Manifestly covariant variational principle for gauge theories of gravity

M. P. Hobson,* A. N. Lasenby,† and W. E. V. Barker‡
Astrophysics Group, Cavendish Laboratory, JJ Thomson Avenue, Cambridge CB3 0HE, United Kingdom

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A variational principle for gauge theories of gravity is presented, which maintains manifest covariance under the symmetries to which the action is invariant, throughout the calculation of the equations of motion and conservation laws. This is performed by deriving explicit manifestly covariant expressions for the Euler-Lagrange variational derivatives and Noether's theorems for a generic action of the form typically assumed in gauge theories of gravity. The approach is illustrated by application to two scale-invariant gravitational gauge theories, namely Weyl gauge theory (WGT) and the recently proposed 'extended' Weyl gauge theory (eWGT), where the latter may be considered as a novel gauging of the conformal group; the method can also be straightforwardly applied to other theories with smaller or larger symmetry groups. In addition, the approach enables one easily to establish the relationship between manifestly covariant forms of variational derivatives obtained when one or more of the gauge field strengths is set to zero either before or after the variation is performed. This is illustrated explicitly for both WGT and eWGT in the case where the translational gauge field strength (or torsion) is set to zero before and after performing the variation, respectively.

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I. INTRODUCTION

For any given action, the process of deriving the manifestly covariant equations of motion for the fields on which it depends can be very time consuming. A key reason is that for an action that is invariant under some set of symmetries, either global or local, the individual terms making up the Euler-Lagrange equations are typically not covariant under those symmetries. One therefore usually obtains equations of motion that, although inevitably covariant, are not manifestly so. One then faces the task of combining terms in various ways to achieve manifest covariance before continuing with further analysis, and this process can require considerable trial and error, often relying on inspired guesswork. Similar difficulties are also encountered when deriving conservation laws, which must again be covariant under the symmetries of the action, but are typically not obtained in a manifestly covariant form when they are derived using the standard forms of Noether's theorems.

Here we present an alternative approach whereby one maintains manifest covariance throughout the calculation of the equations of motion and conservation laws, thereby circumventing the above difficulties. Methods for achieving this, at least for the equations of motion, have been considered previously in the context of gravitational theories that are interpreted in the usual geometrical manner, where the action depends typically on the spacetime metric $g_{\mu\nu}$, together perhaps with some nonmetric connection $\Gamma^{\sigma}{}_{\mu\nu}$ [1–6]. Here we instead focus on developing a manifestly covariant variational principle for gauge theories of gravity [7–11]. In particular, we illustrate the method by application to the scale-invariant Weyl gauge theory (WGT) [12–18] (see also [19]) and its recently proposed 'extended' version (eWGT) [20,21], but the approach presented can be straightforwardly applied to other theories with smaller or larger symmetry groups, such as Poincaré gauge theory (PGT) [10,22–24] or conformal gauge theory (CGT) [25–30]. In addressing WGT and eWGT, we assume the action to depend on a translational gauge field $h_a{}^{\mu}$, a rotational gauge field $A^{ab}{}_{\mu}$ and a dilational gauge field B_{μ}, together with some set of matter fields φ_A, which may include a scalar compensator field (which we occasionally also denote by ϕ). It is worth noting that gauge theories of gravitation are most naturally interpreted as field theories in Minkowski spacetime [31,32], in the same way as the gauge field theories describing the other fundamental interactions, and this is the viewpoint that we shall adopt here. It is common, however, to reinterpret the mathematical structure of gravitational gauge theories geometrically, where in particular the translational gauge field $h_a{}^{\mu}$ is considered as forming the components of a vierbein (or tetrad) system in a more general Weyl-Cartan spacetime,

*mph@mrao.cam.ac.uk
†anthony@mrao.cam.ac.uk
‡wb263@cam.ac.uk

in which $A^{ab}{}_{\mu}$ and B_{μ} then correspond to the spin-connection and Weyl vector, respectively [10]. These issues are discussed in more detail elsewhere [11,20]; we note here merely that the Minkowski spacetime gauge theory approach and the geometric interpretation provide equivalent descriptions up to global topological considerations, so that either may be used to describe local gravitational physics without any limitations.

The manifestly covariant approach presented here also enables one easily to establish the relationship between the forms of variational derivatives, and hence the field equations, obtained by applying first- and second-order variational principles, respectively. A particularly interesting case is provided by comparing the variational derivatives obtained by setting the translational gauge field strength (or torsion) to zero after the variation is performed (first-order approach) with those obtained by setting the torsion to zero in the action before carrying out the variation (second-order approach). In the latter case, the rotational gauge field is no longer an independent field. In WGT (and also PGT and CGT), it may be written explicitly in terms the other gauge fields, whereas in eWGT there exists an implicit constraint relating all the gauge fields. In both cases, one may arrive at simple expressions for the variational derivatives in the second-order approach in terms of those from the first-order approach.

The outline of this paper is as follows. In Sec. II we briefly review the concepts of local symmetries and dynamics in classical field theory. We present our manifestly covariant variational principle in Sec. III, which is applied to WGT and eWGT in Secs. IV and V, respectively. We conclude in Sec. VI. In addition, in the Appendix, we include a brief account of the Bessel-Hagen method [33] for expressing the variation of the vector potential in electromagnetism in a manifestly gauge-invariant form; it is this approach that we generalize to gauge theories of gravity in order to assist in directly obtaining manifestly covariant conservation laws.

II. LOCAL SYMMETRIES AND DYNAMICS IN CLASSICAL FIELD THEORY

We begin by presenting a brief outline of the consequences of local symmetries for classical field theories, focusing in particular on Noether's first and second theorems, the latter being discussed surprisingly rarely in the literature. These considerations allow one also to determine the dynamics of the fields.

Consider a spacetime manifold $\mathscr{M}$, labeled using some arbitrary coordinates x^{μ}, in which the dynamics of some set of (tensor and/or spinor) fields $\chi(x) = \{\chi_A(x)\}$ $(A = 1, 2, \ldots)$ is described by the action[1]

$$S = \int \mathscr{L}(\chi, \partial_{\mu}\chi, \partial_{\mu}\partial_{\nu}\chi) d^4x. \tag{1}$$

It should be understood here that the label A merely enumerates the different fields, although (with some overloading of the notation) can also be considered to represent one or more coordinate and/or local Lorentz frame indices (either as subscripts or superscripts), which we denote by lower-case Greek and Roman letters, respectively. It is worth noting that, in general, each field $\chi_A(x)$ may be either a matter field $\varphi_A(x)$ or gauge field $g_A(x)$. Allowing the Lagrangian density $\mathscr{L}$ in the action (1) to depend on field derivatives up to second order is sufficient to accommodate all the gravitational gauge theories that we will consider (and also general relativity).

Invariance of the action (1) under the infinitesimal coordinate transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$ and form variations $\delta_0\chi_A(x)$ in the fields (where, importantly, the latter need not result solely from the coordinate transformation),[2] implies that

$$\delta S = \int [\delta_0\mathscr{L} + \partial_{\mu}(\xi^{\mu}\mathscr{L})] d^4x = 0, \tag{2}$$

in which the form variation of the Lagrangian density is given by

$$\delta_0\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\chi_A}\delta_0\chi_A + \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\chi_A)}\delta_0(\partial_{\mu}\chi_A) + \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\partial_{\nu}\chi_A)}\delta_0(\partial_{\mu}\partial_{\nu}\chi_A). \tag{3}$$

One should note that δ_0 commutes with partial derivatives and, according to the usual summation convention, there is an implied sum on the label A. The integrand in the invariance condition (2) can be rewritten directly using the product rule to yield

$$\delta S = \int \left(\frac{\delta\mathscr{L}}{\delta\chi_A}\delta_0\chi_A + \partial_{\mu}J^{\mu}\right) d^4x = 0, \tag{4}$$

where the Euler-Lagrange variational derivative $\delta\mathscr{L}/\delta\chi_A$ and the Noether current J^{μ} are given, respectively, by

$$\frac{\delta\mathscr{L}}{\delta\chi_A} = \frac{\partial\mathscr{L}}{\partial\chi_A} - \partial_{\mu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\chi_A)}\right) + \partial_{\mu}\partial_{\nu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\partial_{\nu}\chi_A)}\right), \tag{5a}$$

$$J^{\mu} = \left[\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\chi_A)} - \partial_{\nu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\partial_{\nu}\chi_A)}\right)\right]\delta_0\chi_A + \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\partial_{\nu}\chi_A)}\partial_{\nu}(\delta_0\chi_A) + \xi^{\mu}\mathscr{L}. \tag{5b}$$

[1]In our subsequent discussion, we will typically assume that $\mathscr{M}$ is Minkowski spacetime and x^{μ} are Cartesian inertial coordinates, but this is unnecessary for the analysis in this section.

[2]Adopting Kibble's original notation, for an infinitesimal coordinate transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$, the 'form' variation $\delta_0\chi(x) \equiv \chi'(x) - \chi(x)$ is related to the 'total' variation $\delta\chi(x) \equiv \chi'(x') - \chi(x)$ by $\delta_0\chi(x) = \delta\chi(x) - \xi^{\mu}\partial_{\mu}\chi(x)$.

It is worth noting that the equations of motion for the fields $\chi_A(x)$ are also obtained by considering the behavior of the action under variations of the fields, but with the coordinate system kept fixed, so that $\xi^\mu(x) = 0$. One further assumes that the variations $\delta_0\chi_A(x)$ vanish on the boundary of the integration region of the action, and also that their first derivatives $\partial_\mu(\delta_0\chi_A(x))$ vanish in the case where $\mathscr{L}$ contains second derivatives of the fields. In order for the action to be stationary $\delta S = 0$ with respect to arbitrary such variations $\delta_0\chi_A(x)$ of the fields, one thus requires (4) to hold in these circumstances, which immediately yields the equations of motion $\delta\mathscr{L}/\delta\chi_A = 0$.

Returning to considering (4) as denoting the invariance of the action (1) under some general infinitesimal coordinate transformation $x'^\mu = x^\mu + \xi^\mu(x)$ and form variations $\delta_0\chi_A(x)$ in the fields (which need not vanish on the boundary of the integration region), one sees that if the field equations $\delta\mathscr{L}/\delta\chi_A = 0$ are satisfied for all the fields, then (4) reduces to the (on shell)[3] 'conservation law' $\partial_\mu J^\mu \simeq 0$, which holds up to a total divergence of any quantity that vanishes on the boundary of the integration region of the action (1). This is the content of Noether's first theorem, which applies both to global and local symmetries.

We will focus on the invariance of the action (1) under a local symmetry. In particular, we consider the (usual) case in which the form variations of the fields can be written as

$$\delta_0\chi_A = \lambda^C f_{AC}(\chi, \partial\chi) + (\partial_\mu\lambda^C) f^\mu_{AC}(\chi, \partial\chi), \tag{6}$$

where $\lambda^C = \lambda^C(x)$ are a collection of independent arbitrary functions of spacetime position, enumerated by the label C, and $f_{AC}(\chi, \partial\chi)$ and $f^\mu_{AC}(\chi, \partial\chi)$ are two collections of given functions that, in general, may depend on all the fields and their first derivatives. The general form (6) usually applies only when $\chi_A = g_A$ is a gauge field, whereas typically $f^\mu_{AC}(\chi, \partial\chi) = 0$ if $\chi_A = \varphi_A$ is a matter field. For each value of C, the function $\lambda^C(x)$ represents a set of infinitesimal functions carrying one or more coordinate or local Lorentz frame indices. It is worth noting that on substituting (6) into (5b), one obtains an expression for the current J^μ where the first term is proportional to (6) and, in the event that $\mathscr{L}$ depends on second derivatives of the fields, the second term is proportional to the first derivative of (6), which itself contains second derivatives of the functions $\lambda^C(x)$.

Using the expression (6), and again employing the product rule, the corresponding variation of the action (4) is given by (suppressing functional dependencies for brevity)

$$\delta S = \int \lambda^C\left[f_{AC}\frac{\delta\mathscr{L}}{\delta\chi_A} - \partial_\mu\left(f^\mu_{AC}\frac{\delta\mathscr{L}}{\delta\chi_A}\right)\right] + \partial_\mu(J^\mu - S^\mu)d^4x = 0, \tag{7}$$

where we define the new current $S^\mu \equiv -\lambda^C f^\mu_{AC}\delta\mathscr{L}/\delta\chi_A$. It is worth noting that S^μ depends much more simply than J^μ on the functions λ^C. Since the λ^C are arbitrary functions, for the action to be invariant one requires the separate conditions

$$f_{AC}\frac{\delta\mathscr{L}}{\delta\chi_A} - \partial_\mu\left(f^\mu_{AC}\frac{\delta\mathscr{L}}{\delta\chi_A}\right) = 0, \tag{8a}$$

$$\partial_\mu(J^\mu - S^\mu) = 0, \tag{8b}$$

where the former hold for each value of C separately and the latter holds up to a total divergence of a quantity that vanishes on the boundary of the integration region.

