Squeezing of light from Planck-scale physics

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In this article, the possibility of generating nonclassical light due to Planck-scale effects is considered. For this purpose, a widely studied model of deformation of the Heisenberg uncertainty relation is applied to single-mode and multimode lights. The model leads to a deformed dispersion relation, which manifests in an advancement in the time of arrival of photons. The key finding is that the model also leads to an oscillatory pattern of squeezing of the state of light. Furthermore, while the amplitude of the oscillations is constant for energy eigenstates, it exhibits linear growth over time for coherent states with the annihilation operator eigenvalue $\alpha \neq 0$. This second case leads to the accumulation of squeezing and phase-space displacement, which can be significant for astrophysical photons. In particular, for $\alpha \sim 1$, coherent light in the optical spectrum emitted at megaparsec distances would acquire squeezing with the amplitude of the order unity. This suggests that measurements of the nonclassical properties of light originating from distant astrophysical sources may open a window to test these predictions.

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I. INTRODUCTION

The concept of a minimal length is one of the most widely examined potential manifestations of quantum characteristics of spacetime. The premise here is that there exists a finite limit to the precision at which space can be probed, anticipated to be in the vicinity of the Planck length, $l_{\mathrm{Pl}} \approx 1.62 \times 10^{-35}$ m. This idea originates from quantum gravity considerations, which suggest that at such extreme scales, traditional concepts of spacetime break down and give way to quantum effects.

The idea of a minimal length has been successfully integrated into a self-consistent special-relativistic framework, commonly known as doubly special relativity [1,2]. This paradigm introduces a new invariant scale in addition to the speed of light, typically the Planck length, to reconcile quantum mechanics with relativity. In the broader landscape of quantum gravity theories, some approaches, such as loop quantum gravity (LQG), propose a possible generalization of this concept to a general-relativistic context, thus extending its scope and potential implications. Nevertheless, the actual behavior and impact of this minimal length scale in the dynamical sector remains a subject of ongoing study.

The concept of maximal spatial resolution is naturally implementable in the realm of quantum mechanics, particularly in relation to the Heisenberg uncertainty principle. This principle dictates inherent limits to the precision with which pairs of canonical variables, such as generalized position and momentum, can be simultaneously measured. By employing a generalized uncertainty principle (GUP) [3,4], it is feasible to integrate a minimal length scale into conventional quantum mechanics, thereby suggesting a fundamental limit to our ability to precisely measure position. Nonetheless, this adaptation inevitably implies that the commutation relation between the canonical variables, such as the generalized position and momentum, will undergo alterations.

Modifications to the commutation relation are understood to correlate with the nonlinear geometry of the corresponding phase space. Notably, the existence of a minimum length, a potential hallmark of quantum gravity theories, is intimately associated with a curvature in the momentum component of the phase space. This relationship provides a geometric interpretation for the minimal quantum length [5].

This concept is not confined to the domain of point particles—it extends into the broader field theoretical context. An instance of this expansion can be found in the LQG-inspired polymer quantization. In this scenario, studies such as those presented in Refs. [6,7] analyzed cylindrical (i.e., $\mathbb{R} \times \mathbb{S}$) deformations of the scalar field phase space. Other works in the context of string theory and

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quantum gravity also generalize the notion of phase space to deformed (possibly curved) ones [8–14]. The nonlinear field space theory (NFST) research program, as outlined in Ref. [15], is committed to methodically explore the implications of fields having a nontrivial phase space. In the context of NFST, the case of $\mathbb{S}^2$ phase-space generalization of the scalar field theory has been primarily considered [16,17].

The effects of a GUP can lead to phenomenological consequences, currently under intense investigation in the context of multimessenger astronomy [18]. The physical consequences that are primarily analyzed are deformed dispersion relations and vacuum birefringence.

Here, we focus our attention on the properties of quantum states of light. Specifically, we focus on single and multimode light with GUP in a field theoretical context. Potential manifestations of the GUP have been previously explored in the particle context in Ref. [19]. Our aim is to build upon these existing insights and delve deeper into the understanding of light under the influence of GUP.

This article is structured as follows: In Sec. II, we introduce the model of the GUP and the ensuing single-mode Hamiltonian. Following this, we carry out a perturbative analysis of the time evolution of a single-mode light in Sec. III. Based on these findings, we then examine the quantum squeezing of various states due to the GUP in Sec. IV. As a supplement to the squeezing effect, Sec. V delves into how the GUP modifies the dispersion relation for a multimode light state. In Sec. VI, we consider the potential experimental implications of our derived predictions. Last, in Sec. VII, we summarize our findings and discuss potential future directions.

II. GENERALIZED UNCERTAINTY PRINCIPLE

The most widely studied generalized uncertainty principle leading to a minimal length is of the form [20]

$$\Delta Q \Delta P \geq \frac{\hbar}{2}(1 + \beta \Delta P^2), \tag{1}$$

where ΔQ and ΔP are uncertainties on the generalized position and momenta, respectively. Here, β is a small dimensional parameter, serving as a measure of the strength of quantum gravity effects. This parameter is assumed to be positive definite here. However, negative values of β have also been studied in the literature [3,21].

Importantly, in this article, the framework of field theory is considered, for which $[Q] = E^{-1/2}$, $[P] = E^{1/2}$, and as a consequence $[\beta] = E^{-1}$. Since the modification is considered to be due to Planck-scale physics, it is therefore expected that $\beta \sim 1/E_{\mathrm{Pl}}$, where the Planck energy $E_{\mathrm{Pl}} \approx 1.22 \times 10^{19}$ GeV. Notably, this field theory scenario differs substantially from the usual case of a point particle, for which $[Q] = E^{-1}$, $[P] = E$, and consequently $[\beta] = E^{-2}$, leading to the β parameter being $\beta \sim 1/E_{\mathrm{Pl}}^2$. The differing dimensions of the canonical variables between both scenarios, therefore, suggest that the effects of new physics at the Planck scale transcribed by the GUP could have a more pronounced impact in the field theory context (where they are expected to be suppressed by E/E_{Pl} to some power) compared to the point particle scenario (where they are expected to be suppressed by E^2/E_{Pl}^2 to some power).

Additionally, while the GUP in the context of a point particle implies the existence of a minimal physical length, the situation differs in the field theoretical framework considered here. In this setting, the GUP does not give rise to a minimal length, but rather suggests a minimal field value.

The GUP given by Eq. (1) can be derived from the deformed commutation relation

$$[\hat{Q}, \hat{P}] = i\hbar(\hat{\mathbb{I}} + \beta \hat{P}^2). \tag{2}$$

This commutation relation is predicted by different Planck-scale physics models, such as relative locality [22] or loop quantum gravity [3,23].

An observation made in [24] is that this commutation relation (2) transforms into the standard one $[\hat{q}, \hat{p}] = i\hbar \mathbb{1}$ under the following change of variables:

$$\hat{Q} = \hat{q}, \tag{3}$$

$$\hat{P} = \frac{\tan(\sqrt{\beta}\hat{p})}{\sqrt{\beta}}. \tag{4}$$

This can be proven using the fact that for any function $f(\hat{p})$, $[\hat{q}, f(\hat{p})] = i\hbar \frac{df(\hat{p})}{d\hat{p}}$ if $[\hat{q}, \hat{p}] = i\hbar \hat{\mathbb{I}}$. Since the commutation relation changes, the above change of variables is not a canonical transformation.

This article aims to consider the quantum properties of light, taking into account the deformed commutation relation (2). Some studies in this direction have already been made in Refs. [25,26].

The simplest case we are going to begin with is a single-mode light for which the standard Hamiltonian takes the form

$$\hat{H} = \frac{1}{2}\left(\hat{P}^2 + \omega^2 \hat{Q}^2\right), \tag{5}$$

where ω denotes the frequency of the mode, so that $[\omega] = E$. Consequently, $[\beta\omega\hbar] = 1$. Importantly, the polarization states of light are not considered here but the amplitude of the field solely.

