

## Exploring quantum cosmology within the framework of teleparallel $f(T)$ gravity

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We investigate quantum cosmology in teleparallel  $f(T)$  gravity. We delve extensively into the minisuperspace description within the context of teleparallelism. The  $f(T)$  theory constitutes a second-order theory of gravity, whose cosmological counterpart is delineated by a degenerate pointlike Lagrangian. To formulate the Hamiltonian function encompassing all constraints and degrees of freedom inherent to  $f(T)$  cosmology, we employ the Dirac-Bergmann algorithm. Subsequently, we determine the wave function of the universe and introduce a “probabilistic” interpretation. We perform comparisons to some classical solutions to see to what extent the quantum approach can cure classical singularities.

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### I. INTRODUCTION

Classic cosmology describes the large-scale structure and the dynamics of the Universe. Yet, in order to investigate the quantum phenomena that governed the Universe’s nascent phases, the field of quantum cosmology comes into play. The quantum cosmology we know is based on the treatment of reduced versions of classical gravitational systems, and the absence of a comprehensive theory of quantum gravity leaves room for theoretical discourse regarding its interpretation [1–5]. Nevertheless, quantum cosmology is used in order to understand the initial state of the Universe and the evolution of the quantum fluctuations in the inflationary mechanism [6,7]. Novel attempts in the mathematical construction of a probability density for the possible states of our cosmos, based on the wave function of the Universe, have been presented [8–12]. It has been claimed that the quantum fluctuations can prevent the initial singularity at the very early period of the Universe [13–17]. For more in-depth exploration and examination of (the various realizations of) quantum cosmology, we direct the reader to [18–28] and references therein.

Meanwhile, at the classical level, and in order to explain the acceleration phases of the Universe, cosmologists have proposed various gravitational models that are classified in

two large families. The first family encompasses the modified, or alternative, theories of gravity, wherein new geometric invariants are introduced to alter the Einstein-Hilbert Action of general relativity (GR), which drive the dynamics and provide the necessary acceleration [29–33]. In contrast, the second family consists of models where the gravitational theory remains that of GR, and the cosmic acceleration is explained by introducing specific matter components into Einstein’s field equations [34–40].

In this study, we investigate quantum cosmology in the framework of teleparallel  $f(T)$  gravity. It is a special class of a set of modified gravitational theories comprised by the metric  $f(R)$  theory [41], teleparallel  $f(T)$  theory [42], and symmetric teleparallel  $f(Q)$  theory [43]. In these gravitational frameworks, the gravitational Lagrangian assumes a functional form denoted as  $f$ , utilizing one of the fundamental geometric scalars: the Ricci scalar  $R$  for the Levi-Civita connection, the torsion scalar  $T$  for the Weitzenböck connection [44], and the nonmetricity scalar  $Q$  for a symmetric and flat connection. When the function  $f$  takes on a linear expression, these three gravitational models seamlessly converge to general relativity (GR), the teleparallel equivalent of GR (TEGR) [45], and symmetric teleparallel GR (STGR) [46], respectively [47]. Consequently, these three theories are equivalent in their linear Lagrangian limit. This equivalence can be intuitively understood noting the fact that the three scalars, namely  $R$ ,  $T$ , and  $Q$ , differ by a total derivative term, which can be omitted, as far as the process of variation is concerned [48,49].

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Despite some existing research on this particular topic [50–52], here we follow a different and more extensive procedure. We use the Dirac-Bergmann algorithm [53,54] in order to reveal the constraints of the system and formally obtain the Hamiltonian function to be used in the quantization. In works like [51,52], the Dirac-Bergmann algorithm is ignored altogether, and, as a result, there is no distinction among first and second class constraints, thus, leading to a compromised quantization of a Hamiltonian which contains nonphysical degrees of freedom. This particular issue has also been recently encountered in the exploration of quantum cosmology within  $f(Q)$  gravity [55]. However, in [55], an additional error arises due to an incorrect assumption of gravitational field equations in the presence of nonzero spatial curvature for the background space-time. In [50], on the other hand, the Dirac-Bergmann algorithm is followed correctly for  $f(T)$  cosmology and a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) space-time in the presence of a cosmological constant term. The authors derive the constraints of the theory and choose an approach of promoting, with appropriate reductions, the second class relations to first class, which they later use to quantize the relative system. Here, we adopt a more traditional approach, and for a FLRW space-time with a perfect fluid matter content, we use the second class constraints to introduce Dirac brackets [56] with respect to which we proceed to the canonical quantization.

Quantum cosmology in symmetric teleparallel  $f(Q)$  gravity was the subject of our study in [57] where we presented an approach for constructing a Hamiltonian function in  $f(Q)$  theory that encompasses all first and second class constraints. In many aspects, the process in the  $f(T)$  theory is similar, containing, however, some key difference mostly when the spatial curvature is nonzero. The study of quantum cosmology within  $f(R)$  gravity was conducted in [58,59]. However, the dynamics in  $f(R)$  theory differ from that of  $f(T)$  and  $f(Q)$  theories due to the association of the theory's degrees of freedom with a scalar field. The constraints in this theory align with those of scalar-tensor theory [59]. The structure of the paper is outlined below.

In Sec. II, we introduce the geometric trinity of gravity and discuss the primary differences in the corresponding gravitational theories. The minisuperspace description for the cosmological field equations is discussed in Sec. III, where we review previous studies on the subject and examine their weaknesses. Section IV presents the key findings of this work, where we introduce a complete quantization for the minisuperspace process in teleparallel  $f(T)$  gravity within a FLRW geometry with a matter source. The presence of the matter source in  $f(T)$  cosmology is crucial; otherwise, the unique classical solution of the field equations is that of vacuum GR [60]. Finally, in Sec. V, we provide a summary of our results.

## II. THE GEOMETRICAL TRINITY AND ITS MODIFICATIONS

In the attempt to use geometry to formulate a gravitation theory, there can be employed one or more of three fundamental geometric scalars. Let us start from the definition of a general connection whose components can be split into three parts

$$\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} + K^\lambda{}_{\mu\nu} + L^\lambda{}_{\mu\nu}. \quad (1)$$

The first

$$\tilde{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} \left( \frac{\partial g_{\kappa\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\kappa}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa} \right), \quad (2)$$

given a space-time metric  $g_{\mu\nu}$ , is made up by the Christoffel symbols. The second consists of the contorsion tensor, which is defined as

$$K^\lambda{}_{\mu\nu} = \frac{1}{2} \left( T^\lambda{}_{\mu\nu} + T^\lambda{}_{\nu\mu} - T^\lambda{}_{\mu\mu} \right), \quad (3)$$

where

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu} \quad (4)$$

is the torsion. Finally, the last part

$$L^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} (Q_{\mu\nu\kappa} + Q_{\nu\mu\kappa} + Q_{\kappa\mu\nu}) \quad (5)$$

is the disformation tensor, which emerges due to the presence of nonmetricity

$$Q_{\lambda\mu\nu} = \nabla_\lambda g_{\mu\nu}, \quad (6)$$

where  $\nabla_\lambda$  is the covariant derivative with respect to the connection (1).

In the case of the usual (pseudo-)Riemannian geometry, one considers a torsionless,  $T^\lambda{}_{\mu\nu} = 0$ , connection with zero nonmetricity,  $Q_{\lambda\mu\nu} = 0$ , which leads the connection (1) to be given just in terms of the Christoffel symbols  $\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu}$ . Subsequently, gravitational effects are assigned to the Riemannian curvature and its scalar  $R$ . The action of general relativity is linear in the Ricci scalar  $R$ , and its most famous generalizations are in terms of  $f(R)$  modifications of the Lagrangian density. Similarly, one may adopt a torsionless and flat geometry, thus attributing the gravitational phenomena to the nonmetricity  $Q_{\lambda\mu\nu}$  and its corresponding nonmetricity scalar  $Q$ . In the same manner, a flat geometry (under the assumption of a Weitzenböck connection) can be chosen together with  $Q_{\lambda\mu\nu} = 0$ , leaving this time only the torsion  $T^\lambda{}_{\mu\nu} \neq 0$  and its corresponding scalar to account for gravity. In both of these two last cases, modifications in the form of  $f(Q)$  or  $f(T)$  theories can be taken, respectively. These are the simplest geometrical configurations that can

be considered, where in each case, only one of the three basic geometric scalars is nonzero. The linearized version of this set of theories has been deemed as the geometrical trinity of gravity [61,62].

Here, we are going to consider the case of  $f(T)$  theories constructed with the help of the torsion scalar

$$T = S^{\mu\nu} T^{\rho}_{\mu\nu}, \quad (7)$$

where

$$S^{\rho\mu\nu} = \frac{1}{2} \left( K^{\mu\nu\rho} - g^{\nu\lambda} T^{\lambda\mu}_{\lambda} + g^{\rho\lambda} T^{\lambda\nu}_{\lambda} \right) \quad (8)$$

is called the superpotential. As is well known, theories whose Lagrangian density is linear to the scalar  $T$  defined above, i.e.,  $f(T) \sim T$ , are dynamically equivalent to Einstein's general relativity. The linear in  $T$  theory is referred to as the teleparallel equivalent of general relativity. Similarly, one can define a nonmetricity scalar  $Q$ , in such a manner, so as by taking a theory linear to it the symmetric teleparallel of general relativity is obtained [63].

Although  $f(R)$ ,  $f(Q)$  and  $f(T)$  have a lot in common in the way the theories are constructed, their dynamical behavior is quite different when we deviate from the linear expressions with respect to the fundamental scalars of each case. The  $f(R)$  gravity leads to higher order equations, which effectively correspond to an inclusion of an extra degree of freedom since the theory is dynamically

equivalent to GR with an appropriate addition of a scalar field. In the cases of  $f(Q)$  and  $f(T)$  gravity, we have second order equations, and, when the connection does not play a dynamical role, it leads to important implications for the procedure, which we need to follow in order to perform a minisuperspace quantization. We shall refer to this point more extensively in the subsequent sections.

### III. MINISUPERSPACE LAGRANGIANS

Let us begin by writing the FLRW line element:

$$ds^2 = -N(t)dt^2 + a(t)^2 [d\chi^2 + \chi^2 \text{sinc}^2(\sqrt{k}\chi) \times (d\theta^2 + \sin^2 \theta d\phi^2)], \quad (9)$$

where  $\text{sinc}^2(\sqrt{k}\chi) = \left[ \frac{\sin(\sqrt{k}\chi)}{\sqrt{k}\chi} \right]^2$  is the square of the sine cardinal function. The above line element can describe at the same time the spatially curved  $k = \pm 1$  and the flat  $k = 0$  cases, since  $\lim_{k \rightarrow 0} \left( \frac{\sin(\sqrt{k}\chi)}{\sqrt{k}\chi} \right) = 1 = \text{sinc}(0)$ . In these coordinates,  $\chi$  is considered as a radial angle, while the scale factor  $a(t)$  has units of distance.