The first set of conditions (8a) are usually interpreted as conservation laws, which are covariant under the local symmetry, although not manifestly so in the form given above. The condition (8b) implies that $J^\mu = S^\mu + \partial_\nu Q^{\nu\mu}$, where $Q^{\nu\mu} = -Q^{\mu\nu}$, so the two currents coincide up to a total divergence, which is notable given their very different dependencies on the functions λ^C, f_{AC} and f^μ_{AC}, as described above. By contrast with the case of a global symmetry,[4] if the field equations $\delta\mathscr{L}/\delta\chi_A = 0$ are satisfied for all fields, then the conservation laws (8a) hold identically and the new current vanishes $S^\mu \simeq 0$, so that $J^\mu \simeq \partial_\nu Q^{\nu\mu}$. Thus, the conditions (8a)–(8b) effectively contain no information on shell, which is essentially the content of Noether's second theorem [34].

Nonetheless, the on shell condition that *all* the field equations $\delta\mathscr{L}/\delta\chi_A = 0$ are satisfied can only be imposed if $\mathscr{L}$ is the *total* Lagrangian density, and not if $\mathscr{L}$ corresponds only to some *subset* thereof (albeit one for which the corresponding action should still be invariant under the local symmetry). In particular, suppose one is considering a

[3]We use Dirac's notation $F \simeq 0$ for local functions F that vanish on shell (or *weakly* vanish), i.e., when the equations of motion $\delta\mathscr{L}/\delta\chi_A = 0$ are satisfied for *all* the fields. We further denote by $F \overset{\mathrm{m}}{\simeq} 0$ and $F \overset{\mathrm{g}}{\simeq} 0$ when functions vanish if only the equations of motion of the matter or gauge fields, respectively, need be satisfied.

[4]For a global symmetry, the λ^C are constants and so the second term on the rhs of (6) vanishes. The Noether current (5b) can be then written as

$$J^\mu = \lambda^C\left\{\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\chi_A)} - \partial_\nu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\partial_\nu\chi_A)}\right)\right]f_{AC} + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\partial_\nu\chi_A)}\partial_\nu f_{AC} + \xi^\mu_C\mathscr{L}\right\} \equiv \lambda^C J^\mu_C,$$

where ξ^μ_C are a given set of functions such that $\xi^\mu = \lambda^C\xi^\mu_C$, and we have also defined the further set of functions J^μ_C. One can then replace the two conditions (8) with the following single condition that is not satisfied identically on shell,

$$f_{AC}\frac{\delta\mathscr{L}}{\delta\chi_A} - \partial_\mu J^\mu_C = 0.$$

field theory for which the total Lagrangian density $\mathscr{L}_{\mathrm{T}} = \mathscr{L}_{\mathrm{M}} + \mathscr{L}_{\mathrm{G}}$, where $\mathscr{L}_{\mathrm{G}}$ contains every term that depends *only* on the gauge fields g_A and/or their derivatives, and $\mathscr{L}_{\mathrm{M}}$ contains all the remaining terms. Thus, if $\mathscr{L} = \mathscr{L}_{\mathrm{M}}$, then only the matter field equations $\delta\mathscr{L}/\delta\varphi_A = 0$ can be imposed, whereas if $\mathscr{L} = \mathscr{L}_{\mathrm{G}}$ none of the field equations can be imposed. In either case, the surviving terms in (8a)–(8b) do contain information [35].

III. MANIFESTLY COVARIANT VARIATIONAL PRINCIPLE

In the standard variational approach outlined above, one sees immediately from the plethora of partial derivatives throughout the analysis that the various expressions obtained are not, in general, manifestly covariant under the symmetry group to which the action is invariant. In particular, although the equations of motion $\delta\mathscr{L}/\delta\chi_A = 0$ for each field must be covariant under this symmetry group, it is clear that those derived from (5a) are not manifestly so. Moreover, the conservation laws (8a) suffer from the same shortcoming, but must also be expressible in a manifestly covariant form. By contrast, the currents J^μ and S^μ are not covariant (manifestly or otherwise), in general, since they both contain the arbitrary functions $\lambda^C(x)$, and J^μ also contains their partial derivatives. To obtain manifestly covariant variational derivatives and conservation laws directly, it is expedient to take a different approach that begins afresh by reconsidering the variation of the action in (2).

We are primarily concerned here with gauge theories of gravity. In constructing such theories, one typically begins with an action dependent only on some set of matter fields φ_A, which is defined on Minkowski spacetime $\mathscr{M}$ in Cartesian inertial coordinates x^μ (which we will assume henceforth), and is invariant under some global spacetime symmetry group $\mathcal{G}$, where the coefficients λ^C in (6) are constants. One then gauges the group $\mathcal{G}$ by demanding that the action be invariant with respect to (infinitesimal, passively interpreted) general coordinate transformations (GCTs) and the local action of the subgroup $\mathcal{H}$ (say), obtained by setting the translation parameters of $\mathcal{G}$ to zero (which leaves the origin invariant), and allowing the remaining group parameters to become independent arbitrary functions of position. For example, if one considers global Weyl invariance, then $\{\lambda^1, \lambda^2, \lambda^3\} = \{a^\alpha, \omega^{\alpha\beta}, \rho\}$, which denote a global spacetime translation, rotation and dilation, respectively. The symmetry is then 'promoted' to a local one by allowing $\lambda^C(x)$ to become arbitrary functions of spacetime position x. For local Weyl invariance, one thus has $\{\lambda^1(x), \lambda^2(x), \lambda^3(x)\} = \{a^\alpha(x), \omega^{ab}(x), \rho(x)\}$, where $a^\alpha(x)$ is interpreted as an infinitesimal general coordinate transformation and is usually denoted instead by $\xi^\alpha(x)$, and $\omega^{ab}(x)$ and $\rho(x)$ denote a position-dependent rotation of the local Lorentz frames and a position-dependent dilation, respectively. For the action to remain invariant under the localized symmetry necessitates the introduction of gravitational gauge fields g_A with prescribed transformation properties under the action of the localized symmetry. We will also maintain the somewhat unorthodox viewpoint, albeit hinted at in Kibble's original paper, of considering the gravitational gauge fields as fields in Minkowski spacetime, without attaching any geometric interpretation to them. Consequently, we will adopt a global Cartesian inertial coordinate system x^μ in our Minkowski spacetime, which greatly simplifies calculations, but more general coordinate systems may be straightforwardly accommodated, if required [20].

For an action (1) containing both matter fields $\chi_A = \psi_A$ and gauge fields $\chi_A = g_A$ to be invariant under a local symmetry of the form (6), one requires the Lagrangian density $\mathscr{L}$ to be covariant under this symmetry. One typically always requires invariance of the action under at least (infinitesimal) GCTs, which can be considered as promoting the set of constants λ^C representing global translations to arbitrary functions of position; this necessitates the introduction of the corresponding translational gravitational gauge field, which we will denote by ${h_a}^\mu$ and its inverse by ${b^a}_\mu$ (such that ${h_a}^\mu {b^a}_\nu = \delta^\mu_\nu$ and ${h_a}^\mu {b^c}_\mu = \delta^c_a$). It is therefore convenient to write the Lagrangian density as the product $\mathscr{L} = h^{-1}L$, where $h = \det({h_a}^\mu)$ is a scalar density, since $h^{-1}d^4x$ is an invariant volume element under GCTs.[5] The remaining factor L, which we term the Lagrangian, is also a scalar density constructed from covariant quantities.[6] These typically include the matter fields φ_A themselves and their covariant derivatives, together with the field strength tensors $\mathscr{F}_B$ of the gauge fields g_B, which typically depend both on the gauge fields themselves and their partial derivatives (where we have adopted a 'symbolic' form that suppresses coordinate and local Lorentz frame indices). In this section, we will denote the generic covariant dervative by $\mathscr{D}_a \equiv {h_a}^\mu \mathsf{D}_\mu = {h_a}^\mu(\partial_\mu + \Gamma_\mu)$, where Γ_μ is a linear combination of the generators of the subgroup $\mathcal{H}$ that may depend, in general, on the gauge fields g_A and their first derivatives ∂g_A (note that we will occasionally retain the indices on covariant derivatives, when convenient to do so). In any case, one can thus denote the functional dependence of the Lagrangian symbolically by $L = L(\varphi_A, \mathcal{D}_a\varphi_A, \mathscr{F}_B)$.

A. Manifestly covariant variational derivatives

We begin by rewriting the variation of the action (2) so that one can directly identify manifestly covariant forms for the variational derivatives $\delta\mathscr{L}/\delta\chi_A$. One must first obtain a covariant form for the divergence in (2) by constructing a further covariant derivative operator $\mathfrak{D}_a$ such that, for any coordinate vector V^μ (of the same Weyl weight as the

[5]We will also denote h^{-1} by b where $b \equiv \det({b^a}_\mu)$.

[6]It should be noted that if the set of local symmetries (6) of the action include local scale transformations, then the Weyl weights of the scalar densities b and L should sum to zero, namely $w(b) + w(L) = 0$, so that the action S is invariant.

Lagrangian density $\mathscr{L}$), one has $\partial_\mu V^\mu = h^{-1}\mathfrak{D}_a(h\mathscr{V}^a)$, where we define the local Lorentz frame vector[7] $\mathscr{V}^a = b^a{}_\mu V^\mu$. The construction of such an operator requires one first to define the field strength tensor $\mathcal{T}^a{}_{bc} = 2h_b{}^\mu h_c{}^\nu \mathsf{D}_{[\mu} b^a{}_{\nu]}$ of the translational gauge field, which has the unique (up to a sign) nontrivial contraction $\mathcal{T}_b \equiv \mathcal{T}^a{}_{ba} = h\mathsf{D}_\mu(h^{-1}h_b{}^\mu)$. It is then straightforward to show that the required derivative operator is given by $\mathfrak{D}_a = \mathcal{D}_a + \mathcal{T}_a$. The presence of the (contracted) field strength tensor $\mathcal{T}_a$ within this extended covariant derivative operator occurs because of the generic form $\mathscr{L} = h^{-1}L$ for the Lagrangian density. In order to obtain a manifestly covariant form for the divergence appearing in (2) and thereby arrive below at such a form for Noether's first theorem (4), one must factorize out the invariant volume element $h^{-1}d^4x$ under GCTs. This leads inevitably to the consideration of derivatives of the translational gauge field $h_a{}^\mu$ and hence the occurrence of the corresponding field strength tensor.

One may then rewrite the variation of the action (2) in the alternative form

$$\delta S = \int [\delta_0\mathscr{L} + h^{-1}(\mathcal{D}_a + \mathcal{T}_a)(\xi^a L)]d^4x = 0, \quad (9)$$

in which $\xi^a = b^a{}_\mu \xi^\mu$ and the form variation of the Lagrangian density (3) can be rewritten symbolically as

$$\delta_0\mathscr{L} = h^{-1}\left[\frac{\bar\partial L}{\partial\varphi_A}\delta_0\varphi_A + \frac{\partial L}{\partial(\mathcal{D}_a\varphi_A)}\delta_0(\mathcal{D}_a\varphi_A) + \frac{\partial L}{\partial\mathscr{F}_B}\delta_0\mathscr{F}_B\right] + L\delta_0 h^{-1}, \quad (10)$$

where $\bar\partial L/\partial\varphi \equiv [\partial L(\varphi, \mathcal{D}_a u, \ldots)/\partial\varphi]_{u=\varphi}$, so that φ and $\mathcal{D}_a\varphi$ are treated as independent variables, rather than φ and $\partial_\mu\varphi$. In order to progress further, the variations $\delta_0(\mathcal{D}_a\varphi_A)$, $\delta_0\mathscr{F}_B$ and $\delta_0 h^{-1}$ in (10) must be expressed in terms of the variations $\delta_0\varphi_A$ and $\delta_0 g_B$, respectively, of the matter and gauge fields themselves. In so doing, one typically encounters terms of the (symbolic) form $\mathcal{D}(\delta_0\varphi_A)\partial L/\partial(\mathcal{D}\varphi_A)$ and $\mathcal{D}(\delta_0 g_B)\partial L/\partial\mathscr{F}_B$, which can be accommodated by considering the quantity $(\mathcal{D}_a + \mathcal{T}_a)(h\mathscr{V}^a)$, where (again in symbolic form) $h\mathscr{V} \sim \delta_0\varphi_A\partial L/\partial(\mathcal{D}\varphi_A) + \delta_0 g_B\partial L/\partial\mathscr{F}_B$, and then using the product rule. Following such a procedure, one may rewrite (10) in the general form

$$\delta_0\mathscr{L} = h^{-1}[\alpha^A\delta_0\varphi_A + \beta^B\delta_0 g_B + (\mathcal{D}_a + \mathcal{T}_a)(h\mathscr{V}^a)], \quad (11)$$

where α^A and β^B are manifestly covariant expressions that typically depend on φ_A, $\partial L/\partial\varphi_A$ and $\mathscr{F}_B$, together with $\partial L/\partial(\mathcal{D}\varphi_A)$ and $\partial L/\partial\mathscr{F}_B$ and their covariant derivatives. Inserting (11) into (9), Noether's first theorem (4) becomes

$$\delta S = \int [\alpha^A\delta_0\varphi_A + \beta^B\delta_0 g_B + (\mathcal{D}_a + \mathcal{T}_a)(h\mathscr{J}^a)]h^{-1}d^4x = 0, \quad (12)$$

where the current $h\mathscr{J}^a = h\mathscr{V}^a + \xi^a L$ has the symbolic form

$$h\mathscr{J} \sim \frac{\partial L}{\partial(\mathcal{D}\varphi_A)}\delta_0\varphi_A + \frac{\partial L}{\partial\mathscr{F}_B}\delta_0 g_B + \xi L. \quad (13)$$

By comparing (4) and (12), and noting that $h^{-1}(\mathcal{D}_a + \mathcal{T}_a)\times(h\mathscr{J}^a) = \partial_\mu J^\mu$, one may then immediately identify manifestly covariant expressions for the variational derivatives with respect to the matter and gauge fields, respectively, as

$$\frac{\delta\mathscr{L}}{\delta\varphi_A} = b\alpha^A, \qquad \frac{\delta\mathscr{L}}{\delta g_B} = b\beta^B. \quad (14)$$

If one does not wish to distinguish between matter and gauge fields, one can instead denote the above relations generically by $\delta\mathscr{L}/\delta\chi_A = b\gamma^A$, where γ^A is a manifestly covariant expression.

