

I-Love-Q relations in Einstein-aether theoryKai Vylet,¹ Siddarth Ajith,¹ Kent Yagi¹ ,¹ and Nicolás Yunes²¹*Department of Physics, University of Virginia, Charlottesville, Virginia 22904, USA*²*Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA*

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Although Lorentz symmetry is a staple of general relativity (GR), there are several reasons to believe it may not hold in a more advanced theory of gravity, such as quantum gravity. Einstein-aether theory is a modified theory of gravity that breaks Lorentz symmetry by introducing a dynamical vector field called the aether. The theory has four coupling constants that characterize deviations from GR and that must be determined through observations. Although three of the four parameters have been constrained by various empirical observations and stability requirements, one, called c_ω , remains essentially unconstrained. The aim of this work is to see if a constraint on c_ω can be derived from the I-Love-Q universal relations for neutron stars, which connect the neutron star moment of inertia (I), the tidal Love number (Love), and the quadrupole moment (Q) in a way that is insensitive to uncertainties in the neutron star equation-of-state. To understand if the theory can be constrained through such relations, we model slowly rotating or weakly tidally deformed neutron stars in Einstein-aether theory, derive their I-Love-Q relations, and study how they depend on c_ω . We find that the I-Love-Q relations in Einstein-aether theory are insensitive to c_ω and that they are close to the relations in GR. This means that the I-Love-Q relations in Einstein-aether theory remain universal but cannot be used to constrain the theory. These results indicate that to constrain the theory with neutron stars, it is necessary to investigate relations involving other observables.

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General relativity (GR) is the most successful theory of gravity to date and has stood up to a multitude of tests [1–3]. Despite this, a particular concern for GR is its incompatibility with quantum mechanics, an incompatibility that ought to be resolved by a more fundamental theory of quantum gravity. One proposed feature of quantum gravity theories that are renormalizable and ultraviolet complete is the presence of a local preferred rest frame everywhere in spacetime [4–6]. Since the existence of such a frame would break local Lorentz symmetry, a pillar of GR, searches for gravitational Lorentz symmetry violation provide an avenue for understanding gravity beyond GR.

Einstein-aether theory is a vector-tensor theory of gravity that introduces such preferred frame effects via a dynamical, unit, and time-like vector field called the aether [4,7,8]. The aether selects a preferred time direction at each point in space and affects gravity by coupling to the metric tensor via covariant derivatives in the action. Importantly, the aether does not couple to matter fields directly at the level of the action. This coupling setup helps Einstein-aether theory avoid stringent constraints on Lorentz symmetry violation in the matter sector [9–11], which are stronger than those in the gravitational sector [4,12]. This gap in the gravity sector can be ameliorated by studying specific theories, such as Einstein-aether theory, which provides a

generic framework in which to test low-energy, Lorentz-violating gravity [8,13].

Various gravitational tests have strongly constrained three of the four coupling constants in Einstein-aether theory, ($c_a, c_\theta, c_\omega, c_\sigma$), splitting the theory into two viable regions of parameter space [13,14]. Observations such as the gravitational wave event of a binary neutron star merger GW170817 and its corresponding gamma-ray burst GRB170817A [15], binary pulsars [12,16,17], solar system experiments [2], and light element production during big bang nucleosynthesis [18], as well as theoretical constraints on the mode stability and the absence of the gravitational Cherenkov radiation, all put relatively tight constraints on c_a, c_θ , and c_σ . However, c_ω has been essentially unconstrained thus far [17], with the only bound $c_\omega > 0$ coming from requiring that the vector mode carries positive energy [19]. Recent attempts to develop constraints with gravitational-wave data have been unsuccessful in providing further, more stringent bounds on these parameters [20], and it has also been found that rotating black holes in Einstein-aether theory deviate only slightly from rotating black holes in GR [21]. Hence, an exploration of different tests is needed to further constrain Einstein-aether theory and the c_ω parameter in particular. The goal of this work is to see if better constraints on these coupling constants can be derived from the I-Love-Q relations for neutron stars.

Due to their compactness, neutron stars are excellent systems with which to probe gravity in the strong-field regime. A particularly useful tool for probing gravity with neutron stars is the I-Love-Q relations [22–25]. This acronym refers to relations between the neutron star moment of inertia (I), the tidal Love number (Love), and the spin-induced quadrupole moment (Q), which are insensitive to the nuclear matter equation of state. This latter fact, sometimes referred to as “universality,” allows the I-Love-Q relations to avoid contamination from uncertainties in the internal structure of neutron stars, thus making them especially useful for conducting tests of GR. Additionally, the I-Love-Q relations differ between different gravitational theories, and thus, they can be used to measure deviations from GR in non-GR theories [22,23,26]. The I-Love-Q relations have been studied in several modified gravity theories [26–28]. This includes khronometric gravity [29], a closely related theory that corresponds to the low-energy effective theory of Hořava-Lifshitz gravity [6,30]. The Love number has been constrained by GW170817 [15,31,32] while the moment of inertia has been inferred from NICER observations [33] and is expected to be measured with the double pulsar binary PSR J0737-3039 [34,35].

In this paper, we investigate whether one could use the I-Love-Q relations to constrain Einstein-aether theory. More precisely, we study the structure of slowly rotating and tidally perturbed neutron stars in Einstein-aether theory and focus on how the moment of inertia, the quadrupole moment, and the tidal Love number depend on the coupling constants of the theory. Of particular interest is the dependence of these quantities on the unconstrained constant c_ω . To carry out this study, we follow the same procedure as in GR to construct slowly rotating or weakly tidally deformed neutron stars perturbatively in rotation and tidal deformation. With these solutions, we then derive the moment of inertia, quadrupole moment, and tidal Love number from the asymptotic behavior of the metric [23]. The analysis is similar to previous calculations of stellar sensitivities of slowly moving neutron stars in Einstein-aether theory [12,16]. In addition, we briefly study other components of the metric perturbations unrelated to the I-Love-Q trio but that do determine whether neutron stars in Einstein-aether theory are the same as in GR.

Our main findings are as follows. We focus on the first viable parameter region, called region I, where c_ω is unconstrained, while the other three coupling constants are suppressed by at least $\mathcal{O}(c_a)$, where $c_a \lesssim \mathcal{O}(10^{-5})$ from various observations. This means that the field equations relevant to extracting I-Love-Q quantities are dominantly dependent on c_ω and c_a only. In particular, we study the dependence on c_ω in the relevant field equations and find that it always enters the ϕ -component of the aether field, equivalently the perturbation function $S(r)$. We then

schematically solve the differential equation for $S(r)$ and provide analytic arguments to show that the I-Love-Q relations do not depend strongly on c_ω . Specifically, we argue that deviations from the GR I-Love-Q relations due to c_ω are suppressed or comparable relative to deviations due to c_a , and that the dominant deviation for each of the I-Love-Q quantities is of $\mathcal{O}(c_a)$. This makes it difficult to probe the theory further with neutron star observations through the I-Love-Q relations, as the equation-of-state variation [$\mathcal{O}(1\%)$] is much larger than the Einstein-aether correction of $\mathcal{O}(c_a) \lesssim \mathcal{O}(10^{-5})$. We confirm these findings by calculating the I-Love-Q quantities in Einstein-aether theory numerically. In the second parameter region, region II, where c_ω and c_θ are the only free constants, while the other two are set to vanish, the I-Love-Q relations are exactly identical to those in GR. These findings are similar to those of the khronometric case in [30] and confirm that the I-Love-Q relations are universal not only to the variation in the equations of state but to the variation in Lorentz-violating effects, at least in Einstein-aether and khronometric theory.

The rest of the paper is organized as follows. In Sec. II, we summarize Einstein-aether theory, introduce the field equations, and describe the current bounds on the theory. In Sec. III, we introduce the *Ansatz* for the metric, aether vector field, and the matter stress-energy tensor that we use in this work. In Secs. IV to VI, we present the region I neutron star field equations relevant for the I-Love-Q relations and show that effects due to c_ω , while present, are subdominant to those from c_a . In Sec. VII, we conclude and discuss future directions. In Appendix A, we show that the I-Love-Q quantities in region II reduce exactly to their GR values. Next, in Appendix B, we present the full neutron star field equations relevant to the I-Love-Q relations. In Appendix C, we map and compare our neutron star field equations to those previously found in khronometric gravity. In Appendix D, we briefly study off-diagonal perturbations for tidally deformed neutron stars. Throughout this work, we use the metric signature $(+, -, -, -)$, and geometric units $c = 1 = G_N$, where G_N is the local Newtonian gravitational constant.

II. EINSTEIN-AETHER THEORY

In this section, we present the action and field equations for Einstein-aether theory. We also discuss current constraints on the theory and describe two viable regions for the coupling constants. The action for the metric and aether is given by [17]

$$S_{\mathcal{E}} = -\frac{1}{16\pi G_{\text{bare}}} \int d^4x \sqrt{-g} \left[R + \lambda(U^\mu U_\mu - 1) + \frac{1}{3} c_\theta \theta^2 + c_\sigma \sigma_{\mu\nu} \sigma^{\mu\nu} + c_\omega \omega_{\mu\nu} \omega^{\mu\nu} + c_a A_\mu A^\mu \right]. \quad (1)$$

Here, g is the metric determinant, R is the Ricci scalar, U^μ is the aether vector field, and λ is a Lagrange multiplier that enforces $U^\mu U_\mu = 1$. The other terms come from decomposing the aether congruence into the expansion θ , the shear $\sigma_{\mu\nu}$, the twist $\omega_{\mu\nu}$, and the acceleration A_μ , which are defined via

$$\theta \equiv \nabla_\mu U^\mu, \quad (2)$$

$$A_\mu \equiv U^\nu \nabla_\nu U_\mu, \quad (3)$$

$$\sigma_{\mu\nu} = \nabla_{(\nu} U_{\mu)} + A_{(\mu} U_{\nu)} - \frac{1}{3} \theta (g_{\mu\nu} - U_\mu U_\nu), \quad (4)$$

$$\omega_{\mu\nu} \equiv \nabla_{[\nu} U_{\mu]} + A_{[\mu} U_{\nu]}. \quad (5)$$

The parameters $(c_a, c_\theta, c_\omega, c_\sigma)$ are the coupling constants of the theory, and we can recover GR by taking the limit $(c_a, c_\theta, c_\omega, c_\sigma) \rightarrow 0$. In the action, G_{bare} is the ‘‘bare’’ gravitational constant, which is given by

$$G_{\text{bare}} = G_N \left(1 - \frac{c_a}{2}\right) = \left(1 - \frac{c_a}{2}\right), \quad (6)$$

where G_N is the Newtonian gravitational constant, measured locally in the solar system [4,18]. We shall set $G_N = 1$ throughout this work. The inclusion of matter fields, denoted ψ , produces the full action for Einstein-aether theory

$$S = S_{\text{AE}} + S_{\text{mat}}(g_{\mu\nu}, \psi), \quad (7)$$

with S_{mat} being the matter action, which defines the matter stress-energy tensor

$$T_{\mu\nu}^{(\text{mat})} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{mat}}}{\delta g^{\mu\nu}}. \quad (8)$$

Taking the matter action to be diffeomorphism invariant, it follows that the stress-energy tensor is conserved, $\nabla^\mu T_{\mu\nu}^{(\text{mat})} = 0$.

We now present the equations of motion in Einstein-aether theory. Varying the full action with respect to the metric gives the modified Einstein equations [16,17]

$$E_{\mu\nu} \equiv G_{\mu\nu} - 8\pi G_{\text{bare}} T_{\mu\nu}^{(\text{mat})} - T_{\mu\nu}^{(\text{AE})} = 0. \quad (9)$$

Here, $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}^{(\text{AE})}$ is the stress-energy tensor for the aether field, defined by

$$\begin{aligned} T_{\mu\nu}^{(\text{AE})} &= \nabla_\rho [J_{(\mu}{}^\rho U_{\nu)} - J^\rho{}_{(\mu} U_{\nu)} - J_{(\mu\nu)} U^\rho] \\ &+ \left(c_a - \frac{c_\sigma + c_\omega}{2}\right) [\dot{U}_\mu \dot{U}_\nu - (\dot{U}_\rho \dot{U}^\rho) U_\mu U_\nu] \\ &+ (U_\rho \nabla_\sigma J^{\sigma\rho}) U_\mu U_\nu + \frac{1}{2} M^{\sigma\rho}{}_{\alpha\beta} \nabla_\sigma U^\alpha \nabla_\rho U^\beta g_{\mu\nu}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} M^{\alpha\beta}{}_{\mu\nu} &\equiv \left(\frac{c_\sigma + c_\omega}{2}\right) g^{\alpha\beta} g_{\mu\nu} + \left(\frac{c_\theta - c_\sigma}{3}\right) \delta_\mu^\alpha \delta_\nu^\beta \\ &+ \left(\frac{c_\sigma - c_\omega}{2}\right) \delta_\nu^\alpha \delta_\mu^\beta + \left(c_a - \frac{c_\sigma + c_\omega}{2}\right) U^\alpha U^\beta g_{\mu\nu}, \end{aligned} \quad (11)$$

and

$$J^\alpha{}_\mu \equiv M^{\alpha\beta}{}_{\mu\nu} \nabla_\beta U^\nu, \quad \dot{U}_\nu \equiv U^\mu \nabla_\mu U_\nu. \quad (12)$$

Varying the full action with respect to the aether gives the aether equations

$$\begin{aligned} \mathcal{A}_\mu &\equiv \left[\nabla_\alpha J^{\alpha\mu} - \left(c_a - \frac{c_\sigma + c_\omega}{2}\right) \dot{U}_\alpha \nabla^\mu U^\alpha \right] \\ &\times (g_{\mu\nu} - U_\mu U_\nu) = 0. \end{aligned} \quad (13)$$

In total, there are two sets of dynamical equations for determining the metric $g_{\mu\nu}$ and aether U^μ : the modified Einstein equations and the aether equations, given by Eqs (9) and (13), respectively.