We can of course express the line element in the form

$$ds^2 = \eta_{AB} e^A_{\mu} e^B_{\nu} dx^{\mu} dx^{\nu}, \quad (10)$$

where  $\eta_{AB} = \text{diag}(-1, 1, 1, 1)$  is the Minkowski metric, and the dual vierbein is given by

$$e^A_{\mu} = \begin{pmatrix} N & 0 & 0 & 0 \\ 0 & -a \cos \theta & \frac{a}{\sqrt{k}} (\sin(\sqrt{k}\chi) \cos(\sqrt{k}\chi) \sin \theta) & -\frac{a}{\sqrt{k}} \sin^2(\sqrt{k}\chi) \sin^2 \theta \\ 0 & a \sin \theta \cos \phi & \frac{a}{\sqrt{k}} \sin(\sqrt{k}\chi) (\cos(\sqrt{k}\chi) \cos \theta \cos \phi - \sin(\sqrt{k}\chi) \sin \phi) & -\frac{a}{\sqrt{k}} \sin(\sqrt{k}\chi) (\sin(\sqrt{k}\chi) \cos \theta \cos \phi + \cos(\sqrt{k}\chi) \sin \phi) \sin \theta \\ 0 & -a \sin \theta \sin \phi & -\frac{a}{\sqrt{k}} \sin(\sqrt{k}\chi) (\cos(\sqrt{k}\chi) \cos \theta \sin \phi + \sin(\sqrt{k}\chi) \cos \phi) & \frac{a}{\sqrt{k}} \sin(\sqrt{k}\chi) (\sin(\sqrt{k}\chi) \cos \theta \sin \phi - \cos(\sqrt{k}\chi) \cos \phi) \sin \theta \end{pmatrix}. \quad (11)$$

Note that the above choice is not unique; e.g., one may take the obvious diagonal vierbein, which can be used in (10) to reproduce (9). However, the utilization of (11) is crucial in order to derive a valid minisuperspace Lagrangian. It can be seen that the latter is a generalization of the diagonal vierbein under the action of rotation matrices [42,64].

If we adopt the Weitzenböck connection, we get

$$\Gamma^{\lambda}_{\mu\nu} = e^{\lambda}_{\nu} \partial_{\nu} e^{\mu}_{\lambda}, \quad (12)$$

which together with (11) leads to the usual expression for the torsion scalar in FLRW geometry

$$T = 6 \left( \frac{\dot{a}}{Na} \right)^2 - \frac{6k}{a^2}. \quad (13)$$

The general action in the case of the  $f(T)$  theories of gravity is

$$\mathcal{A} = \mathcal{A}_{\text{grav}} + \mathcal{A}_m = \frac{1}{2\kappa} \int e f(T) d^4x + \mathcal{A}_m, \quad (14)$$

where  $e = \det(e^A_{\mu})$ , and  $\mathcal{A}_m$  is the part of the action that refers to the matter content. The field equations that are obtained by variation with respect to  $e^A_{\nu}$  are

$$\frac{2}{e} \partial_{\mu} (e e^{\rho}_{\nu} S^{\mu\nu}_{\rho} f'(T)) + 2f' S^{\sigma}_{\mu\rho} e^{\rho}_{\nu} S^{\mu\nu}_{\sigma} - \frac{1}{2} e^{\nu}_{\mu} f(T) + \kappa e^{\rho}_{\nu} T^{\nu}_{\rho} = 0, \quad (15)$$

with  $T^\mu_\nu$  expressing the energy momentum tensor for the matter. In our case, we are going to consider a perfect fluid of energy density  $\rho(t)$  and pressure  $p(t)$  with

$$T^\mu_\nu = (\rho + p)v_\mu v_\nu + pg_{\mu\nu}, \quad (16)$$

where  $v^\mu$  is the comoving velocity satisfying  $v^\mu v_\mu = -1$ . For the FLRW metric we consider here, we finally get the well-known expression  $T^\mu_\nu = \text{diag}(-\rho, p, p, p)$ .

The object of writing a minisuperspace Lagrangian is to start from (14), use the ansatz we made for the space-time at

hand, and then obtain a reduced Lagrangian of finite degrees of freedom, which correctly reproduces the result of (15) for the given space-time. For example, let us first take the gravitational part of (14) and substitute in it

$$e = \det(e^A_\mu) = Na^3 \text{sinc}^2(\sqrt{k}\chi) \sin \theta, \quad (17)$$

which is obtained by the use of (11). The gravitational part of the action then factorizes into two subparts

$$\mathcal{A}_{\text{grav}} = \frac{1}{2\kappa} \int e f(T) d^4x = \left[ \int (\chi^2 \text{sinc}^2(\sqrt{k}\chi) \sin \theta) d^3x \right] \overbrace{\frac{1}{2\kappa} \int Na^3 f(T) dt}^{t\text{-dependent dynamical part}}.$$

One part has a spatial dependence  $(\chi, \theta, \phi)$ , while the other is dynamical; i.e., it depends purely on the time variable  $t$ . For the construction of the minisuperspace gravitational Lagrangian  $L_{\text{grav}}$ , we keep the dynamical  $t$ -dependent part of the above relation. The spatial part can be discarded under assumptions of performing an integration in a fixed volume in space, which would turn it into a constant number.

In the dynamical part of the action, we also want to introduce the information that the torsion scalar is related to the scale factor through Eq. (13). We thus complete the gravitation part of the reduced emanating Lagrangian by adding the relation, which gives the torsion scalar together with a Lagrange multiplier  $\lambda(t)$ . In other words, we write

$$L_{\text{grav}} = \frac{1}{2\kappa} Na^3 f(T) + \lambda \left[ T - \left( 6 \left( \frac{\dot{a}}{Na} \right)^2 - \frac{6k}{a^2} \right) \right]. \quad (18)$$

The variation of  $L_{\text{grav}}$  with respect to  $T$  yields the value of the multiplier

$$\frac{\partial L_{\text{grav}}}{\partial T} = 0 \Rightarrow \lambda = -\frac{1}{2\kappa} Na^3 f'(T), \quad (19)$$

which we can replace back into (18) to finally obtain

$$L_{\text{grav}} = \frac{1}{2\kappa} \left[ \frac{6a\dot{a}^2 f'(T)}{N} + Na^3 (f(T) - T f'(T)) - 6kNa f'(T) \right]. \quad (20)$$

Now, the matter part of the action (14) for a perfect fluid is

$$\mathcal{A}_m = \int e p d^4x.$$

In order to variate it, we need to provide an equation of state and take into account the continuity equation  $T^\mu_{\nu;\mu} = 0$ ,

where the semicolon stands for the covariant derivative with respect to the Levi-Civita connection for FLRW.

When  $p = w\rho$ , the  $T^\mu_{\nu;\mu} = 0$  yields

$$T^\mu_{\nu;\mu} = 0 \Rightarrow \dot{\rho} + \frac{\dot{a}}{a}(p + \rho) \Rightarrow \rho = \rho_0 a^{-3(1+w)}, \quad (21)$$

where  $\rho_0$  is a constant of integration. By concentrating again on the  $t$ -dependent part of  $\mathcal{A}_m$ , we can write the minisuperspace contribution due to matter as

$$L_m = N\rho_0 a^{-3w}. \quad (22)$$

By adding the two contributions, we finally write the total Lagrangian as  $L = L_{\text{grav}} + L_m$ , which results in the expression

$$L = \frac{1}{2\kappa} \left[ \frac{6a\dot{a}^2 f'(T)}{N} + N(a^3 (f(T) - T f'(T)) - 6ka f'(T)) \right] + N\rho_0 a^{-3w}. \quad (23)$$

Variation of the above Lagrangian with respect to  $N$  and  $a$  produces equations equivalent to those of (15) for the assumed space-time. Variation with respect to  $T$  gives rise to the result of definition (13). Thus, Lagrangian (23) is a valid minisuperspace Lagrangian that generates correctly the classical dynamics. The Euler-Lagrange equations can be written equivalently as the set:

$$\frac{6\dot{a}^2 f'(T)}{a^2 N^2} - \kappa\rho_0 a^{-3(w+1)} - \frac{1}{2} f(T) = 0 \quad (24a)$$

$$2 \frac{d}{dt} \left( \frac{\dot{a} f'(T)}{aN^2} \right) + \frac{2\dot{a} N f'(T)}{aN^3} + \frac{6\dot{a}^2 f'(T)}{a^2 N^2} - \frac{2kf'(T)}{a^2} + \kappa\rho_0 w a^{-3(w+1)} - \frac{1}{2} f(T) = 0 \quad (24b)$$

$$T - \left( 6 \left( \frac{\dot{a}}{Na} \right)^2 - \frac{6k}{a^2} \right) = 0. \quad (24c)$$

The latter gives of course the expression for the torsion scalar which we see in Eq. (13).

Before proceeding, we need to mention that the use of Lagrangian (23) is not the unique starting point from which one can be led to the Hamiltonian description of this minisuperspace system. An alternative process, which is owed to Schutz [65], would be to consider the degree of freedom of the perfect fluid by introducing variables related to thermodynamic quantities. Later, the application of a canonical transformation at the Hamiltonian level allows for the contribution of the fluid to be written as being linear in the momenta. This is an approach that has been widely used in the canonical quantization of minisuperspace systems [66–69], and it leads to a Schrödinger-like quantum equation for the corresponding system, where the fluid is used as an effective time parameter. This approach is more formal in the sense that it takes into account the perfect fluid degree of freedom, but the use of a classical canonical transformation can raise ambiguities in what regards the quantum equivalence between the two systems, before and after the transformation. To avoid this, we choose here to use Lagrangian (23); in any case, the resulting methodology in what regards the use of fractional derivatives, which we are going to see later, is not to be affected by this choice.

However, prior to discussing the Hamiltonian formulation of (23), it is useful to perform a comparison with the minisuperspace Lagrangians of  $f(R)$  and  $f(Q)$  cosmologies for a FLRW space-time.

### A. Minisuperspace Lagrangians in the rest of the trinity

A similar procedure can be followed in the case of  $f(R)$  gravity with

$$\mathcal{A}_{\text{grav}} = -\frac{1}{2\kappa} \int \sqrt{-g} f(R) d^4x, \quad (25)$$

and  $g = \det g_{\mu\nu}$ . Here, in order to reproduce the relevant minisuperspace Lagrangian, we have to also introduce, with a Lagrange multiplier, the relation for the Ricci scalar as obtained by the—now nonzero—Riemann curvature

$$R = 6 \left( \frac{\ddot{a}}{aN^2} - \frac{\dot{a}\dot{N}}{aN^3} + \frac{\dot{a}^2}{a^2N^2} + \frac{k}{a^2} \right). \quad (26)$$

We avoid the cumbersome details since the procedure can be found easily in the literature [58,59,70–72], and we just present the end result for the minisuperspace Lagrangian, again in the case of a perfect fluid matter source, with a linear barotropic equation of state,  $p = w\rho$ ,

$$L_{f(R)} = \frac{1}{2\kappa} \left[ \frac{6a\dot{a}^2 f'(R)}{N} + \frac{6a^2 \dot{a} \dot{R} f''(R)}{N} - N(a^3(f(R) - Rf'(R)) + 6ka f'(R)) \right] + N\rho_0 a^{-3w}. \quad (27)$$

This Lagrangian is also valid in its context; it reproduces correctly the result of the  $f(R)$  gravity equations for a FLRW space-time. Notice the important difference in comparison to the Lagrangian (23). The latter has no velocity for the degree of freedom  $T$  in contrast to (27), which possesses a velocity with respect to  $R$ . This substantial difference leads to (23) requiring a modified treatment in the construction of its Hamiltonian formulation than the usual treatment applied to Lagrangian (27) and to other minisuperspace Lagrangians.