B. Manifestly covariant conservation laws

We now turn to the direct construction of manifestly covariant expressions for the conservation laws (8a). Clearly, the manifestly covariant expressions (14) may now be used for the variational derivatives, but one encounters two remaining issues, namely the presence of the explicit partial derivative in the second term in (8a), and the fact that the functions f_{AC} and f^μ_{AC} may not be covariant quantities. Indeed, the latter problem always occurs when the functions $\lambda^C(x)$ (say for $C = 1$) correspond to GCTs, such that $\lambda^1(x) = \{\xi^\alpha(x)\}$; this arises because $\delta_0\chi_A = \delta\chi_A - \xi^\alpha\partial_\alpha\chi_A$ for any field and so f_{A1} always contains the noncovariant term $-\partial_\alpha\chi_A$. Other functions from the sets f_{AC} and f^μ_{AC} may also be noncovariant, depending on the gauge theory under consideration.

Nonetheless, it is important to recall that the conservation law (8a) holds for *any* set of form variations of the fields (6) that leave the action invariant. In particular, by generalizing the approach first proposed by Bessel-Hagen for electromagnetism (see Appendix), one can choose specific forms for the functions $\lambda^C(x)$ for $C \neq 1$ in terms of $\lambda^1(x)$ and the nontranslational gauge fields g_B, such that all the functions f^μ_{AC} become (manifestly) covariant (as typically do many of the functions f_{AC}). In this case, one may then write the second term in (8a) by extending the definition of the covariant derivative $(\mathcal{D}_a + \mathcal{T}_a)$ to accommodate any additional free indices represented by the subscript C. In particular, it is convenient to require that for any quantity $V_C{}^\mu$ with this index structure (and the same Weyl weight as the Lagrangian density $\mathscr{L}$), one

[7] We will typically denote a quantity possessing only Roman indices (and its contractions over such indices) as the calligraphic font version of the kernel letter of the corresponding quantity possessing only Greek indices (following [20]), with the exception of quantities having Greek or lower-case kernel letters.

has $h^{-1}(\mathcal{D}_a + \mathcal{T}_a)(hb^a{}_\mu V_C{}^\mu) = \mathsf{D}_\mu V_C{}^\mu = (\partial_\mu + \Gamma_\mu)V_C{}^\mu$, so that in the case where C does not represent any additional indices one recovers the original requirement that $h^{-1}(\mathcal{D}_a + \mathcal{T}_a)(hb^a{}_\mu V^\mu) = \partial_\mu V^\mu$. One may then write the conservation law (8a) as

$$(\mathcal{D}_a + \mathcal{T}_a)(b^a{}_\mu f^\mu_{AC}\gamma^A) - (f_{AC} + \Gamma_\mu f^\mu_{AC})\gamma^A = 0. \quad (15)$$

The first term on the lhs of (15) is now manifestly covariant. Consequently, although the second term on the lhs is not manifestly covariant, it must also be expressible in such a form; indeed, one typically finds that this second term immediately assembles as such, as we will demonstrate in Secs. IV and V where we apply this approach to WGT and eWGT, respectively.

C. Relationship between currents in Noether's second theorem

Finally, we consider the relationship (8b) between the two currents J^μ and S^μ. As noted above, both currents depend on the functions λ^C and so neither is covariant. Nonetheless, from the above discussion, one may rewrite (8b) as $(\mathcal{D}_a + \mathcal{T}_a)[h(\mathscr{J}^a - \mathscr{S}^a)] = 0$, in which

$$\begin{aligned} h\mathscr{S}^a &= -\lambda^C hb^a{}_\mu f^\mu_{AC}\frac{\delta\mathscr{L}}{\delta\chi_A} = -\lambda^C b^a{}_\mu f^\mu_{AC}\gamma^A \\ &= -\lambda^C b^a{}_\mu (f^\mu_{AC}\alpha^A + f^\mu_{BC}\beta^B), \end{aligned} \quad (16)$$

where we have used the relations (14) to write the final expression in terms of the matter fields and gauge fields separately, in keeping with the (symbolic) expression (13) for $h\mathscr{J}^a$. Thus, $h\mathscr{S}^a$ has the form of linear combination of terms that are manifestly covariant (or can be made so using a generalization of the Bessel-Hagen method) with coefficients λ^C. Turning to $h\mathscr{J}^a$, if one substitutes (6) into (13), and recalls that f^μ_{AC} typically vanishes for matter fields, one obtains the (symbolic) expression

$$h\mathscr{J} \sim \lambda^C\left(\frac{\partial L}{\partial(\mathcal{D}\varphi_A)}f_{AC} + \delta^1_C L\right) + \frac{\partial L}{\partial\mathscr{F}_B}(f_{BC}\lambda^C + f^\mu_{BC}\partial_\mu\lambda^C), \quad (17)$$

where we have again assumed that $C = 1$ corresponds to GCTs. One may show, in general, that the forms of the manifestly covariant expressions α^A and β^B obtained in (11) guarantee that the relationship $(\mathcal{D}_a + \mathcal{T}_a)[h(\mathscr{J}^a - \mathscr{S}^a)] = 0$ is satisfied, and so it contains no further information. It is worth noting, however, that for the special case in which L does not depend on the gauge field strengths, such that $\partial L/\partial\mathscr{F}_B = 0$, the relationship takes the form

$$(\mathcal{D}_a + \mathcal{T}_a)\left[\lambda^C\left(\frac{\partial L}{\partial(\mathcal{D}_a\varphi_A)}f_{AC} + \delta^1_C L + b^a{}_\mu f^\mu_{AC}\alpha^A\right)\right] = 0, \quad (18)$$

which may be satisfied by requiring the term in parentheses to vanish for each value of C. In so doing, one obtains a straightforward expression for α^A, which one can show agrees with that obtained in (11).

The procedures presented in this section are best illustrated by example and we apply them to WGT and eWGT in Secs. IV and V, respectively. As we will also show in these examples, the general approach outlined above further lends itself to elucidating the relationship between first- and second-order variational derivatives.

IV. WEYL GAUGE THEORY

It was the gauging of the Poincaré group by Kibble [8] that first revealed how to achieve a meaningful gauging of groups that act on the points of spacetime as well as on the components of physical fields, and so laid the foundations for the construction of gauge theories of gravity. Indeed, the resulting Poincaré gauge theories (PGTs) have since been extensively studied (see, for example, [11] for an accessible summary). Nonetheless, the lack of a clear route to quantizing PGT has led to interest in imposing extra gauge symmetries beyond local Poincaré symmetry. In particular, perhaps the most natural extension of PGT is also to demand local scale invariance, which might provide a clearer route to renormalizability, since such theories contain no absolute energy scale.

The most direct approach to constructing gauge theories of gravity that are invariant under local changes of scale, in addition to local Poincaré transformations, is to gauge the Weyl group [12–18]. As in PGT, the resulting Weyl gauge theories (WGTs) assume the physical model of an underlying Minkowski spacetime in which a continuum matter field (or fields) is distributed continuously and the dynamics are described by a matter action that is invariant under global Weyl coordinate transformations. By then demanding the matter action to be invariant with respect to local Weyl transformations, in which the eleven Weyl group parameters become arbitrary functions of position, one is led to the introduction of the gravitational gauge fields $h_a{}^\mu$, $A^{ab}{}_\mu = -A^{ba}{}_\mu$ and B_μ corresponding to the translational, rotational and dilational parts of the local Weyl transformations, respectively. The field strength tensors of these gauge fields are typically denoted by $\mathscr{R}_{abcd}$, $\mathscr{T}^*_{abc}$ and $\mathscr{H}_{ab}$, respectively.

For WGT, the Lagrangian density has the usual form $\mathscr{L} = h^{-1}L$, where the translational gauge field $h_a{}^\mu$ (with inverse $b_a{}^\mu$) is assigned a Weyl weight $w = -1$, so that $h = \det(h_a{}^\mu)$ and L are scalar densities both of Weyl weight $w = -4$, and hence the action S is invariant under local scale transformations. The Lagrangian has the functional dependencies

$$L = L(\varphi_A, \mathscr{D}^*_a\varphi_A, \mathscr{R}_{abcd}, \mathscr{T}^*_{abc}, \mathscr{H}_{ab}), \quad (19)$$

where φ_A are the matter fields, which typically include a scalar compensator field of Weyl weight $w = -1$ (that we sometimes denote also by ϕ), and their covariant derivatives are denoted in this section by [11,20,21]

$$\mathscr{D}^*_a\varphi_A = h_a{}^\mu D^*_\mu\varphi_A = h_a{}^\mu(\partial_\mu + \Gamma^*_\mu)\varphi_A = h_a{}^\mu\left(\partial_\mu + \frac{1}{2}A^{cd}{}_\mu\Sigma_{cd} + w_A B_\mu\right)\varphi_A, \quad (20)$$

in which $\Sigma_{ab} = -\Sigma_{ba}$ are the generator matrices of the $SL(2,C)$ representation to which the field φ_A belongs.[8] In the expression (20), each field is assumed to have weight w_A (note that this appearance of the index A is purely a label and hence is understood never to be summed over). It is also convenient to define the further derivative operator $\partial^*_\mu\varphi_A = (\partial_\mu + w_A B_\mu)\varphi_A$, of which we will make occasional use.

Under infinitesimal local Weyl transformations consisting of GCTs, rotations of the local Lorentz frames and dilations, which are parametrized by $\xi^\mu(x)$, $\omega^{ab}(x) = -\omega^{ba}(x)$ and $\rho(x)$, respectively, a matter field φ of weight w and the gauge fields transform as [11,21]

$$\delta_0\varphi = -\xi^\nu\partial_\nu\varphi + \left(\frac{1}{2}\omega^{ab}\Sigma_{ab} + w\rho\right)\varphi, \quad (21a)$$

$$\delta_0 h_a{}^\mu = -\xi^\nu\partial_\nu h_a{}^\mu + h_a{}^\nu\partial_\nu\xi^\mu - (\omega^b{}_a + \rho\delta^b_a)h_b{}^\mu, \quad (21b)$$

$$\delta_0 A^{ab}{}_\mu = -\xi^\nu\partial_\nu A^{ab}{}_\mu - A^{ab}{}_\nu\partial_\mu\xi^\nu - 2\omega^{[a}{}_c A^{b]c}{}_\mu - \partial_\mu\omega^{ab}, \quad (21c)$$

$$\delta_0 B_\mu = -\xi^\nu\partial_\nu B_\mu - B_\nu\partial_\mu\xi^\nu - \partial_\mu\rho, \quad (21d)$$

from which one may verify that $\mathscr{D}^*_a\varphi_A$ does indeed transform covariantly under (infinitesimal) local Weyl transformations with weight $w-1$ [20,21].

The field strength tensors $\mathscr{R}_{abcd}$, $\mathscr{T}^*_{abc}$ and $\mathscr{H}_{ab}$ in (19) are defined through the action of the commutator of two covariant derivatives on some field φ of weight w by

$$[\mathscr{D}^*_c, \mathscr{D}^*_d]\varphi = \left(\frac{1}{2}\mathscr{R}^{ab}{}_{cd}\Sigma_{ab} + w\mathscr{H}_{cd} - \mathscr{T}^{*a}{}_{cd}\mathscr{D}^*_a\right)\varphi, \quad (22)$$

which yields the forms $\mathscr{R}^{ab}{}_{cd} = h_a{}^\mu h_b{}^\nu R^{ab}{}_{\mu\nu}$, $\mathscr{H}_{cd} = h_c{}^\mu h_d{}^\nu H_{\mu\nu}$ and $\mathscr{T}^{*a}{}_{bc} = h_b{}^\mu h_c{}^\nu T^{*a}{}_{\mu\nu}$, where

$$R^{ab}{}_{\mu\nu} = 2(\partial_{[\mu}A^{ab}{}_{\nu]} + \eta_{cd}A^{ac}{}_{[\mu}A^{db}{}_{\nu]}), \quad (23a)$$

$$H_{\mu\nu} = 2\partial_{[\mu}B_{\nu]}, \quad (23b)$$

$$T^{*a}{}_{\mu\nu} = 2D^*_{[\mu}b^a{}_{\nu]}. \quad (23c)$$

From the transformation laws (21), it is straightforward to verify that, in accordance with their index structures, the gauge field strength tensors $\mathscr{R}^{ab}{}_{cd}$, $\mathscr{H}_{cd}$ and $\mathscr{T}^{*a}{}_{bc}$ are invariant under GCTs, and transform covariantly under local Lorentz transformations and dilations with Weyl weights $w=-2$, $w=-2$ and $w=-1$, respectively [20,21].