Let us now review existing bounds on the theory [17,20]. The propagation speed of tensor modes is given by $1/(1 - c_\sigma)$, whose deviation from the speed of light has been constrained to be about 10^{-15} from the gravitational-wave observation of GW170817 and its electromagnetic counterpart [15,36]. A similar constraint arises from the lack of observed gravitational Cherenkov radiation in high-energy cosmic rays [17]. Both of these observations lead to the requirement $c_\sigma \lesssim \mathcal{O}(10^{-15})$. Constraints on c_a and c_θ result mainly from bounds on the preferred frame parameters α_1 and α_2 in the parametrized Post-Newtonian (PPN) expansion of the theory. In Einstein-aether theory, the preferred frame parameters relate to the coupling constants via [37]

$$\begin{aligned} \alpha_1 &= 4 \frac{c_\omega(c_a - 2c_\sigma) + c_a c_\sigma}{c_\omega(c_\sigma - 1) - c_\sigma}, \\ \alpha_2 &= \frac{\alpha_1}{2} + \frac{3(c_a - 2c_\sigma)(c_\theta + c_a)}{(2 - c_a)(c_\theta + 2c_\sigma)}. \end{aligned} \quad (14)$$

Solar system experiments combined with binary pulsar observations bound $|\alpha_1| \lesssim 10^{-5}$ and $|\alpha_2| \lesssim 10^{-7}$ [2,17].

These bounds translate into two viable parameter regions for Einstein-aether theory [17,20,21]. The first region, which we will call region I, comes from requiring that $|\alpha_1| \lesssim 10^{-5}$ strictly, i.e., not $|\alpha_1| \ll 10^{-5}$. Combining the constraint on c_σ from tensor mode speed with the bounds on α_1 and α_2 in region I, one is left with $c_\sigma \lesssim \mathcal{O}(10^{-15})$, $c_a \lesssim \mathcal{O}(10^{-5})$, $c_\theta \approx 3c_a[1 + \mathcal{O}(10^{-3})]$, and c_ω unconstrained. The second region, which we will call region II, is obtained by instead requiring $|\alpha_1| \ll 10^{-5}$ so that the bound on $|\alpha_2|$ is automatically satisfied (when $c_\sigma \approx 0$). The parameter space for region II is then given by $c_\sigma \lesssim \mathcal{O}(10^{-15})$, $|c_a| \lesssim \mathcal{O}(10^{-7})$ and (c_θ, c_ω) undetermined. Though the PPN parameters do not constrain c_θ in region II, there is a separate constraint of $|c_\theta| \lesssim 0.3$ from observed light element production during big bang nucleosynthesis [17,18].

To summarize, Einstein-aether theory has two viable spaces for its coupling constants, regions I and II. Region I can be approximated by the parameter space

$$\text{Region I: } (c_a, c_\theta, c_\omega, c_\sigma) \approx (c_a, 3c_a, c_\omega, 0), \quad (15)$$

with $c_a \lesssim \mathcal{O}(10^{-5})$ and c_ω unconstrained. While one could approximate $c_a \approx 0$ in Eq. (15) since it is relatively small, we opt to keep c_a non-zero because the ratio c_a/c_ω could be large (since c_ω is unconstrained) and thus have non-negligible effects. If one uses $c_a \approx 0$, these effects would not be captured. Next, region II can be approximated by the effectively two-dimensional space

$$\text{Region II: } (c_a, c_\theta, c_\omega, c_\sigma) \approx (0, c_\theta, c_\omega, 0), \quad (16)$$

where $|c_\theta| \lesssim 0.3$ and c_ω unconstrained. In the following work, we refer to the approximations given in Eqs (15) and (16) as region I and region II, respectively. We mainly study the I-Love-Q relations in region I. As we will see later, the I-Love-Q relations in region II are identical to those in GR.

III. METRIC, AETHER, AND MATTER PERTURBATIONS

In the following sections, we describe neutron stars perturbed by slow rotation and weak tidal deformation. We first present the metric and aether field *Ansätze*, which can be reduced to the slow rotation case or the tidal deformation case by keeping the relevant free functions and spherical harmonic modes for each case. We then describe the matter stress-energy tensor.

A. Metric and aether *Ansätz*

We form our metric *Ansätz* by adding appropriate parity perturbations in the Regge-Wheeler gauge [38,39] to a generic, static and spherically symmetric background. The metric *Ansätz* includes $l = 1$ odd and $l = 2$ even perturbations and is given by [22]

$$\begin{aligned} ds^2 = & e^{\nu(r)}[1 + \varepsilon^2 \kappa H_0(r) Y_{2m}(\theta, \phi)] d\tilde{t}^2 \\ & - e^{\mu(r)}[1 - \varepsilon^2 \kappa H_2(r) Y_{2m}(\theta, \phi)] dr^2 \\ & - r^2[1 - \varepsilon^2 \kappa K(r) Y_{2m}(\theta, \phi)] \\ & \times \{d\theta^2 + \sin^2\theta[d\phi - \varepsilon[\Omega_\star - \omega(r)P'_1(\cos\theta)]d\tilde{t}]^2\} \\ & + 2\varepsilon^2[\kappa \tilde{H}_1(r) Y_{2m}(\theta, \phi)] d\tilde{t} dr + \mathcal{O}(\varepsilon^3). \end{aligned} \quad (17)$$

Here, ε is a book-keeping parameter denoting the order of the perturbation, $Y_{2m}(\theta, \phi)$ is the $l = 2$ spherical harmonic function, $P'_1(\cos\theta) = dP_1(\cos\theta)/d(\cos\theta)$ where P_1 is the first Legendre polynomial, Ω_\star is the constant angular velocity of the neutron star, and $\kappa = 2\sqrt{\pi/5}$ [chosen so that $\kappa Y_{20}(\theta, \phi) = P_2(\cos\theta)$]. For aether perturbations, we follow Eq. (13) in [40] to form our *Ansätz*

$$\begin{aligned} U_\mu d\tilde{x}^\mu = & e^{\nu/2} \{ [1 + \varepsilon^2 \kappa X(r) Y_{2m}(\theta, \phi)] d\tilde{t} \\ & + \varepsilon^2 \kappa \tilde{W}(r) Y_{2m}(\theta, \phi) dr \\ & + \varepsilon^2 \kappa V(r) \partial_\theta Y_{2m}(\theta, \phi) d\theta + [\varepsilon S(r) \sin^2\theta \\ & + \varepsilon^2 \kappa V(r) \partial_\phi Y_{2m}(\theta, \phi)] d\phi \} + \mathcal{O}(\varepsilon^3), \end{aligned} \quad (18)$$

with $\tilde{x}^\mu = (\tilde{t}, r, \theta, \phi)$. We only consider perturbative terms up to quadratic order in ε in this paper. In the case of tidal perturbations, we also use the above *Ansätze* but only consider even parity perturbations; there is no tidal perturbation at $\mathcal{O}(\varepsilon)$ and the leading perturbation enters at $\mathcal{O}(\varepsilon^2)$ [22].

With initial metric and aether *Ansätze* in hand, we next perform a coordinate transformation and enforce the aether normalization condition $U^\mu U_\mu = 1$. First, we choose the following coordinate transformation

$$t = \tilde{t} + \varepsilon^2 \kappa V(r) Y_{2m}(\theta, \phi), \quad (19)$$

to slightly simplify the aether *Ansätz*. This coordinate transformation was also used in [30] in the context of neutron stars in khronometric gravity, where the transformation was motivated by a more significant simplification of the vector field. Though the simplification is lesser in Einstein-aether theory, implementing this transformation will be useful when comparing our results to those in khronometric gravity. After performing the coordinate transformation, it will be convenient to use the replacements

$$\begin{aligned} H_1(r) & \equiv \tilde{H}_1(r) - e^\nu \partial_r V(r), \\ W(r) & \equiv \tilde{W}(r) - \partial_r V(r), \end{aligned} \quad (20)$$

from here on. Next, using the $U^\mu U_\mu = 1$ constraint on the aether field allows us to solve for $X(r)$ in terms of other functions in the *Ansätz*:

$$X(r) = \frac{1}{6} \left\{ 4S(r)[\Omega_\star - \omega(r)] - \frac{2}{r^2} S(r)^2 e^\nu - 3H_0(r) \right\}. \quad (21)$$

After implementing the coordinate transformation, redefinitions, and solution for $X(r)$ into the initial *Ansätze* in Eqs. (17) and (18), we obtain the metric and aether *Ansätze* which we will use for the rest of this work. The final forms of the aether field and metric *Ansätze* are given by

$$\begin{aligned}
ds^2 = & e^{\nu(r)}[1 + \varepsilon^2 \kappa H_0(r) Y_{2m}(\theta, \phi)] dt^2 - e^{\mu(r)}[1 - \varepsilon^2 \kappa H_2(r) Y_{2m}(\theta, \phi)] dr^2 \\
& - r^2[1 - \varepsilon^2 \kappa K(r) Y_{2m}(\theta, \phi)] \{d\theta^2 + \sin^2\theta [d\phi - \varepsilon[\Omega_\star - \omega(r) P'_1(\cos\theta)] dt]^2\} \\
& + 2\varepsilon^2 \kappa \{H_1(r) Y_{2m}(\theta, \phi) dt dr - e^{\nu(r)} V(r) [\partial_\theta Y_{2m}(\theta, \phi) dt d\theta + \partial_\phi Y_{2m}(\theta, \phi) dt d\phi]\} + \mathcal{O}(\varepsilon^3). \quad (23)
\end{aligned}$$

Note that the aether has lost a θ component while the metric has gained (t, r) and (t, θ) components. Additionally, the metric perturbation $H_0(r)$ has mixed into the aether, and the aether perturbation $V(r)$ has mixed into the metric.

We now summarize which field functions and spherical harmonic modes are considered at each order of perturbation in the slow rotation or tidal deformation case. At $\mathcal{O}(\varepsilon^0)$, we have the radial functions ν and μ , while at $\mathcal{O}(\varepsilon)$ we have ω and S , and at $\mathcal{O}(\varepsilon^2)$ we have H_0, H_1, H_2, K, W , and V . At $\mathcal{O}(\varepsilon^2)$, the functions can be split into two sectors: the diagonal sector (as in the diagonal components of the metric) consisting of H_0, H_2 , and K , and the off-diagonal sector, consisting of H_1, V , and W . For metric perturbations due to slow rotation, only $(l, m) = (1, 0)$ modes are included at $\mathcal{O}(\varepsilon)$ while both $(l, m) = (2, 0)$ and $(l, m) = (0, 0)$ modes are included at $\mathcal{O}(\varepsilon^2)$. Tidal perturbations only enter at $\mathcal{O}(\varepsilon^2)$ with $l = 2$ and all m modes, so we neglect $\mathcal{O}(\varepsilon)$ terms in the metric and aether when considering the tidal deformations. Hence, the free field functions are $\{\nu, \mu, \omega, S, H_0, H_1, H_2, K, V, W\}$ in the slow rotation case and $\{\nu, \mu, H_0, H_1, H_2, K, V, W\}$ in the tidal deformation case.

B. Matter stress-energy

We next turn to the matter stress-energy tensor for the neutron star. We model the neutron star as a uniformly rotating perfect fluid with four velocity given by

$$u^\mu \partial_\mu = u^t (\partial_t + \varepsilon \Omega_\star \partial_\phi), \quad (24)$$

where Ω_\star is the neutron star angular velocity. Using the timelike normalization condition $u^\mu u_\mu = 1$, one finds

$$u^t = e^{\frac{\nu}{2}} + \frac{\varepsilon^2}{2} e^{-\frac{3\nu}{2}} [(\omega r \sin\theta)^2 - \kappa e^\nu H_0 Y_{2m}] + \mathcal{O}(\varepsilon^3). \quad (25)$$

With the neutron star four velocity u^μ , we then have the perfect fluid stress-energy tensor given by

$$\begin{aligned}
U_\mu dx^\mu = & e^{\nu/2} \{ [1 + \varepsilon^2 \kappa X(r) Y_{2m}(\theta, \phi)] dt \\
& + \varepsilon^2 \kappa Y_{2m}(\theta, \phi) W(r) dr + \varepsilon S(r) \sin^2\theta d\phi \} \\
& + \mathcal{O}(\varepsilon^3), \quad (22)
\end{aligned}$$

with $X(r)$ given by Eq. (21), and

$$\begin{aligned}
T_{\mu\nu}^{(\text{mat})} = & [\tilde{\rho}_0 + \tilde{p}_0 + \varepsilon^2 \kappa (\tilde{\rho}_2 + \tilde{p}_2) Y_{2m}] u_\mu u_\nu \\
& - (\tilde{p}_0 + \varepsilon^2 \kappa \tilde{p}_2 Y_{2m}) g_{\mu\nu} + \mathcal{O}(\varepsilon^3), \quad (26)
\end{aligned}$$

where we have implemented our perturbation scheme by introducing $\tilde{\rho}_i$ and \tilde{p}_i , the i th order energy densities and pressures, respectively. One can additionally rescale the pressures and energy densities as

$$\rho_i \equiv \frac{2 - c_a}{2} \tilde{\rho}_i, \quad p_i \equiv \frac{2 - c_a}{2} \tilde{p}_i, \quad (27)$$

to absorb the overall factor introduced by $G_{\text{bare}} = 1 - c_a/2$ in Eq. (9). We will use these rescaled pressure and energy density functions from here on.