In the case of  $f(Q)$  gravity, we start from

$$\mathcal{A}_{\text{grav}} = \frac{1}{2\kappa} \int \sqrt{-g} f(Q) d^4x. \quad (28)$$

Let us require here, for reasons that we will explain shortly, that we treat only the spatially flat case  $k = 0$ . Then, if—among the three different options for a connection—we choose the one that is not dynamical, then the nonmetricity scalar can be written as

$$Q = 6 \left( \frac{\dot{a}}{Na} \right)^2. \quad (29)$$

For  $f(Q)$  cosmology, there is usually a variety of distinct connections that are compatible with the equations of motion [73–80]. In the spatially flat case,  $k = 0$ , there exist three. However, only one of them is nondynamical; i.e., in the coordinate system where the line element is being given by (9), the equation of motion for the connection is identically satisfied. This is the connection whose components vanish (coincident gauge) in Cartesian coordinates, and it is only for this connection that you get relation (29). For the other two connections, in the coordinate system where the metric is diagonal, the dynamical function related to the connection will also appear in the expression for  $Q$  [74,76], meaning that you need to write a Lagrangian that also produces Euler-Lagrange equations for the connection. Anyway, let us consider the simplest case of the nondynamical connection. The resulting minisuperspace Lagrangian is [57]

$$L_{f(Q)} = \frac{1}{2\kappa} \left[ \frac{6a\dot{a}^2 f'(Q)}{N} + Na^3(f(Q) - Qf'(Q)) \right] + N\rho_0 a^{-3w}. \quad (30)$$

Here, there is also no velocity for the additional degree of freedom  $Q$ . This Lagrangian is also successful in

reproducing the result of the field equations under the assumptions we have set  $k = 0$  and considering the non-dynamical connection. The Hamiltonian analysis and an attempt in the canonical quantization of this Lagrangian, for various scenarios of matter content, have been performed in [57].

The similarity of Lagrangian (30) to that of the  $f(T)$  theory, Eq. (23), for  $k = 0$  is obvious. The one becomes the other by interchanging  $T$  with  $Q$ . The same is true for the expressions of the basic scalars (13) and (29). At this point, one may be naively tempted to consider that, in the  $k \neq 0$  case, the nonmetricity scalar is equal to the right-hand side of (13) and write the minisuperspace Lagrangian of  $f(Q)$  cosmology as (23) with  $T \rightarrow Q$ ; this is, for example, what has been done in [55]. However, this is incorrect since

$$Q \neq 6 \left( \frac{\dot{a}}{Na} \right)^2 - \frac{6k}{a^2}, \quad \text{when } k \neq 0. \quad (31)$$

Unlike the  $k = 0$  case, in the coordinate systems where the FLRW metric is diagonal having an obvious homogeneous and isotropic form, there is no nondynamical connection when  $k \neq 0$ . There exists one connection compatible with the symmetries of the space-time; it is a nonzero, dynamical connection, whose time dependence affects the expression of the nonmetricity scalar  $Q$ .

It is a common problem in  $f(Q)$  cosmology that one may be led to the error of thinking that the connection is irrelevant because its flatness implies the existence of a coordinate system where it becomes zero. However, as it has been also stressed in [81], the adoption of a specific type of space-time metric, like the one in Eq. (9), already constitutes a partial gauge fixing. As a result, the coordinate system in which we write the metric may not be compatible with having a zero connection. This is what happens in the  $k \neq 0$  case.

So, we see that, unlike the  $f(T)$  case, in  $f(Q)$  FLRW cosmology, the relevant minisuperspace Lagrangian is not given by (23), with a simple change  $T \rightarrow Q$ , when  $k \neq 0$ . It is only for  $k = 0$  where we can make this identification. It can also be checked that, if one erroneously sets  $T \rightarrow Q$  in (23), with  $k \neq 0$ , and takes the relevant Euler-Lagrange equations, then the latter are not even equivalent to the field equations for  $f(Q)$  theory reduced by the assumption of the case of a spatially nonflat FLRW metric.

#### IV. HAMILTONIAN FORMALISM FOR THE $f(T)$ MINISUPERSPACE

We start by giving a brief description of the usual Hamiltonian formalism for minisuperspace Lagrangians in order to later see in what important way the Lagrangian (23) differentiates from the usual prescription. Minisuperspace Lagrangians are usually of the form

$$L = \frac{1}{2N} G_{ij}(q) \dot{q}^i \dot{q}^j - NV(q), \quad (32)$$

where  $G_{ij}$  is the so-called minisuperspace metric, and  $q^i$  are the rest of the degrees of freedom, beside the lapse  $N$ , which we distinguish. In the case of  $f(R)$  gravity and Lagrangian (27), we have  $q^i = (a, R)$ , in  $f(Q)$  gravity and (30), there is  $q^i = (a, Q)$ , and finally, in the case that we want to treat in this work, and Lagrangian (23), we have  $q^i = (a, T)$ .

The fact that all Lagrangians have in common is that there is no velocity for  $N$ . This already signifies that the Legendre transform is not invertible, and thus, we need to invoke the Dirac-Bergmann algorithm in order to write the Hamiltonian of the system, which ends up to be (given an invertible  $G_{ij}$ )

$$H_T = N\mathcal{H} + u_N p_N = N \left( \frac{1}{2} G^{ij}(q) p_i p_j + V(q) \right) + u_N p_N. \quad (33)$$

The momentum with respect to  $N$ ,  $p_N = \frac{\partial L}{\partial \dot{N}} \approx 0$ , is the primary constraint of the theory. We use the curly equal “ $\approx$ ” in the case of constraints to signify what Dirac mentioned as a weak equality. It basically means that we are allowed to set the constraint equal to zero only when it appears outside of Poisson brackets; i.e.,  $\{N, p_N^2\} = 2\{N, p_N\} p_N = 0$ , but  $\{N, p_N\} = 1$  even though  $p_N = 0$ . The first expression is zero because there appears an overall  $p_N$  outside of a Poisson bracket. In the second example, it would be an error to take  $\{N, p_N\} = \{N, 0\} = 0$ , i.e., set  $p_N$  equal to zero inside a Poisson bracket. This is the meaning of denoting  $p_N \approx 0$ . Finally, an important remark is that, if a phase space quantity is weakly equal to zero, then it is (strongly) equal to a linear combination of the constraints of the system (at least in the case of systems with finite degrees of freedom like the ones we deal here).

Consistency requires that the constraints are to be preserved in time, at least weakly; this means that we need to take

$$\dot{p}_N \approx 0 \Rightarrow \{p_N, H_T\} \approx 0 \Rightarrow \mathcal{H} \approx 0. \quad (34)$$

The

$$\mathcal{H} = \frac{1}{2} G^{ij}(q) p_i p_j + V(q) \approx 0 \quad (35)$$

is the secondary constraint of the theory, and its own conservation in time is identically satisfied. So, no additional constraints emerge. These two constraints commute with each other,  $\{p_N, \mathcal{H}\} = 0$ , and this classifies them as being first class constraints. This is an important distinction: First class constraints commute, at least weakly, with all of the constraints, and their existence is associated with some gauge freedom in the system. In cosmology, the existing manifest freedom is that of performing arbitrary diffeomorphisms in time, which is what evidently remains at the minisuperspace level from the full four-dimensional

diffeomorphism invariance.<sup>1</sup> After having obtained the Hamiltonian, the canonical quantization procedure by following Dirac's idea of enforcing the constraints on the wave function leads to the equations

$$\hat{p}_N \psi = 0, \quad \hat{\mathcal{H}} \psi = 0. \quad (36)$$

The first equation is usually forgotten most of the time, together with  $N$  all together in (32) and  $p_N$  in (33). However, it is the existence of the  $p_N \approx 0$ , which leads us to  $\mathcal{H} \approx 0$  and later to  $\hat{\mathcal{H}} \psi = 0$ , that is the Wheeler-DeWitt equation. The first of (36),  $\hat{p}_N \psi = 0 \Rightarrow -i\hbar \frac{\partial \psi}{\partial N} = 0$ , just tells us that the wave function  $\psi$  does not depend on the lapse  $N$ .

The above procedure is perfectly fine for Lagrangians like (27). However, it is not sufficient for the cases of  $f(Q)$  and of  $f(T)$  cosmologies. The reason lies in the fact that for (30) and (23), the corresponding  $G_{ij}(q)$  matrix is not invertible. We already noticed the reason for this; it is the lack of velocity terms in the Lagrangian for  $Q$  and  $T$ , respectively.

Let us concentrate in Lagrangian (23), whose quantization we want to study. As we mentioned, in (23), we are not only missing a velocity for  $N$ , but we are also missing a velocity for  $T$ ; i.e., there is no  $\dot{T}$  term. So, according to the Dirac-Bergmann algorithm, instead of one, we start off with two primary constraints

$$p_N = \frac{\partial L}{\partial \dot{N}} \approx 0, \quad p_T = \frac{\partial L}{\partial \dot{T}} \approx 0. \quad (37)$$

We may write the canonical part  $N\mathcal{H}$  of the Hamiltonian with what remains of the degrees of freedom; so, we have

$$\begin{aligned} \dot{a}p_a - L = N\mathcal{H} = N & \left[ \frac{\kappa p_a^2}{12a f'(T)} - \frac{a^3}{2\kappa} (f(T) - T f'(T)) \right. \\ & \left. + \frac{3k}{\kappa} a f'(T) - \rho_0 a^{-3w} \right]. \end{aligned} \quad (38)$$

The total Hamiltonian is obtained by adding to it the primary constraints with multipliers

$$H_T = N\mathcal{H} + u_N p_N + u_T p_T. \quad (39)$$

As explained before, consistency requires now that both  $p_N$  and  $p_T$  are preserved in time; thus, we need to require

$$\{p_N, H_T\} \approx 0 \quad \text{and} \quad \{p_T, H_T\} \approx 0. \quad (40)$$

The first relation leads to the expected Hamiltonian constraint because again  $\{p_N, H_T\} = -\mathcal{H}$ , so we need to impose

$$\begin{aligned} \mathcal{H} = \frac{\kappa p_a^2}{12a f'(T)} - \frac{a^3}{2\kappa} (f(T) - T f'(T)) + \frac{3k}{\kappa} a f'(T) - \rho_0 a^{-3w} \\ \approx 0. \end{aligned} \quad (41)$$

The second consistency condition is

$$\{p_T, H_T\} = p_a^2 - \frac{6}{\kappa^2} a^2 f'(T)^2 (a^2 T + 6k) \approx 0. \quad (42)$$

This is nothing more than relation (13) expressed in the phase space formalism and it forms an additional secondary constraint together with the  $\mathcal{H} \approx 0$  of (41). We symbolize this new constraint with  $\varphi$ ,

$$\varphi = p_a^2 - \frac{6}{\kappa^2} a^2 f'(T)^2 (a^2 T + 6k) \approx 0. \quad (43)$$

The further checking of the consistency conditions  $\{\mathcal{H}, H_T\} \approx 0$  and  $\{\varphi, H_T\} \approx 0$  leads to no additional constraints. The first is proportional to  $\varphi \approx 0$ , so it is weakly zero

$$\{\mathcal{H}, H_T\} = \frac{\kappa N f''(T)}{12a f'(T)^2} \varphi \approx 0, \quad (44)$$

while  $\{\varphi, H_T\} \approx 0$  leads to the derivation of the multiplier  $u_T$

$$\{\varphi, H_T\} \approx 0 \Rightarrow u_T \approx \frac{\kappa N p_a (f(T) - 2T f'(T) - 4k f'(T) a^{-2} - 2\kappa \rho_0 w a^{-3(w+1)})}{2f'(T) (a^2 (2T f''(T) + f'(T)) + 12k f''(T))}, \quad (45)$$

if we assume of course that the denominator is different from zero; i.e.

$$a^2 (2T f''(T) + f'(T)) + 12k f''(T) \neq 0. \quad (46)$$

In the expression for  $u_T$ , we have also used relation (43) to eliminate  $p_a^2$  and simplify the end result.

The branch that opens when the expression of the left-hand side of (46) is zero is quite interesting and significantly differentiates the procedure to be followed. For this reason, we will study separately this particular case, where  $u_T$  is not fixed, but first, let us treat the generic case.

<sup>1</sup>The automorphisms of the structure constants of the spatial symmetries account for the freedom particular of mixed spatial-time coordinate transformations [82].

