Singularity at the demise of a black hole

Justin C. Feng[*]

Leung Center for Cosmology and Particle Astrophysics, National Taiwan University, No. 1 Sec. 4, Roosevelt Rd., Taipei 10617, Taiwan, Republic of China

Shinji Mukohyama

Center for Gravitational Physics and Quantum Information (CGPQI), Yukawa Institute for Theoretical Physics (YITP), Kyoto University, Kyoto 606-8502, Japan and Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo Institutes for Advanced Study, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan

Sante Carloni

Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, Prague, V Holešovičkách 2, 180 00 Prague 8, Czech Republic and DIME, Università di Genova, Via all'Opera Pia 15, 16145 Genova, Italy; and INFN Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy

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We consider a class of quasiregular singularities characterized by points possessing two future-directed light cones and two past-directed light cones. Such singularities appear in the $1 + 1$ trousers spacetime and the Deutsch-Politzer spacetime. We argue that these singularities are relevant for describing the end point of an evaporating black hole and show that a class of emergent Lorentz signature theories can provide a microscopic description for these singularities.

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I. INTRODUCTION

About half a century ago, Stephen Hawking considered quantum effects in matter fields on a black hole spacetime, concluding that black holes radiate [1,2]. If black holes radiate and lose mass (i.e., they evaporate), one is led to the question: what is the outcome of black hole evaporation? At least three possible answers have been proposed in the literature [3]: (1) a black hole remnant, (2) a naked singularity [4–7], and (3) a complete evaporation of the black hole to a region of flat spacetime (see, for instance, [8]). The remnant scenario has been studied extensively in literature (see the review [3] and references therein). In this article, we focus on the latter two, which need not be mutually exclusive—in particular, the nature of the naked singularity, which we show to be closely related to so-called "crotch singularity" of the trousers spacetime [9] in the $1 + 1$ dimensional setting, and a scenario in which the singularity is regularized by a candidate ultraviolet (UV) completion for quantum gravity.

Quasiregular singularities [10] are singularities on codimension two surfaces, which may be regarded as generalizations of conical singularities that, in addition to angle surpluses and deficits, include surpluses and deficits in boosts, as well as dislocations [11] and additional light cones (cf. Fig. 3 of [10]); a further generalization can be found in [12]. The latter constitutes a radical change to the causal structure of spacetime on a surface of codimension two—such points are termed *local causal discontinuities* [13], in which the local past and future light cones of an instantaneous observer fail to deform continuously as the observer is displaced in the manifold; the structure of these types of causal discontinuities have been studied in [14–17]. The best-known example of such a singularity is the crotch singularity of the $1 + 1$ trousers spacetime [9] featuring two future-directed light cones and two past-directed light cones. The $1 + 1$ trousers spacetime provides a simple model for studying the effects of topology change on quantum fields and in quantum gravity [9,18–24]. Another example is the singularity present in the Deutsch-Politzer spacetime [25,26] (and the formally homeomorphic teleporter spacetimes [27]), which have a similar causal structure to that of the trousers spacetime at the singular surfaces. The reader can find a discussion of the geometric and topological properties of the Deutsch-Politzer spacetime in [28–30].

Though the study of quasiregular singularities is primarily motivated by topology change in quantum gravity,

[*]jcfeng@ntu.edu.tw

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and a case for studying such singularities was made in [31], interest in the topic has mainly focused on the implications of similar causal structures within the causal set approach to quantum gravity [23,32–35]. Here, we will attempt to bring a sense of urgency to the study of quasiregular singularities by making the case that if no mechanism stops the continued evaporation of a black hole (such as that arising from the generalized uncertainty principle [36], for instance) then one would expect the disappearance of a black hole event horizon at the end point of black hole evaporation. The end state of the black hole will contain a local causal discontinuity closely related to those of quasiregular singularities—in fact, we demonstrate explicitly that in the $1+1$ case, the end point of a black hole that evaporates completely contains a quasiregular singularity of the same type as that of crotch singularity in the trousers spacetime, provided that the Schwarzschild singularity is regularized so that the resulting manifold is Lorentzian everywhere except for the singular point.

Of course, one might expect a microscopic description of causal discontinuities and quasiregular singularities to require a UV completion for quantum gravity—indeed, as the reader can see in the papers referenced thus far, such a possibility has been explored in some detail in the causal set approach to quantum gravity. Here, we will consider a model motivated by a more conservative candidate for a UV completion. The candidate in question is a quadratic curvature, shift-symmetric Euclidean scalar-tensor theory proposed a decade ago in [37]. This theory was shown to be renormalizable [38] and evades the Ostrogradsky instability (one only requires the action to be bounded below since the theory is fundamentally Euclidean, as discussed in [37]). In the long-distance limit, one may recover an effective Lorentz signature metric with an appropriate coupling to the matter degrees of freedom (see Refs. [39,40] for a detailed discussion of this mechanism). In this theory, a scalar field with a saddle profile can mimic the long-distance features of a quasiregular singularity, and we will show that there exists an exact saddle solution for a particular set of parameter choices, and an approximate saddle solution in the vicinity of a point for generic choices of parameters.

This article is organized as follows. In Sec. II, we discuss the singularities of the trousers spacetime, the relationship between the trousers spacetime, and an evaporating $1+1$ black hole. In Sec. III, we review the theory of [37] and obtain solutions that regularize the quasiregular singularity at the end point of an evaporating black hole (assuming an either Lorentzian or Euclidean regularization of the Schwarzschild singularity), and in Sec. IV, we discuss implications and future directions.