IV. MOMENT OF INERTIA

In this and the subsequent two sections, we present analytic arguments and numerical calculations for the I-Love-Q relations in Einstein-aether theory. We focus on region I as the relations in region II turn out to be the same as in GR (see Appendix A for more details). We use the metric and aether *Ansatz* presented in the previous section and consider slowly rotating and weakly tidally deformed neutron stars. We derive the field equations for the *Ansatz* functions order by order and study the appropriate I-Love-Q quantity at each order, namely, the moment of inertia at first order in rotation, the quadrupole moment at second order in rotation, and the tidal Love number at first order in tidal deformation.

A. Background

We first consider the $\mathcal{O}(\varepsilon^0)$, or background, functions in the *Ansatz* and stress-energy tensor: μ , ν , and p_0 . The energy density ρ_0 is determined from p_0 via the neutron star equation of state, which we leave general. Using the ε^0 parts of the modified Einstein equations and the conservation of stress-energy, $\nabla^\mu T_{\mu\nu}^{(\text{mat})} = 0$, we find the modified TOV equations [16,41]

$$\begin{aligned}
 \frac{dM}{dr} &= 4\pi\rho_0 r^2 + \frac{1}{4r(r-2M)} \{-M^2 - 8\pi M r^3 (7p_0 \\
 &\quad + 2\rho_0) - 8\pi r^4 (2\pi p_0^2 - 3p_0 - \rho_0)\} c_a + \mathcal{O}(c_a^2), \\
 \frac{d\nu}{dr} &= \frac{8\pi p_0 r^3 + 2M}{r(r-2M)} - \frac{(4r^3 \pi p_0 + M)^2}{2r(r-2M)^2} c_a + \mathcal{O}(c_a^2), \\
 \frac{dp_0}{dr} &= -\frac{(16\pi p_0 r^3 + M)(p_0 + \rho_0)}{r(r-2M)} \\
 &\quad + \frac{(4r^3 \pi p_0 + M)^2 (p_0 + \rho_0)}{4r(r-2M)^2} c_a + \mathcal{O}(c_a^2), \quad (28)
 \end{aligned}$$

where we have expanded in the small parameter c_a and introduced the function $M(r)$, defined by

$$M(r) \equiv \frac{r}{2} [1 - e^{-\mu(r)}], \quad (29)$$

which we will use in place of $\mu(r)$ from here on. Note that c_a is the only parameter in these background equations and that the leading parts of each equation correspond to the TOV equations in GR [22]. The unexpanded modified TOV equations and their numerical solutions are discussed in Appendix B.

B. First order in spin

Next, we study the first-order rotational perturbations to find the moment of inertia. This quantity can be derived from the asymptotic behavior of the metric perturbation ω far away from the star. The only other *Ansatz* function at this order is S , which comes from the aether field. Using the $\mathcal{O}(\varepsilon)$ parts of the modified Einstein and aether equations, we find coupled equations for S and ω from $E_{t\phi} = 0$ and $\mathcal{E}_\phi = 0$:

$$\frac{d^2\omega}{dr^2} = \beta_0 + \beta_1 c_a + \mathcal{O}(c_a^2), \quad (30)$$

$$\frac{d^2S}{dr^2} = \gamma_0 + \gamma_1 c_a + \gamma_2 \frac{c_a}{c_\omega} + \mathcal{O}\left(c_a^2, \frac{c_a^2}{c_\omega}\right), \quad (31)$$

The coefficients β_i and γ_i are functions of S , ω , and the background functions and do not have an explicit dependence on the parameters c_a and c_ω . Their explicit expressions are given by

$$\begin{aligned}
 \beta_0 &= \frac{4\pi r^3 (p_0 + \rho_0) + 8M - 4r}{r(r-2M)} \frac{d\omega}{dr} + 16\pi r^2 (p_0 + \rho_0) \omega, \\
 \beta_1 &= \frac{8\pi r^3}{(r-2M)^2} \left[\frac{\rho_0 + 3p_0}{4} - \frac{(\rho_0 + 4p_0)M}{2r} - \pi p_0^2 r^2 \right. \\
 &\quad \left. - \frac{M^2}{16\pi r^4} \right] \frac{d\omega}{dr} - \frac{8e^\nu}{r^3(r-2M)} \left[\left(\pi p_0 r^3 + \frac{M}{4} \right) \frac{dS}{dr} \right. \\
 &\quad \left. + \pi r^2 (\rho_0 + 3p_0) S \right], \quad (32)
 \end{aligned}$$

and

$$\begin{aligned}
 \gamma_0 &= \frac{1}{r(r-2M)} \left[(4\pi r^3 \rho_0 - 12\pi r^3 p_0 - 4M) \frac{dS}{dr} + 2S \right], \\
 \gamma_1 &= \frac{8\pi r^3}{(r-2M)^2} \left[\frac{M^2}{16\pi r^4} - \frac{(\rho_0 + 2p_0)M}{2r} + \frac{\rho_0 + 3p_0}{4} \right. \\
 &\quad \left. + \pi p_0^2 r^2 \right] \frac{dS}{dr}, \\
 \gamma_2 &= -\frac{4(4r^3 \pi p_0 + M)(4r^3 \pi p_0 + 3M - r)}{(-r + 2M)^2 r^2} S \\
 &\quad - \frac{2re^{-\nu}(4\pi p_0 r^3 + M)}{r-2M} \frac{d\omega}{dr}. \quad (33)
 \end{aligned}$$

We note that β_0 corresponds to the field equation for ω in GR [22]. We check in Appendix C that Eqs. (30) and (31) correctly reduce to those in khronometric gravity after taking the limit $c_\omega \rightarrow \infty$ and imposing appropriate boundary conditions for neutron stars.

There are a few remarks to make here about Eqs. (30) and (31). First, we have expanded both equations in small c_a ,¹ with full equations presented in Appendix B. Second, Eqs. (30) and (31) were derived assuming that c_ω is nonvanishing, as the aether component \mathcal{A}_ϕ is proportional to c_ω . The need to make this assumption to obtain Eq. (31) is expected because S is undetermined in GR, and GR is the limit of Einstein-aether theory when all coupling constants are taken to zero. Third, the equation for S is the only one with explicit dependence on c_ω [e.g., from the γ_2 term in Eq. (31)], and S first enters into the ω equation through the β_1 term. This means that the largest effect on ω due to c_ω will be at most of the order of $c_a S$ since β_1 corresponds to the $\mathcal{O}(c_a)$ term in Eq. (30).

We now extract the moment of inertia from the asymptotic behavior of ω at large r . To do this, we solve Eqs. (30) and (31) with a power series *Ansatz* in r in the exterior of the neutron star. In this context, the exterior is defined by vanishing pressure and energy density, i.e., setting $p_0 = \rho_0 = 0$ in Eqs. (30) and (31). Additionally, we impose that the exterior solution is regular at $r = \infty$. The exterior behavior for S and ω are then given by

¹No assumptions on the size of c_ω (relative to c_a) were used when calculating the series expansions in Eqs. (30) and (31). To ensure that both series correctly include the leading terms for different scales of c_ω , we substituted $c_a \rightarrow c_a \varepsilon$ and one of the cases (i) $c_\omega \rightarrow c_\omega \varepsilon^{-1}$ (ii) $c_\omega \rightarrow c_\omega \varepsilon^0$ (iii) $c_\omega \rightarrow c_\omega \varepsilon$ (iv) $c_\omega \rightarrow c_\omega \varepsilon^2$ into the equation for S'' (a prime denotes a derivative with respect to the r -coordinate), where ε quantifies the relative size between c_ω and c_a , and each case (i)–(iv) corresponds to a different relative size. For each case, we expanded the rhs of S'' in ε and confirmed that the leading (and sometimes sub-leading) terms are indeed correctly included in the rhs of the S'' equations.

$$S^{(\text{ext})}(r) = \frac{C}{r} + \frac{M_\star}{c_\omega r^2} \left[(c_a + 2c_\omega)C + \frac{3}{2}c_a D \right] + \mathcal{O}\left(\frac{1}{r^3}\right),$$

$$\omega^{(\text{ext})}(r) = \Omega_\star + \frac{D}{r^3} + \frac{c_a C M_\star}{2r^4} + \mathcal{O}\left(\frac{1}{r^5}\right), \quad (34)$$

with C and D being constants of integration, Ω_\star the angular velocity of the star, and $M_\star \equiv M(r)|_{r=\infty}$. The moment of inertia I is then related to the $1/r^3$ coefficient in $\omega^{(\text{ext})}$ via [22]

$$I \equiv -\frac{D}{2\Omega_\star}, \quad (35)$$

and its dimensionless version, \bar{I} , is given by

$$\bar{I} \equiv \frac{I}{M_\star^3}. \quad (36)$$

The integration constants C and D are obtained by imposing appropriate boundary conditions. However, we can determine how I depends on the coupling constants before explicitly calculating it from Eq. (35). Namely, since I is determined from ω , the dependence of I on c_a and c_ω will be the same as that of ω .

I. Analytic scaling

We wish to find how the moment of inertia, I , and ω depend on c_a and c_ω , specifically, how these coupling constants modify I and ω relative to GR. To see this, it is useful to distinguish between the “homogeneous” and “particular” parts of the solution for ω .² The former comes from the homogeneous part of the ω field equation in Eq. (30), i.e., the β_0 term and the $d\omega/dr$ term in β_1 in Eq. (32). The homogeneous solution for ω will have a (largest) modification of $\mathcal{O}(c_a)$ due to the β_1 term in Eq. (30) and the $\mathcal{O}(c_a)$ modifications in the background quantities p_0 , ρ_0 , and M . The particular solution comes from the “source” terms, i.e., the S and dS/dr terms in β_1 in Eq. (32). Thus to study the modification of the particular solution, which is $\mathcal{O}(c_a S)$, we must first estimate how the solution to S depends on the coupling constants. To summarize, the leading modification to the moment of inertia and ω is either $\mathcal{O}(c_a)$, from the homogeneous part, or $\mathcal{O}(c_a S)$, from the particular part.

In the following, we argue that the leading modification to I is at most $\mathcal{O}(c_a)$. To do this, we consider three limits: (1) $c_\omega \gg c_a$, (2) $c_\omega \sim c_a$, and (3) $c_\omega \ll c_a$, with $c_a \ll 1$ in each case, as is appropriate for region I. Since the leading modification is either $\mathcal{O}(c_a)$ or $\mathcal{O}(c_a S)$, all that is left is to estimate the coupling constant dependence of S in each

TABLE I. Leading contribution to S and the leading modification to moment of inertia δI . For δI , we choose the modification that is larger between $\mathcal{O}(c_a)$ and $\mathcal{O}(c_a S)$.

	$c_\omega \gg c_a$	$c_\omega \sim c_a$	$c_\omega \ll c_a$
S	$\mathcal{O}(c_a/c_\omega)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
δI	$\mathcal{O}(c_a)$	$\mathcal{O}(c_a)$	$\mathcal{O}(c_a)$

case. The scalings for S and the leading modification to I are summarized in Table I.

We first consider case (1), where $c_\omega \gg c_a$. In this limit, the field equation for S in Eq. (31) can be approximated as

$$S''(r) \approx \gamma_0 + \gamma_1 c_a. \quad (37)$$

This equation is homogeneous and linear, so we can express the general interior and exterior solutions to S as

$$S^{(\text{int})} = C_1 s_1(r), \quad S^{(\text{ext})} = C_1 s_2(r), \quad (38)$$

where C_1 and C_2 are constants, and where $s_1(r)$ and $s_2(r)$ are two distinct solutions of Eq. (37). Imposing regularity at the neutron star center and infinity allows us to have just one integration constant in $S^{(\text{int})}$ and $S^{(\text{ext})}$, respectively. To obtain the full solution to Eq. (37), we then match $S^{(\text{int})}$ and $S^{(\text{ext})}$ and their first derivative at the neutron star radius R_\star :

$$\begin{aligned} C_1 s_1(R_\star) &= C_2 s_2(R_\star), \\ C_1 s_1'(R_\star) &= C_2 s_2'(R_\star). \end{aligned} \quad (39)$$

Solving this system yields $C_1 = C_2 = 0$, and hence we find that $S \approx 0$ to the order kept in Eq. (37). Thus, to find nonvanishing contributions to S , we must go one order higher and include γ_2 from Eq. (31), which has a nonhomogeneous part. This leads to the coupling constant scaling $S = \mathcal{O}(c_a/c_\omega)$, meaning that the modification to I from the particular part of ω is $\mathcal{O}(c_a S) = \mathcal{O}(c_a^2/c_\omega)$. However, this is smaller than the $\mathcal{O}(c_a)$ homogeneous contribution to ω , and hence the leading correction to I for this case is $\mathcal{O}(c_a)$.