It is worth noting that $\mathscr{R}^{ab}{}_{cd}$ has the same functional form as the rotational field strength in PGT, but that $\mathscr{T}^{*a}{}_{bc} = \mathscr{T}^a{}_{bc} + \delta^a_c\mathscr{B}_b - \delta^a_b\mathscr{B}_c$, where $\mathscr{T}^a{}_{bc}$ is the translational field strength in PGT; we also define $\mathscr{B}_a = h_a{}^\mu B_\mu$. Moreover, using the expression (23c) and defining the quantities $c^{*a}{}_{bc} \equiv 2h_b{}^\mu h_c{}^\nu\partial^*_{[\mu}b^a{}_{\nu]}$, one may show that the fully anholonomic rotational gauge field $\mathscr{A}^{ab}{}_c \equiv h_c{}^\mu A^{ab}{}_\mu$ can be written as [11,20]

$$\mathscr{A}_{abc} = \frac{1}{2}(c^*_{abc} + c^*_{bca} - c^*_{cab}) - \frac{1}{2}(\mathscr{T}^*_{abc} + \mathscr{T}^*_{bca} - \mathscr{T}^*_{cab}). \quad (24)$$

It is also convenient for our later development to obtain the Bianchi identities satisfied by the gravitational gauge field strengths $\mathscr{R}^{ab}{}_{cd}$, $\mathscr{T}^{*a}{}_{bc}$ and $\mathscr{H}_{ab}$ in WGT. These may be straightforwardly derived from the Jacobi identity applied to the generalized covariant derivative, namely $[\mathscr{D}^*_a, [\mathscr{D}^*_b, \mathscr{D}^*_c]]\varphi + [\mathscr{D}^*_c, [\mathscr{D}^*_a, \mathscr{D}^*_b]]\varphi + [\mathscr{D}^*_b, [\mathscr{D}^*_c, \mathscr{D}^*_a]]\varphi = 0$. Inserting the form (20) for the WGT generalized covariant derivative, one quickly finds the three Bianchi identities [20][9]

$$\mathscr{D}^*_{[a}\mathscr{R}^{de}{}_{bc]} - \mathscr{T}^{*f}{}_{[ab}\mathscr{R}^{de}{}_{c]f} = 0, \quad (25a)$$

$$\mathscr{D}^*_{[a}\mathscr{T}^{*d}{}_{bc]} - \mathscr{T}^{*e}{}_{[ab}\mathscr{T}^{*d}{}_{c]e} - \mathscr{R}^d{}_{[abc]} - \mathscr{H}_{[ab}\delta^d_{c]} = 0, \quad (25b)$$

$$\mathscr{D}^*_{[a}\mathscr{H}_{bc]} - \mathscr{T}^{*e}{}_{[ab}\mathscr{H}_{c]e} = 0. \quad (25c)$$

By contracting over various indices, one also obtains the following nontrivial contracted Bianchi identities:

$$\mathscr{D}^*_a\mathscr{R}^{ae}{}_{bc} - 2\mathscr{D}^*_{[b}\mathscr{R}^e{}_{c]} - 2\mathscr{T}^{*f}{}_{a[b}\mathscr{R}^{ae}{}_{c]f} - \mathscr{T}^{*f}{}_{bc}\mathscr{R}^e{}_f = 0, \quad (26a)$$

$$\mathscr{D}^*_a\left(\mathscr{R}^a{}_c - \frac{1}{2}\delta^a_c\mathscr{R}\right) + \mathscr{T}^{*f}{}_{bc}\mathscr{R}^b{}_f + \frac{1}{2}\mathscr{T}^{*f}{}_{ab}\mathscr{R}^{ab}{}_{cf} = 0, \quad (26b)$$

$$\mathscr{D}^*_a\mathscr{T}^{*a}{}_{bc} + 2\mathscr{D}^*_{[b}\mathscr{T}^*_{c]} + \mathscr{T}^{*e}{}_{bc}\mathscr{T}^*_e + 2\mathscr{R}_{[bc]} - 2\mathscr{H}_{bc} = 0. \quad (26c)$$

[8]The asterisks in the definition of the derivative operator are intended simply to distinguish it from the usual notation used [11,20,21] for the covariant derivative $\mathscr{D}_a\varphi_A = h_a{}^\mu D_\mu\varphi_A = h_a{}^\mu(\partial_\mu + \Gamma_\mu)\varphi_A = h_a{}^\mu(\partial_\mu + \frac{1}{2}A^{cd}{}_\mu\Sigma_{cd})\varphi_A$ of Poincaré gauge theory (PGT), and should not be confused with the operation of complex conjugation.

[9]Note that these expressions correct a typographical error in [20] by reversing the sign of each term containing $\mathscr{H}_{ab}$.

A. Manifestly covariant variational derivatives in WGT

We now apply the manifestly covariant variational principle described in Sec. III to WGT. We begin by deriving the variational derivatives, and hence the Euler-Lagrange equations, for the matter fields φ_A and the gravitational gauge fields $h_a{}^\mu$, $A^{ab}{}_\mu$ and B_μ. Using the fact that $\delta_0 h^{-1} = -h^{-1} b^a{}_\mu \delta_0 h_a{}^\mu$, one may write (10) as

$$\begin{aligned} h\delta_0\mathscr{L} &= \delta_0 L - b^a{}_\mu L\delta_0 h_a{}^\mu, \\ &= \frac{\bar{\partial} L}{\partial\varphi_A}\delta_0\varphi_A + \frac{\partial L}{\partial(\mathscr{D}^*_a\varphi_A)}\delta_0(\mathscr{D}^*_a\varphi_A) \\ &\quad + \frac{\partial L}{\partial\mathscr{R}_{abcd}}\delta_0\mathscr{R}_{abcd} + \frac{\partial L}{\partial\mathscr{T}^*_{abc}}\delta_0\mathscr{T}^*_{abc} \\ &\quad + \frac{\partial L}{\partial\mathscr{H}_{ab}}\delta_0\mathscr{H}_{ab} - b^a{}_\mu L\delta_0 h_a{}^\mu. \end{aligned} \tag{27}$$

In order to progress further, one must determine how the variations in (27) depend on the variations of the matter and gravitational gauge fields themselves. This is easily achieved using the definition of the WGT covariant derivative and the expressions (23a)–(23c) for the field strengths. One must also make use of the fact that for any coordinate vector V^μ of weight $w = 0$ (i.e., invariant under local scale transformations, like the Lagrangian density $\mathscr{L}$), one may show that $\partial_\mu V^\mu = h^{-1}(\mathscr{D}^*_a + \mathscr{T}^*_a)(hb^a{}_\mu V^\mu)$ or, equivalently, for any local Lorentz vector $\mathscr{V}^a$ having Weyl weight $w = -3$ one has [20]

$$(\mathscr{D}^*_a + \mathscr{T}^*_a)\mathscr{V}^a = h\partial_\mu(h^{-1}h_a{}^\mu\mathscr{V}^a). \tag{28}$$

Such expressions on the rhs of (27) therefore contribute only surface terms to the variation of the action in (9), but we will retain them nonetheless, as they are required for our later discussion.

We begin by considering together the first two terms on the rhs of (27), for which one obtains

$$\begin{aligned} \frac{\bar{\partial} L}{\partial\varphi_A}\delta_0\varphi_A + \frac{\partial L}{\partial(\mathscr{D}^*_a\varphi_A)}\delta_0(\mathscr{D}^*_a\varphi_A) &= \frac{\bar{\partial} L}{\partial\varphi_A}\delta_0\varphi_A + \frac{\partial L}{\partial(\mathscr{D}^*_a\varphi_A)}\left[\mathscr{D}^*_a(\delta_0\varphi_A) + \delta_0 h_a{}^\mu D^*_\mu\varphi_A + h_a{}^\mu\left(w_A\delta_0 B_\mu + \frac{1}{2}\delta_0 A^{bc}{}_\mu\Sigma_{bc}\right)\varphi_A\right], \\ &= \left[\frac{\bar{\partial} L}{\partial\varphi_A} - (\mathscr{D}^*_a + \mathscr{T}^*_a)\frac{\partial L}{\partial(\mathscr{D}^*_a\varphi_A)}\right]\delta_0\varphi_A + \frac{\partial L}{\partial(\mathscr{D}^*_a\varphi_A)}\left[\delta_0 h_a{}^\mu D^*_\mu\varphi_A\right. \\ &\quad \left. + h_a{}^\mu\left(w_A\delta_0 B_\mu + \frac{1}{2}\delta_0 A^{bc}{}_\mu\Sigma_{bc}\right)\varphi_A\right] + (\mathscr{D}^*_a + \mathscr{T}^*_a)\left[\frac{\partial L}{\partial(\mathscr{D}^*_a\varphi_A)}\delta_0\varphi_A\right], \end{aligned} \tag{29}$$

where the quantity in square brackets in the final term is readily shown to have Weyl weight $w = -3$. Analyzing the further terms containing derivatives on the rhs of (27) in a similar manner, one finds

$$\begin{aligned} \frac{\partial L}{\partial\mathscr{R}_{abcd}}\delta_0\mathscr{R}_{abcd} &= 2\frac{\partial L}{\partial\mathscr{R}_{abcd}}[R_{ab\mu d}\delta_0 h_c{}^\mu + h_d{}^\mu\mathscr{D}^*_c(\delta_0 A_{ab\mu})] \\ &= 2\frac{\partial L}{\partial\mathscr{R}_{abcd}}R_{ab[\mu d]}\delta_0 h_c{}^\mu + [2h_c{}^\mu(\mathscr{D}^*_d + \mathscr{T}^*_d) + h_e{}^\mu\mathscr{T}^{*e}{}_{cd}]\left(\frac{\partial L}{\partial\mathscr{R}_{abcd}}\right)\delta_0 A_{ab\mu} \\ &\quad - 2(\mathscr{D}^*_d + \mathscr{T}^*_d)\left[\frac{\partial L}{\partial\mathscr{R}_{abcd}}h_c{}^\mu\delta_0 A_{ab\mu}\right], \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{\partial L}{\partial\mathscr{T}^*_{abc}}\delta_0\mathscr{T}^*_{abc} &= 2\frac{\partial L}{\partial\mathscr{T}^*_{abc}}[T^*_{a\mu\nu}h_c{}^\nu\delta_0 h_b{}^\mu + h_c{}^\nu\mathscr{D}^*_b(\delta_0 b_{a\nu}) + h_b{}^\mu\delta_0 A_{ac\mu} + \eta_{ac}h_b{}^\mu\delta_0 B_\mu] \\ &= 2\frac{\partial L}{\partial\mathscr{T}^*_{abc}}\left[\left(T^*_{a\mu\nu}h_c{}^\nu\delta^d_b - \frac{1}{2}\mathscr{T}^{*d}{}_{bc}b_{a\mu}\right)\delta_0 h_d{}^\mu + h_b{}^\mu\delta_0 A_{ac\mu} + \eta_{ac}h_b{}^\mu\delta_0 B_\mu\right] \\ &\quad - 2(\mathscr{D}^*_c + \mathscr{T}^*_c)\left(\frac{\partial L}{\partial\mathscr{T}^*_{abc}}\right)b_{a\mu}\delta_0 h_b{}^\mu + 2(\mathscr{D}^*_c + \mathscr{T}^*_c)\left[\frac{\partial L}{\partial\mathscr{T}^*_{abc}}b_{a\mu}\delta_0 h_b{}^\mu\right], \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{\partial L}{\partial\mathscr{H}_{ab}}\delta_0\mathscr{H}_{ab} &= 2\frac{\partial L}{\partial\mathscr{H}_{ab}}[H_{\mu\nu}h_b{}^\nu\delta_0 h_a{}^\mu + h_b{}^\nu\mathscr{D}^*_a(\delta_0 B_\nu)] \\ &= 2\frac{\partial L}{\partial\mathscr{H}_{ab}}\left(H_{\mu\nu}h_b{}^\nu\delta_0 h_a{}^\mu + \frac{1}{2}\mathscr{T}^{*c}{}_{ab}h_c{}^\nu\delta_0 B_\nu\right) + 2(\mathscr{D}^*_b + \mathscr{T}^*_b)\left(\frac{\partial L}{\partial\mathscr{H}_{ab}}\right)h_a{}^\nu\delta_0 B_\nu \\ &\quad - 2(\mathscr{D}^*_b + \mathscr{T}^*_b)\left[\frac{\partial L}{\partial\mathscr{H}_{ab}}h_a{}^\nu\delta_0 B_\nu\right]. \end{aligned} \tag{32}$$

In the above expressions it is assumed that the appropriate antisymmetrizations, arising from the symmetries of the field strength tensors, are performed when the rhs are evaluated. It is also easily shown that the quantity in square brackets in each of the last terms in (30)–(32) has Weyl weight $w = -3$, so according to (28) each such term contributes a surface term to the variation of the action (9).