### A. Generic case

If we assume that (46) holds, we have a four-dimensional set of constraints, spanned by

$$\{p_N, p_T, \mathcal{H}, \varphi\}.$$

The fact that one of the multipliers appearing in (39) is fixed signifies the existence of second class constraints, that is, constraints whose Poisson brackets are not zero even in the weak sense. This implies the existence of “fake” or nonphysical degrees of freedom in the system. The second class constraints in the above set are the  $p_T$  and  $\varphi$ , whose Poisson bracket yields

$$\{p_T, \varphi\} = \frac{6a^2}{\kappa^2} f'(T) (a^2(2Tf''(T) + f'(T)) + 12kf''(T)) \neq 0. \quad (47)$$

The above expression is nonzero due to (46), which we assume that holds in the generic case. Exactly because  $p_T$  and  $\varphi$  are second class, we cannot proceed with the quantization in the same manner we do for first class constraints, for example, the fact that  $p_T \approx 0$  cannot imply now at the quantum level  $\hat{p}_T\psi = 0$ , as it happens for the  $p_N \approx 0$  case. The reason is that this would lead to an inconsistency due to (47) being nonzero. We would have at the quantum level  $\hat{p}_T\psi = 0$  and  $\hat{\varphi}\psi = 0$  with  $[\hat{p}_T, \hat{\varphi}]\psi$  being nonzero by construction, which is impossible. There are two main strategies regarding this predicament. The one requires the enhancement of the phase space with the addition of extra degrees of freedom in such a way so that the second class constraints become first class. In this bigger phase space, one can proceed with the usual quantization of first class constraints. The construction, however, of the necessary Hamiltonian, with the extra degrees of freedom, that will reproduce the same dynamics, is far from trivial.

The second approach, which was what was initially proposed by Dirac, is the elimination of the fake degrees of freedom represented by the second class constraints. This is done with the introduction of the Dirac bracket in place of the Poisson. For two phase space functions  $A, B$ , the former is defined as

$$\{A, B\}_D = \{A, B\} - \{A, X_i\} \Delta_{ij}^{-1} \{X_j, B\}, \quad (48)$$

where  $X_s$  are the elements of the set of second class constraints, and  $\Delta^{-1}$  is the inverse matrix of the one constructed by the Poisson brackets of the second class constraints. In our case, the set of second class constraints is two dimensional (so  $i, j = 1, 2$ ) and made up by  $X_1 = p_T$ ,  $X_2 = \varphi$ . The antisymmetric  $2 \times 2$  matrix  $\Delta$ , whose inverse appears in (48), has components  $\Delta_{ij} = \{X_i, X_j\}$ . The initial idea behind the introduction of Dirac brackets was to

impose all second class constraints strongly equal to zero by making them commute with any phase space function. As a trivial byproduct of this, all constraints, irrespectively of being first or second class, commute with each other through the use of Dirac bracket.

At this point, we can enforce  $p_T = 0$  and  $\varphi = 0$ , as strong equations. That is, we substitute inside the Hamiltonian  $p_T = 0$  and  $p_a$  from (43), thus arriving at the reduced Hamiltonian

$$H_T^{\text{red}} = N\mathcal{H}_{\text{red}} + u_N p_N, \quad (49)$$

where the reduced, stripped from fake degrees of freedom, Hamiltonian constraint is

$$\mathcal{H}_{\text{red}} = \frac{1}{2\kappa} [a^3(2Tf'(T) - f(T)) + 12kaf'(T)] - \rho_0 a^{-3w} \approx 0. \quad (50)$$

The reduced system has only the two remaining first class constraints  $p_N \approx 0$  and  $H_{\text{red}} \approx 0$  and can be quantized in the usual manner, with the exception now that the canonical quantization scheme is to be constructed based on the Dirac brackets and not the Poisson. That is, we need to find functions on the reduced phase space  $q(a, T)$  and  $p(a, T)$  with the property

$$\{q(a, T), p(a, T)\}_D = 1. \quad (51)$$

These are going to be our canonical variables to which we will need to assign the basic quantum operators of position  $\hat{q}\psi = q\psi$  and momentum  $\hat{p}\psi = -i\frac{d}{dq}\psi$  (from now on we assume units  $\hbar = 1$ ).

For  $k = 0$ , a very convenient choice of functions satisfying (51) is

$$q(a, T) = \pm(2Tf'(T) - f(T)) \quad \text{and} \quad p(a, T) = \frac{a^3}{\kappa\sqrt{6T}}. \quad (52)$$

In particular, we have

$$\{q(a, T), p(a, T)\}_D = \mp \frac{\kappa p_a}{\sqrt{6T}a^2 f'(T)} = 1, \quad (53)$$

with the last equation being true by virtue of the constraint (43), which we substitute as  $p_a = \mp \sqrt{6T}a^2 f'(T)/\kappa$ . Note that the arbitrariness in the sign of  $p_a = \frac{\partial L}{\partial \dot{a}} = 6af'(T)\dot{a}/(\kappa N)$  is related with the arbitrariness in the sign of  $N$  (only  $N^2$  appears in the metric). This carries over an arbitrariness in the sign that we may use to define  $q$  in (52). As we are going to see, this does not really affect the end result, but it may be useful in allowing us to expand the



domain of definition of the  $q$  variable depending on the  $f(T)$  theory that we adopt.

In the new variables, the reduced Hamiltonian constraint reads

$$\mathcal{H}_{\text{red}} = \pm \sqrt{\frac{3}{2}} q \sqrt{T(q)} p - 6^{-\frac{w}{2}} \rho_0 \kappa^{-w} T(q)^{-\frac{w}{2}} p^{-w} \approx 0, \quad (54)$$

where  $T(q)$  is the inverse of the first of (52), which is to be found for some given  $f(T)$  theory. We can exploit the time-reparametrization invariance of the system in order to scale the lapse function as

$$N \mapsto \bar{N} = \sqrt{\frac{3}{2}} T^{-\frac{w}{2}} p^{-w} N, \quad (55)$$

which allows us to write the reduced Hamiltonian (49) as

$$H_T^{\text{red}} = \bar{N} \bar{\mathcal{H}}_{\text{red}} + u_{\bar{N}} p_{\bar{N}}, \quad (56)$$

where we introduced the rescaled Hamiltonian constraint

$$\bar{\mathcal{H}}_{\text{red}} = \pm q T(q)^{\frac{w+1}{2}} p^{1+w} - \frac{2^{\frac{1-w}{2}} \rho_0}{3^{\frac{1+w}{2}} \kappa^w} \approx 0. \quad (57)$$

Having all the dynamical dependence incorporated in one term helps significantly in the construction of an appropriate quantum operator for the  $\bar{\mathcal{H}}_{\text{red}}$  and allows us to interpret the quantum version of (57) as an eigenvalue equation. Note that the scaling we did in (55) is equivalent to a change in the lapse in the metric of the form  $N = \bar{N} \frac{a^{3w}}{6^{w/2} \kappa^w}$  due to  $p$  being given by (52).

From what we observe in (57), the momentum is raised in a power, which is not necessarily a natural number. Such Hamiltonians are the object of study in the theory of fractional quantum mechanics [83–85]. Several different implementations of fractional derivatives exist in the literature [86,87]. In [57], we made use of the Katugampola fractional derivative, which has been introduced in [88]. This derivative has the advantage of producing a differential equation for the quantum version of (57), instead of an integro-differential equation as happens in the case of other fractional derivatives. This significantly simplifies the derivation of the relative quantum solutions. The Katugampola fractional derivative with index  $\alpha$  is defined as

$$D^\alpha(\Psi(q)) = \lim_{\epsilon \rightarrow 0} \frac{\Psi(qe^{\epsilon q^{-\alpha}}) - \Psi(q)}{\epsilon}, \quad (58)$$

when  $0 < \alpha \leq 1$  and  $q > 0$ . Here, the arbitrariness of the sign in the definition of  $q$  in (52) can come into play since for different regions in the domain of  $T$ , we may choose that definition of  $q$  so that the latter remains positive ( $q > 0$ ) in the relative domain. Remember that we have the

ability to define  $q$  from (52) with different overall signs with respect to the expression  $2Tf'(T) - f(T)$ , depending on whether the latter is positive or negative.

The action of (58) upon a given function gives

$$D^\alpha(\Psi(q)) = q^{1-\alpha} \frac{d\Psi}{dq}. \quad (59)$$

Although, initially the derivative is introduced for  $0 < \alpha \leq 1$ , the definition can be modified to be extended for other values of  $\alpha$ . For example, for  $1 < \alpha \leq 2$ , one obtains the action

$$D^\alpha(\Psi(q)) = q^{2-\alpha} \frac{d^2\Psi}{dq^2}. \quad (60)$$

These are the two operators that are of interest to us.

We have a Hamiltonian constraint given by Eq. (57), which is of the general form

$$\bar{\mathcal{H}}_{\text{red}} = \pm A(q) p^\alpha - \Sigma^2 \approx 0, \quad (61)$$

where

$$A(q) = qT(q)^{\frac{w+1}{2}}, \quad \alpha = 1 + w \quad \text{and} \quad \Sigma^2 = \frac{2^{\frac{1-w}{2}} \rho_0}{3^{\frac{1+w}{2}} \kappa^w}. \quad (62)$$

When the power of the momentum  $\alpha = 1 + w$  is in the region  $0 < \alpha \leq 1$ , i.e.  $-1 < w \leq 0$ , we assign to  $\bar{\mathcal{H}}_{\text{red}}$  the operator

$$\hat{\mathcal{H}} = \mp \frac{i}{2\mu(q)} \left[ \mu(q) A(q) q^{1-\alpha} \frac{d}{dq} + \frac{d}{dq} (\mu(q) A(q) q^{1-\alpha}) \right] - \Sigma^2. \quad (63)$$

The  $\hat{\mathcal{H}}$  of (63) has been constructed so that at the limit  $\alpha = 1$ , it gives the most general, linear, first order differential operator, which is Hermitian under a measure  $\mu(q)$ . The equivalent of the Wheeler-DeWitt equation in this case is given by taking  $\hat{\mathcal{H}}\Psi = 0$ , which results to the eigenvalue equation

$$\mp \frac{i}{2\mu(q)} \left[ \mu(q) A(q) q^{1-\alpha} \frac{d\Psi}{dq} + \frac{d}{dq} (\mu(q) A(q) q^{1-\alpha} \Psi) \right] = \Sigma^2 \Psi. \quad (64)$$

The latter has the general solution

$$\Psi = C \frac{q^{\frac{\alpha-1}{2}}}{\mu(q)^{\frac{1}{2}} A(q)^{\frac{1}{2}}} \exp \left[ \pm i \Sigma^2 \int \frac{q^{\alpha-1}}{A(q)} dq \right], \quad (65)$$

where  $C$  denotes the constant of integration, and the  $A(q)$ ,  $\alpha$  and  $\Sigma$  are given by (62). We can write the corresponding probability amplitude to be

$$\begin{aligned}
 P &= \mu(q)\Psi^*\Psi = CC^* \frac{q^{\frac{\alpha-1}{2}}}{A(q)^{\frac{1}{2}}} \left( \frac{q^{\frac{\alpha-1}{2}}}{A(q)^{\frac{1}{2}}} \right)^* \\
 &= \frac{|C|^2}{T^{\frac{1+w}{2}} |2Tf'(T) - f(T)|^{1-w}}, \quad (66)
 \end{aligned}$$

where in the last equality, we used the definition of  $q$ , as a function of  $T$ , from (52). Note that choosing a particular expression for the measure function in (65) is irrelevant since it is eliminated from the final expression of the probability amplitude. Of course, for a well-defined probabilistic interpretation, the integral of  $P$ , for all  $q$  (or similarly  $T$ ), should be finite. However, as usually happens in cosmology, this is not in general the case. Nevertheless, we are able to provide a normalization up to a delta function if we write, for two different values  $\Sigma$  and  $\Sigma'$ , the inner product

$$\begin{aligned}
 \int \mu(q)\Psi_{\Sigma}(q)\Psi_{\Sigma'}^*(q) dq &= |C|^2 \int \frac{q^{\alpha-1}}{A(q)} e^{[i \int \frac{q^{\alpha-1}}{A(q)} dq (\Sigma^2 - \Sigma'^2)]} dq \\
 &= |C|^2 \int e^{i(\Sigma^2 - \Sigma'^2)u} du,
 \end{aligned}$$

where we performed the change of variable  $u = \int \frac{q^{\alpha-1}}{A(q)} dq$ . If the variable  $u$  in the last integral runs in the whole real line, then the above expression becomes equal to  $2\pi|C|^2\delta(\Sigma^2 - \Sigma'^2)$  providing a normalization value for the  $C$  constant,  $C = \frac{1}{\sqrt{2\pi}}$ . Of course, the domain of definition of  $q$  and consequently of  $u$  depends on the assumed  $f(T)$  theory that we consider.