Employing the change of variables (3) and (4) and expanding the obtained expression up to the linear order in β leads to

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2) + \frac{\beta}{3}\hat{p}^4 + \mathcal{O}(\beta^2). \tag{6}$$

Worth mentioning is that only even powers of $\hat{p}$ are contributing to the series. From now on all $\mathcal{O}(\beta^2)$ contributions will be neglected, and we will focus only on the leading effect.

For further convenience, the full Hamiltonian (6) is decomposed into a free $\hat{H}_0 = \frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2)$, and an interaction $\hat{H}_1 = \frac{\beta}{3}\hat{p}^4$ parts, such that

$$\hat{H} = \hat{H}_0 + \hat{H}_1. \tag{7}$$

At this point, it is useful to introduce the standard creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ defined in the usual way:

$$\hat{q} \coloneqq \sqrt{\frac{\hbar}{2\omega}}(\hat{a}^\dagger + \hat{a}), \tag{8}$$

$$\hat{p} \coloneqq i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}), \tag{9}$$

so that $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbb{I}}$. The free Hamiltonian then reads $\hat{H}_0 = \frac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)$ and verifies $[\hat{H}_0, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger$ and $[\hat{H}_0, \hat{a}] = -\hbar\omega\hat{a}$. Furthermore, the interaction Hamiltonian reads

$$\hat{H}_1 = \frac{\beta}{3}\left(\frac{\hbar\omega}{2}\right)^2(\hat{a}^\dagger - \hat{a})^4. \tag{10}$$

By applying the standard perturbation theory, one can find that at the first order of the perturbative expansion, the Hamiltonian (7) eigenvalues are

$$\begin{aligned} E_n^{(1)} &= \langle n^{(0)}|\hat{H}_0 + \hat{H}_1|n^{(0)}\rangle \\ &= \langle n|\hbar\omega\left(\hat{N} + \frac{1}{2}\hat{\mathbb{I}}\right)|n\rangle + \langle n|\frac{\beta}{3}\hat{p}^4|n\rangle \\ &= \hbar\omega\left(n + \frac{1}{2}\right) + \beta\frac{\hbar^2\omega^2}{4}(2n^2 + 2n + 1), \end{aligned} \tag{11}$$

in which $\hat{N} \coloneqq \hat{a}^\dagger\hat{a}$. One easily recovers from this expression that the difference of energy levels at zeroth order in β is $E_n^{(0)} - E_m^{(0)} = \hbar\omega(n - m)$.

The associated eigenstates are, at first order in β,

$$\begin{aligned} |n^{(1)}\rangle &\coloneqq |n\rangle + \sum_{m\neq n}\frac{\langle m|\hat{H}_1|n\rangle}{E_n^{(0)} - E_m^{(0)}}|m\rangle \\ &= |n\rangle + \frac{\beta\hbar\omega}{12}\left[-\frac{1}{4}\sqrt{\frac{(n+4)!}{n!}}|n+4\rangle \right. \\ &\quad + \sqrt{n+1}\sqrt{n+2}(2n+3)|n+2\rangle \\ &\quad - \sqrt{n}\sqrt{n-1}(2n-1)|n-2\rangle \\ &\quad \left. + \frac{1}{4}\sqrt{\frac{n!}{(n-4)!}}|n-4\rangle\right]. \end{aligned} \tag{12}$$

In particular, the first-order vacuum energy is

$$E_0^{(1)} = \frac{\hbar\omega}{2}\left(1 + \beta\frac{\hbar\omega}{2}\right), \tag{13}$$

and the first-order vacuum state

$$|0^{(1)}\rangle = |0\rangle + \frac{\beta\hbar\omega}{12}\left(3\sqrt{2}|2\rangle - \sqrt{\frac{3}{2}}|4\rangle\right). \tag{14}$$

Therefore, the GUP correction slightly leverages the ground state energy.

III. TIME EVOLUTION

The time evolution of a single-mode light can be studied by introducing the following operator:

$$\hat{F}(t) \coloneqq \hat{U}_0^{-1}(t)\hat{U}(t), \tag{15}$$

where the unitary operator $\hat{U}_0(t) \coloneqq \exp(-\frac{i}{\hbar}\hat{H}_0 t)$ utilizes the free part of the Hamiltonian (7), whereas $\hat{U}(t) = \exp(-\frac{i}{\hbar}\hat{H}t)$. The operator $\hat{F}$ satisfies the equation

$$\frac{d\hat{F}(t)}{dt} = -\frac{i}{\hbar}\hat{H}_1^I(t)\hat{F}(t), \tag{16}$$

which has a solution in the form of a Dyson series:

$$\begin{aligned} \hat{F}(t) &= \hat{T}\exp\left(-\frac{i}{\hbar}\int_0^t \hat{H}_1^I(t')dt'\right) \\ &= \hat{\mathbb{I}} - \frac{i}{\hbar}\int_0^t \hat{H}_1^I(t')dt' + \mathcal{O}(1/\hbar^2), \end{aligned} \tag{17}$$

where

$$\hat{H}_1^I(t) \coloneqq \hat{U}_0^{-1}(t)\hat{H}_1\hat{U}_0(t) \tag{18}$$

is the interaction Hamiltonian in the interaction picture. Furthermore, $\hat{T}$ is the time ordering operator, and the operator $\hat{F}(t)$ satisfies the initial condition $\hat{F}(0) = \hat{\mathbb{I}}$.

The time evolution of an initial state $|\Psi(0)\rangle$ is given by

$$\begin{aligned} |\Psi(0)\rangle &= \hat{U}(t)|\Psi(0)\rangle \\ &= \hat{U}_0(t)\hat{F}(t)|\Psi(0)\rangle \\ &= \hat{U}_0(t)|\Psi(0)\rangle \\ &\quad -\frac{i}{\hbar}\hat{U}_0(t)\int_0^t \hat{H}_1^I(t')dt'|\Psi(0)\rangle + \mathcal{O}(1/\hbar^2). \end{aligned} \tag{19}$$

From now on, all $\mathcal{O}(1/\hbar^2)$ contributions will be neglected, and we will focus on the leading effect only.

Employing the Baker-Campbell-Hausdorff formula, one can find that

$$\hat{U}_0^{-1}(t)\hat{a}^\dagger\hat{U}_0(t) = \hat{a}^\dagger e^{i\omega t}, \tag{20}$$

$$\hat{U}_0^{-1}(t)\hat{a}\hat{U}_0(t) = \hat{a}e^{-i\omega t}. \tag{21}$$

The use of the above leads to

$$\hat{H}_1^I(t) = \frac{\beta}{3}\left(\frac{\hbar\omega}{2}\right)^2\left(\hat{a}^\dagger e^{i\omega t} - \hat{a}e^{-i\omega t}\right)^4, \tag{22}$$

for $\hat{H}_1$ given by the interaction term in Eq. (6).