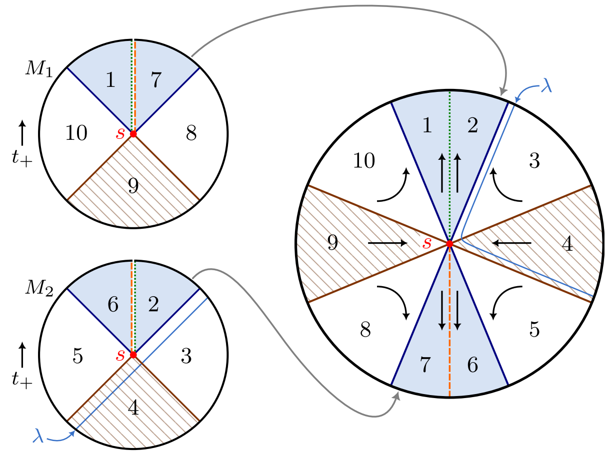

FIG. 1. Cut-and-paste procedure for the construction of an SCDS. On the left are illustrations of two regions of $1+1$ Minkowski spacetime M_1 and M_2. A caricature of the result is on the right (in which angles are *not* preserved—null directions on the right are *not* 45° lines, as illustrated by the behavior of the null line λ). The solid black arrows indicate the positive time direction in each region.

II. SADDLELIKE SINGULARITIES AND BLACK HOLE EVAPORATION

A. Construction

The simplest causally discontinuous quasiregular singularity can be constructed by performing a simple cut-and-paste procedure [41] in two regions of $1+1$ Minkowski space, which we call M_1 and M_2 [each equipped with coordinates (x,t)], as indicated in Fig. 1; one can find similar diagrams in [10] and [43] [cf. Fig. 4(e) in the former and Figs. 17 and 18 in the latter]. In the regions M_1 and M_2 on the left side of Fig. 1, one performs a cut along the positive time axis indicated by the dotted lines, and are glued so that region 1 is glued to region 2 and region 6 is glued to region 7. Regions 1,2 and 6,7 correspond to future light cones of s, and regions 9 and 4 correspond to past light cones of s. A caricature of the result is on the right of Fig. 1. As one can infer, the origin s is causally discontinuous, as points in the neighborhood of s other than s itself have one future-directed light cone and one past-directed light cone, while the point s has two future-directed and two past-directed light cones. From this point, we will refer to a naked singularity with such a causal structure as a *saddlelike causally discontinuous singularity* (SCDS); in particular, we define a SCDS to be a singular surface s with a spacelike quasiregular singularity (as defined in [10]) such that each point on s has two opposed future-directed light cones and two opposed past-directed light cones. The name refers the fact that the surfaces of constant t behave as contours of a saddle, as one might infer from the direction of increasing time indicated on the right-hand side of Fig. 1.

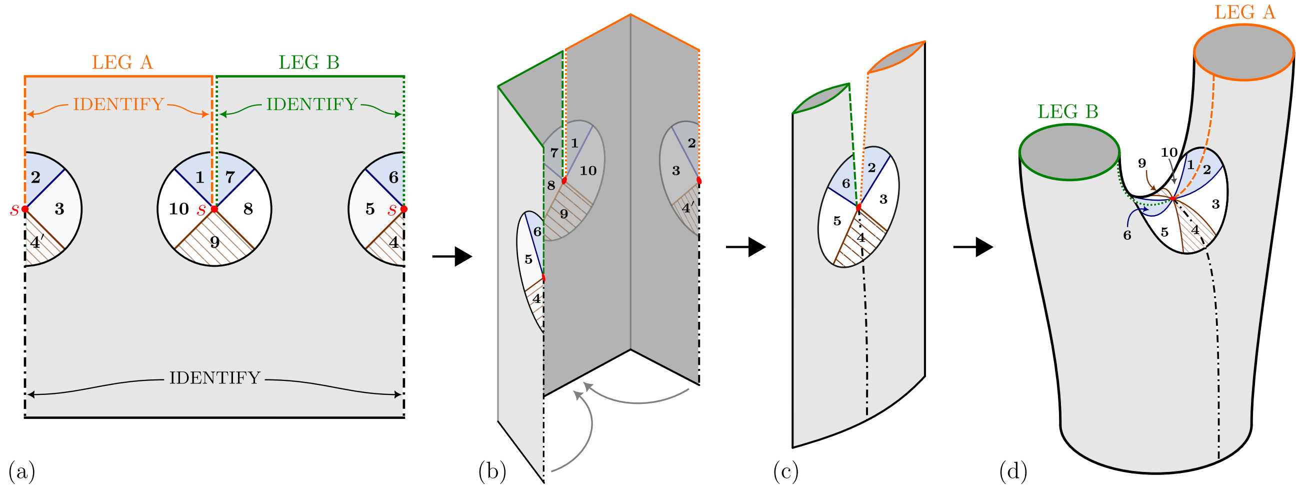

FIG. 2. (a) depicts a flat $1+1$ trousers spacetime and the neighborhood N_s of the crotch singularity s, with the time direction oriented vertically. (b) and (c), respectively, illustrate a folding and gluing procedure to construct a paper model of a $1+1$ trousers spacetime [observe that regions $4'$ and 4 of (a) and (b) have been combined into a single region 4 here to facilitate comparison with Fig. 1]. (d) illustrates a homeomorphic deformation of the $1+1$ trousers spacetime to illustrate the neighborhood N_s more clearly. In all diagrams, the solid shaded regions of N_s indicate the future light cones of s, and the crosshatched regions indicate the past light cones of s.