From what we observe in Eq. (66), we expect the probability amplitude to favor values of  $T$  that lie on the roots of  $T^{\frac{1+w}{2}}(2Tf'(T) - f(T))^{1-w} = 0$ . Take, for example, theories that imply a power-law modification added to the original GR dynamics; that is  $f(T) = T + \beta T^{\mu}$ . Then, we see that the probability amplitude goes to infinity if

$$\begin{aligned}
 T^{\frac{1+w}{2}}(2Tf'(T) - f(T))^{1-w} &= T^{\frac{3+w}{2}}(\beta(2\mu - 1)T^{\mu-1} + 1)^{1-w} \\
 &= 0. \quad (67)
 \end{aligned}$$

Assuming that  $T \geq 0$  and given that the amplitude corresponds to the case  $-1 < w \leq 0$ , the above equation is satisfied at  $T = 0$ , and if  $\beta < 0$  and  $\mu > 0$ , also at  $T = [\beta(1 - 2\mu)]^{\frac{1}{1-\mu}}$ . We are going to see some examples of this in the subsequent analysis.

Now, if the power of the momentum is  $1 < \alpha \leq 2$  in (61), that is  $0 < w \leq 1$  for (57), we use as the Hamiltonian constraint operator

$$\hat{\mathcal{H}} = \mp \frac{1}{\mu(q)} \frac{d}{dq} \left[ \mu(q)A(q)q^{2-\alpha} \frac{d}{dq} \right] - \Sigma^2, \quad (68)$$

in which the measure function is  $\mu = |q^{\frac{\alpha}{2}-1}A(q)^{-\frac{1}{2}}|$ . The differential part of the above expression, for  $\alpha = 2$ ,

becomes the one-dimensional Laplacian, which we would normally use in the case of a quadratic Hamiltonian. The general solution to the eigenvalue equation

$$\mp \frac{1}{\mu(q)} \frac{d}{dq} \left[ \mu(q)A(q)q^{2-\alpha} \frac{d}{dq} \Psi \right] = \Sigma^2 \Psi \quad (69)$$

is

$$\begin{aligned}
 \Psi &= C_1 \exp \left[ (\mp 1)^{\frac{1}{2}} \Sigma \int \frac{q^{\frac{\alpha}{2}-1}}{A(q)^{\frac{1}{2}}} dq \right] \\
 &+ C_2 \exp \left[ -(\mp 1)^{\frac{1}{2}} \Sigma \int \frac{q^{\frac{\alpha}{2}-1}}{A(q)^{\frac{1}{2}}} dq \right], \quad (70)
 \end{aligned}$$

which, in the case of the upper sign  $(-1)^{\frac{1}{2}} = i$ , can be seen as a linear combination of ‘‘ingoing’’ and ‘‘outgoing’’ waves. If we choose either one of the two waves, we can impose a normalizability up to a delta function. For example, take for two different values  $\Sigma$  and  $\Sigma'$  the integral

$$\int \mu(q)\Psi_{\Sigma}(q)\Psi_{\Sigma'}^*(q) dq = |C_1|^2 \int e^{i(\Sigma - \Sigma')x} dx, \quad (71)$$

where we have set  $x = \int \frac{q^{\frac{\alpha}{2}-1}}{A(q)^{\frac{1}{2}}} dq$ . The above integral results in  $2\pi|C_1|^2\delta(\Sigma - \Sigma')$  if we consider now the  $x$  to run in the whole set of the real numbers.

Similarly to (66), the probability amplitude, for a single state given by the first branch of (70), becomes

$$P = \mu(q)\Psi^*\Psi = C_1 C_1^* \frac{q^{\frac{\alpha}{2}-1}}{A(q)^{\frac{1}{2}}} = \frac{|C_1|^2}{T^{\frac{1+w}{2}} (2Tf'(T) - f(T))^{\frac{2-w}{2}}}, \quad (72)$$

which again, we may claim that it can assume its maximum (diverging) values at the points where the denominator goes to zero. This holds for the upper sign of (69) and (70), which corresponds to having  $q = 2Tf'(T) - f(T) > 0$ . When  $2Tf'(T) - f(T) < 0$ , however, we use the opposite definition for  $q$  from (52) with  $q = -(2Tf'(T) - f(T)) > 0$ , which leads to real exponents in (70). The probability amplitude in this case can be taken to be

$$P = \mu(q)\Psi^*\Psi = |C_2|^2 \frac{q^{\frac{\alpha}{2}-1}}{A(q)^{\frac{1}{2}}} \exp \left[ -2\Sigma \int \frac{q^{\frac{\alpha}{2}-1}}{A(q)^{\frac{1}{2}}} dq \right], \quad (73)$$

where we have chosen the branch that (in the following examples) decays as  $T \rightarrow \infty$ .

In Fig. 1, we present the graphs of the probability amplitudes, in various cases, for theories of the form  $f(T) = T + \beta T^2$ . In the first part of Fig. 1, the fact that we take  $\beta < 0$  leads the common denominator in (66), (72), and (73) to become zero at two points:  $T = 0$  and  $T = -\frac{1}{3\beta}$ ;

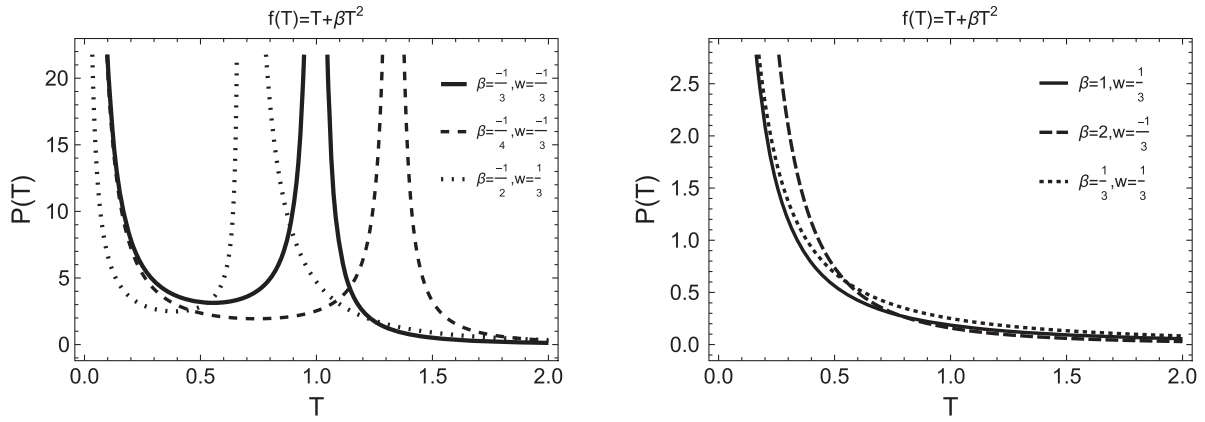


FIG. 1. The probability amplitude  $P$  for different values of  $\beta$  and  $w$  in the  $f(T) = T + \beta T^2$  theory. One can distinguish between two ( $T = 0$ ,  $T = T_0 \neq 0$ ) and a single ( $T = 0$ ) most probable values for the torsion scalar  $T$ . Expressions (66), (72), and (73) have been used for  $P$ , with  $C = C_1 = C_2 = \frac{1}{\sqrt{2\pi}}$ , respectively, and, in the case of (73), we have additionally set, for reasons of facilitating the depiction of the graphs,  $\Sigma = 1/6$ .

we see the probability amplitude diverging at the respective points. Notice that for the cases  $w = -1/3$ , (66) has been used since it belongs to the case  $0 < \alpha \leq 1$ , while for  $w = 1/3$  ( $1 < \alpha \leq 2$ ), we used (72) when  $2Tf'(T) - f(T) > 0$  (that is  $0 < T < -\frac{1}{3\beta}$ ) and (73) when  $2Tf'(T) - f(T) < 0$ , i.e.,  $T > -\frac{1}{3\beta}$ . From the definition of the fractional derivative we use, we need to have  $q > 0$ . One needs to be careful in the calculation of the various probability amplitudes regarding this point. The  $q$  introduced in Eq. (52) has in its definition an arbitrariness in the sign that does not affect the functional form of the wave function. We make use of this to split regions of  $T$ , where we assign the appropriate expression for  $q$ , from (52), which remains positive in that region.

In the second set of graphs appearing in Fig. 1, we depict cases with positive  $\beta$ ; here, the highest probability amplitude is encountered in the region  $T \rightarrow 0$ . Unlike the  $\beta < 0$  case of the first part, here, there is only one dominating value, implying that the torsion scalar is expected to be almost zero.

It is interesting to note that the classical equations (24), for  $k = 0$  and for  $f(T) = T + \beta T^2$ , are solved by the combination

$$N^2 = \frac{6\dot{a}^2}{a^2 T}, \quad T = \frac{-1 \pm \sqrt{1 + 24\beta\kappa\rho_0 a^{-3(1+w)}}}{6\beta}. \quad (74)$$

The latter implies that, for  $\beta < 0$ , and as long as  $w > -1$ , the torsion scalar  $T$  is bounded by  $0 \leq T \leq -\frac{1}{3\beta}$ . Both values are reached asymptotically as  $a \rightarrow +\infty$ , the first for the upper sign appearing in the expression for  $T$  in (74) and the other for the lower. Thus, from what we may observe in Fig. 1, there is a quantum region assigning nonzero amplitudes at classically restricted values of  $T$ : the region  $T > -\frac{1}{3\beta}$  in the first set of plots.

In Fig. 2, we generalize the previous analysis in the case of  $f(T) = T + \beta T^\mu$  theories. Here, we see how the different values of  $\mu$  affect the probability amplitude. The first set of graphs is similar to what we have seen before in Fig. 1: theories that lead to two distinct most probable values for the torsion scalar  $T$ . In the second plot, we see theories with negative  $\mu$  in which  $T = 0$  no longer maximizes the probability amplitude. There is a finite nonzero value of  $T$  with that property. Similarly to Fig. 1, we have used (66), (72), or (73) for the relevant regions of  $T$  and the values of the parameters.

In both Figs. 1 and 2, we recognize as a relative good sign the fact that in all cases,  $P(T) \rightarrow 0$  as  $T \rightarrow +\infty$  (of course, in the  $0 < w \leq 1$  case, we chose the branch of the wave function that would guarantee this property). This in a sense assigns a zero probability amplitude to values that would correspond to a divergence of the geometric scalar of the theory, the torsion scalar  $T$ . Avoiding thus, the “equivalent” of a classical curvature singularity, which is encountered in metric theories of gravity.