We next consider case (2): $c_\omega \sim c_a$. Here, we can approximate the field equation for S as

$$S''(r) \approx \gamma_0 + \gamma_2 + \gamma_1 c_a + \mathcal{O}(c_a). \quad (40)$$

Notice that the “source” term in γ_2 is now $\mathcal{O}(1)$ (with respect to the coupling constants), meaning that S will have a non-vanishing $\mathcal{O}(1)$ part. Thus, $\mathcal{O}(c_a S) = \mathcal{O}(c_a)$, the same order as the homogeneous contribution to I , and the dominant modification to ω and I is also $\mathcal{O}(c_a)$ in this case.

Lastly, we consider case (3): $c_\omega \ll c_a$. To solve Eq. (31) in this limit, let us first consider the following toy model:

$$\frac{d^2 y}{dx^2} = -y + \frac{1}{\epsilon}(y - x), \quad (41)$$

²Strictly speaking, S also depends on ω , so Eqs. (30) and (31) form a set of homogeneous equations and there is no source term.

where $\epsilon \ll 1$ is a small constant. The solution to this equation is

$$y(x) = \frac{x}{1-\epsilon} + C_1 \exp\left(\frac{x\sqrt{1-\epsilon}}{\sqrt{\epsilon}}\right) + C_2 \exp\left(-\frac{x\sqrt{1-\epsilon}}{\sqrt{\epsilon}}\right). \quad (42)$$

Taking the small ϵ limit of the solution $y(x)$ gives

$$y_\epsilon(x) = x + C_1 \exp\left(\frac{x}{\sqrt{\epsilon}}\right) + C_2 \exp\left(-\frac{x}{\sqrt{\epsilon}}\right). \quad (43)$$

Note that one can also obtain the solution y_ϵ by first taking the appropriate limits in Eq. (41) and solving those equations. Namely, the first term in Eq. (42) (corresponding to the particular solution) is obtained by taking the $\epsilon \rightarrow 0$ limit in Eq. (41):

$$0 = \frac{1}{\epsilon}(y - x), \quad (44)$$

and then solving for y . The other two terms in Eq. (42) (corresponding to the homogeneous solutions) are obtained by taking the small ϵ limit of the homogeneous part Eq. (41):

$$\frac{d^2 y}{dx^2} = \frac{1}{\epsilon} y. \quad (45)$$

Now, we apply this procedure from the toy model to the field equation for S in case (3) with c_ω acting as the small parameter ϵ . First, we take the $c_\omega \rightarrow 0$ limit in Eq. (31) to obtain $\gamma_2 = 0$. This equation for γ_2 gives us the algebraic solution

$$S(r) = \frac{e^{-\nu} r^3 (r - 2M)}{8\pi p_0 r^3 + 6M - 2r} \frac{d\omega}{dr}. \quad (46)$$

This solution is continuous and smooth at the neutron star radius, so we do not need to impose any additional conditions. Next, we solve the homogeneous part of the S field equation. However, from an identical calculation as done in case (1) (imposing regularity and matching the exterior and interior solutions), we find that the homogeneous solutions vanish. Hence, Eq. (46) is the full solution for S in this case, leading to $S = \mathcal{O}(1)$ and $\mathcal{O}(c_a S) = \mathcal{O}(c_a)$. So, as in the other cases, we find that the leading modification to I scales as $\mathcal{O}(c_a)$.

2. Numerical results

In this section, we solve S and ω numerically to confirm that the leading modification to the moment of inertia, δI , scales as $\mathcal{O}(c_a)$. For the numerical calculation, we use an adaptive 4th-order Runge-Kutta algorithm and begin solving near the neutron star center. By imposing regularity at $r = 0$, we find the behavior of S and ω near the center:

$$\begin{aligned} S^{(\text{int})}(r) &= Ar^2 + \frac{2\pi Ar^4}{5} \left[\frac{2\rho_c}{3} - 6p_c + \left(p_c + \frac{\rho_c}{3} \right) c_a \right. \\ &\quad \left. + 4 \left(p_c + \frac{\rho_c}{3} \right) \frac{c_a}{c_\omega} \right] + \mathcal{O}(r^5), \\ \omega^{(\text{int})}(r) &= B + r^2 \left[\frac{8\pi B(p_c + \rho_c)}{5} - \frac{4\pi c_a e^{\nu_c} B(\rho_c + 3p_c)}{3} \right] \\ &\quad + \mathcal{O}(r^4). \end{aligned} \quad (47)$$

where A and B are integration constants. The constants p_c , ρ_c , and ν_c are the values of $p(r)$, $\rho(r)$, and $\nu(r)$ evaluated at the center (see Appendix B for more detail). To determine the integration constants A , B , C , and D , where C and D are from the exterior solutions in Eq. (34), one can require that S and ω are continuous and differentiable at the neutron star surface. These conditions are imposed via the matchings

$$\begin{aligned} S^{(\text{int})}(R_\star) &= S^{(\text{ext})}(R_\star), \quad \omega^{(\text{int})}(R_\star) = \omega^{(\text{ext})}(R_\star), \\ S'^{(\text{int})}(R_\star) &= S'^{(\text{ext})}(R_\star), \quad \omega'^{(\text{int})}(R_\star) = \omega'^{(\text{ext})}(R_\star), \end{aligned} \quad (48)$$

where R_\star is the neutron star radius.

For the numerical calculation, we employ a slightly different scheme described in Ref. [16]. We begin by choosing two sets of arbitrary values for (A, B) in Eq. (47). For each of these two sets of initial conditions, we numerically solve the field equations for Eqs. (30) and (31) from a small radius near the center to the neutron star surface at R_\star . Then, we evaluate these interior solutions at the surface and use them as initial conditions for numerically finding the exterior solutions. After this process, we have two sets of solutions $Z_i^{(1)}$ and $Z_i^{(2)}$, where $Z_i \equiv (S, \omega)$. Since Eqs. (30) and (31) are linear and homogeneous in S and ω , we can write the general solution Z_i as the linear combination

$$Z_i = A' Z_i^{(1)} + B' Z_i^{(2)}, \quad (49)$$

where A' and B' are new constants. These new constants, along with C and D from Eq. (34), are determined by requiring that the solutions Z_i match the $r = \infty$ asymptotic behavior of S and ω at some large radius $r_b \gg R_\star$:

$$\begin{aligned} S(r_b) &= S^{(\text{ext})}(r_b), \quad \omega(r_b) = \omega^{(\text{ext})}(r_b), \\ S'(r_b) &= S'^{(\text{ext})}(r_b), \quad \omega'(r_b) = \omega'^{(\text{ext})}(r_b). \end{aligned} \quad (50)$$

After fixing the integration constants with the above matching, we have the full solutions for S and ω and can calculate the moment of inertia using Eq. (35).

Figure 1 shows the relative fractional difference between \bar{I} in GR and Einstein-aether theory as a function of stellar compactness, $\mathcal{C} = M_\star/R_\star$, for various combinations of c_ω and c_a in the $c_\omega \ll c_a$ regime. We used the APR equation of state (EoS) [42] for the calculations presented here.

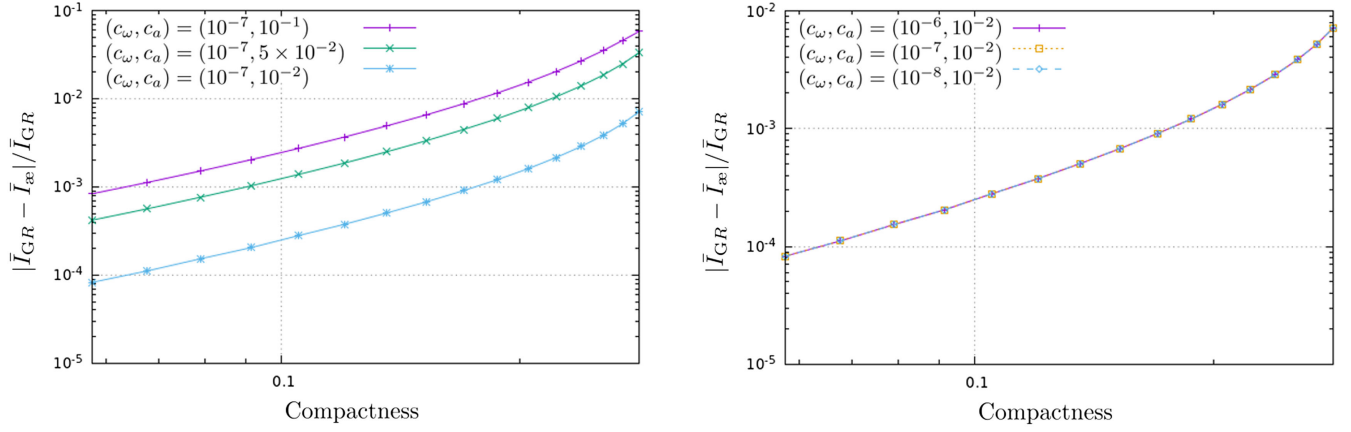


FIG. 1. The normalized relative difference between \bar{I} , defined in Eq. (36), in GR and Einstein-aether theory in region I (\bar{I}_{GR} and \bar{I}_{E} , respectively). We either fix c_ω and vary c_a (left) or fix c_a and vary c_ω (right). Values for c_a chosen here are beyond the current constraint and are used only for demonstration purposes. These plots show that the correction to \bar{I} is independent of c_ω while it scales linearly with c_a .

For the results in the left panel, we fixed $c_\omega = 10^{-7}$ and numerically calculate \bar{I} for three cases³: $c_a = 10^{-1}$, 10^{-2} , and 10^{-3} . For a given choice of (c_a, c_ω) , the modification to GR can be represented through the fractional relative difference, and we find that the latter scales close to linearly with c_a in the left panel. In the right panel, we fixed $c_a = 10^{-2}$ and calculated \bar{I} for $c_\omega = 10^{-6}$, 10^{-7} , and 10^{-8} . Unlike the cases in which the value of c_a was varied, changes in the fractional relative difference due to c_ω have no effect. In addition to the $c_a \ll c_\omega$ regime, we completed similar calculations for the $c_a \sim c_\omega$ and $c_a \gg c_\omega$ regimes and confirm identical scalings to the ones presented in Fig. 1. These findings confirm that this leading modification to I always scales as $\mathcal{O}(c_a)$, in accordance with the analytic results in Table I.

V. QUADRUPOLE MOMENT

We next study second-order rotational perturbations of neutron stars and the spin-induced quadrupole moment. The quadrupole moment is encoded in the (t, t) component of the metric at large radius r , which is, in turn, determined by H_0 at this order.

The function H_0 is one of three functions that control the diagonal metric perturbations: $\{H_0, H_2, K\}$. Together, their field equations form a closed system. To find these equations, we use the $\mathcal{O}(\varepsilon^2)$ parts of the modified Einstein equations. We first algebraically solve for $H_2(r)$ in terms of $H_0(r)$ from $E_{\phi\phi} - E_{\theta\theta} = 0$. We then substitute this equation for $H_2(r)$ into $E_{r\theta} = 0$ and $E_{rr} = 0$ to obtain the following field equations for H_0 and K :

$$\frac{dH_0}{dr} = \xi_0 + \xi_1 c_a + \xi_2 c_\omega + \mathcal{O}(c_a^2), \quad (51)$$

$$\frac{dK}{dr} = \zeta_0 + \zeta_1 c_a + \zeta_2 c_\omega + \mathcal{O}(c_a^2). \quad (52)$$

Here, we have expanded both equations in small c_a ,⁴ and have used the abbreviations ξ_i and ζ_i in the same way as in the first-order equations. The unexpanded equations are given in Appendix B. Note that the c_ω dependence in both equations is strictly linear, while c_a enters at both linear and higher orders. The terms ξ_0 and ξ_2 in Eq. (51), and the terms ζ_0 and ζ_2 in Eq. (52) are given by

$$\begin{aligned} \xi_0 = & \frac{1}{6r(r-2M)(4r^3\pi p_0 + M)} \left\{ 12H_0M^2 + 32e^{-\nu}r^3 \left[(r-2M) \left(\frac{M^2}{16} + \frac{\pi p_0 M r^3}{2} + \frac{Mr}{16} + \pi^2 2p_0^2 r^6 - \frac{r^2}{32} \right) \left(\frac{d\omega}{dr} \right)^2 \right. \right. \\ & + 16\pi r \left(\frac{M^2}{16} + \frac{\pi p_0 M r^3}{2} - \frac{Mr}{16} + \pi^2 p_0^2 r^6 + \frac{r^2}{32} \right) (p_0 + \rho_0) \omega^2 \left. \right] - 144Mr \left(\pi p_0 H_0 r^2 + \frac{\pi \rho_0 H_0 r^2}{3} + \frac{K}{6} - \frac{H_0}{12} \right) \\ & \left. - 192r^2 \left(\pi^2 p_0^2 H_0 r^4 - \frac{\pi(p_0 + \rho_0)H_0 r^2}{8} - \frac{K}{16} + \frac{H_0}{16} \right) \right\}, \end{aligned}$$

³Although these values are larger than the constraint on c_a , they are enough to demonstrate the scaling of the modification.