One may then substitute the expressions (29)–(32) into (27), which may itself subsequently be substituted into (9) to obtain an expression of the general form (12) for Noether's first theorem, which may be written as

$$\delta S = \int [\upsilon^A \delta_0 \varphi_A + \tau^a{}_\mu \delta_0 h_a{}^\mu + \sigma_{ab}{}^\mu \delta_0 A^{ab}{}_\mu + \zeta^\mu \delta_0 B_\mu + h^{-1}(\mathscr{D}^*_p + \mathscr{T}^*_p)(h\mathscr{J}^p)]d^4x = 0, \tag{33}$$

where the current $h\mathscr{J}^p$ is given by

$$h\mathscr{J}^p = \frac{\partial L}{\partial(\mathscr{D}^*_p\varphi_A)}\delta_0\varphi_A + 2\left(\frac{\partial L}{\partial \mathscr{T}^*_{abp}} b_{a\mu}\delta_0 h_b{}^\mu - \frac{\partial L}{\partial \mathscr{R}_{abcp}} h_c{}^\mu \delta_0 A_{ab\mu} - \frac{\partial L}{\partial \mathscr{H}_{ap}} h_a{}^\mu \delta_0 B_\mu\right) + b^p{}_\mu \xi^\mu L, \tag{34}$$

and we have defined the variational derivative $\upsilon^A \equiv \delta\mathscr{L}/\delta\varphi_A$ with respect to the matter field φ_A, and the total dynamical energy-momentum $\tau^a{}_\mu \equiv \delta\mathscr{L}/\delta h_a{}^\mu$, spin-angular-momentum $\sigma_{ab}{}^\mu \equiv \delta\mathscr{L}/\delta A^{ab}{}_\mu$ and dilation current $\zeta^\mu \equiv \delta\mathscr{L}/\delta B_\mu$ of both the matter and gravitational gauge fields. Manifestly covariant forms for these variational derivatives may be read off from the expressions (29)–(32). Converting all Greek indices to roman and defining the quantities $\tau^a{}_b \equiv \tau^a{}_\mu h_b{}^\mu$, $\sigma_{ab}{}^c \equiv \sigma_{ab}{}^\mu b^c{}_\mu$ and $\zeta^a \equiv \zeta^\mu b^a{}_\mu$, one then makes the following identifications:

$$h\upsilon^A = \frac{\bar{\partial} L}{\partial \varphi_A} - (\mathscr{D}^*_a + \mathscr{T}^*_a)\frac{\partial L}{\partial(\mathscr{D}^*_a\varphi_A)}, \tag{35a}$$

$$h\tau^a{}_b = \frac{\partial L}{\partial(\mathscr{D}^*_a\varphi_A)}\mathscr{D}^*_b\varphi_A + 2\frac{\partial L}{\partial \mathscr{R}_{pqra}}\mathscr{R}_{pqrb} + 2\frac{\partial L}{\partial \mathscr{H}_{pa}}\mathscr{H}_{pb} + 2\frac{\partial L}{\partial \mathscr{T}^*_{pqa}}\mathscr{T}^*_{pqb} - [\mathscr{T}^{*a}{}_{qr} + 2\delta^a_q(\mathscr{D}^*_r + \mathscr{T}^*_r)]\frac{\partial L}{\partial \mathscr{T}^{*b}{}_{qr}} - \delta^b_a L, \tag{35b}$$

$$h\sigma_{ab}{}^c = \frac{1}{2}\frac{\partial L}{\partial(\mathscr{D}^*_c\varphi_A)}\Sigma_{ab}\varphi_A + [\mathscr{T}^{*c}{}_{pq} + 2\delta^c_p(\mathscr{D}^*_q + \mathscr{T}^*_q)]\frac{\partial L}{\partial \mathscr{R}^{ab}{}_{pq}} - 2\frac{\partial L}{\partial \mathscr{T}^{*[ab]}{}_c}, \tag{35c}$$

$$h\zeta^a = \frac{\partial L}{\partial(\mathscr{D}^*_a\varphi_A)} w_A\varphi_A + [\mathscr{T}^{*a}{}_{pq} + 2\delta^a_p(\mathscr{D}^*_q + \mathscr{T}^*_q)]\frac{\partial L}{\partial \mathscr{H}_{pq}} + 2\frac{\partial L}{\partial \mathscr{T}^{*p}{}_{qr}}\delta^a_q\delta^p_r, \tag{35d}$$

where, once again, it is assumed that the appropriate antisymmetrizations, arising from the symmetries of the field strength tensors, are performed when the rhs are evaluated. The expressions (35) constitute the completion of our first goal. One sees immediately that, unlike (5a), the above forms for the variational derivative of each field (and hence the equations of motion obtained by setting each rhs to zero) are manifestly covariant. Moreover, they are straightforward to evaluate, since they require one only to differentiate the Lagrangian L with respect to the matter fields, their covariant derivatives and the field strengths, respectively. One may easily confirm that the above expressions lead to precisely the same variational derivatives as those obtained by using the standard (but much longer) approach of evaluating (5a) for each field and then reassembling the many resulting terms into manifestly covariant forms.

The expressions (35) not only provide a significant calculational saving in obtaining the variational derivatives in WGT, but also yield a useful insight into their general form. In particular, one notes that for a Lagrangian L that does not contain the gauge field strength tensors, but depends only on the matter fields and their covariant derivatives, the variational derivatives with respect to the gauge fields reduce to the *covariant canonical currents* [11,21] of the matter fields. For Lagrangians that do depend on the gauge field strengths, also of interest are the analogous forms of the penultimate terms on the rhs of (35b)–(35d), which are the only terms capable of producing a dependence on the covariant derivatives of the field strength tensors; in each case, the corresponding term depends on the covariant derivative of the field strength tensor for the gauge field with respect to which the variational derivative is taken. It is also worth pointing out that we have not assumed the equations of motion to be satisfied in deriving (35a)–(35d). Thus, one may calculate the corresponding variational derivatives for *any subset* of terms in L that is a scalar density of weight $w = -4$. Individually, however, such quantities do *not* vanish, in general. Rather, each equation of motion requires only the vanishing of the sum of such quantities, when derived from disjoint subsets that exhaust the total Lagrangian L.

Finally, we note that the above approach is easily adapted to other gravitational gauge theories. For example, to apply it to PGT one needs simply to 'remove the asterisks', thereby replacing the WGT covariant derivative and torsion by their PGT counterparts, and set $B_\mu \equiv 0$, so that ζ^a and $\mathscr{H}_{ab}$ also vanish identically. Indeed, the above approach is of even greater use in PGT than WGT, since the functional dependence of the PGT Lagrangian on the matter fields, their covariant derivatives and the field strengths can be more complicated than in WGT, as in PGT one does not require L to have Weyl weight $w = -4$ [11,20].

B. Relationship between first- and second-order variational derivatives in WGT

Before turning our attention to the direct derivation of manifestly covariant conservation laws for WGT, we first briefly demonstrate how the analysis in the previous section is well suited to comparing first- and second-order variational derivatives. In particular, we will focus on the example of the variational derivatives obtained by setting the WGT torsion to zero *after* the variation is performed (first-order approach) with those obtained by setting the torsion to zero in the action *before* carrying out the variation (second-order approach).

Let us begin by considering the simpler case of the first-order approach, where one merely sets $\mathscr{T}^{*a}{}_{bc} = 0$ (which is a properly WGT-covariant condition) in the expressions (35a)–(35d). The condition $\mathscr{T}^{*a}{}_{bc} = 0$ results in the rotational gauge field $A^{ab}{}_\mu$ no longer being an independent field, but one determined explicitly by the other gauge fields $h_a{}^\mu$ and B_μ, which we thus denote by ${}^0A^{*ab}{}_\mu$ and term the 'reduced' A-field [20,21]. From (24), these quantities are given explicitly by ${}^0A^*_{ab\mu} = b^c{}_\mu {}^0\mathscr{A}^*_{abc}$, where

$$ {}^0\mathscr{A}^*_{abc} = \frac{1}{2}(c_{abc} + c_{bca} - c_{cab}) + \eta_{ac}\mathscr{B}_b - \eta_{bc}\mathscr{B}_a, \qquad (36)$$

in which $c^a{}_{bc} \equiv h_b{}^\mu h_c{}^\nu(\partial_\mu b^a{}_\nu - \partial_\nu b^a{}_\mu)$. Under a local Weyl transformation, the quantities ${}^0A^{*ab}{}_\mu$ transform in the same way as $A^{ab}{}_\mu$, so one may construct the 'reduced' WGT covariant derivative ${}^0\mathscr{D}^*_a\varphi = h_a{}^{\mu 0}D^*_\mu\varphi = h_a{}^\mu(\partial_\mu + \frac{1}{2}{}^0A^{*cd}{}_\mu\Sigma_{cd} + wB_\mu)\varphi$, which transforms in the same way as $\mathscr{D}^*_a\varphi$, but depends only on the h field, its first derivatives, and the B-field. Thus, the corresponding quantities to (35a)–(35d) are obtained simply by evaluating the rhs with $\mathscr{T}^{*a}{}_{bc}$ (and its contractions) set to zero, which also implies $\mathscr{D}^*_a \to {}^0\mathscr{D}^*_a$. This yields

$$ h^0\upsilon^A = \frac{\bar{\partial} L}{\partial\varphi_A}\bigg|_0 - {}^0\mathscr{D}^*_a \frac{\partial L}{\partial(\mathscr{D}^*_a\varphi_A)}\bigg|_0, \qquad (37a)$$

$$ h^0\tau^a{}_b = \frac{\partial L}{\partial(\mathscr{D}^*_a\varphi_A)}\bigg|_0 {}^0\mathscr{D}^*_b\varphi_A + 2\frac{\partial L}{\partial\mathscr{R}_{pqra}}\bigg|_0 {}^0\mathscr{R}_{pqrb} + 2\frac{\partial L}{\partial\mathscr{H}_{pa}}\bigg|_0 \mathscr{H}_{pb} + 2{}^0\mathscr{D}^*_r\frac{\partial L}{\partial\mathscr{T}^{*b}{}_{ar}}\bigg|_0 - \delta^b_a L|_0, \qquad (37b)$$

$$ h^0\sigma_{ab}{}^c = \frac{1}{2}\frac{\partial L}{\partial(\mathscr{D}^*_c\varphi_A)}\bigg|_0 \Sigma_{ab}\varphi_A + 2\delta^c_r {}^0\mathscr{D}^*_s \frac{\partial L}{\partial\mathscr{R}^{ab}{}_{rs}}\bigg|_0 - 2\frac{\partial L}{\partial\mathscr{T}^{*[ab]}{}_c}\bigg|_0, \qquad (37c)$$

$$ h^0\zeta^a = \frac{\partial L}{\partial(\mathscr{D}^*_a\varphi_A)}\bigg|_0 w_A\varphi_A + 2\delta^a_p {}^0\mathscr{D}^*_q\frac{\partial L}{\partial\mathscr{H}_{pq}}\bigg|_0 + 2\frac{\partial L}{\partial\mathscr{T}^{*p}{}_{qr}}\bigg|_0 \delta^a_q\delta^p_r, \qquad (37d)$$

where $|_0$ denotes that the quantity to its immediate left is evaluated assuming $\mathscr{T}^*_{abc} = 0$. The equations of motion from the first-order approach are then given simply by equating each of (37a)–(37d) to zero. Once again, it is worth noting that we have not assumed any equations of motion to be satisfied in deriving the quantities (37a)–(37d). Thus, one may derive corresponding quantities for *any subset* of terms in L that are a scalar density with weight $w = -4$, and these quantities do not vanish, in general.

We now consider the second-order approach, where one imposes $\mathscr{T}^*_{abc} = 0$ at the level of action, prior to evaluating the variational derivatives. In this case, the rotational gauge field $A^{ab}{}_\mu$ is again determined explicitly by $h_a{}^\mu$ and B_μ according to (36), and so now the action depends only on these other gauge fields. From (36), one readily finds that

$$ \begin{aligned} \delta_0 A_{ab\mu} = b^c{}_\mu(&h_{[c}{}^{\nu 0}\mathscr{D}^*_{b]}\delta_0 b_{a\nu} + h_{[a}{}^{\nu 0}\mathscr{D}^*_{c]}\delta_0 b_{b\nu} \\ &- h_{[b}{}^{\nu 0}\mathscr{D}^*_{a]}\delta_0 b_{c\nu} + 2\eta_{c[a}h_{b]}{}^\nu\delta_0 B_\nu), \end{aligned} \qquad (38)$$

from which one may show that (up to terms that are the divergence of a quantity that vanishes on the boundary of the integration region) the integrand in the expression (2) for the variation of the action is given by

$$\frac{\delta\mathcal{L}}{\delta\chi_A}\delta_0\chi_A = {}^0v^A\delta_0\varphi_A + {}^0\tilde{\tau}^a{}_\mu\delta_0 h_a{}^\mu - bb^f{}_\mu(\eta_{fa}\delta^e_{[b}{}^0\mathscr{D}^*_{c]} + \eta_{fb}\delta^e_{[c}{}^0\mathscr{D}^*_{a]} - \eta_{fc}\delta^e_{[a}{}^0\mathscr{D}^*_{b]})(h^0\tilde{\sigma}^{abc})\delta_0 h_e{}^\mu + 2{}^0\tilde{\sigma}^{abc}\eta_{c[a}h_{b]}{}^\nu\delta_0 B_\mu + {}^0\tilde{\zeta}^\mu\delta_0 B_\mu, \quad (39)$$

$$\equiv v^A\delta_0\varphi_A + t^a{}_\mu\delta_0 h_a{}^\mu + j^\mu\delta_0 B_\mu, \quad (40)$$

where we have again made use of (28) and ${}^0\tilde{\tau}^a{}_\mu$, ${}^0\tilde{\sigma}^c_{ab}$ and ${}^0\tilde{\zeta}^\mu$ denote quantities analogous to (37b)–(37d), respectively, but *without* the terms containing $\partial L/\partial\mathcal{T}^*_{abc}|_0$. In the last line, we have also defined the total dynamical energy-momentum $t^a{}_\mu$ and dilation current j^μ of both the matter and gravitational gauge fields, and the matter field variational derivatives v^A, in the second-order approach. By comparing (39) and (40), and converting all indices to Roman, one finds that the second-order variational derivatives are given in terms of the first-order ones by

$$hv^A = h^0v^A, \quad (41)$$

$$ht_{ab} = h^0\tilde{\tau}_{ab} + {}^0\mathscr{D}^*_c(h^0\tilde{\sigma}_{ab}{}^c - h^0\tilde{\sigma}^c{}_{ab} - h^0\tilde{\sigma}^c{}_{ba}), \quad (42)$$

$$hj^a = h({}^0\tilde{\zeta}^a - 2{}^0\tilde{\sigma}^{ab}{}_b). \quad (43)$$

It is clear that the forms of the matter variational derivatives are identical in the two approaches, but those of the gravitational gauge fields $h_a{}^\mu$ and B_μ differ, in general. In particular, the form for the energy-momentum tensor t_{ab} in the second-order approach is clearly the gauge theory equivalent of the Belinfante tensor [36]. By analogy, the expression (43) may be considered to define an associated Belinfante dilation current, which is clearly related to the 'field virial' that is relevant to the invariance of an action under special conformal transformations [21,37].