The analysis we performed up to now was dedicated to the generic case, where (46) holds, but strictly for  $k = 0$ . The generic case of  $k \neq 0$  is plagued by an important difficulty. Although it is quite easy to find a pair of canonical variables with respect to Dirac brackets satisfying Eq. (51), it is highly nontrivial to produce a Hamiltonian that can be straightforwardly assigned to a quantum operator. One usually obtains complicate functions  $\mathcal{H}(q, p)$  for which we do not currently have a clear prescription to propose on how to proceed.

In the next section, we treat the particular case, where (46) does not hold and the relevant expression is zero. As we are going to see, this case makes sense (for modified dynamics from GR) only if  $k \neq 0$ . In order to avoid the general complication of the  $k \neq 0$  case, we shall follow a different procedure, exploiting the fact that classically you can easily obtain a relation between  $T$  and  $a$ .

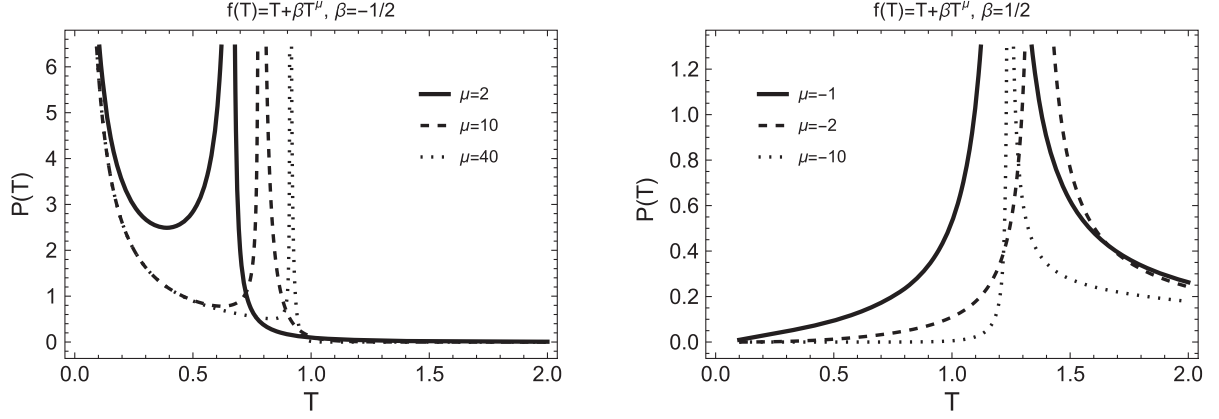


FIG. 2. Probability amplitude  $P$  in theories  $f(T) = T + \beta T^\mu$  for different values of  $\mu$ . The first set of plots corresponds to  $\mu > 0$ , leading to two most probable values, while in the second case, we can see theories with  $\mu < 0$  leading to a single dominant non-zero value for  $T$ . Once more, expressions (66), (72), and (73) have been used appropriately, with  $C = C_1 = C_2 = \frac{1}{\sqrt{2\pi}}$  and  $\Sigma = 1/6$ .

### B. The particular case

In this section, we study what happens when

$$a^2(2Tf''(T) + f'(T)) + 12kf''(T) = 0. \quad (75)$$

First, let us notice that this case is of interest only when  $k \neq 0$ . If  $k = 0$ , the reduced expression

$$(2Tf''(T) + f'(T)) = 0 \quad (76)$$

can be satisfied either if  $f(T) \sim \sqrt{T}$ , which leads to a Lagrangian that is a total derivative, or if  $T$  is a constant, whose value is such that satisfies algebraically equation (76). In this latter case, the resulting space-time is the well-known de Sitter solution of general relativity, where now the cosmological constant is given in terms of  $T$ . As a result, we see that interesting novel solutions may emerge only in the context of  $k \neq 0$ , and this is what we will study here.

The treatment that we will apply is different than before. This has to do with the enhanced complication that the factor  $k \neq 0$  brings into play. Even though we can find some canonical conjugate variables with respect to Dirac brackets, the Hamiltonian becomes so involved in these variables that we do not have a clear recipe on how to map them to quantum operators. We shall thus exploit here the fact that we can relate easily  $a$  to  $T$  at the classical level.

If we use (75), together with the known expression for the torsion scalar (13), we can derive from the former an expression for the lapse

$$N^2 = -\frac{12\dot{a}^2 f''(T)}{a^2 f'(T)}. \quad (77)$$

By utilizing (75), its time derivative, and the lapse given by (77), in the Euler-Lagrange equations (24), we arrive at the supplementary conditions

$$f'(T) = \frac{a^2 f(T) + 2\kappa\rho_0 a^{-3w-1}}{2a^2 T + 12k} \quad (78a)$$

$$f(T) = \frac{\kappa\rho_0 a^{-3(w+1)}}{2k} [(w+1)a^2 T + 2k(3w+1)]. \quad (78b)$$

We now take the time derivative of the last relation, divide with  $\dot{T}$ , and set it equal to the first relation. Thus, we obtain a differential equation connecting  $T$  with  $a$

$$2(3w+1)\dot{a}(a^2 T + 6k) - a^3 \dot{T} = 0. \quad (79)$$

Its solution is

$$T(a) = \begin{cases} -\frac{6(3kw+k)}{(3w+2)a^2} + T_0 a^{2(3w+1)}, & \text{when } w \neq -\frac{2}{3} \\ \frac{T_0}{a^2} - 12k \frac{\ln a}{a^2}, & \text{when } w = -\frac{2}{3} \end{cases}, \quad (80)$$

where  $T_0$  is a constant of integration. Up to now, we have guaranteed the consistency of (78), and we need also to check the consistency with (75) and, of course, with the Euler-Lagrange equations. This can be done easily if we use (80) to write the  $f(T)$  as a function of the scale factor  $f(a)$ . From (78a), we obtain

$$f(a) = \begin{cases} \frac{T_0 \kappa\rho_0 (w+1) a^{3w+1}}{2k} - \frac{\kappa\rho_0 (3w+1) a^{-3(w+1)}}{3w+2}, & \text{when } w \neq -\frac{2}{3} \\ \frac{\kappa\rho_0}{a} \left( \frac{T_0}{6k} - 1 - 2 \ln a \right), & \text{when } w = -\frac{2}{3}. \end{cases} \quad (81)$$

By making the change  $f(T) \rightarrow f(a)$  in the Euler-Lagrange equations, we can easily see that the latter are satisfied with the substitution of (77), (80), and (81). From the first, we obtain the expression for the lapse function

$$N(t) = \begin{cases} \left[ \frac{6(3w+2)}{T_0(3w+2)a^{2(3w+2)}+6k} \right]^{\frac{1}{2}} \dot{a}, & \text{when } w \neq -\frac{2}{3} \\ \left( \frac{6}{T_0+6k-12k \ln a} \right)^{\frac{1}{2}} \dot{a}, & \text{when } w = -\frac{2}{3}. \end{cases} \quad (82)$$

Finally, the resulting space-time has the line element

$$ds_{w \neq -\frac{2}{3}}^2 = -\frac{6}{T_0 a^{2(3w+2)} + \frac{6k}{3w+2}} da^2 + a^2 \left[ d\chi^2 + \chi^2 \text{sinc}^2(\sqrt{k}\chi)(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (83)$$

or

$$ds_{w = -\frac{2}{3}}^2 = -\frac{6}{T_0 + 6k - 12k \ln a} da^2 + a^2 \left[ d\chi^2 + \chi^2 \text{sinc}^2(\sqrt{k}\chi)(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (84)$$

respectively. In this form, where the solution has been obtained in terms of the lapse, the scale factor  $a$  becomes effectively the ‘‘time’’ variable of the system.

Unfortunately, the generic expression (80) giving  $T(a)$  for  $w \neq -2/3$  cannot be easily inverted for all values of the parameters. Thus, it is not trivial to obtain the explicit form of the relative  $f(T)$  function for which the line element (83) is the solution. A case for which this is easily possible is when  $T_0 = 0$ . Then, the first of (80) is easily invertible, and we obtain from (81)

$$f(T) = \frac{\kappa \rho_0}{6^{\frac{3(w+1)}{2}} k} \left( -\frac{3w+2}{k(3w+1)} \right)^{\frac{3w+1}{2}} T^{\frac{3(w+1)}{2}}, \quad (85)$$

which corresponds to a power-law theory, with the power depending on the equation of state parameter. Thus, we can infer that the nonzero  $T_0$  signifies in this case deviations from this particular class of power-law theories.

The situation is easier in the  $w = -2/3$  case, where the relative expression in (80) can be inverted to give

$$a = \left[ \frac{6k}{T} W \left( \frac{e^{\frac{T_0}{6k}}}{6k} T \right) \right]^{\frac{1}{2}}, \quad (86)$$

where  $W(z)$  is the Lambert  $W$  function, which is defined as the solution to the equation  $W e^W = z$ . The corresponding  $f(T)$  theory can simply be given by substituting the above expression in the second branch of (81). Let us now proceed to see how to quantize this reduced system in each of the two cases.

### 1. The $w \neq -2/3$ case

In the attempt to construct a quantization scheme for these particular classes of  $f(T)$  theories, we will consider as a given the relations  $T = T(a)$  obtained in (80) and try to write a Lagrangian for the remaining degrees of freedom. We start by considering the generic case  $w \neq -2/3$ . We notice that by crudely substituting the corresponding relations from (80) and (81) into the Lagrangian (23), we are led to a new, reduced Lagrangian, which reads

$$L_{w \neq -\frac{2}{3}} = \frac{(3\rho_0(w+1)a^{-3w})}{4kN} \dot{a}^2 - N \frac{\rho_0(w+1)a^{-3w}(T_0(3w+2)a^{2(3w+2)}+6k)}{8k(3w+2)}. \quad (87)$$

Interestingly enough, the Euler-Lagrange equations of  $L_{w \neq -\frac{2}{3}}$ ,

$$N^2(T_0(3w+2)a^{2(3w+2)}+6k) - 6(3w+2)\dot{a}^2 = 0 \quad (88)$$

$$2\ddot{a} - 3w\frac{\dot{a}^2}{a} - 2\dot{a}\frac{\dot{N}}{N} + \frac{N^2}{a} \left( \frac{3kw}{3w+2} - \frac{T_0}{6}(3w+4)a^{6w+4} \right) = 0 \quad (89)$$

give rise to the correct lapse function  $N(t)$ , as seen in the first of (82). The latter satisfies both of the above equations. Hence, we have a Lagrangian giving rise to a dynamically equivalent solution to that of the gravitational system for the class of theories characterized by the principal branch of (81).

Lagrangian (87) describes a constrained system, whose Hamiltonian is (we avoid repeating in detail the formal Dirac procedure since in this case the treatment is the typical one encountered in most minisuperspace systems)

$$H_{w \neq -\frac{2}{3}} = N\mathcal{H}_{w \neq -\frac{2}{3}} + u_N p_N, \quad (90)$$

where  $p_N \approx 0$  is the primary constraint, and

$$\mathcal{H}_{w \neq -\frac{2}{3}} = \frac{\kappa a^{3w}}{3\rho_0(1+w)} p_a^2 - \frac{\rho_0(w+1)}{4} \left[ \frac{3}{(3w+2)a^{3w}} + \frac{T_0}{2k} a^{3w+4} \right] \approx 0 \quad (91)$$

is the secondary.