B. Trousers spacetime and black hole evaporation

The $1+1$ trousers spacetime provides a simple example of topology change and contains a "crotch" singularity s featuring two distinct future-directed light cones and two distinct past-directed light cones, making s an SCDS. Figure 2 illustrates an example of a trousers spacetime, and the SCDS s. Figures 2(a) through 2(c) illustrates how one can construct a trousers spacetime from appropriate identifications in $1+1$ Minkowski spacetime [9], and Fig. 2(d) is a homeomorphic deformation included to clearly illustrate the neighborhood of the singular point (cf. Fig. 3 of [34]).

We offer here a physical motivation for considering the $1+1$ trousers spacetime and its SCDS s in detail. In particular, we point to the observation that the full conformal diagram for a $1+1$ evaporating Schwarzschild-like black hole can be pasted into the region indicated in Fig. 3(b) on the flat trousers spacetime [44], demonstrating that up to the singularities, such a $1+1$ black hole is conformal to a subset of the flat trousers spacetime. One

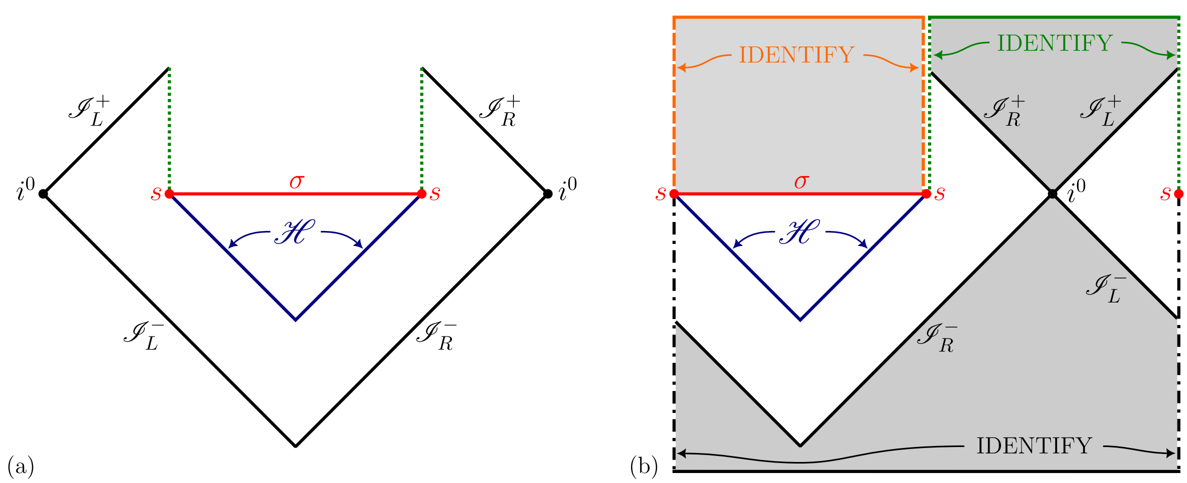

FIG. 3. A full conformal diagram describing the formation and evaporation of a $1+1$ Schwarzschild-like black hole (a), accompanied by an illustration demonstrating a conformal embedding of the $1+1$ evaporating black hole in a region of the $1+1$ trousers spacetime (b). In the $1+1$ case, the respective future and past null infinity $\mathcal{I}^+$ and $\mathcal{I}^-$ are split into two distinct lines distinguished by the subscripts L and R. The lines labeled $\mathcal{H}$ represent the horizon and σ represents the classical singularity, which we suppose is regularized by quantum gravity effects. In (b), the dotted lines are identified, as are the points labeled s and i^0, which are the evaporation point and spatial infinity, respectively.

