becomes infinite at the finite proper time $\tau = \tau_{\rm fin}$

$$\frac{1}{2}\tau_0 \le \tau_{\rm fin} \le \tau_0. \tag{5.22}$$

2. Spin evolution

Let us consider the spin of the black hole as a function of its mass, $J = J(M)$. Then using (5.12) and (5.14), one obtains

$$\frac{1}{J}\frac{dJ}{dM} = -\frac{1}{6}\frac{P_\perp^2 M^2}{r_+(M^2+P_0^2)^2}. \tag{5.23}$$

Substituting $r_+ = bM$ and solving the obtained equation with fixed value of b, one finds

$$J_b(M) = J_0 \exp\left[-\frac{1}{6b}\frac{P_\perp^2(M^2-M_0^2)}{(M^2+P_0^2)(M_0^2+P_0^2)}\right]. \tag{5.24}$$

The exact solution $J(M)$ of (5.23) satisfies the inequalities

$$J_1(M) \le J(M) \le J_2(M). \tag{5.25}$$

For $M \to \infty$,

$$J_{b{\rm fin}} = J_0 \exp\left[-\frac{P_\perp^2}{6b(M_0^2+P_0^2)}\right]. \tag{5.26}$$

Hence, at the moment when the mass of the black hole becomes infinitely large, its spin remains finite $J = J_{\rm fin}$

$$J_{1,{\rm fin}} \le J_{\rm fin} \le J_{2,{\rm fin}}. \tag{5.27}$$

VI. SPECIAL CASES

A. Black hole at rest in $\tilde{K}$ frame

If the black hole is at rest, then $P_0 = 0$, and the spin of the black hole remains constant. The equation for the black hole mass takes the form

$$\frac{dM}{d\tau} = \beta M r_+ = \beta\left(M^2 + \sqrt{M^4 - J^2}\right). \tag{6.1}$$

If the spin J does not vanish, one can denote

$$\mu = M/\sqrt{J}, \tag{6.2}$$

and write a solution of the equation (6.1) in the form

$$\sqrt{J}\beta\tau = \hat{\tau}, \qquad \hat{\tau} = \int \frac{d\mu}{\mu^2 + \sqrt{\mu^4 - 1}}. \tag{6.3}$$

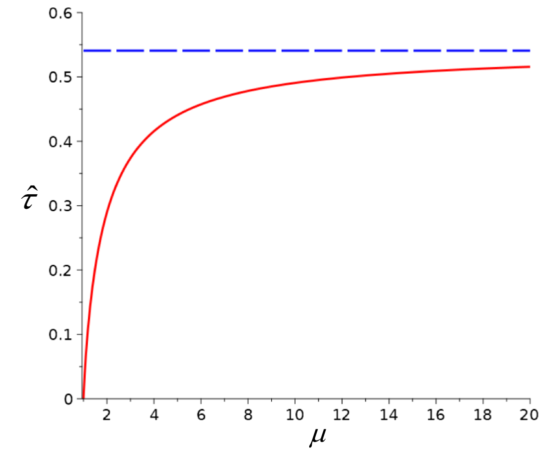

FIG. 3. $\hat{\tau}$ as a function of μ. Dashed line shows the limiting value of $\hat{\tau}$ for $\mu \to \infty$, which is equal to 0.54068.

Taking the integral, one obtains

$$\hat{\tau} = \frac{1}{3}\left[\mu^3 - 1 - \mu\sqrt{\mu^4 - 1} + 2(F(i\mu, i) - F(i, i))\right]. \tag{6.4}$$

Here $F(a, b)$ is the incomplete elliptic integral of the first kind, and the integration constant in this expression is chosen so that $\hat{\tau} = 0$ for $\mu = 1$. For $\mu \to \infty$, the parameter $\hat{\tau}$ has a limit 0.54068. The plot of $\hat{\tau}$ as a function of μ is shown in Fig. 3.