When developing this expression and using the commutation relation between the creation and annihilation operators, one gets the following form of the interaction Hamiltonian in the interaction picture[1]:

$$\begin{aligned} \hat{H}_1^I(t) = \frac{\beta\hbar^2\omega^2}{12}&\left[(\hat{a}^\dagger)^4 e^{4i\omega t} - \left(4(\hat{a}^\dagger)^3\hat{a} + 6(\hat{a}^\dagger)^2\right)e^{2i\omega t}\right. \\ &+ \left(6(\hat{a}^\dagger)^2(\hat{a})^2 + 12\hat{a}^\dagger\hat{a} + 3\hat{\mathbb{I}}\right) \\ &\left.- \left(4\hat{a}^\dagger(\hat{a})^3 + 6(\hat{a})^2\right)e^{-2i\omega t} + (\hat{a})^4 e^{-4i\omega t}\right]. \end{aligned} \tag{25}$$

The time integral contributing to the series formula (19) can be analytically evaluated and decomposed into real and imaginary parts:

$$\int_0^t \hat{H}_1(t')dt' = \beta\frac{\hbar^2\omega}{24}\left(\hat{\mathcal{R}}(t) + i\hat{\mathcal{I}}(t)\right), \tag{26}$$

with

$$\begin{aligned} \hat{\mathcal{R}}(t) = 2\omega t&\left[6(\hat{a}^\dagger)^2(\hat{a})^2 + 12\hat{a}^\dagger\hat{a} + 3\hat{\mathbb{I}}\right] \\ &+ \frac{1}{2}\sin(4\omega t)\left[(\hat{a}^\dagger)^4 + (\hat{a}^4)\right] \\ &- \sin(2\omega t)\left[4(\hat{a}^\dagger)^3\hat{a} + 6(\hat{a}^\dagger)^2\right. \\ &\left.+ 4\hat{a}^\dagger(\hat{a})^3 + 6(\hat{a})^2\right] \end{aligned} \tag{27}$$

and

$$\begin{aligned} \hat{\mathcal{I}}(t) = \frac{1}{2}\left(1 - \cos(4\omega t)\right)&\left[(\hat{a}^\dagger)^4 - (\hat{a}^4)\right] + \left(1 - \cos(2\omega t)\right) \\ &\times\left[-4(\hat{a}^\dagger)^3\hat{a} - 6(\hat{a}^\dagger)^2 + 4\hat{a}^\dagger(\hat{a})^3 + 6(\hat{a})^2\right]. \end{aligned} \tag{28}$$

IV. SQUEEZING

In the phase-space formulation of quantum mechanics the system is described by a quantum state in a space defined by generalized position and momentum coordinates, known as phase space. Quantum squeezing corresponds in this context to the distortion of the quantum state, represented, e.g., by its Wigner quasiprobability function $W(q,p)$, in this phase space. More specifically, it refers to the reduction/spread of the uncertainty in one direction (say, position) at the expense of increasing/decreasing it in the conjugate dimension (momentum), ensuring compliance with the Heisenberg or more generally the Robertson-Schrödinger uncertainty relation. The squeezed state is represented as an ellipse rather than a circle (the latter corresponding to equal uncertainty in both dimensions) in the phase space. Worth emphasizing is that quantum squeezing is of particular interest in quantum optics and quantum information science, where it can improve measurement precisions and information processing capabilities by reducing quantum noise and enhancing signal strength in a specific direction (see, e.g., Refs. [27,28]).

The squeezing prompted by the GUP given by Eq. (1) is examined in the next section, focusing on two classes of states: quantum harmonic oscillator energy eigenstates $|n\rangle$ and Glauber's coherent states $|\alpha\rangle$, which play key roles in quantum optics and quantum information sciences.

Squeezing properties are quantified by the first and second moments of the $\hat{q}$ and $\hat{p}$ operators. We introduce

$$\langle\hat{q}\rangle \coloneqq \langle\Psi(t)|\hat{q}|\Psi(t)\rangle, \tag{29}$$

$$\langle\hat{p}\rangle \coloneqq \langle\Psi(t)|\hat{p}|\Psi(t)\rangle, \tag{30}$$

$$\Delta\hat{q} \coloneqq \sqrt{\langle\Psi(t)|\hat{q}^2|\Psi(t)\rangle - (\langle\Psi(t)|\hat{q}|\Psi(t)\rangle)^2}, \tag{31}$$

$$\Delta\hat{p} \coloneqq \sqrt{\langle\Psi(t)|\hat{p}^2|\Psi(t)\rangle - (\langle\Psi(t)|\hat{p}|\Psi(t)\rangle)^2}, \tag{32}$$

[1]It may be useful to observe that

$$\hat{a}(\hat{a}^\dagger)^n = j(\hat{a}^\dagger)^{n-1} + (\hat{a}^\dagger)^j\hat{a}(\hat{a}^\dagger)^{n-j}, \tag{23}$$

$$(\hat{a})^n\hat{a}^\dagger = j(\hat{a})^{n-1} + (\hat{a})^{n-j}\hat{a}^\dagger(\hat{a})^j, \tag{24}$$

for all integers $j \in [0,n]$.

$$C_{qp} \coloneqq \langle\Psi(t)|(\hat{q} - \langle\hat{q}\rangle)(\hat{p} - \langle\hat{p}\rangle)|\Psi(t)\rangle_{\text{Weyl}} = \frac{1}{2}\langle\Psi(t)|(\hat{q}\,\hat{p} + \hat{p}\,\hat{q})|\Psi(t)\rangle - \langle\hat{q}\rangle\langle\hat{p}\rangle, \tag{33}$$

so that the Robertson-Schrödinger uncertainty principle holds:

$$(\Delta\hat{q})^2(\Delta\hat{p})^2 - C_{qp}^2 \geq \hbar^2/4. \tag{34}$$

In the definition of the covariance C_{qp} the Weyl symmetrization is applied. Furthermore, we will add superscript $^{(0)}$ in case of the zeroth-order formulas ($\beta \to 0$).

Alternatively, the Robertson-Schrödinger uncertainty can be written as

$$(\Delta\hat{q})^2(\Delta\hat{p})^2(1 - \rho^2) \geq \hbar^2/4, \tag{35}$$

where ρ is the dimensionless correlation coefficient

$$\rho \coloneqq \frac{C_{qp}}{\Delta\hat{q}\Delta\hat{p}}. \tag{36}$$

When ρ is different from zero, the semiaxes of the ellipsoid of covariance do not overlap with the directions q and p.

To quantify this effect, it is convenient to introduce the covariance matrix

$$\mathbf{\Sigma} \coloneqq \begin{bmatrix} \omega(\Delta\hat{q})^2 & C_{qp} \\ C_{qp} & (\Delta\hat{p})^2/\omega \end{bmatrix}, \tag{37}$$

where the ω factor has been introduced for dimensional reasons. The eigenvalues of the previous matrix are

$$\lambda_\pm = \frac{1}{2}\left[\mathrm{tr}\mathbf{\Sigma} \pm \sqrt{(\mathrm{tr}\mathbf{\Sigma})^2 - 4\det\mathbf{\Sigma}}\right], \tag{38}$$

where $\mathrm{tr}\mathbf{\Sigma} = \omega(\Delta\hat{q})^2 + (\Delta\hat{p})^2/\omega$ and $\det\mathbf{\Sigma} = (\Delta\hat{q})^2 \times (\Delta\hat{p})^2(1-\rho^2)$. Because of the square root in Eq. (38), the $\mathcal{O}(\beta^2)$ factors could in principle bring a contribution of the β order in the squeezing amplitude. However, the terms of the order β^2 in $(\Delta\hat{q})^2$, in $(\Delta\hat{p})^2$, and in C_{qp} bring no contribution of the order β^2 in $[(\mathrm{tr}\mathbf{\Sigma})^2 - 4\det\mathbf{\Sigma}]$ whatever the state considered (see Appendix). In consequence, the unknown factors do not contribute to the linear in β expressions for the eigenvalues $\lambda_\pm$.