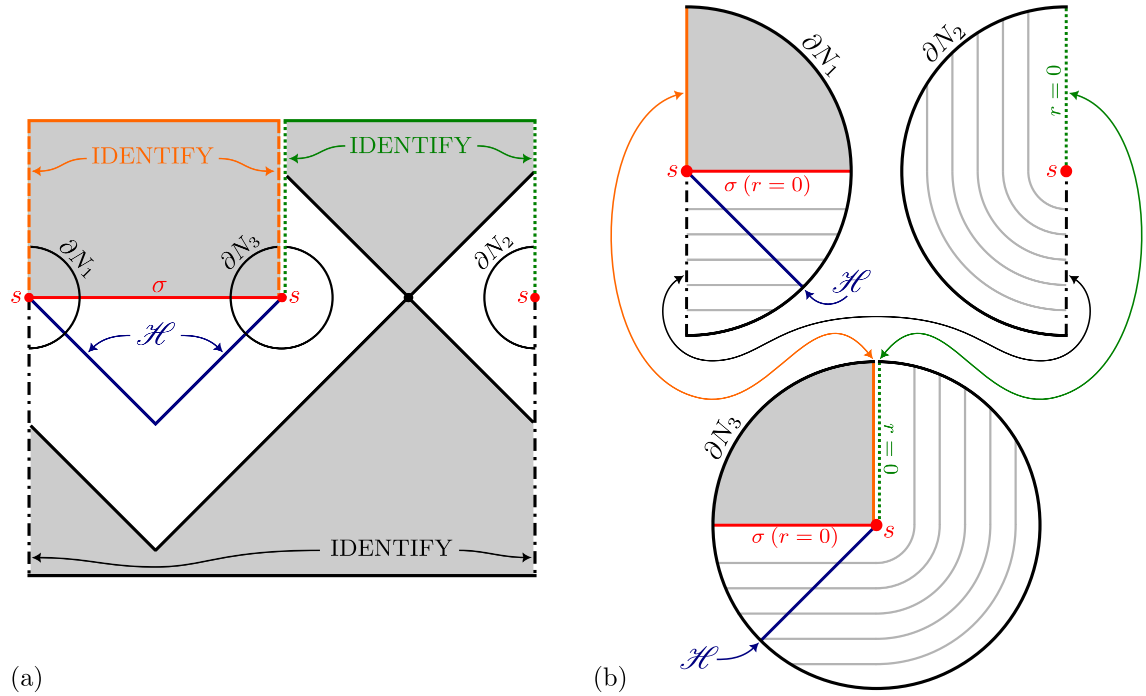

FIG. 4. Neighborhood of a $d + 1$ dimensional black hole end point s on a surface Σ_Ω. In (a), the neighborhood of s in the conformal diagram [cf. Fig. 3(b)], is the region enclosed by the boundaries ∂N_i, $i \in \{1, 2, 3\}$. (b) is a detailed diagram of the neighborhood (identifications indicated by arrows) with the gray contours describing surfaces of constant areal radius $r > 0$.

finds that the singular point s coincides with the SCDS of a trousers spacetime. This is a rather suggestive result, as one might imagine that even if one has in hand a $1 + 1$ Lorentzian quantum gravity theory that regularizes the usual classical Schwarzschild singularity σ, the external causal structure of an evaporating black hole requires the presence of an SCDS at s. If one demands a regularization of the Schwarzschild singularity σ, Fig. 3(b) suggests that upon regularization, the Schwarzschild singularity will be replaced by a bounce and formation of a new universe [46], and from the results in [9,18], one might also expect some rather extreme behavior (a "thunderbolt" [7,48,49]) arising from the interaction between quantum fields and the SCDS at s, though it has been argued that this problem can be avoided [21,50]. Of course, one can avoid these issues outright by forbidding topology change, fixing the topology of spacetime from the outset by fiat, and in such an approach, one must either accept that the end state of a black hole is a remnant or discard the notion of a global black hole event horizon altogether (which would likely require an effective violation of energy conditions to avoid singularities); though these are interesting scenarios, we shall not consider them further in this article.

The reader might wonder about the extension of the aforementioned discussion to the evaporation of a spherically symmetric $d + 1$ dimensional black hole. One might proceed in the $d + 1$ dimensional case by reinterpreting the conformal diagram in Fig. 3(a) as representing a surface Σ_Ω of codimension two with two sets of antipodally related angular values—that is, points reflected about the vertical axis of symmetry represent antipodal points. The evaporation point s then forms an SCDS in the surface Σ_Ω as one might infer from Fig. 4(a). Of course, the reader may wish to gain a more complete understanding of the nature of the SCDS at the evaporation end point of a spherically symmetric black hole and the properties of its neighborhood. Though a full discussion is well beyond the scope of this article, we offer a first step in that direction. In particular, one may think of the areal radius r as defining a scalar field $r(x)$ [x being a point on the manifold], then consider possible contours of constant $r > 0$ in the vicinity of an SCDS in Σ_Ω. Points in Fig. 4 represent $d - 1$ dimensional spheres, and Fig. 4(b) illustrates possible constant $r > 0$ contours in a neighborhood N_s of the evaporation point s for a $d + 1$ dimensional spherically symmetric evaporating black hole. If σ is regularized, it may no longer correspond to an $r = 0$ surface, and the green dotted $r = 0$ contour represents a "valley" in the scalar field profile $r(x)$ that ends at the point s.

C. Time machines and teleporters

Before proceeding, it is perhaps appropriate to remark on the implications of SCDSs, if they are permitted to exist. One would be the formation of Deutsch-Politzer spacetimes

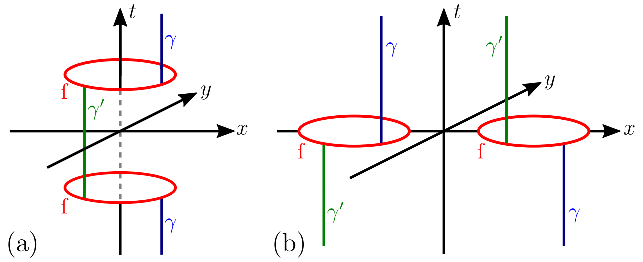

FIG. 5. (a) is an illustration of a $2+1$ Deutsch-Politzer spacetime, and (b) is an illustration of a $2+1$ teleporter spacetime. In each case, the spacetimes are assumed to be flat everywhere except the surface $\mathfrak{f}$, which contain a SCDS. Both γ and γ' are timelike curves [note that γ' in (a) is a closed timelike curve].