If $J = 0$, one obtains the following solution of the equation (6.1) for the mass

$$M = \frac{M_0}{1 - 2\beta M_0 \tau}. \tag{6.5}$$

B. Motion of a nonrotating black hole

When the spin of the black hole vanishes, one can always chose the orientation of the axes so that $P_\perp = 0$ and $P_\parallel = P_0$. One also has $r_+ = 2M$. The equation (5.12) simplifies and takes the form

$$\frac{dM}{d\tau} = 2\beta(M^2 + P_0^2). \tag{6.6}$$

It can be easily integrated with the following result:

$$2\beta P_0 \tau = \arctan(M/P_0). \tag{6.7}$$

C. Motion in the spin direction

For the motion in the spin direction, one has $P_\perp = 0$, and, as a result, the spin of the black hole remains constant. If its initial value does not vanish, we denote

$$\mu = M/\sqrt{J}, \qquad p = P_0/\sqrt{J}. \tag{6.8}$$

Then one has

$$\beta\sqrt{J}\tau = \int \frac{\mu^2 d\mu}{(\mu^2+p^2)(\mu^2+\sqrt{\mu^4-1})}. \tag{6.9}$$

The integral in the right-hand side of this relation can be expressed in terms of the incomplete elliptic integrals.

D. Transverse to the spin motion

For this case, $P_\parallel = 0$ and $P_0 = P_\perp$. We denote

$$M = P_\perp m, \qquad J = P_\perp j, \qquad \hat{\tau} = \beta P_\perp \tau. \tag{6.10}$$

Then the equations for the mass and spin evolution take the form

$$\begin{aligned}
\frac{dm}{d\hat{\tau}} &= \frac{\hat{r}_+(m^2+1)}{m},\\
\frac{dj}{d\hat{\tau}} &= -\frac{1}{6}\frac{mj}{m^2+1},\\
\hat{r}_+ &= m+\sqrt{m^2-j^2/m^2}.
\end{aligned} \tag{6.11}$$

VII. DISCUSSION

In this paper, we discussed a motion of a rotating black hole in the homogeneous massless scalar field. For this purpose, we used the Kerr-Schild form of the Kerr metric. We introduced two frames. One that we denoted by $\tilde{K}$ is the frame in which the asymptotic scalar field does not depend on the spatial coordinates. The other frame, which we denoted by K, is a frame moving with respect to $\tilde{K}$ with a constant velocity V and in which the black hole is at rest. We first solved the scalar field equation in K frame and found a solutions satisfying the condition of the regularity at the black hole horizon and proper behavior at the infinity. After this, we calculated the fluxes of the energy, momentum and angular momentum through a surface surrounding the black hole. This allowed us to find the force acting on the black hole and to obtain the equation of its motion in $\tilde{K}$ frame.

The main results of the analysis of the solutions of these equations are the following:

(i) For a general type of motion, the components of the momentum of the black hole both in the direction of its spin $P_\parallel$ and in the transverse plane $P_\perp$ are conserved.

(ii) The black hole mass M monotonically grows and formally becomes infinitely large in a finite interval of time $\tau_{\rm fin}$, which depends both on the strength of the scalar field and initial value of the mass, spin and velocity of the black hole.

(iii) The spin J of the black hole monotonically decreases but does not vanish at $\tau = \tau_{\rm fin}$. However, since the mass of the black hole becomes infinitely large, the rotation parameter $s = J/M$ vanishes in this limit, and the Kerr metric reduces to its Schwarzschild limit.

Let us note that these results are obtained in the adiabatic approximation in which the mass M and the spin J of the black hole change slowly in time. At time close to $\tau_{\rm fin}$, where $\dot{M}$ is not small, this approximation is broken.

In this paper, we considered a simple model and assumed that the scalar field obeys a linear equation $\Box\Phi = 0$. This approximation can be violated in application to some interesting cases, e.g., for the scalar field of the ghost condensate, where the effects of the nonlinearity might become important. In application to the cosmology, the scalar field may depend on the general expansion of the Universe, which would affect the imposed boundary conditions at the infinity. For the black hole in the expanding universe, the adopted form (1.4) of the metric should also be modified. It is interesting to study how these modification of the model would affect the interaction of the black hole with a scalar field and its motion.