Both of them are first class, so in this case, we may proceed in the usual manner and try to construct a quantum operator for  $\mathcal{H}_{w \neq -\frac{2}{3}}$ . It is quite convenient in these cases to exploit the inherent parametrization invariance in these systems to give to the Hamiltonian constraint a form that

will be easier to lead to a quantum solution. For example, if we adopt the lapse reparametrization

$$N \mapsto \tilde{N} = \frac{\rho_0}{\frac{1}{8}(w+1)a^{-3w}\left(\frac{T_0 a^{6w+4}}{k} + \frac{6}{3w+2}\right)} N, \quad (92)$$

then we may write the same Hamiltonian as

$$H_{w \neq -\frac{2}{3}} = \tilde{N} \tilde{\mathcal{H}}_{w \neq -\frac{2}{3}} + u_{\tilde{N}} P_{\tilde{N}}, \quad (93)$$

where now

$$\tilde{\mathcal{H}}_{w \neq -\frac{2}{3}} = \frac{1}{h(a)} p_a^2 - \rho_0^2 \approx 0, \quad (94)$$

in which we have

$$h(a) = \frac{3T_0(w+1)^2}{8k^2} a^4 + \frac{9(w+1)^2}{4k(3w+2)a^{6w}}. \quad (95)$$

The Hamiltonian constraint (94) can be easily assigned to a Hermitian operator whose action on the wave function will result in a typical eigenequation. Such an operator is the following:

$$\hat{\mathcal{H}}_{w \neq -\frac{2}{3}} = \frac{1}{\mu(a)} \frac{\partial}{\partial a} \left( \mu(a) \frac{1}{h(a)} \frac{\partial}{\partial a} \right) - \rho_0^2, \quad (96)$$

which is Hermitian under the measure  $\mu(a)$ . For the natural measure  $\mu(a) = \sqrt{h(a)}$ , the differential part of the above operator becomes a one-dimensional Laplacian [89]. In this case, the solution to  $\hat{\mathcal{H}}_{w \neq -\frac{2}{3}} \Psi = 0$ , yields the wave function

$$\begin{aligned} \Psi(a) = & C_1 \exp\left(i\rho_0 \int \sqrt{h(a)} da\right) \\ & + C_2 \exp\left(-i\rho_0 \int \sqrt{h(a)} da\right). \end{aligned} \quad (97)$$

Assuming that we choose either an outgoing or an ingoing wave, i.e., have either one of the  $C_i$ 's zero, then the probability amplitude,  $P$ , ends up to be proportional to

$$P(a) \propto \mu(a) \Psi^* \Psi = \sqrt{h(a)}. \quad (98)$$

In Fig. 3, we present the graphs of the probability amplitude (98) and of the torsion scalar [first of Eq. (80)] with respect to the scale factor  $a$ . The latter is the classical relation that we enforced in order to construct the reduced minisuperspace Lagrangian. The values of the parameters have been chosen so that the solution (83) represents a Lorentzian line element for all values of  $a$ . In general, by inspecting (83), we have the following possibilities for a Lorentzian metric: (i)  $T_0 > 0$ ,  $k > 0$  and  $w > -2/3$ , (ii)  $T_0 > 0$ ,  $k < 0$  and  $w < -2/3$ , (iii)  $T_0 > 0$ ,  $k > 0$ ,  $w < -2/3$ , with an upper bound on  $a(t)$ , (iv)  $T_0 > 0$ ,  $k < 0$ ,  $w > -2/3$ , with a nonzero lower bound on  $a(t)$ , (v)  $T_0 < 0$ ,  $k > 0$ ,  $w > -2/3$  with a fully bounded  $a(t)$ , and (vi)  $T_0 < 0$ ,  $k < 0$ ,  $w < -2/3$ , again with a totally bounded scale factor. In Fig. 3, we present graphs belonging to cases i) and ii), where the scale factor can take values in the whole real line. The plots of the (classical)  $T(a)$  shows us the possibly problematic points, where the geometric scalar goes to infinity. In the first couple of graphs, we see a  $T(a)$  plot where the torsion scalar diverges both at  $a \rightarrow 0$  and at  $a \rightarrow +\infty$ . At the same point, the probability amplitude seem to also diverge. In the second pair, there are also two points of divergence—with an exception of the  $T_0 = 0$  case—for the  $T(a)$ . The problematic points are again,  $a \rightarrow 0$  and  $a \rightarrow +\infty$ ; the first divergence seems to be “cured” in the quantum description in the sense that the probability amplitude goes to zero at  $a = 0$ . However, for  $a \rightarrow +\infty$ , we also see a divergence at  $P(a)$ . Only the  $T_0 = 0$  case is free of this latter singularity. Finally, the last couple of graphs show that in the hyperbolic case of  $k = -1$ , the quantum probability amplitude becomes zero at the point of the classical singularity, which is now only at  $a \rightarrow 0$ . We need to mention, however, that the fact that we partly imposed the classical solution (80), in order to arrive at a quantum minisuperspace description, can possibly undermine the effort of lifting the classical singularity at the quantum level.

As in the previous cases, we can also enforce a crude normalization in terms of a delta function by extending the domain of the scale factor  $a(t)$  to negative values as well and thus consider  $a \in \mathbb{R}$ . The classical solution is in any case insensitive to the sign of  $a(t)$  since only  $a(t)^2$  appears inside the metric. With this consideration for the probability integral, we would have, for two states of matter  $\rho_0$  and  $\tilde{\rho}_0$ ,

$$\begin{aligned} |C_1|^2 \int_{-\infty}^{+\infty} \sqrt{h(a)} \Psi_{\rho_0}^* \Psi_{\rho_0} &= |C_1|^2 \int_{-\infty}^{+\infty} \sqrt{h(a)} e^{-i\tilde{\rho}_0 \int h(a)^{1/2} da} e^{i\rho_0 \int h(a)^{1/2} da} da \\ &= |C_1|^2 \int_{-\infty}^{+\infty} e^{-i(\tilde{\rho}_0 - \rho_0)s} ds = 2\pi |C_1|^2 \delta(\rho_0 - \tilde{\rho}_0), \end{aligned} \quad (99)$$

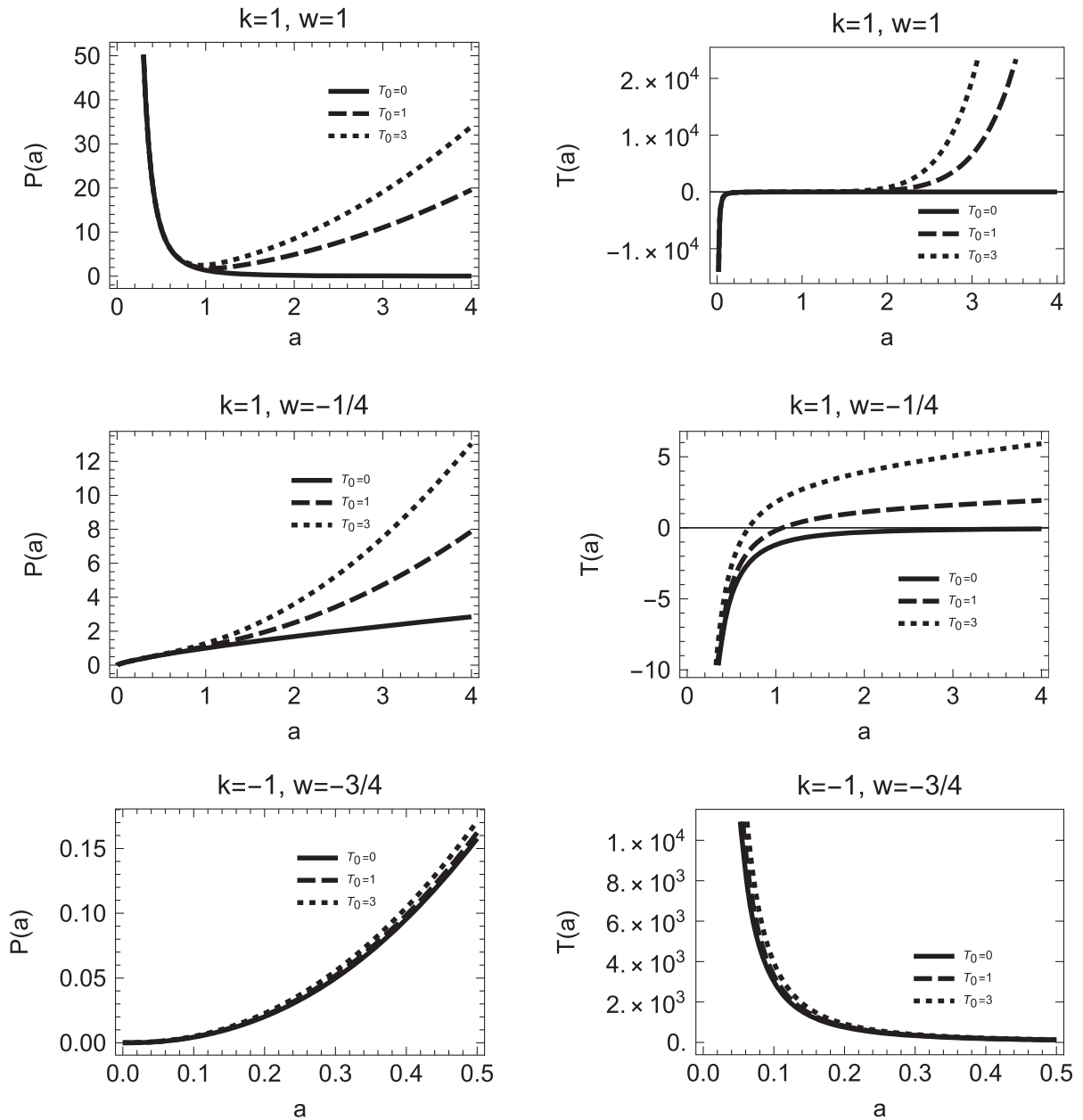


FIG. 3. A comparison of the probability amplitude  $P$  and the torsion scalar  $T$ , as functions of the scale factor  $a$ , for different  $f(T)$  theories characterized by different values of  $T_0$ .

where we made use of the change of variables  $a \rightarrow s = \int h(a)^{1/2} da$ ; it can be seen for the various values of the parameters that when  $a \in (-\infty, +\infty)$ , then also  $s \in (-\infty, +\infty)$ .

For completeness, we include the graphs of the Hubble function

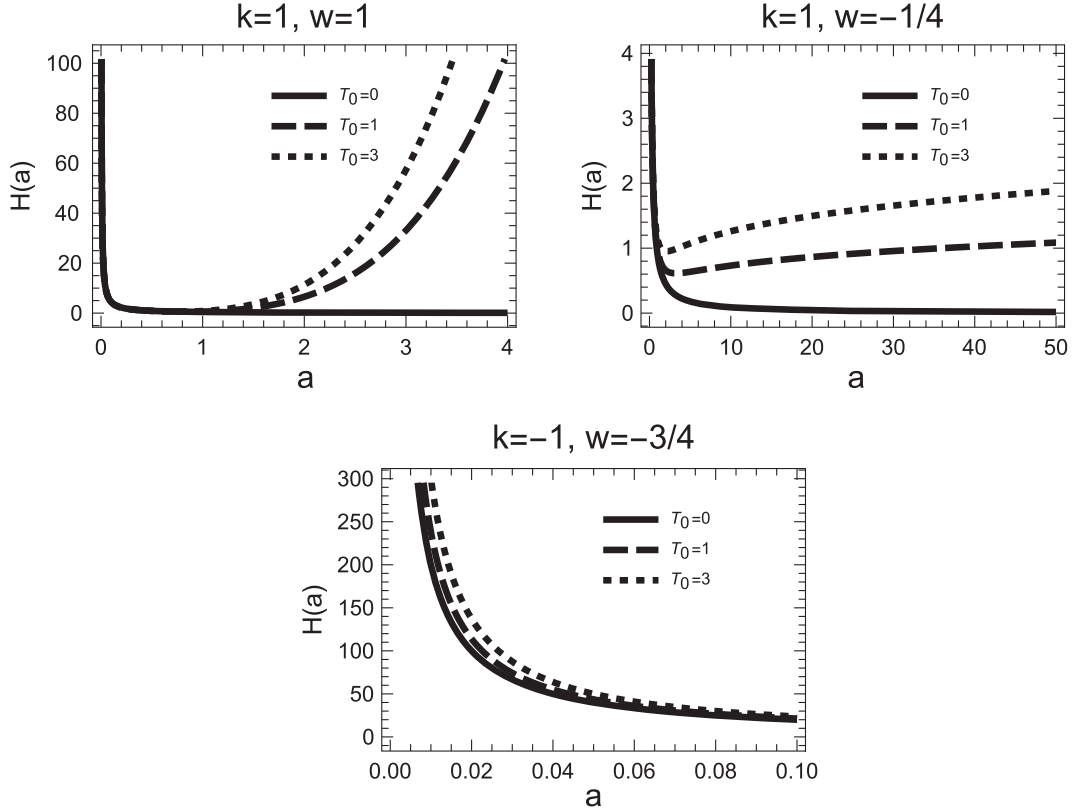
$$H(a) = \frac{1}{N} \frac{\dot{a}}{a} = \left( \frac{1}{6} T_0 a^{6w+2} + \frac{k}{a^2(3w+2)} \right)^{\frac{1}{2}}, \quad (100)$$