⁴To ensure that Eqs. (51) and (52) are correct regardless of the size of c_ω , we repeated the same check done at first order, described in and confirm that the series give the correct leading order and next-to-leading order expressions.

$$\begin{aligned}
 \xi_2 &= \frac{16e^\nu}{12\pi p_0 r^5 + 3Mr^2} \left\{ \left(\frac{M^2}{16} + \frac{\pi p_0 M r^3}{2} - \frac{Mr}{16} + \pi^2 p_0^2 r^6 + \frac{r^2}{32} \right) \left(\frac{dS}{dr} \right)^2 + \left[\pi p_0 r^3 + \frac{r}{4} - \frac{M}{4} \right] \frac{S}{2} \frac{dS}{dr} + \frac{S^2}{8} \right\}, \\
 \zeta_0 &= \frac{1}{24\pi p_0 r^4 + 6Mr} \left\{ -8e^{-\nu} r^3 \left[(r-2M) \left(\pi p_0 r^3 + \frac{r}{8} + \frac{M}{4} \right) \left(\frac{d\omega}{dr} \right)^2 + 16\pi r (p_0 + \rho_0) \left(\pi p_0 r^3 - \frac{r}{8} + \frac{M}{4} \right) \omega^2 \right] \right. \\
 &\quad \left. + 24\pi p_0 H_0 r^3 - 12H_0 M + 24r \left(\pi \rho_0 H_0 r^2 + \frac{K}{2} - \frac{H_0}{2} \right) \right\}, \\
 \zeta_2 &= \frac{8e^\nu}{24\pi p_0 r^5 + 6Mr^2} \left\{ (r-2M) \left[\pi p_0 r^3 - \frac{r}{8} + \frac{M}{4} \right] \left(\frac{dS}{dr} \right)^2 - (r-2M) \frac{S}{2} \frac{dS}{dr} - \frac{S^2}{2} \right\}. \tag{53}
 \end{aligned}$$

We forego presenting the expressions for ξ_1 and ζ_1 as they are lengthy and their exact forms are unimportant for our analysis of the quadrupole moment. Note that ξ_2 and ζ_2 , the terms that control the c_ω dependence in Eqs. (51) and (52), are both quadratic in S and S' . We also note that the expressions for ξ_0 and ζ_0 correspond to the GR parts of H'_0 and K' [22].

With the field equations in hand, we can now derive the quadrupole moment from the asymptotic behavior of H_0 far away from the star. As done at first order in Eq. (34), we solve Eqs. (51) and (52) for the exterior behavior of H_0 and K by using a power-series *Ansatz* that is regular at $r = \infty$. We find

$$\begin{aligned}
 H_0^{(\text{ext})}(r) &= \frac{F}{r^3} + \frac{1}{r^4} \left[\frac{((c_\omega - 4)C^2 - 6CD - 3D^2)c_a}{6} \right. \\
 &\quad \left. + \frac{c_\omega C^2}{3} + 3FM_\star - D^2 \right] + \mathcal{O}\left(\frac{1}{r^5}\right), \\
 K^{(\text{ext})}(r) &= \frac{F}{r^3} + \frac{1}{r^4} \left[\left(\frac{(c_\omega - 4)C^2}{6} - CD + \frac{FM_\star}{4} \right) c_a \right. \\
 &\quad \left. - \frac{c_a D^2}{2} + \frac{c_\omega C^2}{4} + \frac{5FM_\star}{2} - \frac{D^2}{4} \right] + \mathcal{O}\left(\frac{1}{r^5}\right), \tag{54}
 \end{aligned}$$

where F is an integration constant and C and D are the integration constants that arise from the solutions to the first-order analysis in Eq. (34). The quadrupole moment Q is the coefficient of the $P_2(\cos\theta)/r^3$ term in the Newtonian potential [43], which in our case is

$$Q = -\frac{F}{2}. \tag{55}$$

As with the moment of inertia, it is convenient to define the dimensionless quadrupole moment

$$\bar{Q} = -\frac{QM_\star}{(\Omega_\star I)^2}, \tag{56}$$

where I is the moment of inertia.

A. Analytic scaling

As was done with the moment of inertia, we can first study how H_0 , or Eq. (51), depends on c_a and c_ω before explicitly calculating Q using Eq. (55). Understanding the coupling constant dependence of H_0 will then allow us to estimate that of Q . Similar to the moment of inertia case, the leading modifications are either of $\mathcal{O}(c_a)$ coming from ξ_0 (due to modifications of the background functions) and ξ_1 , or of $\mathcal{O}(c_\omega S^2)$ arising from ξ_2 . The latter type of modification comes from the fact that ξ_2 is quadratic in S and S' . Additional modifications could come from K , since it couples to H_0 starting with the ξ_0 term. However, the leading modifications to K are also either $\mathcal{O}(c_a)$ or $\mathcal{O}(c_\omega S^2)$, since the field equation for K has the same coupling-constant dependence and form as H_0 , as seen in Eqs. (51)–(53).

We now argue that the leading-order modification to Q is at most $\mathcal{O}(c_a)$. To do so, we consider separately case (1) $c_\omega \gg c_a$, case (2) $c_\omega \sim c_a$, and case (3) $c_\omega \ll c_a$ and assume $c_a \ll 1$ in each. Since the leading modification is either $\mathcal{O}(c_a)$ or $\mathcal{O}(c_\omega S^2)$, we just need to find the leading modification to S in each case. To this end, we use the behavior for S found in the previous section, which is presented in Table I. The results for the quadrupole moment are summarized in Table II.

B. Numerical results

We now present numerical calculations of Q to show explicitly that δQ indeed scales as $\mathcal{O}(c_a)$, confirming our analytic scaling arguments. To do this, we numerically solve Eqs. (51) and (52) in a similar fashion as done for the first order functions. We begin by finding the behavior of H_0 and K near the center:

TABLE II. Leading contribution to S and the leading modification to moment of inertia δQ . For δQ , we choose the modification that is larger between $\mathcal{O}(c_a)$ and $\mathcal{O}(c_a S^2)$.

	$c_\omega \gg c_a$	$c_\omega \sim c_a$	$c_\omega \ll c_a$
S	$\mathcal{O}(c_a/c_\omega)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
δQ	$\mathcal{O}(c_a)$	$\mathcal{O}(c_a)$	$\mathcal{O}(c_a)$

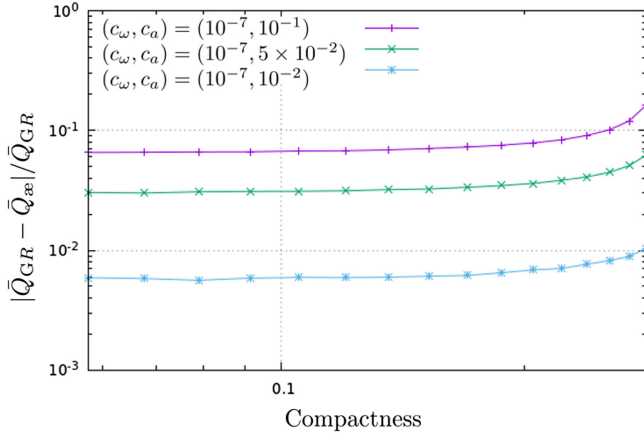


FIG. 2. Similar to the left panel of Fig. 1 but for the normalized quadrupole moment \bar{Q} . The correction to \bar{Q} scales linearly with c_a .

$$\begin{aligned} H_0^{(\text{int})} &= \left(\frac{4c_\omega e^{\nu} c A^2}{3} + G \right) r^2 + \mathcal{O}(r^4), \\ K^{(\text{int})} &= Gr^2 + \mathcal{O}(r^4). \end{aligned} \quad (57)$$

Here, G is an integration constant (not to be confused with a gravitational coupling constant) and A is the constant that arises from the first-order solution in our analysis of Eq. (47).

Now, since Eqs. (51) and (52) both have homogeneous terms and source terms, we can write their general solutions as the sum of a particular solution and an undetermined constant times a homogeneous solution:

$$Z_i = Z_{\text{part},i} + C' Z_{\text{hom},i}, \quad (58)$$

where $Z_i \equiv (H_0, K)$ here. We first find a particular solution by choosing an arbitrary value for G in Eq. (57) and numerically solving in the interior. With the interior solution as initial data, we then numerically solve from the star surface outward to find the exterior solution. For the homogeneous solution, we choose an arbitrary test value for G and follow the same procedure. To complete our solution for Z_i , we must impose boundary conditions and find C' and F , the exterior integration constant from Eq. (54). As in the moment of inertia case, we require that our solutions Z_i match the asymptotic exterior behavior, Eq. (54), at a large radius $r_b \gg R_\star$:

$$H_0(r_b) = H_0^{(\text{ext})}(r_b), \quad K(r_b) = K^{(\text{ext})}(r_b). \quad (59)$$

After imposing the above boundary conditions, we have a solution for F and can now calculate the quadrupole moment using Eq. (55).

Figure 2 shows the relative fractional difference between \bar{Q} in GR and Einstein-aether theory as a function of compactness for various combinations of c_ω and c_a in

the $c_\omega \ll c_a$ regime. We investigate the relative fractional difference when c_ω is fixed and c_a is varied. As in the numerical results for \bar{I} , we use the APR EoS for the numerical calculations presented here. We find the same scaling behavior as in the moment of inertia case, namely, the curves scale about linearly with c_a . In addition to the results presented in Fig. 2, we confirm that $\delta\bar{Q}$ is independent of c_ω and these scalings hold for the $c_a \sim c_\omega$ and $c_a \gg c_\omega$ regimes. These findings confirm that the leading modification to Q always scales as $\mathcal{O}(c_a)$, in accordance with the analytic results of Table II.

VI. TIDAL LOVE NUMBER

We finish our study of the I-Love-Q trio with the tidal Love number. Like the quadrupole moment, the Love number is derived from the behavior of H_0 . However, unlike the quadrupole moment, the Love number is determined from the asymptotic behavior in a region called the *buffer zone*. This zone is characterized by the neutron star radius and the source of the tidal perturbation (the companion); namely, it is the radial region defined by $R_\star \ll r \ll \mathcal{R}$, where R_\star is the neutron star radius and \mathcal{R} is the curvature radius of the source causing the tidal perturbation. Another difference is that the leading-order tidal perturbation enters at $\mathcal{O}(\varepsilon^2)$. This means that we can ignore $\mathcal{O}(\varepsilon)$ contributions to the metric and aether in the tidal case.

To find the field equations for H_0 and K in the tidal case, we can use the equations found for spin perturbations, Eqs. (51) and (52), but with the $\mathcal{O}(\varepsilon)$ contributions set to zero, i.e., setting $S = \omega = 0$. We then have a homogeneous system for H_0' and K' , which we can further decouple to obtain separate equations for H_0'' and K'' . Since the Love number is derived from H_0 , we just focus on the H_0'' equation, given by

$$\frac{d^2 H_0}{dr^2} = \varphi_0 + \varphi_1 c_a + \mathcal{O}(c_a^2). \quad (60)$$

We leave the expressions for φ_0 and φ_1 to Appendix B. Notice that, unlike in the rotation case, H_0 now obeys a homogeneous equation and is completely independent of c_ω . This is expected since c_ω always couples to S in that case, and we have set $S = 0$ here. Hence, the leading-order modification to H_0 , and thus the tidal Love number, is at most $\mathcal{O}(c_a)$.

To determine the scaling of the leading modification, we next define the Love number and calculate it using numerical methods. First, we find the behavior of H_0 in the buffer zone, which is done by using a power series *Ansatz* for Eq. (60) in the exterior. Unlike the case for rotational perturbation, we do not require that the growing modes vanish since we expand only at $r \gg R_\star$ and not $r = \infty$. The behavior of H_0 in the buffer zone is then given by

$$H_0^{(\text{buf})} = \tilde{C} \left(r^2 - 2M_\star r - \frac{c_a M_\star^3}{6r} - \frac{c_a M_\star^4}{3r^2} \right) + \frac{\tilde{D}}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right), \quad (61)$$

where \tilde{C} and \tilde{D} are integration constants. The tidal Love number is then extracted from the buffer zone behavior of the Newtonian potential. The latter can be written as [22,44]

$$\begin{aligned} \frac{1 - g_{tt}}{2} &= \mathcal{O}\left(\frac{r^3}{\mathcal{R}^3}\right) + \frac{1}{3} \mathcal{E}^{(\text{tid})} P_2(\cos\theta) r^2 + \dots \\ &\quad - \frac{P_2(\cos\theta)}{r^3} Q^{(\text{tid})} + \mathcal{O}\left(\frac{R_\star^4}{r^4}\right), \end{aligned} \quad (62)$$

where the coefficients $\mathcal{E}^{(\text{tid})}$ and $Q^{(\text{tid})}$ are the tidal potential and tidally induced quadrupole moment, respectively. We have only included $l = 2$ modes in Eq. (62), as is appropriate for the tidal perturbations we consider. The tidal Love number (more commonly called the tidal deformability in the context of gravitational waves) is then defined as the ratio

$$\lambda \equiv -\frac{Q^{(\text{tid})}}{\mathcal{E}^{(\text{tid})}}. \quad (63)$$

Using Eqs. (23) and (61), we find that the Love number in our case is

$$\lambda = \frac{1}{3} \frac{\tilde{D}}{\tilde{C}} - \frac{c_a M_\star^5}{80} \left(c_a - \frac{176}{9} \right). \quad (64)$$