It is again worth noting that the expressions (41)–(43) have been derived without assuming any equations of motion are satisfied. One may therefore obtain analogous relations between corresponding first- and second-order variational derivatives derived from *any subset* of the terms in the total Lagrangian L that are a scalar density of weight $w = -4$. If one does consider the total Lagrangian L, however, then the second-order equations of motion for the matter and gauge fields are obtained simply by setting the expressions (41)–(43) to zero. In this case, provided the terms of the form $\partial L/\partial\mathscr{T}^*_{abc}|_0$ vanish in the first-order equations of motion obtained by setting (37)–(37d) to zero, then this implies that the second-order equations of motion obtained by setting (41)–(43) to zero are also satisfied, but the contrary does not necessarily hold.

C. Manifestly covariant conservation laws in WGT

We now turn our attention to deriving the conservation laws for WGT in a manner that maintains manifest covariance throughout, by applying the general method outlined in Sec. III. One may begin by considering the general form of the conservations laws given in (15). As discussed above, the key issue to address is the forms of the functions f_{AC} and f^μ_{AC} that appear in this expression and define the form variations (6) of the fields, since these are typically not covariant. For (15) to be valid, one requires at least the functions f^μ_{AC} to be (manifestly) covariant, although many of the functions f_{AC} may also be made so; as outlined in Sec. III, this is performed by generalizing the approach introduced by Bessel-Hagen for electromagnetism, which is reviewed in Appendix, and developed further below.

The form variations of the fields in WGT are given in (21). By comparing these transformation laws with the generic form (6), one may read off the functions f_{AC} and f^μ_{AC} in the latter from the coefficients of $\{\lambda^C\} = \{\lambda^1,\lambda^2,\lambda^3\} = \{\xi^\alpha,\omega^{ab},\rho\}$ and their partial derivatives, respectively. As anticipated, one immediately finds that many of the functions f_{AC} and f^μ_{AC} are not covariant quantities. In the context of the Bessel-Hagen method, the form variations (21) are already in the most general form that leaves the generic WGT action invariant (ignoring the possibility of additional accidental symmetries occurring). Following the general methodology outlined for electromagnetism in Appendix, we consider separately the conservation laws that result from the invariance of the WGT action under infinitesimal GCTs, local rotations and local dilations, respectively.

Considering first the infinitesimal GCTs characterized by $\xi^\alpha(x)$ (which we take to correspond to $C = 1$), one may make use of the invariance of the action under the transformations (21) for arbitrary functions $\omega^{ab}(x)$ and $\rho(x)$ by choosing them in a way that yields covariant forms for the new functions f^μ_{A1} (and also f_{A1} in this case) in the resulting form variations. This is achieved by setting $\omega^{ab} = -A^{ab}{}_\nu\xi^\nu$ and $\rho = -B_\nu\xi^\nu$ (where the minus signs are included for later convenience), which yields transformation laws of a much simpler form than in (21), given by

$$\delta_0\varphi = -\xi^\nu D^*_\nu\varphi, \quad (44a)$$

$$\delta_0 h_a{}^\mu = -\xi^\nu D^*_\nu h_a{}^\mu + h_a{}^\nu\partial_\nu\xi^\mu, \quad (44b)$$

$$\delta_0 A^{ab}{}_\mu = \xi^\nu R^{ab}{}_{\mu\nu}, \quad (44c)$$

$$\delta_0 B_\mu = \xi^\nu H_{\mu\nu}. \quad (44d)$$

From these form variations, one may immediately read off the new forms of the functions f_{A1} and f^μ_{A1}, all of which are now manifestly covariant. Inserting these expressions into the general form (15), one directly obtains the manifestly covariant conservation law

$$(\mathscr{D}_c^* + \mathscr{T}_c^*)(h\tau^c{}_\nu) - h(\sigma_{ab}{}^\mu R^{ab}{}_{\mu\nu} + \zeta^\mu H_{\mu\nu} - \tau^a{}_\mu D_\nu^* h_a^\mu - \upsilon^A D_\nu^* \varphi_A) = 0, \tag{45}$$

where it is worth noting that $h\upsilon^A = \delta L/\delta\varphi_A$. On multiplying through by $h_d{}^\nu$, one may rewrite the conservation law wholly in term of quantities possessing only Roman indices as

$$(\mathscr{D}_c^* + \mathscr{T}_c^*)(h\tau^c{}_d) - h(\sigma_{ab}{}^c \mathscr{R}^{ab}{}_{cd} + \zeta^c \mathscr{H}_{cd} - \tau^c{}_b \mathscr{T}^{*b}{}_{cd} - \upsilon^A \mathscr{D}_d^* \varphi_A) = 0. \tag{46}$$

We next consider invariance of the action under infinitesimal local Lorentz rotations characterized by $\omega^{ab}(x)$ (which we take to correspond to $C = 2$). In this case, the functions f^μ_{A2} in the original set of transformation laws (21) are already manifestly covariant. One may thus insert the functions f^μ_{A2} and f_{A2} read off from (21) directly into the general form (15), without employing the Bessel-Hagen method. On recalling that $\Gamma^*_\beta \sigma_{pq}{}^\beta = -A^r{}_{p\beta}\sigma_{rq}{}^\beta - A^r{}_{q\beta}\sigma_{pr}{}^\beta$ (since $\sigma_{ab}{}^\mu$ has Weyl weight $w = 0$) one finds that the final set of terms on the lhs of (15) vanish when γ^A corresponds to $h\sigma_{ab}{}^\mu$, and one immediately obtains the manifestly covariant conservation law

$$(\mathscr{D}_c^* + \mathscr{T}_c^*)(h\sigma_{ab}{}^c) + h\tau_{[ab]} + \frac{1}{2} h\upsilon^A \Sigma_{ab}\varphi_A = 0. \tag{47}$$

Finally, we consider invariance of the action under infinitesimal local dilations characterized by $\rho(x)$ (which we take to correspond to $C = 3$). Once again, the relevant functions f^μ_{A3} in the original set of transformation laws (21) are already manifestly covariant. One may thus insert f^μ_{A3} and f_{A3} read off from (21) directly into the general form (15), which immediately yields the manifestly covariant conservation law

$$(\mathscr{D}_c^* + \mathscr{T}_c^*)(h\zeta^c) - h\tau^c{}_c + h\upsilon^A w_A \varphi_A = 0. \tag{48}$$

It is straightforward to verify that the manifestly covariant conservations WGT laws (46)–(47) have the correct forms [20,21] and match those derived (albeit at considerably greater length) using the standard form of Noether's second theorem (8a). Before moving on to consider the further condition (8b) arising from Noether's second theorem, in the context of WGT, we note that the conservation law (47) may be used to simplify the expression (42) for the second-order variational derivative with respect to $h_a{}^\mu$ in terms of first-order variational derivatives. Imposing the condition $\mathscr{T}^*_{abc} = 0$, the conservation law (47) becomes

$${}^0\mathscr{D}_c^*(h^0\tilde{\sigma}_{ab}{}^c) + h^0\tilde{\tau}_{[ab]} + \frac{1}{2} h^0\tilde{\upsilon}^A \Sigma_{ab}\varphi_A = 0. \tag{49}$$

If one assumes the *matter* equations of motion ${}^0\tilde{\upsilon}^A = 0$ are satisfied (or, equivalently, that the Lagrangian L does not depend on matter fields), the expression (42) can thus be written in the simpler and manifestly symmetric form

$$ht_{ab} \overset{\mathrm{m}}{\simeq} h^0\tilde{\tau}_{(ab)} - 2{}^0\mathcal{D}_c^*(h^0\tilde{\sigma}^c{}_{(ab)}). \tag{50}$$

D. Relationship between currents in Noether's second theorem in WGT

We conclude this section by considering the relationship in WGT between the two currents that appear in Noether's second theorem (8b). As discussed in Sec. III C, this equation may be rewritten as $(\mathscr{D}_a^* + \mathscr{T}_a^*)[h(\mathscr{J}^a - \mathscr{S}^a)] = 0$, where $h\mathscr{J}^a$ for WGT is given by (34) and the expression for $h\mathscr{S}^a$ may be obtained from the general form (16), which on using the original WGT field variations (21) yields

$$h\mathscr{S}^p = h[-\xi^\mu(\tau^p{}_\mu - \sigma_{ab}{}^p A^{ab}{}_\mu - \zeta^p B_\mu) + \omega^{ab}\sigma_{ab}{}^p + \rho\zeta^p]. \tag{51}$$

It is worth noting that this expression does not depend on the variational derivatives $\upsilon^A \equiv \delta\mathscr{L}/\delta\psi_A$ with respect to the matter fields since, as expected, the functions f^μ_{AC} vanish in this case, as can be read off from the field variations (21). Thus, in order for $h\mathscr{S}^p$ to vanish, it is sufficient that just the equations of motion of the gauge fields are satisfied.

If one substitutes the original form variations (21) into the expression (34) for $h\mathscr{J}^p$, one finds after a long calculation,[10] which requires careful use of the definition (22) of the field strength tensors, the contracted Bianchi identity (26c) and the manifestly covariant expressions (35b)–(35d) for the variational derivatives with respect to the gravitational gauge fields, that

$$\begin{aligned}(\mathscr{D}_p^* + \mathscr{T}_p^*)(h\mathscr{J}^p) &= (\mathscr{D}_p^* + \mathscr{T}_p^*)[-\xi^\mu h(\tau^p{}_q b^q{}_\mu \\ &\quad - \sigma_{ab}{}^p A^{ab}{}_\mu - \zeta^p B_\mu) \\ &\quad + \omega^{ab} h\sigma_{ab}{}^p + \rho h\zeta^p] \\ &= (\mathscr{D}_p^* + \mathscr{T}_p^*)(h\mathscr{S}^p),\end{aligned} \tag{52}$$

thereby verifying explicitly the relationship between the two currents that is implied by Noether's second theorem (8b).

[10]The calculation can be somewhat shortened, better organized and carried out in a largely manifestly covariant manner if one assumes the local Weyl transformaton parameters in (21) to have the forms $\xi^\mu(x)$, $\omega^{ab}(x) = \bar{\omega}^{ab}(x) - A^{ab}{}_\nu\xi^\nu$ and $\rho(x) = \bar{\rho}(x) - B_\nu\xi^\nu$, where $\xi^\mu(x)$, $\bar{\omega}^{ab}(x)$ and $\bar{\rho}(x)$ are arbitrary functions of position, and considers separately the three cases: (i) $\bar{\omega}^{ab} = 0 = \bar{\rho}$; (ii) $\xi^\mu = 0 = \bar{\rho}$; and (iii) $\xi^\mu = 0 = \bar{\omega}^{ab}$. This is a similar approach to that used in Sec. IV C to derive directly the manifestly covariant forms of the WGT conservation laws and, in particular, allows one in case (i) to make use again of the manifestly covariant form variations (44) derived using the Bessel-Hagen method.

Thus, as expected for an action that is invariant under a set of local symmetries, this relationship contains no further information, but nonetheless provides a useful check of the derivation of the expressions (35b)–(35d). Indeed, the requirement $(\mathscr{D}^*_a + \mathscr{T}^*_a)[h(\mathscr{J}^a - \mathscr{S}^a)] = 0$ from Noether's second theorem can thus be used as an alternative (albeit rather longer) means of deriving the expressions (35b)–(35d) for the variational derivatives with respect to the gravitational gauge fields; it has been demonstrated, however, that this equivalence between the Noether and Hilbert (variational) approaches does not hold in general for all modified gravity theories [38].