containing closed timelike curves [25,26], and another would be teleporter spacetimes, which feature dislocations that can instantaneously (that is, along a spacelike hypersurface) "transport" timelike geodesics across large distances [27]. One can easily construct such spacetimes from a cut-and-paste procedure in Minkowski spacetime. An illustration in the $2+1$ case is provided in Fig. 5. The cut-and-paste procedure is performed along circular disks on constant t planes such that the curve γ passes through the disks in the manner indicated in Fig. 5. More generally, the gluing in each case is performed to obtain a spacetime with topology $\mathbb{S}^2 \times \mathbb{S}^1 - \{0\}$ (which generalizes to $\mathbb{S}^d \times \mathbb{S}^1 - \{0\}$ in the $d+1$ dimensional case); the difference between the Deutsch-Politzer and teleporter spacetimes being the presence of closed timelike curves in the former. Teleporter spacetimes are defined so that all pairs of points on the surface $\mathfrak{f}$ are spacelike separated. Since such structures have not yet been observed in nature, a theory which admits SCDSs should provide some mechanism to suppress these types of structures at macroscopic scales.

III. REGULARIZED SCDS IN EMERGENT LORENTZ SIGNATURE THEORIES

Up to this point, we have considered the nature of SCDS at the end point of an evaporating black hole. One might expect SCDSs to be ultimately described by some UV completion for quantum gravity or some appropriate modified gravity theory. In this section, we will consider one such theory [37], showing that it yields a regularization for SCDS even at the classical level.

A. Emergent Lorentz signature theories

We consider a class of theories described in [39], which we will henceforth refer to as *emergent Lorentz signature theories* (ELSTs). ELSTs are scalar-tensor theories constructed from a Euclidean signature metric g_{ab} and a scalar φ so that one has an effective metric $\mathbf{g}_{ab}$ (and inverse $\bar{\mathbf{g}}^{ab}$) of the form,

$$\mathbf{g}_{ab} = g_{ab} - \frac{\nabla_a\varphi\nabla_b\varphi}{X_C}, \qquad \bar{\mathbf{g}}^{ab} = g^{ab} - \frac{\nabla^a\varphi\nabla^b\varphi}{X - X_C}, \tag{1}$$

where X_C is a positive constant that appears in the matter couplings, $X \coloneqq \nabla^a\varphi\nabla_a\varphi$, and we employ the convention that indices are raised and lowered with the Euclidean signature metric g_{ab}. The proposal in ELSTs is that in the long-distance limit, matter couples to the metric g_{ab} and scalar φ degrees of freedom exclusively through the effective metric $\mathbf{g}_{ab}$, which has a Lorentzian signature for sufficiently large scalar field gradients [51]. In this manner, Lorentz signature emerges through matter couplings, as described in [39,40].

In [37], an ELST is proposed on a four-dimensional manifold M in which the action has the following form:

$$S = \int_M d^4x\sqrt{|g|}L, \qquad L \coloneqq L_0 + L_2 + L_4, \tag{2}$$

where (defining $\varphi_a \coloneqq \nabla_a\varphi$, $\varphi_{ab} \coloneqq \nabla_a\nabla_b\varphi$)

$$\begin{aligned}
L_0 &\coloneqq c_{11}, \\
L_2 &\coloneqq c_9 R + c_{10} X, \\
L_4 &\coloneqq c_1 R^2 + c_2 R_{ab}R^{ab} + c_3 R_{abcd}R^{abcd} + c_4 XR \\
&\quad + c_5 R^{ab}\varphi_a\varphi_b + c_6 X^2 + c_7(\Box^2\varphi)^2 + c_8\varphi_{ab}\varphi^{ab}.
\end{aligned} \tag{3}$$

The couplings c_I ($I \in \{1,\ldots,11\}$) can be chosen so that the action is bounded below. The action is constructed so that it is shift symmetric and invariant under $\varphi \to -\varphi$. Higher derivatives of the scalar φ and quadratic curvature terms are introduced so that the theory is power-counting renormalizable [52]; it was later shown in [38] that the theory defined by S is indeed perturbatively (super-) renormalizable in three and four dimensions. In the long-distance limit, the theory reduces to a Lorentzian scalar-tensor theory [37]. The renormalizability of the resulting theory and the boundedness of the action below (since the theory is fundamentally Euclidean the latter allows it to evade the Ostrogradsky instability and its associated ghosts, a pathology that would plague a Lorentzian higher derivative theory [53,54]) makes it a potential candidate as a UV completion for quantum gravity. Moreover, the renormalizability of the theory makes it useful as an effective field theory even if the UV completion takes a radically different form, so long as it retains symmetries that reduce to the shift symmetry, diffeomorphism invariance and local $SO(4)$ invariance seen in the action described by Eqs. (2) and (3). Of course, while this discussion is motivated by quantum gravitational considerations, for simplicity, we shall focus on *classical* solutions of the theory.