We note that the second terms in Eq. (64) come from the multiplication of e^ν and $H_0^{(\text{buf})}$ [Eqs. (61) and (B4)] in g_{tt} . As usual, there is also a dimensionless Love number, defined by

$$\bar{\lambda} = \frac{\lambda}{M_\star^2}. \quad (65)$$

We now present explicit calculations for the Love number. To do so, we solve Eq. (60) numerically by following a similar procedure as outlined for the cases at first and second order in spin. We first need the interior behavior of H_0 , which we can find from Eq. (57) in the second-order spin case. As noted previously, we can map the second-order-spin case to the first-order-tidal case by setting quantities related to S and ω to zero. In Eq. (57), this mapping is achieved by setting A , the integration constant from $S^{(\text{int})}$, to zero. We thus find the interior behavior for H_0 in the tidal case to be

$$H_0^{(\text{int})} = \tilde{G} r^2 + \mathcal{O}(r^4), \quad (66)$$

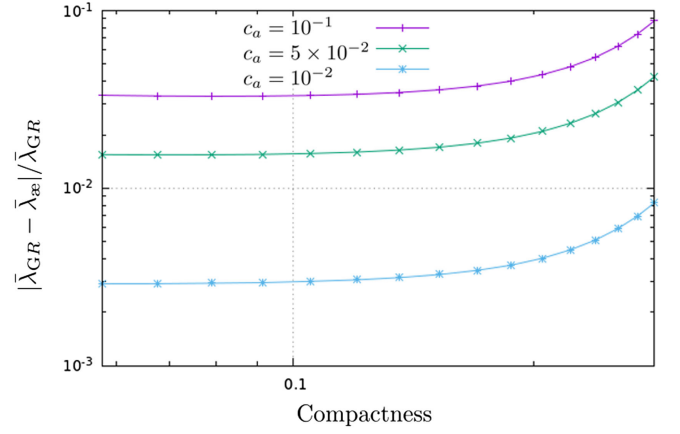


FIG. 3. Similar to the left panel of Fig. 1 but for the normalized tidal deformability $\bar{\lambda}$. The tidal deformability is independent of c_ω in region I, and the modification scales as $\mathcal{O}(c_a)$.

where we have renamed the integration constant from G to \tilde{G} to distinguish it from that used in the spin case.

One may be worried that we have three integration constants in our setup, $\{\tilde{C}, \tilde{D}, \tilde{G}\}$, but only two boundary conditions from matching,

$$\begin{aligned} H_0^{(\text{int})}(R_\star) &= H_0^{(\text{buf})}(R_\star), \\ H_0^{(\text{int})}{}_{,l}(R_\star) &= H_0^{(\text{buf})}{}_{,l}(R_\star). \end{aligned} \quad (67)$$

However, this is not a problem if we are only interested in calculating λ . To see why, we first note that since Eq. (60) is homogeneous, $H_0^{(\text{int})}$ in Eq. (66) is proportional to \tilde{G} . This means that changing \tilde{G} will only change the interior solution by an overall factor, and hence \tilde{C} and \tilde{D} , obtained from solving Eq. (67), will both have the same prefactor of \tilde{G} . Since λ only depends on the ratio \tilde{D}/\tilde{C} in Eq. (63), the factors of \tilde{G} will cancel, and \tilde{G} will have no effect on λ .

We are now ready to solve Eq. (60). As in the previous cases, we begin by numerically solving in the interior with initial conditions near the center, given by Eq. (66). Following the previous discussion, we choose an arbitrary value of \tilde{G} . We then evaluate the interior solution at R_\star , the surface, and use the boundary conditions in Eq. (67) to solve for \tilde{C} and \tilde{D} . With solutions for \tilde{C} and \tilde{D} in hand, we then use Eq. (63) to calculate the Love number. Figure 3 shows the relative fractional difference between $\bar{\lambda}$ in GR and Einstein-aether theory for several values of c_a . As expected, the modification to $\bar{\lambda}$ scales as $\mathcal{O}(c_a)$ and is independent of c_ω as this coupling constant does not enter the tidal field equation for H_0 .

Figure 4 shows the I-Love and Q-Love relations in GR and Einstein-aether theory for $(c_\omega, c_a) = (10^{-5}, 10^{-5})$ and $(c_\omega, c_a) = (10^{-7}, 10^{-1})$. For both of these parameter choices, the Einstein-aether I-Love relation has no discernible difference from the one in GR. Although the Q-Love relation for $(c_\omega, c_a) = (10^{-5}, 10^{-5})$ is also similar to

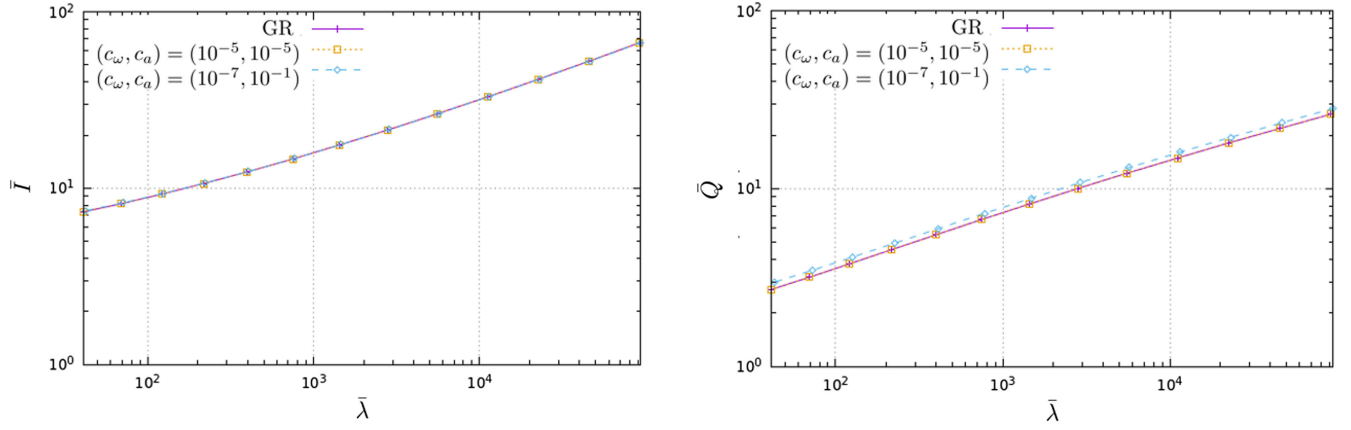


FIG. 4. The I-Love (left) and Q-Love (right) relations in GR and Einstein-aether theory (using the normalized I-Love-Q quantities). The Q-Love relation is very close to GR for $(c_\omega, c_a) = (10^{-5}, 10^{-5})$, while the $(c_\omega, c_a) = (10^{-7}, 10^{-1})$ relation illustrates how Einstein-aether theory could deviate noticeably from GR. This value for c_a is larger than the current constraint, however. On the other hand, the I-Love relation is close to GR for both parameter choices.

GR, the Q-Love relation for $(c_\omega, c_a) = (10^{-7}, 10^{-1})$ noticeably deviates from GR. This aligns with the observation that \bar{Q} has larger deviations from GR than \bar{I} , which can be seen by comparing Fig. 1 and 2.

Although the tidal Love number closely follows its behavior in GR, it is still possible that tidally deformed neutron stars have noticeable deviations in Einstein-aether theory. Such deviations could come from the off-diagonal perturbations $\{H_1, V, W\}$, which are not relevant for the I-Love-Q quantities but which may be used to define non-GR Love numbers as done in khronometric gravity [30]. We briefly investigate these perturbations in Appendix D.

VII. CONCLUSION AND DISCUSSION

We have derived the I-Love-Q quantities in Einstein-aether theory. Although the parameter c_ω appears in the field equations relevant to the moment of inertia and quadrupole moment, we find these contributions are at most $\mathcal{O}(c_a) \approx \mathcal{O}(10^{-5})$ in region I. Similarly, we find that the Love number has no dependence on c_ω . Hence, it would be difficult to use the I-Love-Q relations in Einstein-aether theory to constrain c_ω . However, since the deviations from GR are small or vanishing, this means that the relations remain universal in Einstein-aether theory. Additionally, combining with the results found in khronometric gravity [30], our findings may give further indications that the I-Love-Q relations are also insensitive to Lorentz-violating effects.

To obtain these results, we began by perturbatively constructing slowly rotating and weakly tidally deformed neutron stars in Einstein-aether theory via *Ansätze* for the metric and aether fields. From there, we derived the neutron star field equations and studied their solutions in both the rotating and tidal cases. We focused on the parameter space denoted as region I. At first order in rotation, we found that the perturbation functions depended on both c_ω and c_a , but

that the dependence on c_ω was controlled solely by the first-order perturbation function $S(r)$, which originates from the ϕ -component of the aether. Through an analysis of the $S(r)$ field equation, we showed that the modifications to the GR moment of inertia due to c_ω are at most $\mathcal{O}(c_a)$ for any size of c_ω . Similarly, at second order in rotation, we found that the c_ω dependence in the field equations was proportional to $S(r)$ or $S'(r)$, and that, again, the modifications to the quadrupole moment were at most $\mathcal{O}(c_a)$ for any size of c_ω . At first order in tidal deformations, we found that the c_ω dependence drops out due to $S(r)$ vanishing in this case. To confirm these analyses, we numerically solved the neutron star field equations and found that the dominant modification and scaling for all three I-Love-Q quantities was $\mathcal{O}(c_a)$ in region I. In region II, we found that field equations relevant to I-Love-Q are independent of the coupling constants and hence reduce to GR exactly.

Reference [21] argued that the remaining viable parameter space of Einstein-aether theory (regions I and II) admit weak-field solutions in which the coupling constants c_ω and c_θ are large but have a small effect on the gravitational dynamics. The authors also found that Einstein-aether rotating black holes in regions I and II display similar behavior in which c_ω and c_θ can be large, but the twist and expansion of the aether field remain small enough to make deviations from GR of $\mathcal{O}(c_a)$. Our results lead to similar conclusions for slowly rotating and weakly tidally deformed neutron stars, where c_ω may be large, but its effects on gravity are highly suppressed due to the relevant part of the twist, $S(r)$ (see Appendix C), being small.

There are several ways in which this work may be extended. In our study, we only considered time-independent perturbations. One possibly interesting route for future work would be to introduce time-dependent perturbations for modeling stellar oscillation frequencies. In this case, it is possible that c_ω could enter other components of the field

equations and affect observables like the fundamental-mode frequency that is known to have universal relations with the Love number in GR [45]. Another avenue is to remove the ambiguity in the definition of the Love number that is known to exist with the method adopted in this paper [46,47]. In particular, the growing mode in $H_0^{(\text{buf})}$ in Eq. (61) does not terminate at a finite order, which indicates that the way we separate growing and decaying modes is not unique unless we provide a prescription. For example, one can compute the Love number for black holes in Einstein-aether theory and use that as a reference to remove the ambiguity as proposed in GR [46], or one may use the wave scattering technique to determine the Love number in a unique way [48].

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APPENDIX A: I-LOVE-Q IN REGION II

In this appendix, we focus on the I-Love-Q quantities in region II, which is characterized by the coupling constants c_θ and c_ω [Eq. (16)]. We find that c_θ does not enter any of the relevant equations for the I-Love-Q trio while c_ω enters the equations for the quadrupole moment. Despite this latter point, we find that the I-Love-Q relations in region II are identical to those in GR. To show this, we follow the same procedure as outlined in the main text.

Let us first study the background equations. Using the $\mathcal{O}(\varepsilon^0)$ parts of the modified Einstein equations and the conservation of stress-energy, $\nabla^\mu T_{\mu\nu}^{(\text{mat})} = 0$, we find the TOV equations to be identical to the GR ones [22]. Hence, neither of the coupling constants enter into the background quantities, and thus these quantities will not provide any modifications to the I-Love-Q trio, unlike in region I.

Next, we look at first order in spin. From the components $E_{t\phi} = 0$ and $\mathcal{A}_\phi = 0$, we find the following uncoupled field equations for S and ω :

$$\frac{d^2 S}{dr^2} = \gamma_0, \quad (\text{A1})$$

$$\frac{d^2 \omega}{dr^2} = \beta_0, \quad (\text{A2})$$

where γ_0 and β_0 are the same functions defined earlier in Eqs. (32) and (33). As with the background quantities, we find that S and ω are unaffected by the coupling constants, and hence the moment of inertia is the same as in GR. Further, we can show that S vanishes by following the argument given in case (1) in Sec. IV B 1. Namely, we can use the fact that Eq. (A1) is homogeneous and then impose boundary conditions to find that $S = 0$.

We next find the $\mathcal{O}(\varepsilon^2)$ equations for H_0 and K . We find

$$\begin{aligned} \frac{dH_0}{dr} &= \xi_0 + \xi_2 c_\omega, \\ \frac{dK}{dr} &= \zeta_0 + \zeta_2 c_\omega, \end{aligned} \quad (\text{A3})$$

where the ξ_i and ζ_i are the same functions as defined previously in Eq. (53) (for tidally deformed neutron stars, we set $\omega = 0$ and $S = 0$). However, as noted previously, ξ_2 and ζ_2 are proportional to S and S' , meaning that $\xi_2 = \zeta_2 = 0$ per our finding that $S = 0$ in region II. It then follows that H_0 and K are independent of the coupling constants. In turn, this means that the quadrupole moment and tidal Love number are also independent of the coupling constants and hence are the same as in GR.