V. EXTENDED WEYL GAUGE THEORY

We now move on to consider eWGT [20], which proposes an 'extended' form for the transformation laws of the rotational and dilational gauge fields under local dilations. In particular, under infinitesimal local Weyl transformations consisting of GCTs, rotations of the local Lorentz frames and dilations, parametrized by $\xi^\mu(x)$, $\omega^{ab}(x) = -\omega^{ba}(x)$ and $\rho(x)$, respectively, a matter field φ of weight w and the gauge fields transform as

$$\delta_0\varphi = -\xi^\nu\partial_\nu\varphi + \left(\frac{1}{2}\omega^{ab}\Sigma_{ab} + w\rho\right)\varphi, \tag{53a}$$

$$\delta_0 h_a{}^\mu = -\xi^\nu\partial_\nu h_a{}^\mu + h_a{}^\nu\partial_\nu\xi^\mu - (\omega^b{}_a + \rho\delta^b_a)h_b{}^\mu, \tag{53b}$$

$$\begin{aligned}\delta_0 A^{ab}{}_\mu = {} & -\xi^\nu\partial_\nu A^{ab}{}_\mu - A^{ab}{}_\nu\partial_\mu\xi^\nu - 2\omega^{[a}{}_c A^{b]c}{}_\mu \\ & - \partial_\mu\omega^{ab} + 2\theta\eta^{c[a}b^{b]}{}_\mu h_c{}^\nu\partial_\nu\rho,\end{aligned} \tag{53c}$$

$$\delta_0 B_\mu = -\xi^\nu\partial_\nu B_\mu - B_\nu\partial_\mu\xi^\nu - \theta\partial_\mu\rho, \tag{53d}$$

where θ is an arbitrary parameter that can take any value. The proposed form for the transformation law (53c) of the rotational gauge field is motivated by the observation that the WGT (and PGT) matter actions for the massless Dirac field and the electromagnetic field (neither of which depends on the dilation gauge field) are invariant under local dilations even if one assumes this 'extended' transformation law for the rotational gauge field. A complementary motivation for introducing the extended transformation law (53c) is that under local dilations it places the transformation properties of the PGT rotational gauge field strength $\mathscr{R}^{ab}{}_{cd}$ and translational gauge field strength $\mathscr{T}^a{}_{bc}$ on a more equal footing: for general values of θ, neither $\mathscr{R}^{ab}{}_{cd}$ nor $\mathscr{T}^a{}_{bc}$ transforms covariantly, but $\mathscr{R}^{ab}{}_{cd}$ does transform covariantly and $\mathscr{T}^a{}_{bc}$ transforms inhomogeneously for $\theta = 0$, and vice versa for $\theta = 1$. It is also worth noting that the extended transformation law for the rotational gauge field reduces to that in WGT for $\theta = 0$, whereas the extended transformation law (53d) for the dilational gauge field reduces to the WGT form for $\theta = 1$; thus there is no value of θ for which both transformation laws reduce to their WGT forms.

In eWGT, the covariant derivative, denoted by $\mathscr{D}^\dagger_a$, has a somewhat different form to that shown in (20) for WGT. In particular, one does not adopt the standard approach of introducing each gauge field as the linear coefficient of the corresponding generator. Rather, in order to accommodate our proposed extended transformation law (53c) under local dilations, one is led to introduce the 'rotational' gauge field $A^{ab}{}_\mu(x)$ and the 'dilational' gauge field $B_\mu(x)$ in a very different way, so that[11]

$$\begin{aligned}\mathscr{D}^\dagger_a\varphi_A &= h_a{}^\mu D^\dagger_\mu\varphi_A = h_a{}^\mu(\partial_\mu + \Gamma^\dagger_\mu)\varphi_A \\ &= h_a{}^\mu\left[\partial_\mu + \frac{1}{2}A^{\dagger ab}{}_\mu\Sigma_{ab} + w_A\left(B_\mu - \frac{1}{3}T_\mu\right)\right]\varphi_A,\end{aligned} \tag{54}$$

where we have introduced the modified A-field

$$A^{\dagger ab}{}_\mu \equiv A^{ab}{}_\mu + 2b^{[a}{}_\mu\mathcal{B}^{b]}, \tag{55}$$

in which $\mathcal{B}_a = h_a{}^\mu B_\mu$ and $T_\mu = b^a{}_\mu\mathscr{T}_a$, where $\mathscr{T}_a \equiv \mathscr{T}^b{}_{ab}$ is the trace of the PGT torsion.[12] It is straightforward to show that, if φ has Weyl weight w, then (54) does indeed transform covariantly with Weyl weight $w-1$, as required. Unlike the transformation laws for $A^{ab}{}_\mu$ and B_μ, the covariant derivative (54) does not explicitly contain the parameter θ. Consequently, it does *not* reduce to the standard WGT covariant derivative $\mathscr{D}^*_a\varphi_A$ in either special case $\theta = 0$ or $\theta = 1$, while retaining its covariant transformation law for *any* value of θ.

The derivative (54) does in fact transform covariantly under the much *wider* class of gauge field transformations in which $\theta\partial_\mu\rho(x)$ is replaced in (53c)–(53d) by an *arbitrary* vector field $Y_\mu(x)$. Indeed, one finds that the WGT (and PGT) matter actions for the massless Dirac field and the electromagnetic field are still invariant under local dilations after such a replacement, although the discussion above regarding the transformation properties of $\mathscr{R}^{ab}{}_{cd}$ and $\mathscr{T}^a{}_{bc}$ following requires appropriate modification, since neither transforms covariantly if $\theta\partial_\mu\rho(x)$ is replaced by an arbitrary vector $Y_\mu(x)$. The covariance of $\mathscr{D}^\dagger_a\varphi_A$ under this wider class of transformations allows one to identify a further gauge symmetry of eWGT, namely under the simultaneous transformations

[11]The daggers in the definition of the derivative operator are intended simply to distinguish it from the usual notation used [11,20,21] for the covariant derivatives of PGT and WGT, and should not be confused with the operation of Hermitian conjugation.

[12]It is worth noting that $A^{\dagger ab}{}_\mu$ is not considered to be a fundamental field (notwithstanding the variational approach adopted below), but merely a shorthand for the above combination of the gauge fields $h_a{}^\mu$ (or its inverse), $A^{ab}{}_\mu$ and B_μ. Similarly, T_μ is merely a shorthand for the corresponding function of the gauge fields $h_a{}^\mu$ (or its inverse) and $A^{ab}{}_\mu$.

$$A^{ab}{}_{\mu} \to A^{ab}{}_{\mu} + 2b^{[a}{}_{\mu}\mathscr{Y}^{b]}, \qquad B_{\mu} \to B_{\mu} - Y_{\mu}, \tag{56}$$

where $\mathscr{Y}_a = h_a{}^{\mu} Y_{\mu}$ and Y_{μ} is an arbitrary vector field. Under this symmetry, both $A^{\dagger ab}{}_{\mu}$ and $B_{\mu} - \frac{1}{3}T_{\mu}$ remain unchanged and thus $\mathscr{D}^{\dagger}_a \varphi$ is invariant, as too are the eWGT field strengths and action discussed below. One may make use of this symmetry of eWGT to choose a gauge in which either B_{μ} or T_{μ} is self-consistently set to zero, which can considerably simplify subsequent calculations.

It was noted in [20] that the extended transformation laws (53c)–(53d) implement Weyl scaling in a novel way that may be related to gauging of the full conformal group. This is discussed in more detail in [21], where it is shown that eWGT does indeed constitute a valid novel gauge theory of the conformal group. We briefly summarize below the aspects of eWGT that are relevant to our present discussion, and refer the reader to [20,21] for further details.

By analogy with WGT, the Lagrangian density in eWGT has the usual form $\mathscr{L} = h^{-1}L$, where the translational gauge field $h_a{}^{\mu}$ is assigned a Weyl weight $w = -1$, so that $h = \det(h_a{}^{\mu})$ and L are scalar densities both of Weyl weight $w = -4$, and hence the action S is invariant under local scale transformations. The Lagrangian has the functional dependencies

$$L = L(\varphi_A, \mathscr{D}^{\dagger}_a\varphi_A, \mathscr{R}^{\dagger}_{abcd}, \mathscr{T}^{\dagger}_{abc}, \mathscr{H}^{\dagger}_{ab}), \tag{57}$$

where the quantities $\mathscr{R}^{\dagger}_{abcd}$, $\mathscr{T}^{\dagger}_{abc}$, $\mathscr{H}^{\dagger}_{ab}$ are the eWGT 'rotational', 'translational' and 'dilational' gauge field strengths, respectively, which are defined through the action of the commutator of two eWGT covariant derivatives on some field φ of weight w by

$$[\mathscr{D}^{\dagger}_c, \mathscr{D}^{\dagger}_d]\varphi = \left(\frac{1}{2}\mathscr{R}^{\dagger ab}{}_{cd}\Sigma_{ab} + w\mathscr{H}^{\dagger}_{cd} - \mathscr{T}^{\dagger a}{}_{cd}\mathscr{D}^{\dagger}_a\right)\varphi. \tag{58}$$

The field strengths have the forms $\mathscr{R}^{\dagger ab}{}_{cd} = h_a{}^{\mu}h_b{}^{\nu}R^{\dagger ab}{}_{\mu\nu}$, $\mathscr{H}^{\dagger}_{cd} = h_c{}^{\mu}h_d{}^{\nu}H^{\dagger}_{\mu\nu}$ and $\mathscr{T}^{\dagger a}{}_{bc} = h_b{}^{\mu}h_c{}^{\nu}T^{\dagger a}{}_{\mu\nu}$, where

$$R^{\dagger ab}{}_{\mu\nu} = 2(\partial_{[\mu}A^{\dagger ab}{}_{\nu]} + \eta_{cd}A^{\dagger ac}{}_{[\mu}A^{\dagger db}{}_{\nu]}), \tag{59}$$

$$H^{\dagger}_{\mu\nu} = 2\left(\partial_{[\mu}B_{\nu]} - \frac{1}{3}\partial_{[\mu}T_{\nu]}\right), \tag{60}$$

$$T^{\dagger a}{}_{\mu\nu} = 2D^{\dagger}_{[\mu}b^a{}_{\nu]}. \tag{61}$$

From the transformation laws (53), it is straightforward to verify that, in accordance with their index structures, the gauge field strength tensors $\mathscr{R}^{\dagger ab}{}_{cd}$, $\mathscr{H}^{\dagger}_{cd}$ and $\mathscr{T}^{\dagger a}{}_{bc}$ are invariant under GCTs, and transform covariantly under local Lorentz transformations and dilations with Weyl weights $w = -2$, $w = -2$ and $w = -1$, respectively [20,21], similarly to their WGT counterparts.

It is worth noting, however, that $\mathscr{R}^{\dagger ab}{}_{cd}$ and $\mathscr{T}^{\dagger a}{}_{bc}$ differ in form substantially from those in WGT, and are given in terms of the PGT field strengths $\mathscr{R}^{ab}{}_{cd}$ and $\mathscr{T}^a{}_{bc}$ by

$$\begin{aligned}\mathscr{R}^{\dagger ab}{}_{cd} &= \mathscr{R}^{ab}{}_{cd} + 4\delta^{[b}_{[c}\mathscr{D}_{d]}\mathscr{B}^{a]} - 4\delta^{[b}_{[c}\mathscr{B}_{d]}\mathscr{B}^{a]} \\ &\quad - 2\mathscr{B}^2\delta^{[a}_c\delta^{b]}_d - 2\mathscr{B}^{[a}\mathscr{T}^{b]}{}_{cd}, \\ \mathscr{T}^{\dagger a}{}_{bc} &= \mathscr{T}^a{}_{bc} + \frac{2}{3}\delta^a_{[b}\mathscr{T}_{c]},\end{aligned} \tag{62}$$

where $\mathscr{B}^2 \equiv \mathscr{B}^a\mathscr{B}_a$ and $\mathscr{D}_a \equiv h_a{}^{\mu}D_{\mu} \equiv h_a{}^{\mu}(\partial_{\mu} + \frac{1}{2}A^{ab}{}_{\mu}\Sigma_{ab})$ is the PGT covariant derivative operator. It is particularly important to note that the trace of the eWGT translational field strength tensor vanishes identically, namely $\mathscr{T}^{\dagger}_b \equiv \mathscr{T}^{\dagger a}{}_{ba} = 0$, so that $\mathscr{T}^{\dagger a}{}_{bc}$ is completely trace-free (contraction on any pair of indices yields zero). Moreover, using the expression (61) and defining the quantities $c^{*a}{}_{bc} \equiv 2h_b{}^{\mu}h_c{}^{\nu}\partial^{\dagger}_{[\mu}b^a{}_{\nu]}$, where $\partial^{\dagger}_{\mu} = \partial_{\mu} + w(B_{\mu} - \frac{1}{3}T_{\mu})$, one may show that the fully anholonomic modified A-field $\mathcal{A}^{\dagger ab}{}_c \equiv h_c{}^{\mu}A^{\dagger ab}{}_{\mu}$ can be written as [20]

$$\mathscr{A}^{\dagger}_{abc} = \frac{1}{2}(c^{\dagger}_{abc} + c^{\dagger}_{bca} - c^{\dagger}_{cab}) - \frac{1}{2}(\mathscr{T}^{\dagger}_{abc} + \mathscr{T}^{\dagger}_{bca} - \mathscr{T}^{\dagger}_{cab}). \tag{63}$$