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APPENDIX A: COMPLEX NULL TETRADS

In this appendix, we collect useful formulas and expressions for the complex null tetrads in the Kerr-Schild geometry. Let us denote

$$\begin{aligned}
m^\mu &= \sqrt{\frac{\Delta_y^{(0)}}{2\Sigma}}\left(1,0,i,\frac{a}{\Delta_y^{(0)}}\right),\\
\bar{m}^\mu &= \sqrt{\frac{\Delta_y^{(0)}}{2\Sigma}}\left(1,0,-i,\frac{a}{\Delta_y^{(0)}}\right).
\end{aligned} \tag{A1}$$

These vectors satisfy the following relations:

$$\begin{aligned}
g_{\mu\nu}m^\mu m^\nu &= \eta_{\mu\nu}m^\mu m^\nu = 0,\\
g_{\mu\nu}\bar{m}^\mu \bar{m}^\nu &= \eta_{\mu\nu}\bar{m}^\mu \bar{m}^\nu = 0,\\
g_{\mu\nu}m^\mu \bar{m}^\nu &= \eta_{\mu\nu}m^\mu \bar{m}^\nu = 1.
\end{aligned} \tag{A2}$$

The complex null vectors $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ are orthogonal to $\boldsymbol{l}$ both in $\boldsymbol{g}$ and $\boldsymbol{\eta}$ metrics.

The forth vector of the null tetrad has a slightly different form for $\boldsymbol{g}$ and $\boldsymbol{\eta}$ metrics. We denote

$$K^\mu = \frac{\Delta_r^{(0)}}{2\Sigma}\left(1,1,0,\frac{a}{r^2+a^2}\right). \tag{A3}$$

This vector is null in the metric $\boldsymbol{\eta}$ and normalized so that

$$\eta_{\mu\nu} l^\mu K^\nu = -1. \tag{A4}$$

A similar vector $\boldsymbol{k}$ for $\boldsymbol{g}$ metric is

$$k^\mu = K^\mu + \frac{1}{2}\Phi l^\mu. \tag{A5}$$

It satisfies the relations

$$g_{\mu\nu} k^\mu k^\nu = 0, \quad g_{\mu\nu} l^\mu k^\nu = -1, \quad g_{\mu\nu} m^\mu k^\nu = g_{\mu\nu} \bar{m}^\mu k^\nu = 0. \tag{A6}$$

The complex null tetrad in the metric $\boldsymbol{g}$ regular at the future horizon of the Kerr black hole is

$$z^a = (\boldsymbol{l}, \boldsymbol{k}, \boldsymbol{m}, \bar{\boldsymbol{m}}). \tag{A7}$$

The index a enumerating the basis vectors takes the values 0, 1, 2, 3. By the construction, the vectors of the basis z^a are regular the future event horizon.

APPENDIX B: ENERGY AND ANGULAR MOMENTUM FLUXES THROUGH THE HORIZON

Let ξ^μ be a Killing vector and $T_{\mu\nu}$ be a conserved stress-energy tensor, $T^{\mu\nu}{}_{;\nu} = 0$. Then the following vector,

$$\mathcal{P}^\mu = T^{\mu\nu}\xi_\nu, \tag{B1}$$

is conserved

$$\mathcal{P}^\mu{}_{;\mu} = 0. \tag{B2}$$

Let us consider a rotating black hole and use the coordinates (T, r, y, ϕ). We denote by $\Sigma_\pm$ two 3D slices determined by the equations $T = T_\pm$ and restricted from one side by the horizon $r = r_+$ and from the other side by a surface $r = r_0 > r_+$. We denote by Σ_H a part of the horizon surface between $T = T_-$ and T_+. Similarly, we denote be Σ_0 a 3D surface $r = r_0$ between $T = T_-$ and T_+ (see Fig. 4).

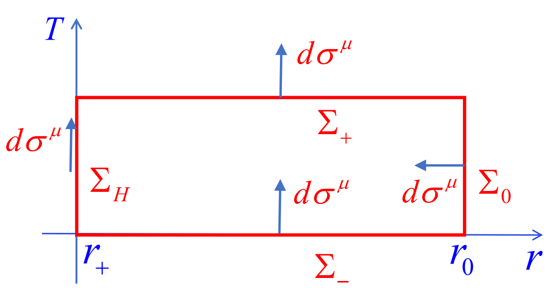

FIG. 4. Illustration to the Stockes' theorem.

Let $\mathcal{V}$ be a four volume restricted by $\Sigma_\pm$, Σ_H and Σ_0. Using the Stockes' theorem, one can write

$$\begin{aligned} 0 &= \int_{\mathcal{V}} \sqrt{-g} d^4 x \mathcal{P}^\mu{}_{;\mu} \\ &= \left[\int_{\Sigma_+} - \int_{\Sigma_-}\right]\mathcal{P}^\mu d\sigma_\mu + \left[\int_{\Sigma_0} - \int_{\Sigma_H}\right]\mathcal{P}^\mu d\sigma_\mu. \end{aligned} \tag{B3}$$

The surface elements $d\sigma^\mu$ are chosen so that for $\Sigma_\pm$ and at the horizon Σ_H, they are both future directed, while at Σ_0, it is directed into this surface's interior.