APPENDIX B: NEUTRON STAR FIELD EQUATIONS

In this section, we present the full, unexpanded neutron star field equations relevant to the moment of inertia, quadrupole moment, and tidal Love number in region I. We also study the interior and exterior behavior of the background functions.

1. Background

We first look at the modified TOV equations for the background quantities p , ν , and M . They are obtained from the $\mathcal{O}(\varepsilon^0)$ parts of the $E_{\theta\theta}$, E_{rr} , and E_{tt} components of the modified Einstein equations as well as the $\nabla^\mu T_{\mu r}$ component of the stress-energy conservation equation. We find

$$\begin{aligned} \frac{dM}{dr} &= \frac{-2(c_a - 2)\sqrt{r - 2M}\sqrt{(c_a - 2)M + 4\pi p_0 c_a r^3 + r} - 8\pi p_0 c_a r^3(2c_a - 1) + (6c_a - c_a^2 - 8)M - 8\pi \rho_0 c_a r^3 - 2r(c_a - 2)}{(c_a - 2)c_a r}, \\ \frac{d\nu}{dr} &= \frac{-4}{(r - 2M)c_a r} \left\{ \sqrt{r - 2M}\sqrt{(c_a - 2)M + 4\pi p_0 c_a r^3 + r} + 2M - r \right\}, \\ \frac{dp_0}{dr} &= \frac{2}{(r - 2M)c_a r} \left\{ \left[\sqrt{r - 2M}\sqrt{(c_a - 2)M + 4\pi p_0 c_a r^3 + r} + 2M - r \right] (p_0 + \rho_0) \right\}, \end{aligned} \quad (\text{B1})$$

and $\rho_0(r)$ is found from the equation of state. Expanding these equations in small c_a gives the series in Eq. (28). We next find the interior and exterior behaviors of the background quantities. These behaviors are used when deriving the appropriate interior and exterior behaviors for the $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon^2)$ functions (i.e., $S^{(\text{int})}$, $S^{(\text{ext})}$, $H_0^{(\text{int})}$, etc.). For the interior, we have

$$\begin{aligned} M^{(\text{int})}(r) &= \frac{4\pi(3c_a p_c + 2\rho_c)}{3(c_a - 2)} r^3 + \mathcal{O}(r^5), \\ \nu^{(\text{int})}(r) &= \nu_c - \frac{8\pi(\rho_c + 3p_c)}{3(c_a - 2)} r^2 + \mathcal{O}(r^4), \\ p_0^{(\text{int})}(r) &= p_c + \frac{4\pi(\rho_c^2 + 4\rho_c p_c + 3p_c^2)}{3(c_a - 2)} r^2 + \mathcal{O}(r^4), \\ \rho_0^{(\text{int})}(r) &= \rho_c + \rho_2 r^2 + \mathcal{O}(r^4). \end{aligned} \quad (\text{B2})$$

The constants ν_c , p_c , and ρ_c are the respective function evaluated at the neutron star center, $r=0$, while the constant ρ_2 can be expressed in terms of p_c and ρ_c by using the $p_0^{(\text{int})}$ equation and the neutron star EoS.

When solving the modified TOV equation in the interior, we specify the central pressure ρ_c by hand and then use it with the EoS to find p_c . The value for ν_c is determined by boundary conditions. Next, the exterior behaviors are found to be

$$\begin{aligned} M^{(\text{ext})}(r) &= M_\star + \frac{c_a M_\star^2}{4r} + \frac{c_a M_\star^3}{4r^2} \\ &+ \frac{c_a M_\star^4}{3r^3} + \frac{c_a(c_a + 48)M_\star^5}{96r^4} + \mathcal{O}\left(\frac{1}{r^5}\right), \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} e^{\nu^{(\text{ext})}(r)} &= 1 - \frac{2M_\star}{r} - \frac{c_a M_\star^3}{6r^3} - \frac{c_a M_\star^4}{3r^4} \\ &- \left(\frac{3c_a^2}{80} + \frac{3c_a}{5}\right) \frac{M_\star^5}{r^5} + \mathcal{O}\left(\frac{1}{r^6}\right), \end{aligned} \quad (\text{B4})$$

where $M_\star \equiv M(r)|_{r=\infty} = G_N M_{\text{obs}}$ is the neutron star mass observed by a Keplerian experiment. In Eq. (B4), we have used the fact that ν' is shift-invariant to fix the constant term to unity. Eqs. (B3) and (B4) also describe the respective buffer zone behaviors of $M(r)$ and $\nu(r)$.

As done with the higher-order functions, we numerically solve the background quantities by beginning with the initial conditions in Eq. (B2) and solving Eq. (B1) outward. For the functions that are non-vanishing in the exterior, M and ν , we then use the interior solutions evaluated at the star surface as initial conditions for solving Eq. (B1) in the exterior. At a large radius we match these numerical solutions to Eqs. (B3) and (B4) to find M_\star and ν_c .

2. First order in rotation

Next, we present the full $\mathcal{O}(\varepsilon)$ equations. From the Einstein-aether field equations $E_{i\phi} = 0$ and $\mathcal{A}_\phi = 0$, we find

$$\frac{d^2 S}{dr^2} = \frac{1}{a_0} \left(a_1 \frac{dS}{dr} + a_1 S + a_3 \frac{d\omega}{dr} \right), \quad (\text{B5})$$

where

$$\begin{aligned} a_0 &= c_\omega c_a (c_a - 2)(r - 2M)r^2, \\ a_1 &= 2c_\omega r \left\{ 2r(c_a - 2) - 2\sqrt{r - 2M}(c_a - 2)\sqrt{M(c_a - 2) + 4r^3 \pi p_0 c_a + r} - 4\pi c_a r^3 [\rho_0 + p_0(2c_a - 1)] + (-c_a^2 - 2c_a + 8)M \right\}, \\ a_2 &= 8(c_a - 2) \left(\sqrt{r - 2M}(c_a + 4)\sqrt{M(c_a - 2) + 4r^3 \pi p_0 c_a + r} - 8r^3 \pi p_0 c_a + 8M + \left(\frac{c_\omega}{4} - 1\right) r c_a - 4r \right), \\ a_3 &= 4 \left(-\sqrt{r - 2M}\sqrt{M(c_a - 2) + 4r^3 \pi p_0 c_a + r} + r - 2M \right) e^{-\nu} (c_a - 2) c_a r^3, \end{aligned} \quad (\text{B6})$$

and

$$\frac{d^2 \omega}{dr^2} = \frac{1}{b_0} \left(b_1 \frac{d\omega}{dr} + b_2 \omega + b_3 \frac{dS}{dr} + b_4 S \right), \quad (\text{B7})$$

with

$$\begin{aligned}
 b_0 &= (r - 2M)r^3 c_a (c_a - 2), \\
 b_1 &= 4r^2 \left[\sqrt{r - 2M}(c_a - 2) \sqrt{M(c_a - 2) + 4r^3 \pi p_0 c_a + r - 4\pi c_a \left(c_a - \frac{1}{2}\right) r^3 p_0 + \left(\frac{3}{2}c_a^2 - c_a - 4\right) M} \right. \\
 &\quad \left. - r(2\pi p_0 c_a r^2 + c_a^2 - c_a - 2) \right], \\
 b_2 &= 16c_a \pi r^4 (p_0 + \rho_0)(c_a - 2), \\
 b_3 &= 4c_a e^\nu (c_a - 2) (-\sqrt{r - 2M} \sqrt{M(c_a - 2) + 4r^3 \pi p_0 c_a + r} + r - 2M), \\
 b_4 &= 16c_a^2 \pi (\rho_0 + 3p_0) e^\nu r^2.
 \end{aligned} \tag{B8}$$

3. Second order in rotation

For the quadrupole moment, we consider the $\mathcal{O}(\varepsilon^2)$ diagonal perturbation functions H_0 , H_2 , and K . From $E_{\theta\theta} - E_{\phi\phi} = 0$, we first find an algebraic equation for H_2 :

$$\begin{aligned}
 H_2(r) &= \frac{1}{3r^2(c_a - 2)} \left\{ (2 - c_a) e^{-\nu} r^5 \left[(r - 2M) \left(\frac{d\omega}{dr} \right)^2 + r\omega^2 \pi (p_0 + \rho_0) \right] - r e^\nu c_\omega (c_a - 2) (r - 2M) \left(\frac{dS}{dr} \right)^2 \right. \\
 &\quad - 4e^\nu (c_a - 2) S (r - 2M) \left(\sqrt{2} \sqrt{\frac{(-4 + 2c_a)M + 8r^3 \pi p_0 c_a + 2r}{r - 2M}} - 2 \right) \frac{dS}{dr} + 16r^2 \left[S^2 \pi c_a (\rho_0 + 3p_0) e^\nu \right. \\
 &\quad \left. \left. + \frac{3H_0(c_a - 2)}{16} \right] \right\}.
 \end{aligned} \tag{B9}$$

From $E_{rr} = 0$ and $E_{r\theta} = 0$, we find H_0 and K equations given by

$$\begin{aligned}
 \frac{dK}{dr} &= \frac{1}{3c_a r^3} \left\{ -3\sqrt{2} \left[\frac{2e^\nu S^2 c_a}{3} + r^2 (c_a - 1) H_0 - r^2 H_2 \right] \sqrt{\frac{(-4 + 2c_a)M + 8r^3 \pi p_0 c_a + 2r}{r - 2M}} + 3c_a r^3 \left(\frac{dH_0}{dr} \right) \right. \\
 &\quad \left. - 2e^\nu S \left(\frac{dS}{dr} \right) c_\omega r c_a + 4e^\nu S^2 c_a + 3r^2 (c_a - 2) (H_0 + H_2) \right\},
 \end{aligned} \tag{B10}$$

$$\begin{aligned}
 \frac{dH_0}{dr} &= \frac{1}{6c_a r^3 \sqrt{r - 2M} \sqrt{M(c_a - 2) + 4r^3 \pi p_0 c_a + r}} \left\{ 12\sqrt{r - 2M} \left[\frac{2S c_a r^3}{3} \frac{d\omega}{dr} + r^3 \left(\frac{dK}{dr} \right) - \frac{2e^\nu S^2 (c_a + 4)}{3} \right] \right. \\
 &\quad \times \sqrt{M(c_a - 2) + 4r^3 \pi p_0 c_a + r} - \left[(2M - r) \left(\frac{d\omega}{dr} \right)^2 + 16\pi r \omega^2 (p_0 + \rho_0) \right] r^6 c_a e^{-\nu} - 8r^3 c_a S (r - 2M) \left(\frac{d\omega}{dr} \right) \\
 &\quad + 6(c_a - 2) r^3 (r - 2M) \left(\frac{dK}{dr} \right) - e^\nu c_a c_\omega r^2 (r - 2M) \left(\frac{dS}{dr} \right)^2 + 64S^2 e^\nu \left[-M + r^3 \pi p_0 c_a + \frac{r}{2} + \left(-\frac{c_\omega}{16} + \frac{1}{8} \right) c_a r \right] \\
 &\quad \left. - 24r^3 \left[\pi r^2 (H_0 + 2H_2) p_0 + r^2 \pi H_0 \rho_0 - \frac{3H_0}{4} + \frac{H_2}{4} + \frac{K}{2} \right] c_a \right\}.
 \end{aligned} \tag{B11}$$

To obtain the unexpanded H_0 and K equations from which Eqs. (51) and (52) are derived, one just needs to substitute Eq. (B9) into Eqs. (B10) and (B11) and disentangle K' and H_0' with some algebra.