Since the action S is shift symmetric in φ, the field equation corresponding to variations in φ has the form $\nabla_a(J^a - \Phi^a) = 0$, where Φ^a is the contribution from the matter degrees of freedom and the current J^a is

$$J^a = (c_7 + c_8)\Box\varphi^a - (c_5 + c_7)R^{ab}\varphi_b - \varphi^a(c_{10} + c_4 R + 2c_6 X). \tag{4}$$

The field equation corresponding to variations in g_{ab} takes the form,

$$Q^R_{ab} + Q^\varphi_{ab} + g_{ab}Q = \frac{1}{2}T_{ab} \tag{5}$$

where T_{ab} is the stress tensor for matter degrees of freedom (defined with respect to variations in g_{ab}), and

$$\begin{aligned} Q^R_{ab} &= -2c_9 R_{ab} - 4c_2 R_{ac}R_b{}^c - 4c_1 R_{ab}R - 4c_3 R_a{}^{cde}R_{bcde} + 4(c_2 + 2c_3)(R_{ac}R_b{}^c - R^{cd}R_{acbd}) \\ &\quad + 2(2c_1 + c_2 + 2c_3)\nabla_a\nabla_b R - 2(c_2 + 4c_3)\Box R_{ab}, \\ Q^\varphi_{ab} &= -2c_{10}\varphi_{ab} + 4(c_5 - c_8)\varphi_{ab}\Box\varphi + 4c_4\varphi_{cb}\varphi^c{}_a + 4(c_7 + c_8)(\varphi_{(a}\Box\varphi_{b)}) + 2(2c_4 + c_5 - c_8)\varphi^c\nabla_c\varphi_{ab} \\ &\quad - 2c_4(R\varphi_a\varphi_b + R_{ab}X - 2R_{acbd}\varphi^c\varphi^d) - 4c_6\varphi_a\varphi_b X - 4\varphi^c(c_5 + c_7)R_{c(a}\varphi_{b)}, \\ Q &= L - (c_5 + 2c_7)(\Box\varphi)^2 + (c_5 + 2c_7)R^{cd}\varphi_c\varphi_d - (4c_1 + c_2)\Box R - 2(2c_4 + c_5 + c_7)\varphi^c\Box\varphi_c \\ &\quad - (4c_4 + c_5)\varphi_{cd}\varphi^{cd}. \end{aligned} \tag{6}$$

Combined with $\nabla_a(J^a - \Phi^a) = 0$, Eqs. (4)–(6) yield the field equations for the action given in Eqs. (2) and (3). These expressions were obtained using the *xTras* add-on of the *xAct* package for *Mathematica* [55].

B. Microscopic model for SCDSs

We now ask whether the field equations admit regular solutions and resemble SCDSs at large distances. Since the gradient of the scalar field φ_a corresponds to a timelike direction in the Lorentzian phase, one can treat φ as a time coordinate; from the direction of time indicated in Fig. 1, one may infer that the level surfaces of φ for a regularized SCDS should resemble those of a saddle, as illustrated in Fig. 6. If one chooses $g_{\mu\nu} = \mathrm{diag}(1,1,1,1)$ with coordinates (u, v, y, z), then a simple saddle profile for φ is a harmonic one of the form,

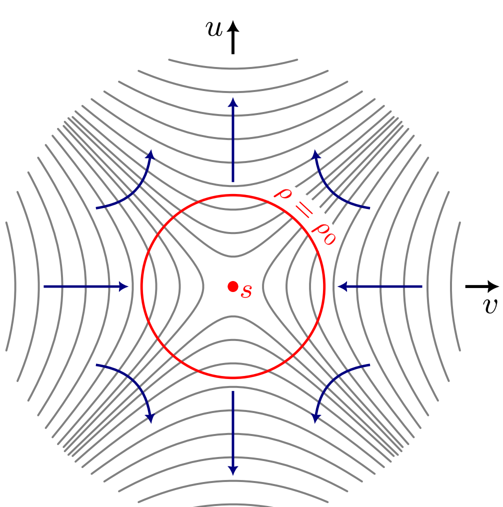

FIG. 6. Contour plot for scalar field near a saddle point s in the u-v plane. The solid blue arrows indicate the direction of increasing scalar field φ, and the gray contours indicate level surfaces of the scalar field profile in Eq. (7). The circle represents the surface of degenerate $\mathbf{g}_{\mu\nu}$ dividing regions where $\mathbf{g}_{\mu\nu}$ has Euclidean signature $\rho < \rho_0$ from Lorentzian signature regions $\rho > \rho_0$.

$$\varphi = (u^2 - v^2)/(2l_0), \tag{7}$$

which for the flat metric satisfies $\Box\varphi = 0$, with $\varphi_{\mu\nu}$ being constant. The contours of φ are illustrated in Fig. 6. One finds that the Ricci tensor for the effective metric $\mathbf{g}_{\mu\nu}$ vanishes for large $\rho := \sqrt{u^2 + v^2}$, and that $\mathbf{g}_{\mu\nu}$ becomes degenerate for $\rho = \rho_0 := \sqrt{X_C}l_0$, with Euclidean signature for $\rho < \rho_0$ and Lorentzian signature for $\rho > \rho_0$. It should be remarked that the singularity has only been regularized with regard to the variables $g_{\mu\nu}$ and φ; the effective metric $\mathbf{g}_{\mu\nu}$ becomes singular at the degenerate surface $\rho = \rho_0$, which we refer to as s_d. From the perspective of the effective Lorentzian geometry, s_d is a spacelike curvature singularity (which follows from the vanishing of the determinant of $\mathbf{g}_{ab}$ as the surface s_d is approached). Thus, we have provided here a simple microscopic model for an SCDS in terms of regular quantities; this is the sense in which an ELST regularizes an SCDS.