with respect to the scale factor  $a$ , which can be seen in Fig. 4. We can thus have a sense of the expansion rates that

each classical solution implies for different  $f(T)$  theories, characterized by  $T_0$ , in conjunction to the spatial curvature and the matter content. We observe that the closed universes, for  $T_0 \neq 0$ , give rise to ever faster late time acceleration rates, while in the  $k = -1$  configuration, we get a  $H(a)$  function, which tends to zero as  $a \rightarrow +\infty$ . It is a matter of which term dominates in Eq. (100).

## 2. The $w = -2/3$ case

We follow a similar procedure for the  $w = -2/3$  case. This time, the reduced minisuperspace Lagrangian that we construct as a function of  $N(t)$ ,  $a(t)$  and its derivative is


 FIG. 4. The Hubble function  $H(a)$ , for different  $f(T)$  theories,  $k$  and  $w$ .

$$L_{w=-\frac{2}{3}} = \frac{\rho_0 a^2}{4kN} \dot{a}^2 - N\rho_0 a^2 \left( \frac{12k \ln a + 18k - T_0}{24k} - 1 \right). \quad (101)$$

Once more, the Euler-Lagrange equations, which are equivalent to

$$N^2 - \frac{6\dot{a}^2}{-12k \ln a + 6k + T_0} = 0 \quad (102)$$

$$\ddot{a} - \frac{\dot{a}\dot{N}}{N} + \frac{\dot{a}^2}{a} + \frac{N^2}{a} \left( 2k \ln a - \frac{T_0}{6} \right) = 0, \quad (103)$$

are satisfied by the gravitational result, i.e., the second of (82). Thus, we have written a Lagrangian generating the expected dynamics.

In the exact same manner as before, we write the Hamiltonian and perform a reparametrization of the lapse function as

$$N \mapsto \tilde{N} = \frac{24k\rho_0}{a^2(6k + T_0 - 12k \ln a)} N \quad (104)$$

to obtain

$$H_{w=-\frac{2}{3}} = \tilde{N} \tilde{\mathcal{H}}_{w=-\frac{2}{3}} + u_{\tilde{N}} P_{\tilde{N}}, \quad (105)$$

where the Hamiltonian constraint assumes the same form as in (94)

$$\tilde{\mathcal{H}}_{w \neq -\frac{2}{3}} = \frac{1}{h(a)} p_a^2 - \rho_0^2 \approx 0, \quad (106)$$

where this time, the  $h(a)$  function is

$$h(a) = \frac{a^4(6k + T_0 - 12k \ln a)}{24k^2}. \quad (107)$$

The same analysis as before leads to an operator of the form (96) leading to a wave function given by (97), where, of course, this time, the  $h(a)$  function is the one we see in Eq. (107).

In Fig. 5, we give once more the probability amplitudes, the torsion scalar and the Hubble function with relation to the scale factor  $a$ . We have chosen to depict the  $k = -1$  case, which leads to the scale factor being bound from below to a nonzero value. The geometric singularity of  $T \rightarrow -\infty$  for this value seems to be resolved since it corresponds to zero probability amplitudes.

The procedure we followed in these last sections is strikingly different from the one which led as to the introduction of fractional operators in Sec. IV A. This is owed to the fact that, in the  $k \neq 0$  case, we were not able to assign a quantum operator to the obtained Hamiltonian



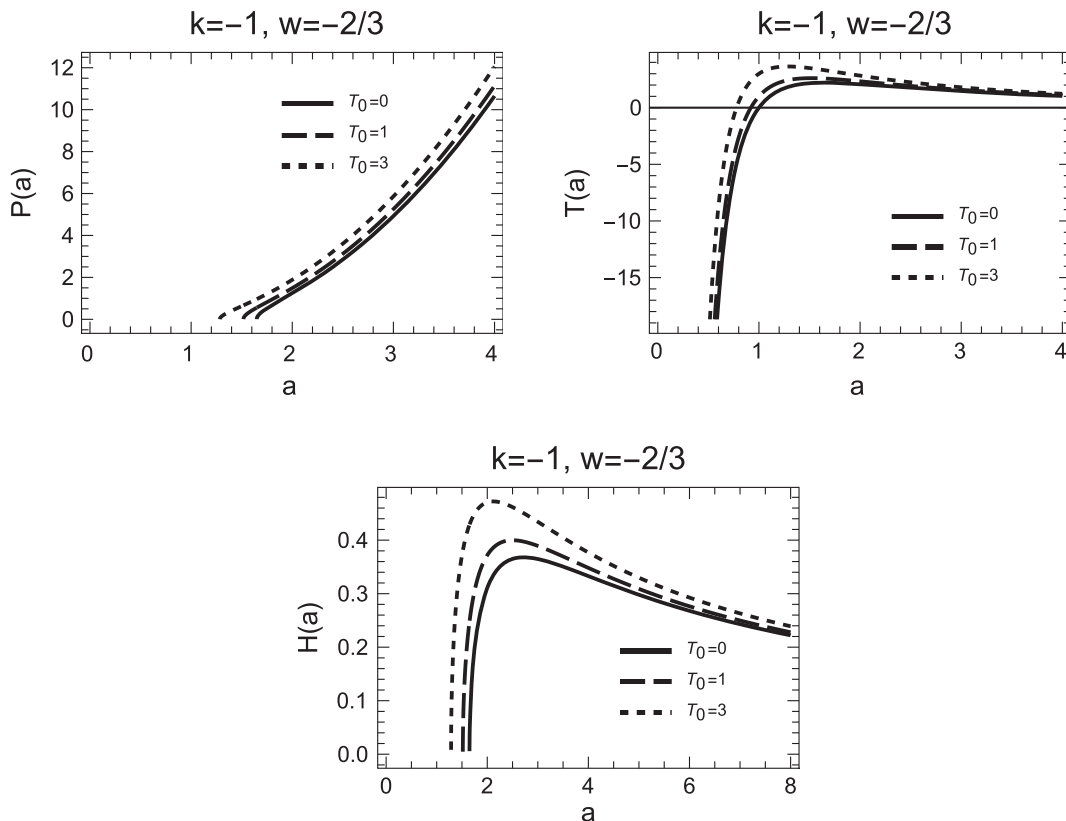


FIG. 5. The probability amplitude  $P$ , the torsion scalar  $T$  and Hubble function  $H$ , as functions of the scale factor  $a$ , for  $k = -1$ ,  $w = -2/3$  and for different  $T_0$  values.

after a pair of canonical variables with respect to Dirac brackets had been introduced. What we did instead here is exploit the fact that we deal with a particular subcase of the  $k \neq 0$  case, which leads to a specific relation between the scale factor and the torsion scalar. To unveil this specific relation, we partially integrated the classical system, and we afterward quantized the remaining, reduced system. We would intuitively expect that this second approach we follow, for the particular  $k \neq 0$  case, is an even cruder approximation of the quantum gravitational system (comparing with the usual quantum cosmology). This is due to the aforementioned successive enforcement of classical relations before quantization. Such an approximation could be possibly applied also to the case of Sec. IV A but not for a generic  $f(T)$  function since some information should be needed to obtain a specific  $T(a)$  relation. Unfortunately, the result of this section is not directly comparable with that of the generic  $k = 0$  case either. This is because, when setting  $k = 0$  to the starting relation (75), we obtain the case a trivial theory  $f(T) \sim \sqrt{T}$  in  $k = 0$ , whose Lagrangian is a total derivative.

## V. CONCLUSIONS

We performed a detailed analysis of canonical quantum cosmology in  $f(T)$ -gravity. With the help of the

Dirac-Bergmann algorithm, the constraints of the theory are revealed and classified into first and second class. We use the Dirac brackets as the basic relations with respect to which we proceed to the canonical quantization of the system.

Similarly to what happens in the  $f(Q)$ -theory [57], in the case of a perfect fluid and for a spatially flat FLRW metric, we arrive in a Hamiltonian where the momentum is raised in a power that is not necessarily an integer number, thus implying the necessity of putting in use the theory of fractional quantum mechanics. We use some examples in our attempt to interpret the square of the modulus of the wave function in a probabilistic manner, which gets maximized in certain values of the torsion scalar  $T$ , depending on the theory and its parameters. It is quite positive in the examples that we examined that probability amplitudes can be constructed that tend to zero as the scalar  $T$  diverges. In comparison with classical dynamics, we also revealed that the quantum description may even allow for values of  $T$  that are classically forbidden.

For the  $k \neq 0$  case, we were not able to treat the generic theory at the quantum level due to not being able to produce an appropriate mapping from the reduced Hamiltonian to a quantum operator. In the presence of spatial curvature, we restricted our study in the specific case, where Eq. (75) holds, leading to a specific relation  $T(a)$ . We exploited this

relation to reduce even further the classical system and perform a canonical quantization on what remains of it. Here, a typical quantization based purely on first class constraints can be applied. The singularity avoidance, in the sense of  $P \rightarrow 0$  as  $T \rightarrow \infty$ , was not as consistent as in the previous case. We cannot infer if this is an effect of reducing further the system by the application of the classical  $T(a)$  relation.

Apart from presenting our findings, we meant this work to also have a pedagogical approach. For this reason, we also made a comparison among the minisuperspace Lagrangians emerging in the modifications of the gravitational trinity. We stressed the idiosyncrasy of the Hamiltonian formulation in the case of teleparallel cosmology. The application of the Dirac-Bergmann algorithm requires a more careful treatment for systems with second class constraints; one cannot just blindly put in use the exact same process reserved for systems with purely first class constraints. The algorithm is a very important tool not only because it offers a way to proceed in a rigorous manner to the Hamiltonian formulation and subsequently to the canonical quantization, but also because it allows us to distinguish the physical degrees of freedom in a given problem.

In the case of field theories, the realization of the Dirac-Bergmann algorithm often requires the circumvention of certain difficulties of technical nature [90], and this is

something that has recently been underlined for the case of teleparallel theories of gravity [91] (see more recently [92] for an account of how surpassing the inherent difficulties in the case of  $f(Q)$  theory to offer a formal count of the physical degrees of freedom). In cosmology, however, we only have to deal with systems of finite degrees of freedom, where the application of the algorithm is straightforward. The main difficulty here lies in the quantization method and finding an appropriate mapping yielding the corresponding quantum operators.

This work completes a series of studies on quantum cosmology for the modified theories of gravity described by the scalar fields which form the trinity of general relativity. In the future, we plan to extend our analysis in the case of astrophysical objects.

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