4. First order in tidal deformation

First-order tidal deformations enter at $\mathcal{O}(\varepsilon^2)$. This means that we may use the second-order rotation equations of the

previous subsection but with all $\mathcal{O}(\varepsilon)$ quantities, namely S and ω , set to zero. It is also convenient to decouple the K' and H_0' and just consider the resulting equation for H_0'' . Decoupling Eqs. (B10) and (B11) and substituting H_2 from Eq. (B9), we find

$$\frac{d^2 H_0}{dr^2} = \frac{1}{c_0} \left(c_1 \frac{dH_0}{dr} + c_2 H_0 \right), \tag{B12}$$

where

$$\begin{aligned}
c_0 &= -r^2(r-2M)c_a^2(c_a-2) \left\{ \sqrt{M(c_a-2) + 4r^3\pi p_0 c_a + r((c_a-4)M + 4r^3\pi p_0 c_a + 2r)} \sqrt{r-2M} \right. \\
&\quad \left. - \frac{r-2M}{2} \left[\left(\frac{c_a}{4} - \frac{1}{2} \right) M + r^3\pi p_0 c_a + \frac{r}{4} \right] \right\}, \\
c_1 &= 64rc_a^2 \left[\left(-\frac{c_a}{8} + \frac{1}{4} \right) M + r \left(\pi c_a r^2 - \frac{\pi p_0 r^2}{2} + \frac{\pi \rho_0 r^2}{2} + \frac{c_a}{8} - \frac{1}{4} \right) \right] \left\{ -2(r-2M) \left[\left(\frac{c_a}{4} - \frac{1}{2} \right) M + r^3\pi p_0 c_a + \frac{r}{4} \right] \right. \\
&\quad \left. + \frac{1}{4} \sqrt{M(c_a-2) + 4r^3\pi p_0 c_a + r((c_a-4)M + 4r^3\pi p_0 c_a + 2r)} \sqrt{r-2M} \right\}, \\
c_2 &= 128\sqrt{r-2M} \left\{ \frac{(c_a^2 - 16c_a + 32)(c_a - 2)^2 M^2}{16} - \frac{\pi c_a^3 r^4 (r-2M)}{32} \frac{d\rho_0}{dr} + \frac{rM}{2} \left[\pi p_0 c_a r^2 \left(c_a^3 - \frac{105}{8} c_a^2 + \frac{81}{2} c_a - 32 \right) \right. \right. \\
&\quad \left. \left. - \frac{5}{8} \left(\pi \rho_0 c_a^2 r^2 + \frac{3(c_a - 8)(c_a - \frac{8}{3})(c_a - 2)}{20} \right) (c_a - 4) \right] + r^2 \left[\pi^2 c_a^2 r^4 \left(c_a^2 - \frac{25}{4} c_a + 4 \right) p_0^2 - \frac{5r^2 c_a \pi p_0}{4} \left(\pi \rho_0 c_a^2 r^2 \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{3}{20} c_a^3 - \frac{19}{10} c_a^2 + \frac{73}{10} c_a - \frac{32}{5} \right) - 2c_a - \frac{3c_a^3}{32} + \frac{11c_a^2}{16} - \frac{5\pi \rho_0 c_a^2 r^2}{8} + 2 \right] \right\} \sqrt{M(c_a-2) + 4r^3\pi p_0 c_a + r} \\
&\quad + 16(r-2M) \left[\left(\frac{c_a}{4} - \frac{1}{2} \right) M + r^3\pi p_0 c_a + \frac{r}{4} \right] \left\{ r^4 \pi c_a^3 \left(\frac{d\rho_0}{dr} \right) - 8(c_a - 4)(c_a - 2)^2 M \right. \\
&\quad \left. - 32r \left[r^2 \pi p_0 c_a \left(c_a^2 - \frac{41}{8} c_a + 4 \right) - 2c_a - \frac{3c_a^3}{32} + \frac{11c_a^2}{16} - \frac{5\pi \rho_0 c_a^2 r^2}{8} + 2 \right] \right\}. \tag{B13}
\end{aligned}$$

The term $d\rho_0/dr$ is calculated by using the solution for $p_0(r)$ and an EoS. With Eq. (B12) in hand, we can now find its expansion in small c_a . This gives the following expressions for φ_0 and φ_1 in Eq. (60):

$$\begin{aligned}
\varphi_0 &= \frac{1}{r^2(4r^3\pi p_0 + M)(r-2M)^2} \left\{ -16 \left(r^3\pi p_0 - \pi r^3\rho_0 + \frac{1}{2}r - \frac{1}{2}M \right) \left(r^3\pi p_0 + \frac{M}{4} \right) r(r-2M) \left(\frac{dH_0}{dr} \right) \right. \\
&\quad \left. + 64H_0 \left[\frac{r^4\pi(r-2M)^2}{16} \frac{d\rho_0}{dr} + (4r^3\pi p_0 + M) \left[\frac{M^2}{16} + \frac{13r^3\pi p_0 M}{8} + \frac{5\pi r^3\rho_0 M}{8} - \frac{3rM}{16} + r^2 \left(\pi^2 p_0^2 r^4 - \frac{9}{16} \pi p_0 r^2 \right. \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{5}{16} \pi \rho_0 r^2 + \frac{3}{32} \right) \right] \right] \right\}, \\
\varphi_1 &= -24\pi r^4 \left(p_0 + \frac{\rho_0}{3} \right) \left(r^3\pi p_0 + \frac{M}{4} \right) (r-2M)^2 \left(\frac{dH_0}{dr} \right) + 512H_0 \left\{ -\frac{r^4\pi(r-2M)^2}{128} \left(r^3\pi p_0 + \frac{1}{2}r - \frac{3}{4}M \right) \frac{d\rho_0}{dr} \right. \\
&\quad \left. + \left[-\frac{M^3}{64} + \left(\frac{1}{2}r^3\pi p_0 + \frac{5}{16}\pi r^3\rho_0 + \frac{1}{64}r \right) M^2 + \frac{r^4\pi M}{4} \left(\pi p_0^2 r^2 - \frac{7}{4}p_0 - \frac{5}{4}\rho_0 \right) \right. \right. \\
&\quad \left. \left. + r^5\pi \left(r^4\pi^2 p_0^3 + \frac{1}{4}\pi p_0^2 r^2 + \frac{9}{64}p_0 + \frac{5}{64}\rho_0 \right) \right] \times \left(r^3\pi p_0 + \frac{M}{4} \right) \right\}. \tag{B14}
\end{aligned}$$

APPENDIX C: KHROMETRIC GRAVITY LIMIT

Khronometric gravity is another vector-tensor gravity theory that can be thought of as Einstein-aether theory with the additional restriction, at the level of the action, that the aether is hypersurface orthogonal [12]. Even after varying the action, khronometric gravity can be reached from Einstein-aether theory by taking a certain limit in the Einstein-aether coupling constants [49]. In this appendix,

we check our region I neutron star field equations by taking this limit and comparing them with the known field equations in khronometric gravity [30]. To be more specific, we first take the limit $c_\omega \rightarrow \infty$ (holding everything else constant) in the aether equations, Eq. (13), and impose appropriate boundary conditions to obtain a set of constraints. To maintain regular solutions, taking this limit is equivalent to setting terms proportional to c_ω to zero.

Let us first return to the modified Einstein equations. To obtain Eq. (9) from the action in Eq. (1), U^μ was kept

constant while varying with respect to the metric. To successfully take the khronometric gravity limit, however, it is necessary to use the modified Einstein equations where U_μ was kept constant during variation [50]. This leads to an additional term in Eq. (9):

$$E_{\mu\nu} - 2\mathcal{E}_{(\mu}U_{\nu)} = 0. \quad (\text{C1})$$

$$\begin{aligned} \frac{d^2S}{dr^2} = & \frac{1}{r(c_a - 2)c_a(2M - r)} \left\{ 4(c_a - 2)\sqrt{r - 2M}\sqrt{M(c_a - 2) + 4r^3\pi p_0 c_a} + r \left(\frac{dS}{dr} \right) \right. \\ & \left. + \left[16\pi p_0 c_a r^3 \left(c_a - \frac{1}{2} \right) + (2c_a^2 + 4c_a - 16)M + 8r^3\pi p_0 c_a - 4(c_a - 2)r \right] \frac{dS}{dr} - 2c_a(c_a - 2)S \right\}. \quad (\text{C2}) \end{aligned}$$

As we show below, the relevant solution is $S = 0$ (i.e. the one in khronometric gravity) by imposing appropriate boundary conditions for neutron stars.⁵ To do so while keeping our analysis analytically tractable, we work in the post-Minkowskian (weak-field) approximation and consider the exterior and interior regions of neutron stars separately. We impose asymptotic flatness and regularity at the center in the exterior and interior, respectively, and then match the solutions at the surface order by order.

Now, we shall describe the details of this analysis and show that S vanishes by induction. To carry this out, we expand the field in orders of stellar compactness $\mathcal{C} = M_\star/R_\star$, where M_\star is the stellar mass and R_\star is the stellar radius. This gives the expansion

$$S = \sum_{i=0} s_i(r) \mathcal{C}^i. \quad (\text{C3})$$

Here, \mathcal{C} is a book-keeping parameter that denotes order of compactness. We note that the background fields enter at the following orders of compactness:

$$M(r) = \mathcal{O}(\mathcal{C}^0), \quad \nu(r) = \mathcal{O}(\mathcal{C}^0), \quad p_0 = \mathcal{O}(\mathcal{C}^2). \quad (\text{C4})$$

Let us first study the leading order. Using the above *Ansatz*, we find that the field equation at $\mathcal{O}(\mathcal{C}^0)$ for both the interior and exterior is given by

$$s_0'' = \frac{2s_0}{r}, \quad (\text{C5})$$

which yields the general solution

$$s_0(r) = \frac{A_0}{r} + B_0 r^2, \quad (\text{C6})$$

With this in mind, we next substitute the constraints obtained from the aether equations into Eq. (C1) and then take $c_\omega \rightarrow \infty$. The field equations one obtains from this limit are the khronometric gravity field equations.

We next discuss the constraint equation from the aether equations. At first order in spin, we take the $c_\omega \rightarrow \infty$ limit in $\mathcal{E}_\phi = 0$, which gives

for some integration constants A_0 and B_0 . Imposing regularity at the center for the interior and asymptotic flatness in the exterior yields the solution

$$s_0^{(\text{int})}(r) = B_0 r^2, \quad s_0^{(\text{ext})}(r) = \frac{A_0}{r}. \quad (\text{C7})$$

We also impose that this function is continuous and smooth, giving the conditions

$$s_0^{(\text{int})}(R_\star) = s_0^{(\text{ext})}(R_\star), \quad s_0'^{(\text{int})}(R_\star) = s_0'^{(\text{ext})}(R_\star), \quad (\text{C8})$$

yielding $A_0 = B_0 = 0$.

We will now carry out a proof by induction to conclude that S vanishes in the khronometric limit. Above, we showed the base case of $i = 0$, so now we assume that this holds for $i \leq n - 1$. Then, at $\mathcal{O}(\mathcal{C}^n)$ we find the solutions and boundary conditions

$$s_n^{(\text{int})}(r) = B_n r^2, \quad s_n^{(\text{ext})}(r) = \frac{A_n}{r}. \quad (\text{C9})$$

$$s_n^{(\text{int})}(R_\star) = s_n^{(\text{ext})}(R_\star), \quad s_n'^{(\text{int})}(R_\star) = s_n'^{(\text{ext})}(R_\star). \quad (\text{C10})$$

Upon imposing the boundary conditions, we find $B_n = A_n = 0$. Therefore, $S = 0$ by induction. We note that no equation of state was needed since the matter terms couple to lower orders of s_i in the field equations at each order. Thus, each order was described by a homogeneous equation whose only solution is $s_i = 0$. This proves that the relevant solution for us is $S = 0$.

We note that the condition $S = 0$ comes from the extra constraint in khronometric gravity that the aether must be hypersurface orthogonal, which is equivalent to requiring that the twist, defined in Eq. (5), vanishes [49]. Using our aether and metric forms defined in Eqs. (22) and (23) [which were also used for khronometric gravity in [30] up to the redefinition of W in Eq. (20)], the twist up to $\mathcal{O}(\epsilon^2)$ is linear in either S or W , meaning that vanishing twist is

⁵This is similar to the example of the rotating black hole in [49], where the correct khronometric limit was recovered only after imposing the correct boundary condition (asymptotic flatness).

equivalent to $S = 0$ and $W = 0$. By simply setting $S = 0$ in Eqs. (B7), (B9), (B10), and (B11), we recover the correct equations for ω , H_2 , K , and H_0 in khronometric gravity [30].

Since S vanishes in region II, the field equations in region II (Appendix A) also reach khronometric gravity in the $c_\omega \rightarrow \infty$ limit.

APPENDIX D: OFF-DIAGONAL PERTURBATIONS

In this appendix, we briefly study the $\mathcal{O}(\varepsilon^2)$, off-diagonal perturbations $\{W, V, H_1\}$ in region I in the tidal case. Although this sector is not relevant for the I-Love-Q quantities, these perturbations may be used to define shift and vector Love numbers in Einstein-aether theory in a similar fashion as done in khronometric gravity [30].

From $E_{rr} = 0$, we first find an algebraic equation for $W(r)$:

$$W(r) = \frac{1}{c_\omega r} \left[r e^{-\nu} H_1 + 4V \sqrt{\frac{(c_a - 2)M + 4\pi p_0 c_a r^3 + r}{r - 2M}} - 4V + r \frac{dV}{dr} \right]. \quad (\text{D1})$$

Substituting this equation for $W(r)$ into $\mathcal{A}_\theta = 0$ and $\mathcal{A}_r = 0$ allows us to then find coupled, second-order

equations for $H_1(r)$ and $V(r)$. These equations are given in a supplemental *Mathematica* notebook [51].

We find the behavior near $r = 0$ in the usual way:

$$H_1^{(\text{int})} = Ar + Br^3 + \psi_5(A, B)r^5 + \mathcal{O}(r^7), \quad (\text{D2})$$

$$V^{(\text{int})} = \chi_2(A)r^2 + \chi_4(A, B)r^4 + \mathcal{O}(r^6), \quad (\text{D3})$$

where A and B are integration constants. The coefficients ψ_5 , χ_2 , and χ_4 are functions of A and B and are given in the supplemental *Mathematica* notebook [51]. For the buffer zone, we find the solutions using series *Ansätze* in M_\star :

$$H_1^{(\text{buf})} = \eta_0 + \eta_1 M_\star + \eta_2 M_\star^2 + \mathcal{O}(M_\star^3), \quad (\text{D4})$$

$$V^{(\text{buf})} = \lambda_0 + \lambda_1 M_\star + \lambda_2 M_\star^2 + \mathcal{O}(M_\star^3). \quad (\text{D5})$$

Expressions for a few orders of η_i and λ_i are also given in the supplemental notebook. We note that the λ_i here are simply expansion coefficients, not to be confused with the tidal Love number studied in the main text. The exterior integration constants $\{C, D, F, G\}$ can be summarized from λ_0 :

$$\lambda_0 = Cr^4 + Dr^2 + \frac{F}{r} + \frac{G}{r^3}. \quad (\text{D6})$$

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