Importantly, the square roots of the eigenvalues have interpretations of major and minor diameters of the ellipsoid of covariance, respectively, so that the uncertainty relation (34) takes the form

$$\sqrt{\lambda_+}\sqrt{\lambda_-} \geq \hbar/2. \tag{39}$$

In the eigenframe the correlation vanishes, and the relative values of $\sqrt{\lambda_+}$ and $\sqrt{\lambda_-}$ can be used to quantify the squeezing of the state. Specifically, after suitable normalization of the variables, we can write

$$\sqrt{\lambda_+} = \sqrt{\frac{\hbar}{2}}e^r, \tag{40}$$

$$\sqrt{\lambda_-} = \sqrt{\frac{\hbar}{2}}e^{-r}, \tag{41}$$

where r is the squeezing amplitude related to the complex squeezing parameter $\xi = |r|e^{i\gamma}$. Geometrically, $\gamma/2$ is the angle between the minor axes of the ellipsoid and the q axis. The ξ parameter enters the squeezing operator $\hat{S}(\xi)$ as follows:

$$\hat{S}(\xi) \coloneqq \exp\left(\frac{1}{2}\left(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2}\right)\right). \tag{42}$$

A. Squeezing of quantum harmonic oscillator eigenstates: $|\Psi(0)\rangle = |n\rangle$

The free evolution of a $|n\rangle$ state under the harmonic oscillator Hamiltonian is given by

$$\hat{U}_0|n\rangle = e^{-i\omega t(n+\frac{1}{2})}|n\rangle. \tag{43}$$

To compute the evolution under the full Hamiltonian at first order in β, one needs to use

$$\hat{H}_1^I|n\rangle = \frac{\beta\hbar^2\omega^2}{12}\left[e^{4i\omega t}\sqrt{n+1}\sqrt{n+2}\sqrt{n+3}\sqrt{n+4}|n+4\rangle - e^{2i\omega t}\sqrt{n+1}\sqrt{n+2}(4n+6)|n+2\rangle + (6n^2+6n+3)|n\rangle - e^{-2i\omega t}\sqrt{n}\sqrt{n-1}(4n-2)|n-2\rangle + e^{-4i\omega t}\sqrt{n}\sqrt{n-1}\sqrt{n-2}\sqrt{n-3}|n-4\rangle\right], \tag{44}$$

obtained from Eq. (25). Recall that $n \geq 0$ for the initial state to be well defined. Similarly, any state $|n-m\rangle$ is well defined if and only if $(n-m) \geq 0$.

Under the evolution given Eq. (19) the state $|n\rangle$ at any time boils down to

$$
\begin{aligned}
|\Psi(t)\rangle = e^{-i\omega t(n+\frac{1}{2})}|n\rangle - \frac{\beta\hbar\omega}{24}\Bigg[&\frac{1}{2}\left(e^{-i\omega t(n+\frac{1}{2})} - e^{-\frac{i\omega t}{2}(2n+9)}\right)\frac{\sqrt{(n+4)!}}{\sqrt{n!}}|n+4\rangle \\
&- \left(e^{-i\omega t(n+\frac{1}{2})} - e^{-\frac{i\omega t}{2}(2n+5)}\right)\frac{\sqrt{(n+2)!}}{\sqrt{n!}}(4n+6)|n+2\rangle + 2i\omega t e^{-i\omega t(n+\frac{1}{2})}(6n^2+6n+3)|n\rangle \\
&+ \left(e^{-i\omega t(n+\frac{1}{2})} - e^{-\frac{i\omega t}{2}(2n-3)}\right)\frac{\sqrt{n!}}{\sqrt{(n-2)!}}(4n-2)|n-2\rangle - \frac{1}{2}\left(e^{-i\omega t(n+\frac{1}{2})} - e^{-\frac{i\omega t}{2}(2n-7)}\right)\frac{\sqrt{n!}}{\sqrt{(n-4)!}}|n-4\rangle\Bigg]. \quad (45)
\end{aligned}
$$

It should be emphasized here that the normalization $\langle\Psi|\Psi\rangle = 1 + \mathcal{O}(\beta^2)$ differs from unity at second order in β only, not at the $\mathcal{O}(\beta)$ order.

For an evolution governed by the corrected Hamiltonian $\hat{H}$, the q and p first moments, computed using the state (45), vanish at any time:

$$\langle\hat{q}\rangle = 0 = \langle\hat{p}\rangle. \quad (46)$$

Because of the $\hat{p}^4$-type of the interaction term, this is satisfied at any order in β. For the same reason, any odd power of $\hat{q}$ and $\hat{p}$, including the mixed terms, will also vanish.

The mean value of the $|n\rangle$ state in the phase space is therefore not modified by the new dynamics. Its dispersions in q and p, however, vary as

$$(\Delta\hat{q})^2 = \frac{\hbar}{2\omega}(1+2n) + \beta\hbar^2\sin^2(\omega t)(2n^2+2n+1) + \mathcal{O}(\beta^2), \quad (47)$$

$$(\Delta\hat{p})^2 = \frac{\hbar\omega}{2}(1+2n) - \beta\hbar^2\omega^2\sin^2(\omega t)(2n^2+2n+1) + \mathcal{O}(\beta^2). \quad (48)$$

The covariance evaluated at any time writes

$$C_{qp} = \frac{\beta\hbar^2\omega}{2}\sin(2\omega t)(2n^2+2n+1) + \mathcal{O}(\beta^2). \quad (49)$$

One can easily verify that the leading order in the β contribution to the Robertson-Schrödinger relation cancels out so that

$$\frac{\hbar^2}{4}(1+2n)^2 + \mathcal{O}(\beta^2) \geq \frac{\hbar^2}{4}. \quad (50)$$

Importantly, the first order in β corrections to the quadratic moments of $\hat{q}$ and $\hat{p}$ oscillate in time at the frequency ω. There is, therefore, no accumulative contribution to the dispersions of the state at this order.

For the case under consideration, the square roots of the eigenvalues of the covariance matrix (37) can be expressed as follows:

$$\sqrt{\lambda_+} = \sqrt{\frac{\hbar}{2}}\sqrt{1+2n}e^{r}, \quad (51)$$

$$\sqrt{\lambda_-} = \sqrt{\frac{\hbar}{2}}\sqrt{1+2n}e^{-r}, \quad (52)$$

where the squeezing amplitude r for an arbitrary state $|n\rangle$ is given by

$$r = \frac{1+2n+2n^2}{1+2n}\beta\hbar\omega|\sin(\omega t)| + \mathcal{O}(\beta^2). \quad (53)$$

Consequently, for large n, the squeezing amplitude grows as $\mathcal{O}(n)$.

For the vacuum state $|\Psi(0)\rangle = |0\rangle$ the dispersions in q and p and the covariance reduce to

$$(\Delta\hat{q})^2 = \frac{\hbar}{2\omega}\left[1 + 2\beta\hbar\omega\sin^2(\omega t) + \mathcal{O}(\beta^2)\right], \quad (54)$$

$$(\Delta\hat{p})^2 = \frac{\hbar\omega}{2}\left[1 - 2\beta\hbar\omega\sin^2(\omega t) + \mathcal{O}(\beta^2)\right], \quad (55)$$

$$C_{qp} = \beta\frac{\hbar^2\omega}{2}\sin(2\omega t) + \mathcal{O}(\beta^2), \quad (56)$$

and the squeezing amplitude Eq. (53) boils down to

$$r = \beta\hbar\omega|\sin(\omega t)| + \mathcal{O}(\beta^2). \quad (57)$$

The maximal squeezing magnitude expected for the corrected vacuum state at first order in β expansion is therefore

$$r_{\max,|0\rangle} \approx \beta\hbar\omega. \quad (58)$$

B. Squeezing of Glauber's coherent states: $|\Psi(0)\rangle = |\alpha\rangle$

The second example we are going to consider is a state that is initially Glauber's coherent state of a harmonic oscillator:

$$|\Psi(0)\rangle = |\alpha\rangle \coloneqq e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad (59)$$

introduced such that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where $\alpha = |\alpha|e^{i\theta} \in \mathbb{C}$. Let us remind the reader that the free evolution of such a state is given by

$$|\Psi^{(0)}(t)\rangle \coloneqq \hat{U}_0(t)|\alpha\rangle = e^{-i\frac{\omega t}{2}}|\alpha e^{-i\omega t}\rangle. \tag{60}$$

In this state, the $\hat{q}$ and $\hat{p}$ mean values are

$$\langle \hat{q}\rangle^{(0)} = |\alpha|\sqrt{\frac{2\hbar}{\omega}}\cos(\theta - \omega t), \tag{61}$$

$$\langle \hat{p}\rangle^{(0)} = -|\alpha|\sqrt{2\omega\hbar}\sin(\theta - \omega t), \tag{62}$$

and the standard deviations and the covariance write

$$\Delta\hat{q}^{(0)} = \sqrt{\frac{\hbar}{2\omega}}, \tag{63}$$

$$\Delta\hat{p}^{(0)} = \sqrt{\frac{\hbar\omega}{2}}, \tag{64}$$

$$C_{qp}^{(0)} = 0, \tag{65}$$

which are constant in time.