However, Eq. (7) is only an exact solution of the field equations for flat $g_{\mu\nu}$ for the parameter choice,

$$c_4 = c_6 = c_{10} = 0, \qquad l_0^2 = (c_5 - c_8)/c_{11}. \tag{8}$$

This parameter choice is potentially problematic, as one might wish to have some dynamical mechanism that drives X to a finite value in the long-distance limit and a simple way to achieve this is by adjusting the parameters c_6 and

c_{10} so that the corresponding terms form an effective potential for X. Setting $c_6 = c_{10} = 0$ excludes this possibility. It is nonetheless important to note that the analysis here at the very least provides a proof-of-concept that the microscopic resolution of SCDSs is possible in the context of ELSTs.

If one wishes to avoid such parameter restrictions, it is still possible to construct from Eq. (7) a class of approximate solutions in the neighborhood of the origin for general parameter choices using the Riemann normal coordinate expansion, in which the metric takes the following form [56]:

$$g_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{3}[R_{\mu\alpha\nu\beta}]_0 x^\alpha x^\beta - \frac{1}{6}[\nabla_\gamma R_{\mu\alpha\nu\beta}]_0 x^\alpha x^\beta x^\gamma - \left[\frac{2}{45}R_{\mu\alpha\lambda\beta}R^\lambda{}_{\gamma\delta\nu} + \frac{1}{20}\nabla_\gamma\nabla_\delta R_{\mu\alpha\nu\beta}\right]_0 x^\alpha x^\beta x^\gamma x^\delta + O(x^{\cdot 5}), \tag{9}$$

where $[\cdot]_0$ indicates evaluation at the origin $x^\mu = 0$. Upon evaluating the field equations at the origin, one finds that for Eq. (7) $[\Box\varphi_a]_0 = [\varphi_a]_0 = 0$, and it follows that $[J^a]_0 = 0$. With the addition of an $O(x^{\cdot 3})$ term to Eq. (7), one can choose the corresponding coefficient to satisfy the field equation $[\nabla_a(J^a - \Phi^a)]_0 = 0$, so that the scalar equations are satisfied at the origin. From the gravity equations, the coefficients have to satisfy

$$\begin{aligned}
&[4\{(c_2+4c_3)\Box R_{\mu\nu} - (2c_1+c_2+2c_3)\nabla_\mu\nabla_\nu R\} + 4c_{10}\varphi_{\mu\nu} \\
&\quad + 8\{c_3 R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma} - (c_2+2c_3)(R_{\mu\alpha}R_\nu{}^\alpha - R^{\alpha\beta}R_{\mu\alpha\nu\beta})\} \\
&\quad + 8\{c_2 R_{\mu\alpha}R_\nu{}^\alpha - c_4\varphi_{\alpha\mu}\varphi^\alpha{}_\nu\} + 4(c_9 + 2c_1 R)R_{\mu\nu} \\
&\quad + 2g_{\mu\nu}\{(4c_1+c_2)\Box R - c_{11} - c_9 R - c_1 R^2 - c_2 R_{\alpha\beta}R^{\alpha\beta} \\
&\quad - c_3 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - (c_8 - 4c_4 - c_5)\varphi_{\alpha\beta}\varphi^{\alpha\beta}\} + T_{\mu\nu}]_0 \\
&= 0.
\end{aligned} \tag{10}$$

Now since $(\nabla\varphi)^2/X_C \sim (x^\cdot)^2/\rho_0^2$, one may require that the scale of the curvature satisfies $[R_{\mu\alpha\nu\beta}]_0 \ll 1/\rho_0^2$ within some neighborhood of the origin $x^\mu = 0$, and in the same neighborhood, the effective metric $\mathbf{g}_{\mu\nu}$ has Euclidean signature for $\rho \lesssim \rho_0$ and Lorentzian signature for $\rho \gtrsim \rho_0$. This can in principle be done in a neighborhood where $O(x^{\cdot 2})$ terms are dominant, since the derivatives $[\Box R_{\mu\nu}]_0$ can be chosen independently of $[R_{\mu\alpha\nu\beta}]_0$. Of course, the conditions under which the solution continues to resemble an SCDS requires a more complete solution, which will be considered elsewhere.

As we remarked earlier, if a theory admits regularized SCDS solutions, it should also provide a mechanism for suppressing topological structures such as those arising in Deutsch-Politzer and teleporter spacetimes. Here, we briefly describe one possible mechanism for the ELST we have considered here. The action defined in Eqs. (2) and (3), being quadratic in the curvature, contains a topological term, which yields the Euler characteristic of the integration domain and would contribute a large change in the value of the action for such structures compared to a simply connected domain. Moreover, one might expect Deutsch-Politzer or teleporter structures to lower the Euler characteristic (such structures would, for instance, correspond to an increase in genus for a closed manifold). Such topological configurations may be suppressed, provided that the coefficient in front of the Euler characteristic in the action is properly chosen and that the SCDS is regularized as prescribed above.