For the problem under consideration, the gradient of the scalar field Ψ does not depend on time, and, since the metric is also time independent, the stress-energy tensor has the same property. As a result, the expression in the first square bracket in (B3) vanishes, and one has

$$\int_{\Sigma_0} \mathcal{P}^\mu d\sigma_\mu = \int_{\Sigma_H} \mathcal{P}^\mu d\sigma_\mu. \tag{B4}$$

This relation shows that

(i) The flux of $\mathcal{P}$ inside Σ_0 during the time interval $T_+ - T_-$ is equal to the flux through the horizon for the same interval of time T.

(ii) The flux of $\mathcal{P}$ inside Σ_0 in fact does not depend in the choice of the radius r_0.

Let us emphasize that at the surface of constant radius $r = r_0$, the coordinates T, v, and t differ only by constant values. For this reason, one has

$$T_+ - T_- = v_+ - v_- = t_+ - t_-. \tag{B5}$$

These remarks can be used to confirm the results (4.6) for the energy and angular momentum fluxes through Σ_0.

Let us calculate the energy and angular momentum fluxes through the horizon of the black hole. Denote by

$$\eta^\mu = \xi^\mu + \Omega\zeta^\mu, \qquad \Omega = \frac{a}{2Mr_+}. \tag{B6}$$

Then for the horizon surface Σ_H, one has

$$d\sigma^\mu = -\eta^\mu \frac{2Mr_+}{a} dy\, d\phi\, dT. \tag{B7}$$

The energy and angular momentum fluxes through the horizon are

$$\begin{aligned} \mathcal{E} &= -\frac{1}{T_+ - T_-}\frac{2Mr_+}{a}\int_{\Sigma_H} \xi^\mu \eta^\nu T_{\mu\nu} dy d\phi dT \\ &= \frac{2Mr_+}{a}\int_{-a}^{a} (T_{TT} + \Omega T_{T\phi}) dy d\phi, \\ \mathcal{J} &= \frac{1}{T_+ - T_-}\frac{2Mr_+}{a}\int_{\Sigma_H} \zeta^\mu \eta^\nu T_{\mu\nu} dy\, d\phi\, dT \\ &= -\frac{2Mr_+}{a}\int_{-a}^{a} (T_{T\phi} + \Omega T_{\phi\phi}) dy\, d\phi. \end{aligned} \tag{B8}$$

Calculations give

$$T_{TT} + \Omega T_{T\phi} \stackrel{H}{=} \bar{\Psi}_0^2 \left[1 - \frac{(r_- \cos\phi + a \sin\phi)\sqrt{a^2 - y^2}}{2\sqrt{2Mr_+}a} V_X - \frac{(r_- \sin\phi - a\cos\phi)\sqrt{a^2 - y^2}}{2\sqrt{2Mr_+}a} V_Y \right]. \tag{B9}$$

After integration of this expression over the angle ϕ, the terms that depend on the velocity components V_X and V_Y vanish. The further integration over y gives

$$\mathcal{E} \stackrel{H}{=} 8\pi\bar{\Psi}_0^2 M r_+. \tag{B10}$$

This expression correctly reproduces the result (4.6) as it should be.

To calculate the angular momentum flux through the horizon, one needs first to find the value of $T_{T\phi} + \Omega T_{\phi\phi}$ on the horizon. The corresponding expression is rather long, and we do not reproduce it here. Instead of this, we give the expression that is obtained after integration of this object over the angle ϕ

$$\int_0^{2\pi} (T_{T\phi} + \Omega T_{\phi\phi}) d\phi = -\bar{\Psi}^2 \frac{\pi M}{2ar_+} (V_X^2 + V_Y^2)(a^2 - y^2). \tag{B11}$$

Using (B8) and performing the integration over y, one obtains

$$\mathcal{J} \stackrel{H}{=} -\frac{4}{3}\pi\bar{\Psi}_0^2 a M^2 (V_X^2 + V_Y^2). \tag{B12}$$

This result correctly reproduces the expression obtained earlier in (4.6). The calculations presented in this appendix provide an additional check of the results presented in Sec. IV.

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