The right-hand action of the interaction Hamiltonian on the coherent state $|\alpha\rangle$ is given by

$$\hat{H}_1^I(t)|\alpha\rangle = \frac{\beta\hbar^2\omega^2}{12}\big[(\hat{a}^\dagger)^4 e^{4i\omega t} - \big(4\alpha(\hat{a}^\dagger)^3 + 6(\hat{a}^\dagger)^2\big)e^{2i\omega t} + \big(6\alpha^2(\hat{a}^\dagger)^2 + 12\alpha\hat{a}^\dagger + 3\mathbb{1}\big) - \big(4\alpha^3\hat{a}^\dagger + 6\alpha^2\big)e^{-2i\omega t} + (\alpha)^4 e^{-4i\omega t}\big]|\alpha\rangle, \tag{66}$$

which allows one to evaluate the mean values of $\hat{q}$ and $\hat{p}$ operators:

$$\langle\hat{q}\rangle = |\alpha|\sqrt{\frac{2\hbar}{\omega}}\cos(\theta - \omega t) + \frac{1}{6\sqrt{2}}|\alpha|\beta\omega\hbar\sqrt{\frac{\hbar}{\omega}}\Big(12(|\alpha|^2+1)\omega t\sin(\theta)\cos(\omega t) - 2\sin(\omega t)\big[-6\sin(\theta) + |\alpha|^2\big(-6\sin(\theta) + \sin(3\theta) + 3\sin(3\theta - 2\omega t)\big) + 6(|\alpha|^2+1)\omega t\cos(\theta)\big]\Big) + \mathcal{O}(\beta^2), \tag{67}$$

$$\langle\hat{p}\rangle = -|\alpha|\sqrt{2\omega\hbar}\sin(\theta - \omega t) + \frac{1}{3}\sqrt{2}|\alpha|\beta(\omega\hbar)^{3/2}\big[|\alpha|^2\sin^2(\omega t)\sin(3\theta - \omega t) - 3(|\alpha|^2+1)(\omega t\cos(\theta - \omega t) - \cos(\theta)\sin(\omega t))\big] + \mathcal{O}(\beta^2). \tag{68}$$

A displacement of the mean values of $\hat{q}$ and $\hat{p}$ due to the GUP correction is exhibited. Importantly, the correction is not only oscillatory in time but, due to the multiplicative ωt factor, a time-cumulative effect appears.

The phase factor θ plays no important role in our discussion and thus can be fixed at $\theta = 0$ for the simplicity of the further analysis. Then, at late times the formulas (67) and (68) are well approximated by

$$\langle\hat{q}\rangle \approx |\alpha|\sqrt{\frac{2\hbar}{\omega}}\big[\cos(\omega t) - \beta\omega\hbar(\omega t)(|\alpha|^2+1)\sin(\omega t)\big], \tag{69}$$

$$\langle\hat{p}\rangle \approx |\alpha|\sqrt{2\omega\hbar}\big[\sin(\omega t) - \beta\omega\hbar(\omega t)(|\alpha|^2+1)\cos(\omega t)\big]. \tag{70}$$

The expectation value of the annihilation operator $\hat{a}$ (which gives the mean location of the state on the complex plane representation of the phase space) becomes

$$\langle\hat{a}\rangle = \sqrt{\frac{2\omega}{\hbar}}\langle\hat{q}\rangle + i\sqrt{\frac{1}{2\hbar\omega}}\langle\hat{p}\rangle \approx |\alpha|\big(1 - \beta\omega\hbar(\omega t)(|\alpha|^2+1)\sin(2\omega t)\big)e^{i\varphi}, \tag{71}$$

where

$$\varphi \approx \omega t\big[1 - 2\beta\omega\hbar(\omega t)(|\alpha|^2+1)\big]. \tag{72}$$

At leading order in β, the displacement of the state from the origin of the phase space follows oscillations whose amplitude grows linearly in time.

The variances and the covariance of the canonically conjugated variables q and p can be evaluated with the use of Eq. (66), leading to

$$(\Delta\hat{q})^2 = \frac{\hbar}{2\omega} - \frac{1}{6}\beta(\hbar^2(|\alpha|^2(|\alpha|(-9\cos(\theta) - 4|\alpha|\sin(\omega t)(\sin(4\theta - 3\omega t) - |\alpha|\sin(5\theta - 3\omega t)) + 6|\alpha|\omega t\sin(2\theta - 2\omega t) + 9\cos(\theta - 2\omega t)) - 3\cos(2\omega t)(2\omega t\sin(2\theta) + 1) + 3\sin(2\omega t)(\sin(2\theta) + 2\omega t\cos(2\theta) + \omega t)) + 3(|\alpha|^2 + \cos(2\omega t) - 1))) + \mathcal{O}(\beta^2), \tag{73}$$

$$(\Delta\hat{p})^2 = \frac{\hbar\omega}{2} + \frac{1}{6}\beta\omega^2(\hbar^2(|\alpha|^2(|\alpha|(-9\cos(\theta) - 4\alpha\sin(\omega t)(\sin(4\theta - 3\omega t) - |\alpha|\sin(5\theta - 3\omega t)) + 6|\alpha|\omega t\sin(2\theta - 2\omega t) + 9\cos(\theta - 2\omega t)) - 3\cos(2\omega t)(2\omega t\sin(2\theta) + 1) + 3\sin(2\omega t)(\sin(2\theta) + 2\omega t\cos(2\theta) + \omega t)) + 3(|\alpha|^2 + \cos(2\omega t) - 1))) + \mathcal{O}(\beta^2), \tag{74}$$

$$C_{qp} = -\frac{1}{6}\beta\omega\hbar^2(-4|\alpha|^4\sin(\omega t)\cos(4\theta - 3\omega t) + 6(2|\alpha|^2 + 3)|\alpha|^2\omega t\cos(2\theta - 2\omega t) + (|\alpha|^2(|\alpha|^2 + 4)\cos(2\theta) - 3(2|\alpha|^4 + 4|\alpha|^2 + 1))\sin(2\omega t)) + \mathcal{O}(\beta^2). \tag{75}$$

In the $|\alpha| \to 0$ limit the case of the vacuum state $|0\rangle$, for which Eqs. (54)–(56) hold, is correctly recovered. Furthermore, the Robertson-Schrödinger relation gains no correction in the order linear in β, i.e.,

$$\frac{\hbar^2}{4} + \mathcal{O}(\beta^2) \geq \frac{\hbar^2}{4}. \tag{76}$$

By applying the formula (41) and the results of Appendix, the squeezing amplitude r for the coherent state with the β correction can be found. Following Appendix the formula can be written as

$$r = \frac{\beta}{\hbar}\sqrt{X^2 + Y^2} + \mathcal{O}(\beta^2), \tag{77}$$

where the X and Y functions can be read out from the expressions (73)–(75).