IV. SUMMARY AND DISCUSSION

In this article, we considered the end point of an evaporating $1+1$ black hole. We have shown how given a theory that regularizes the Schwarzschild singularity in the Lorentzian regime, the end state of an evaporating $1+1$ black hole can be characterized by a particular type of quasiregular singularity, which we have termed a saddlelike causally discontinuous singularity (SCDS). We have described how such singularities can be described in terms of regular quantities within the framework of emergent Lorentz signature theories (ELSTs) [39,40], in which the notion of time and Lorentz signature emerge from a Euclidean scalar-tensor theory. In particular, we consider the Euclidean signature quadratic curvature scalar-tensor theory of [37], which evades the Ostrogradsky instability (as the theory is fundamentally Euclidean) and is perturbatively renormalizable [38]. In particular, we showed that the classical field equations for the purely gravitational part of the aforementioned ELST admits an exact solution corresponding to a regularization of SCDSs for a particular choice of parameters, and for more general parameter choices, a class of approximate solutions in the neighborhood of a saddle point in the scalar field profile can be obtained. These results provide a proof-of-concept that the microscopic resolution of SCDSs is possible in the context of ELSTs. In addition to their utility for describing the end point of black hole evaporation and topological structures such as Deutsch-Politzer and teleporter spacetimes, the solutions we have presented may be of use for regularizing other types of naked singularities [57,58].

Of course, there are plenty of issues to resolve before our main proposals, namely that the solutions we have found describe SCDSs at long-distance scales and that the regularized SCDS solutions describe the end state of an evaporating black hole, can be conclusively established. A resolution of the former will ultimately require a more complete numerical or analytical effort; an appropriate starting point, perhaps, would be an exploration of the solution space neighborhood about the solutions we have considered—in particular, one might begin by performing a linear perturbation analysis (accompanied by an

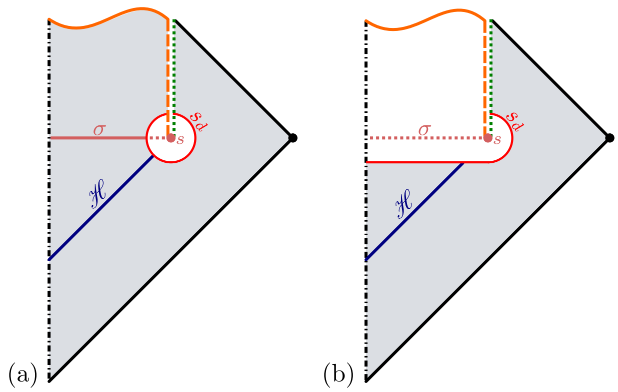

FIG. 7. Possible regularization scenarios for an evaporating black hole in an ELST. As in Figs. 3 and 4, $\mathcal{H}$ is the horizon, and σ and s indicate where the respective Schwarzschild and end point singularities would otherwise be. The effective metric $\mathbf{g}_{ab}$ is Lorentzian in the shaded regions, and the red contour s_d is the surface where $\mathbf{g}_{ab}$ becomes degenerate.

investigation into questions of linearization stability, the matter of whether solutions of the linearized equations are in fact tangent to the solution space for the nonlinear equations).

Whether the field equations of an ELST does indeed regularize the singularities of an evaporating spherically symmetric [59] black hole will also require additional effort. We have only shown that this is a possibility in ELSTs, but a more complete analysis is required to conclusively demonstrate this. The solutions presented in Sec. III assume the Schwarzschild singularity is regularized so that the effective metric is $\mathbf{g}_{ab}$ only has Euclidean signature in a compact neighborhood of the end point s of the black hole; this scenario is illustrated in Fig. 7(a); $\mathbf{g}_{ab}$ is Lorentzian in the shaded region. However, another possibility, illustrated in Fig. 7(b), is that the regularization of both the end point s and Schwarzschild singularity occur in a region where $\mathbf{g}_{ab}$ has Euclidean signature beyond the boundary s_d of region where $\mathbf{g}_{ab}$ has Lorentzian signature. One can easily imagine plenty of other regularization scenarios; for instance, one can imagine "islands" or noncompact regions in Fig. 7(b) below the surface s_d in which the effective metric $\mathbf{g}_{ab}$ has Lorentzian signature, or "islands" of Euclidean signature in Fig. 7(a). In either scenario, such a regularization would yield a "baby universe" resolution [61–63] to the famed black hole information paradox [64] (see also [65] for a discussion of possible resolutions), though the resulting universe need not admit a Lorentz signature metric.

We have only considered one possible scenario in which a candidate ultraviolet completion for quantum gravity can regularize SCDSs (though we emphasize that our analysis is entirely classical). Various approaches to signature change have been extensively explored in the literature [66–108], and from time to time, it has been suggested, based on quantum gravitational considerations, that the metric should be complex valued [22,109–114]; allowability criteria for complex metrics have recently become a topic of interest [115–122]. Another possible avenue to consider may be a generalization or modification of the Callan-Giddings-Harvey-Strominger model [123] or related toy models [124,125] to incorporate signature change or some emergent Lorentz signature mechanism. It may be of interest to consider the regularization of SCDSs in these alternative approaches, as any quantitative or qualitative differences between the different approaches may eventually lead to distinct predictions for any signals from the final stages of black hole evaporation.

Nevertheless, for the scenarios in which the end state of an evaporating black hole is an apparent naked singularity or a region of flat spacetime, the approach based on the ELST we have considered in this paper is of general interest for two reasons. First, the ELST of [37] can be regarded as the perturbatively renormalizable part for a wide class of effective field theories, so it may be relevant even if the ultimate UV completion takes a radically different form. Second, the solutions we have presented are *classical* and may either provide a starting point for a saddle point analysis in the full quantum theory or a semiclassical description for a possible end state of an evaporating black hole.

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