Similar to the study of the first-order moments, the phase θ is of no importance for our discussion and is therefore fixed at $\theta = 0$ for the simplicity of the analysis. Introducing $\phi \coloneqq \omega t$, it follows:

$$\frac{r}{\beta\omega\hbar} = \frac{1}{6}\Big[4\sin^2(\phi)\big[|\alpha|^2(2(|\alpha| - 1)|\alpha|^2\sin(3\phi) + 3(2|\alpha|^2 - 3)\phi\cos(\phi) + (9|\alpha| - 3)\sin(\phi)) + 3\sin(\phi)\big]^2 + \big[\sin(2\phi)(3|\alpha|^4 + 8|\alpha|^2 + 4|\alpha|^4\cos(2\phi) + 3) - 6|\alpha|^2(2|\alpha|^2 + 3)\phi\cos(2\phi)\big]^2\Big]^{1/2} + \mathcal{O}(\beta). \tag{78}$$

Importantly, this expression exhibits a factor linear in ϕ, resulting in the accumulation of squeezing over time. This is in contrast to the case of energy eigenstates $|n\rangle$ discussed before. At late times, i.e., $\phi \gg 1$, the term with the contribution linear in ϕ dominates over the accompanying oscillatory ones, and the formula (78) is well approximated by

$$\frac{r}{\beta\omega\hbar} \approx \frac{\phi|\alpha|^2}{2\sqrt{2}}\sqrt{45 + 36|\alpha|^2 + 20|\alpha|^4 + 3(4|\alpha|^4 + 20|\alpha|^2 + 9)\cos(4\phi)}. \tag{79}$$

In consequence, at late times, the squeezing exhibits oscillatory behavior with frequency 4ω, and the amplitude of the oscillations belonging to the range $[r_{\min}, r_{\max}]$, where

$$r_{\min,|\alpha\rangle} \approx \beta\omega\hbar\frac{1}{2}|(2|\alpha|^2 - 3)||\alpha|^2\omega t \tag{80}$$

and

$$r_{\max,|\alpha\rangle} \approx \beta\omega\hbar|(2|\alpha|^2 + 3)||\alpha|^2\omega t. \tag{81}$$

This formula suggests that the squeezing can grow to a relevant magnitude after a sufficiently long time. The possible associated empirical consequences will be examined in Sec. VI. Care should, however, be taken here as this cumulative effect, when integrated over long times, leads to a breakdown of the perturbative approach used in this study. This happens for times typically greater than $(\beta\omega^2\hbar)^{-1}$. Naturally, the lower the photon energy is compared to the energy scale of new physics β^{-1}, the longer the perturbative approach remains valid. The extension of this work to the nonperturbative regime is beyond the scope of this paper and left for future work.

In Fig. 1, we demonstrate the time dependence given by the formula (78) alongside its approximation (79). The approximation proves to be highly effective, exhibiting only minor deviations from the original formula at short timescales. Additionally, Fig. 1 includes the approximations for $r_{\min,|\alpha\rangle}$ and $r_{\max,|\alpha\rangle}$, as described by Eqs. (80) and (81), respectively. These approximations also maintain a high degree of accuracy, closely fulfilling their intended roles.

V. DEFORMED DISPERSION RELATION

Most of the relevant properties of light (except for the polarization states) can be captured by considering a

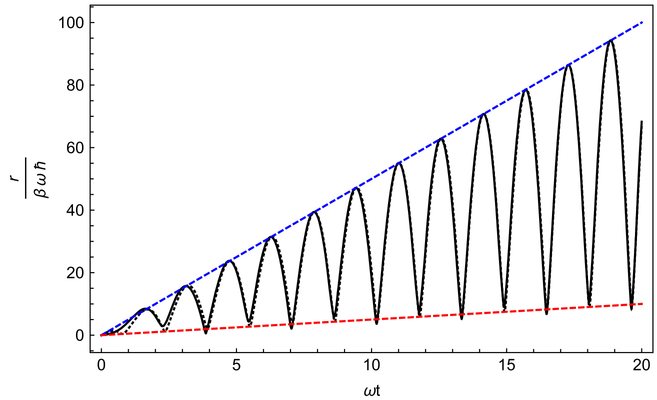

FIG. 1. Time evolution of the squeezing amplitude r for $|\alpha| = 1$. Here, the solid (black) line corresponds to the formula given by Eq. (78). The approximation provided in Eq. (79) is depicted as a dotted (black) line. The upper dashed (blue) line corresponds to Eq. (81), and the bottom dashed (red) line corresponds to Eq. (80).

massless scalar field model. This allows one to consider the multimode light state and derive the associated dispersion relation.

The multimode light Hamiltonian generalizes Eq. (5) to

$$\hat{H} = \sum_{\mathbf{k}} \hat{H}_{\mathbf{k}} = \frac{1}{2}\sum_{\mathbf{k}}\left(\hat{P}_{\mathbf{k}}^2 + \omega^2 \hat{Q}_{\mathbf{k}}^2\right) = \frac{1}{2}\sum_{\mathbf{k}}\left(\hat{p}_{\mathbf{k}}^2 + \omega^2 \hat{q}_{\mathbf{k}}^2\right) + \frac{\beta}{3}\sum_{\mathbf{k}} \hat{p}_{\mathbf{k}}^4 + \mathcal{O}(\beta^2), \quad (82)$$

where $\omega^2 = c^2 \mathbf{k}\cdot\mathbf{k}$, which is just a sum of single-mode Hamiltonians for different values of ω. As a consequence, previous results, and in particular the energy levels and perturbed eigenstates, can be adopted here.

The $\hat{q}_{\mathbf{k}}$ and $\hat{p}_{\mathbf{k}'}$ satisfy the canonical commutation relation $[\hat{q}_{\mathbf{k}}, \hat{p}_{\mathbf{k}}] = i\hbar \hat{\mathbb{I}}\delta_{\mathbf{k},\mathbf{k}}$.

Employing the expression for the first-order vacuum state $|0^{(1)}\rangle$ given by Eq. (14), one can evaluate the action of the field operator on that state:

$$\hat{q}_{\mathbf{k}}(0)|0^{(1)}\rangle = \sum_n c_n |n^{(1)}\rangle. \quad (83)$$

At first order in β,

$$c_1 = \sqrt{\frac{\hbar}{2\omega}}\left(1 + \frac{\beta\hbar\omega}{2}\right), \quad (84)$$

$$c_3 = -\frac{\sqrt{3}}{4}\beta\hbar^{3/2}\omega^{1/2}, \quad (85)$$

and the remaining coefficients c_n are equal to zero. Single-particle states are therefore not the only states created by an elementary excitation of the vacuum: three-particle states are also expected. While Eq. (14) may suggest that also five-particle states are created, this contribution cancels out at first order in β. This property has further consequences on the propagation of the quanta.

To quantify this effect, the two-point correlation function of $\hat{q}(\vec{x}, t)$ on the first-order vacuum state $|0\rangle \coloneqq \otimes_{\mathbf{k}} |0_{\mathbf{k}}^{(1)}\rangle \in \mathcal{H}$ is considered:

$$\langle 0|\hat{q}(\mathbf{x},t)\hat{q}(\mathbf{y},t')|0\rangle = \frac{1}{V}\sum_{\mathbf{k},n}|c_n|^2 e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) - i\frac{\Delta E_n^{(1)}}{\hbar}(t-t')} = \frac{1}{V}\sum_{\mathbf{k}}\int\frac{d\omega}{2\pi}G_p e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) - i\omega(t-t')}, \quad (86)$$

where

$$\Delta E_n^{(1)} = E_n^{(1)} - E_0^{(1)} = \hbar\omega n\left(1 + \frac{\beta\hbar\omega}{2}(n+1)\right). \quad (87)$$

The G_p entering Eq. (86) is a propagator. Introducing the four-momentum $p = (\frac{\hbar\omega}{c}, \hbar\mathbf{k})$, of pseudonorm $p^2 = -\frac{\hbar^2\omega^2}{c^2} + \hbar^2\mathbf{k}\cdot\mathbf{k}$, it writes

$$G_p \coloneqq \sum_n \frac{2i\Delta E_n^{(1)}|c_n|^2}{p^2 + \left(\frac{\Delta E_n^{(1)}}{c}\right)^2 - \mathbf{k}\cdot\mathbf{k} - i\epsilon} = \frac{i(\hbar^2 + 2\beta\hbar^3\omega)}{p^2 + \frac{2\beta\hbar^3\omega^3}{c^2} - i\epsilon} + \mathcal{O}(\beta^2). \quad (88)$$

Because of the c_n coefficients appearing with a modulus squared, only $n = 1$ contributes to the previous expression at first order in β, the $n = 3$ term being of the order β^2. In the $\beta \to 0$ limit the standard Feynman propagator of a scalar field is recovered: $G_p = \frac{i}{p^2 - i\epsilon}$.

As we see, the propagator (88) exhibits a pole at

$$p^2 c^2 + 2\beta\hbar^3\omega^3 = 0, \quad (89)$$

which corresponds to the dispersion relation

$$\omega = ck + \beta\hbar c^2 k^2 + \mathcal{O}(\beta^2), \quad (90)$$

where k denotes the norm of the wave vector $\mathbf{k}$. This expression exhibits an additional term $\propto k^2$. As expected, the usual linear form is recovered in the $\beta \to 0$ limit.

This quantum correction to the dispersion relation leads to an energy dependence of the photon group velocity:

$$v_{\mathrm{gr}} \coloneqq \frac{\partial\omega}{\partial k} = c + 2\beta\hbar c^2 k. \quad (91)$$

For positive values of β, the higher the photon energy, the faster it moves. This cumulative effect leads to a time advancement as the photon energy increases. For two photons 1 and 2 of energies E_1 and E_2 the difference in time arrival $\Delta t \coloneqq t_2 - t_1$ writes

$$c\Delta t \approx -2\beta \Delta E L, \tag{92}$$

L being the traveled distance and $\Delta E \coloneqq E_2 - E_1$. The correct limit $\Delta t \to 0$ when $\beta \Delta E \to 0$ is recovered. Please note that the formula (92) does not take into account any effect of cosmological expansion.

VI. OBSERVABILITY

Testing quantum gravity effects using astrophysical photons is an intensively explored avenue in quantum gravity phenomenology [18,29–34]. By studying the behavior of photons, particularly those originating from distant astrophysical sources such as gamma ray bursts (GRBs), one can attempt to probe the fundamental nature of gravity at the quantum level [35–39]. These experiments involve examining subtle deviations from classical predictions, such as the energy-dependent speed of photons and the resulting time lags or advancement accumulated over long distances. An example of such an accumulative effect has been provided in the previous section for the GUP-type effects.

To estimate the magnitudes of both the squeezing and time advancement of photons, it has to be remembered that the parameter β has the dimension $[\beta] = [E^{-1}]$. Since the considered modification to the usual uncertainty principle is expected to be a manifestation of Planck-scale physics, β^{-1} is expected to be around the Planck scale:

$$\beta \sim \frac{1}{E_{\mathrm{Pl}}}, \tag{93}$$

where $E_{\mathrm{Pl}} \approx 1.22 \times 10^{19}$ GeV.

The time-advancement established equation (92) can be written as

$$\Delta t \sim -\left(\frac{\Delta E}{E_{\mathrm{Pl}}}\right) \frac{L}{c}. \tag{94}$$

Two high-energy photons in the energy range of the Cherenkov Telescope Array (CTA) [40], typically of 1 and 10 TeV, respectively,[2] would undergo a time advancement of $|\Delta t| \sim 7$ s after a traveled distance of 100 Mpc.

[2]The full energy range covered by CTA being comprised between 20 GeV and 300 TeV.

This article aimed to explore the potential cumulative effects on phase-space properties of quantum states, specifically squeezing. As shown in Sec. IV, cumulative effects are not expected for energy eigenstates at linear order in the β parameter. They are, however, predicted for coherent states with $\alpha \neq 0$.

Indeed, the expected maximal squeezing of the vacuum state $|0\rangle$ is, according to Eq. (58),

$$r_{\max,|0\rangle} \sim \frac{E}{E_{\mathrm{Pl}}}, \tag{95}$$

which, for $E \sim 1$ TeV photons, gives a very small value $r_{|0\rangle} \sim 10^{-16}$.

On the other hand, for coherent states, the formula (81) can be recast into

$$r_{\max,|\alpha\rangle} \sim |(2|\alpha|^2 + 3)||\alpha|^2 \left(\frac{E}{E_{\mathrm{Pl}}}\right)^2 \frac{L}{l_{\mathrm{Pl}}}, \tag{96}$$

which for $|\alpha| \sim 1$ reduces to the following estimate:

$$r_{\max,|\alpha\rangle} \sim \left(\frac{E}{E_{\mathrm{Pl}}}\right)^2 \frac{L}{l_{\mathrm{Pl}}}. \tag{97}$$

The validity of the perturbative approach used in this work requires the previous estimation of $r_{\max}$ to be smaller than unity. For photons of fixed energy, this imposes a maximum distance of $l_{\mathrm{Pl}}(E_{\mathrm{Pl}}/E)^2$. For 1 eV photons, this distance is around 10^5 parsecs, and its value quickly decreases as the photon energy goes up. For higher-energy photons at this distance, although the perturbative expansion cannot be used anymore, important squeezing effects are still to be expected.

The important takeaway message is, therefore, that photons of astrophysical origin emitted in a coherent state should undergo important squeezing due to the GUP, although the precise computation of the value of the effect for high-energy photons requires one to go beyond the perturbative approach.

Homodyne detectors (see, e.g., [41]) offer a promising means to measure squeezing in quantum systems. These detectors are capable of detecting both the amplitude and the phase information of a quantum state, making them well-suited for investigating the squeezing phenomenon. By employing homodyne detection techniques, one can measure the variances of both quadrature components of the quantum state, which directly relates to the degree of squeezing present. Employing this technique in analyzing photons of astrophysical origin could serve as a new window of constraining quantum gravity effects.

Furthermore, analysis of the statistics of light may also provide an opportunity to constrain quantum gravitational effects. In particular, the projection in the interacting theory of the coherent states onto the occupation number ones [in the leading order—$\mathcal{O}(\beta)$] leads to the following probabilities:

$$P_n \coloneqq |\langle n^{(1)}|\alpha(t)\rangle|^2 = \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2} + \frac{\beta\hbar\omega}{12}e^{-|\alpha|^2} \times \left[-\frac{1}{2}\frac{|\alpha|^{2n+4}}{n!}\cos(4\theta) + \frac{4n+6}{n!}|\alpha|^{2n+2}\cos(2\theta) - \frac{4n-2}{(n-2)!}|\alpha|^{2n-2}\cos(2\theta) + \frac{1}{2}\frac{|\alpha|^{2n-4}}{(n-4)!}\cos(4\theta)\right], \tag{98}$$

exhibiting a correction that is linear in β. No accumulation of the correction is, however, expected at this order.

VII. SUMMARY

This article explored potential effects of Planck-scale physics on the Heisenberg uncertainty principle. The focus has been put on the time evolution of quantum states of single-mode light under the influence of quantum gravitational corrections.

Findings show that for both energy eigenstates and coherent states of light, the leading order quantum gravity corrections result in squeezing following an oscillatory pattern. There is, however, a significant difference between the two cases.

For the energy eigenstates, there is no net cumulative effect at first order in the parameter β governing the strength of Planck-scale effects. Those states remain stable under the influence of the proposed quantum gravity effects, and no significant contribution from the Planck scale is expected in this case.

For the coherent states such that $\alpha \neq 0$, there is, however, a net accumulation of the amplitude of squeezing (and displacement) over time. The effect is predicted to be strong in the case of photons traveling on astrophysical or cosmological distances and may have potential empirical constraints. For high-energy photons, the predictions require going beyond the linear-perturbation approach used here, which will be the object of a future paper. Measurements of nonclassical properties of light originating from distant astrophysical sources may, therefore, open a window to test Planck-scale physics through such predictions.

A corrected form of the dispersion relation of light has also been derived that provides additional insights, including the analysis of corresponding advancements of photons. Interestingly, both the constraints on the time of arrival and the squeezing can be used simultaneously, hopefully leading to tighter bounds on the effects. This possibility opens another interesting path for further studies.

Future research in this direction may also expand the investigation upon considering the generalized extended uncertainty principle [42]:

$$\Delta Q\Delta P \geq \frac{\hbar}{2}(1+\beta\Delta P^2+\alpha\Delta Q^2). \tag{99}$$

This generalized form of the uncertainty principle introduces additional terms β and α, which modify the trade-off between the uncertainties in position (ΔQ) and momentum (ΔP). To go even beyond, one could also include linear terms in (ΔP) and/or in (ΔQ), in the spirit of [43,44]. An interesting objective is also to investigate the fate of the thermal states, not considered here, and the effect on the polarization degrees of freedom of light.

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APPENDIX

Let us consider the following form of the variances and of the covariance of the operators $\hat{q}$ and $\hat{p}$:

$$\omega(\Delta\hat{q})^2 = \frac{\hbar}{2} + X\beta + A\beta^2 + \mathcal{O}(\beta^3), \tag{A1}$$

$$(\Delta\hat{p})^2/\omega = \frac{\hbar}{2} - X\beta + B\beta^2 + \mathcal{O}(\beta^3), \tag{A2}$$

$$C_{qp} = Y\beta + C\beta^2 + \mathcal{O}(\beta^3), \tag{A3}$$

where X, Y, A, B, C are some functions, not being dependent on β. In this article, the forms of X and Y are derived, while the form of the A, B, C functions is not known.

By applying the above expressions to Eq. (38) one finds that

$$\lambda_\pm = \frac{\hbar}{2} \pm \sqrt{X^2+Y^2}\beta + \mathcal{O}(\beta^2). \tag{A4}$$

Therefore, the $\mathcal{O}(\beta^2)$ terms, related to the unknown functions A, B, and C, entering the square root in Eq. (38), cancel out and do not contribute in the linear order in β to the eigenvalues $\lambda_\pm$.

Consequently, by expressing the square roots of the eigenvalues as follows:

$$\sqrt{\lambda_\pm} = \sqrt{\frac{\hbar}{2}}e^{\pm r}, \tag{A5}$$

we find that the amplitude of squeezing can be written as

$$r = \frac{\beta}{\hbar}\sqrt{X^2+Y^2} + \mathcal{O}(\beta^2). \tag{A6}$$

This formula is used to derive the expressions for the amplitude of squeezing in Sec. IV.

[1] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002).
[2] G. Amelino-Camelia, Nature (London) **418**, 34 (2002).
[3] M. Bojowald and A. Kempf, Phys. Rev. D **86**, 085017 (2012).
[4] P. Bosso, G. G. Luciano, L. Petruzziello, and F. Wagner, Classical Quantum Gravity **40**, 195014 (2023).
[5] F. Wagner, Phys. Rev. D **104**, 126010 (2021).
[6] G. M. Hossain, V. Husain, and S. S. Seahra, Classical Quantum Gravity **27**, 165013 (2010).
[7] G. M. Hossain, V. Husain, and S. S. Seahra, Phys. Rev. D **82**, 124032 (2010).
[8] L. Freidel, R. G. Leigh, and D. Minic, Phys. Lett. B **730**, 302 (2014).
[9] L. Freidel, R. G. Leigh, and D. Minic, J. Phys. Conf. Ser. **804**, 012032 (2017).
[10] A. Riello, Phys. Rev. D **97**, 025003 (2018).
[11] B. Dittrich, J. High Energy Phys. 05 (2017) 123.
[12] H. M. Haggard, M. Han, and A. Riello, Ann. Henri Poincare **17**, 2001 (2016).
[13] V. Bonzom, M. Dupuis, F. Girelli, and E. R. Livine, arXiv: 1402.2323.
[14] M. Dupuis, E. R. Livine, and Q. Pan, Classical Quantum Gravity **37**, 025017 (2020).
[15] J. Mielczarek and T. Trześniewski, Phys. Lett. B **759**, 424 (2016).
[16] D. Artigas, S. Crowe, and J. Mielczarek, Classical Quantum Gravity **38**, 085007 (2021).
[17] D. Artigas, J. Bilski, S. Crowe, J. Mielczarek, and T. Trześniewski, Phys. Rev. D **102**, 125029 (2020).
[18] A. Addazi, J. Alvarez-Muniz, R. Alves Batista, G. Amelino-Camelia, V. Antonelli, M. Arzano, M. Asorey, J. L. Atteia, S. Bahamonde, F. Bajardi *et al.*, Prog. Part. Nucl. Phys. **125**, 103948 (2022).
[19] I. Pikovski, M. R. Vanner, M. Aspelmeyer, M. Kim, and C. Brukner, Nat. Phys. **8**, 393 (2012).
[20] S. Das and E. C. Vagenas, Phys. Rev. Lett. **101**, 221301 (2008).
[21] Y. C. Ong, J. Cosmol. Astropart. Phys. 09 (2018) 015.
[22] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, Phys. Rev. D **84**, 084010 (2011).
[23] J. Mielczarek, Europhys. Lett. **108**, 40003 (2014).
[24] P. Pedram, Phys. Rev. D **85**, 024016 (2012).
[25] C. Conti, arXiv:1805.11716.
[26] P. Bosso, S. Das, and R. B. Mann, Phys. Rev. D **96**, 066008 (2017).
[27] B. J. Lawrie, P. D. Lett, A. M. Marino, and R. C. Pooser, ACS Photonics **6**, 1307 (2019).
[28] G. Frascella, S. Agne, F. Y. Khalili, and M. V. Chekhova, npj Quantum Inf. **7**, 72 (2021).
[29] G. Amelino-Camelia, G. D'Amico, G. Rosati, and N. Loret, Nat. Astron. **1**, 0139 (2017).
[30] Y. J. Ng and E. S. Perlman, Universe **8**, 382 (2022).
[31] J. Ellis and N. E. Mavromatos, Astropart. Phys. **43**, 50 (2013).
[32] P. He and B. Q. Ma, Universe **8**, 323 (2022).
[33] A. Barrau, B. Bolliet, M. Schutten, and F. Vidotto, Phys. Lett. B **772**, 58 (2017).
[34] A. Abdo, M. Ackermann, M. Ajello *et al.*, Nature (London) **462**, 331 (2009).
[35] G. Amelino-Camelia, J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar, Nature (London) **393**, 763 (1998).
[36] G. Amelino-Camelia, G. D'Amico, F. Fiore, S. Puccetti, and M. Ronco, Symmetry **13**, 541 (2021).
[37] J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and A. S. Sakharov, Astron. Astrophys. **402**, 409 (2003).
[38] D. J. Bartlett, H. Desmond, P. G. Ferreira, and J. Jasche, Phys. Rev. D **104**, 103516 (2021).
[39] M. G. Bernardini, G. Ghirlanda, S. Campana, P. D'Avanzo, J. L. Atteia, S. Covino, G. Ghisellini, A. Melandri, F. Piron, R. Salvaterra *et al.*, Astron. Astrophys. **607**, A121 (2017).
[40] B. S. Acharya *et al.* (CTA Consortium Collaboration), *Science with the Cherenkov Telescope Array* (World Scientific Publishing, Singapore, 2018), ISBN 978-981-327-008-4, 10.1142/10986.
[41] A. I. Lvovsky, arXiv:1401.4118.
[42] S. Hossenfelder, Living Rev. Relativity **16**, 2 (2013).
[43] S. Das, E. C. Vagenas, and A. F. Ali, Phys. Lett. B **690**, 407 (2010); **692**, 342(E) (2010).
[44] E. C. Vagenas, S. M. Alsaleh, and A. Farag, Europhys. Lett. **120**, 40001 (2017).