As in our discussion of WGT, it is convenient to list the Bianchi identities satisfied by the gravitational gauge field strengths $\mathscr{R}^{\dagger ab}{}_{cd}$, $\mathscr{T}^{\dagger a}{}_{bc}$ and $\mathscr{H}^{\dagger}_{ab}$ in eWGT. These may again be straightforwardly derived from the Jacobi identity, but now applied to the eWGT covariant derivative. One quickly finds the three Bianchi identities [20]

$$\mathscr{D}^{\dagger}_{[a}\mathscr{R}^{\dagger de}{}_{bc]} - \mathscr{T}^{\dagger f}{}_{[ab}\mathscr{R}^{\dagger de}{}_{c]f} = 0, \tag{64a}$$

$$\mathscr{D}^{\dagger}_{[a}\mathscr{T}^{*d}{}_{bc]} - \mathscr{T}^{\dagger e}{}_{[ab}\mathscr{T}^{\dagger d}{}_{c]e} - \mathscr{R}^{\dagger d}{}_{[abc]} - \mathscr{H}^{\dagger}_{[ab}\delta^d_{c]} = 0, \tag{64b}$$

$$\mathscr{D}^{\dagger}_{[a}\mathscr{H}^{\dagger}_{bc]} - \mathscr{T}^{\dagger e}{}_{[ab}\mathscr{H}^{\dagger}_{c]e} = 0. \tag{64c}$$

By contracting over various indices, one also obtains the following nontrivial contracted Bianchi identities:

$$\mathscr{D}^{\dagger}_a\mathscr{R}^{\dagger ae}{}_{bc} - 2\mathscr{D}^{\dagger}_{[b}\mathscr{R}^{\dagger e}{}_{c]} - 2\mathscr{T}^{\dagger f}{}_{a[b}\mathscr{R}^{\dagger ae}{}_{c]f} - \mathscr{T}^{\dagger f}{}_{bc}\mathscr{R}^{\dagger e}{}_f = 0, \tag{65a}$$

$$\mathscr{D}^{\dagger}_a\left(\mathscr{R}^{\dagger a}{}_c - \frac{1}{2}\delta^a_c\mathscr{R}^{\dagger}\right) + \mathscr{T}^{\dagger f}{}_{bc}\mathscr{R}^{\dagger b}{}_f + \frac{1}{2}\mathscr{T}^{\dagger f}{}_{ab}\mathscr{R}^{\dagger ab}{}_{cf} = 0, \tag{65b}$$

$$\mathscr{D}^{\dagger}_a\mathscr{T}^{\dagger a}{}_{bc} + 2\mathscr{R}^{\dagger}_{[bc]} - 2\mathscr{H}^{\dagger}_{bc} = 0, \tag{65c}$$

which are somewhat simpler than their WGT counterparts (26a)–(26c) on account of the condition $\mathscr{T}^\dagger_a = 0$.

A. Manifestly covariant variational derivatives in eWGT

As in WGT, we begin by considering directly the variation of the action. In particular, by analogy with (27), one may immediately write

$$h\delta_0\mathscr{L} = \frac{\bar{\partial} L}{\partial\varphi_A}\delta_0\varphi_A + \frac{\partial L}{\partial(\mathscr{D}^\dagger_a\varphi_A)}\delta_0(\mathscr{D}^\dagger_a\varphi_A) + \frac{\partial L}{\partial\mathscr{R}^\dagger_{abcd}}\delta_0\mathscr{R}^\dagger_{abcd} + \frac{\partial L}{\partial\mathscr{T}^\dagger_{abc}}\delta_0\mathscr{T}^\dagger_{abc} + \frac{\partial L}{\partial\mathscr{H}^\dagger_{ab}}\delta_0\mathscr{H}^\dagger_{ab} - b^a{}_\mu L\delta_0 h_a{}^\mu. \tag{66}$$

In eWGT, however, there is an additional subtlety compared with WGT: although the dynamical energy-momentum tensor $\tau^a{}_\mu \equiv \delta\mathscr{L}/\delta h_a{}^\mu$ derived from the *total* Lagrangian density is covariant, this does *not* necessarily hold for the corresponding quantities obtained from *subsets* of the terms in L, even if they transform covariantly with weight $w=-4$ [20]. This leads one to the construct an alternative quantity for which this more general covariance property does hold. This may be arrived at more directly from an alternative variational principle, in which one makes a change of field variables from the set φ_A, h^μ_a, $A^{ab}{}_\mu$ and B_μ to the new set φ_A, h^μ_a, $A^{\dagger ab}{}_\mu$ and B_μ. It is worth noting that one is simply making a change of field variables here, rather than considering $A^{\dagger ab}{}_\mu$ to be an independent field variable; in other words, one still considers $A^{\dagger ab}{}_\mu$ to be given in terms of h^μ_a, $A^{ab}{}_\mu$, B_μ by its defining relationship (55), rather than an independent quantity whose relationship to the other variables would be determined from the variational principle. Moreover, as shown in [20], the eWGT covariant derivative can be expressed wholly in terms of the fields h^μ_a (or its inverse) and $A^{\dagger ab}{}_\mu$, and thus so too can the eWGT field strengths. In particular, if one defines the (noncovariant) derivative operator $\mathscr{D}^\natural_a\varphi \equiv h_a{}^\mu D^\natural_\mu\varphi \equiv h_a{}^\mu(\partial_\mu + \frac{1}{2}A^{\dagger bc}{}_\mu\Sigma_{bc})\varphi$ and the quantities $\mathscr{T}^{\natural a}{}_{bc} \equiv 2h_b{}^\mu h_c{}^\nu D^\natural_{[\mu}b^a{}_{\nu]}$, then one may easily show that $\mathscr{D}^\dagger_a\varphi = (\mathscr{D}^\natural_a - \frac{1}{3}w\mathscr{T}^\natural_a)\varphi$. Consequently, in the new set of field variables, the Lagrangian L in (57) has no explicit dependence on B_μ.

Following the general procedure used for WGT, one must now determine how the variations in (66) depend on the variations the new set of fields φ_A, h^μ_a and $A^{\dagger ab}{}_\mu$ themselves. This is easily achieved using the definition of the eWGT covariant derivative and the expressions (59)–(61) for the field strengths. By analogy with the approach adopted for WGT, one must also make use of the fact that for any coordinate vector V^μ of weight $w=0$ (i.e., invariant under local scale transformations, like the Lagrangian density $\mathscr{L}$), one may show that $\partial_\mu V^\mu = h^{-1}\mathscr{D}^\dagger_a(hb^a{}_\mu V^\mu)$ or, equivalently, for any local Lorentz vector $\mathscr{V}^a$ having Weyl weight $w=-3$ one has [20]

$$\mathscr{D}^\dagger_a\mathscr{V}^a = h\partial_\mu(h^{-1}h_a{}^\mu\mathscr{V}^a), \tag{67}$$

which is somewhat simpler than its WGT counterpart (67) because of the condition $\mathscr{T}^\dagger_a = 0$. Expressions of the form (67) on the rhs of (66) therefore contribute only surface terms to the variation of the action in (9), but we will retain them nonetheless, as they are required for our later discussion.

We begin by considering together the first two terms on the rhs of (66), for which one obtains (after a rather lengthy calculation)

$$\begin{aligned}
\frac{\bar{\partial} L}{\partial\varphi_A}\delta_0\varphi_A + \frac{\partial L}{\partial(\mathscr{D}^\dagger_a\varphi_A)}\delta_0(\mathscr{D}^\dagger_a\varphi_A) &= \frac{\bar{\partial} L}{\partial\varphi_A}\delta_0\varphi_A + \frac{\partial L}{\partial(\mathscr{D}^\dagger_a\varphi_A)}\left[\mathscr{D}^\dagger_a(\delta_0\varphi_A) + \delta_0 h_a{}^\mu D^\dagger_\mu\varphi_A + \left(\frac{1}{2}h_a{}^\mu\Sigma_{bc} + \frac{1}{3}w_A\eta_{a[c}h_{b]}{}^\mu\right)\varphi_A\delta_0 A^{\dagger bc}{}_\mu\right. \\
&\quad \left. + \frac{2}{3}w_A\varphi_A\left(h_{[a}{}^\mu\mathscr{D}^\dagger_{b]} + \frac{1}{2}h_c{}^\mu\mathscr{T}^{\dagger c}{}_{ab}\right)\delta_0 b^b{}_\mu\right], \\
&= \left[\frac{\bar{\partial} L}{\partial\varphi_A} - \mathscr{D}^\dagger_a\frac{\partial L}{\partial(\mathscr{D}^\dagger_a\varphi_A)}\right]\delta_0\varphi_A + \left[\frac{\partial L}{\partial(\mathscr{D}^\dagger_a\varphi_A)}D^\dagger_\mu\varphi_A + \frac{2}{3}w_A b^c{}_\mu\delta^a_{[b}\mathscr{D}^\dagger_{c]}\left(\frac{\partial L}{\partial(\mathscr{D}^\dagger_b\varphi_A)}\varphi_A\right)\right]\delta_0 h_a{}^\mu \\
&\quad + \frac{\partial L}{\partial(\mathscr{D}^\dagger_a\varphi_A)}\left(\frac{1}{2}h_a{}^\mu\Sigma_{bc} + \frac{1}{3}w_A\eta_{a[c}h_{b]}{}^\mu\right)\varphi_A\delta_0 A^{\dagger bc}{}_\mu, \\
&\quad + \mathscr{D}^\dagger_a\left[\frac{\partial L}{\partial(\mathscr{D}^\dagger_a\varphi_A)}\delta_0\varphi_A + \frac{2}{3}\frac{\partial L}{\partial(\mathscr{D}^\dagger_{[a}\varphi_A)}w_A\varphi_A b^{b]}{}_\mu\delta_0 h_b{}^\mu\right],
\end{aligned} \tag{68}$$

where both terms in square brackets in the last line are readily shown to have Weyl weight $w=-3$, with the second one having no analog in the corresponding expression (29) in WGT. Analyzing the further terms containing derivatives on the rhs of (66) in a similar manner, one finds (again after lengthy calculations in each case)

$$\begin{aligned}\frac{\partial L}{\partial \mathscr{R}^\dagger_{abcd}}\delta_0\mathscr{R}^\dagger_{abcd} &= 2\frac{\partial L}{\partial \mathscr{R}^\dagger_{abcd}}[R^\dagger_{ab\mu d}\delta_0 h_c{}^\mu + h_d{}^\mu \mathscr{D}^\dagger_c(\delta_0 A^\dagger_{ab\mu})],\\ &= 2\frac{\partial L}{\partial \mathscr{R}^\dagger_{abcd}}R^\dagger_{ab[\mu d]}\delta_0 h_c{}^\mu + (h_e{}^\mu \mathscr{T}^{\dagger e}{}_{cd} + 2h_c{}^\mu \mathscr{D}^\dagger_d)\left(\frac{\partial L}{\partial \mathscr{R}^\dagger_{abcd}}\right)\delta_0 A^\dagger_{ab\mu} - 2\mathscr{D}^\dagger_d\left[\frac{\partial L}{\partial \mathscr{R}^\dagger_{abcd}}h_c{}^\mu\delta_0 A^\dagger_{ab\mu}\right],\end{aligned} \tag{69}$$

$$\begin{aligned}\frac{\partial L}{\partial \mathscr{T}^\dagger_{abc}}\delta_0\mathscr{T}^\dagger_{abc} &= 2\frac{\partial L}{\partial \mathscr{T}^\dagger_{abc}}[T^\dagger_{a\mu\nu}h_c{}^\nu\delta_0 h_b{}^\mu + h_c{}^\nu\mathscr{D}^\dagger_b(\delta_0 b_{a\nu}) + h_b{}^\mu\delta_0 A^\dagger_{ac\mu}\\ &\quad -\frac{1}{3}\eta_{ac}(\eta_{b[p}h_{q]}{}^\mu\delta_0 A^{\dagger pq}{}_\mu + 2h_{[q}{}^\mu\mathscr{D}^\dagger_{b]}(\delta_0 b^q{}_\mu) + h_p{}^\mu\mathscr{T}^{\dagger p}{}_{qb}\delta_0 b^q{}_\mu)],\\ &= 2\frac{\partial L}{\partial \mathscr{T}^\dagger_{abc}}\left[\left(T^\dagger_{a\mu\nu}h_c{}^\nu\delta^d_b - \frac{1}{2}\mathscr{T}^{\dagger d}{}_{bc}b_{a\mu}\right)\delta_0 h_d{}^\mu + h_b{}^\mu\delta_0 A^\dagger_{ac\mu}\right] - 2\mathscr{D}^\dagger_c\left(\frac{\partial L}{\partial \mathscr{T}^\dagger_{abc}}\right)b_{a\mu}\delta_0 h_b{}^\mu\\ &\quad -\frac{2}{3}\eta_{ac}\left[\left(b^p{}_\mu\mathscr{D}^\dagger_b - \delta^p_b b^q{}_\mu\mathscr{D}^\dagger_q\right)\left(\frac{\partial L}{\partial \mathscr{T}^\dagger_{abc}}\right)\delta_0 h_p{}^\mu + \frac{\partial L}{\partial \mathscr{T}^\dagger_{abc}}\eta_{b[p}h_{q]}{}^\mu\delta_0 A^{\dagger pq}{}_\mu\right]\\ &\quad + 2\mathscr{D}^\dagger_c\left[\left(\frac{\partial L}{\partial \mathscr{T}^\dagger_{abc}}b_{a\mu} - \frac{2}{3}\eta_{pq}\frac{\partial L}{\partial \mathscr{T}^\dagger_{pq[c}}b^{b]}{}_\mu\right)\delta_0 h_b{}^\mu\right],\end{aligned} \tag{70}$$

$$\begin{aligned}\frac{\partial L}{\partial \mathscr{H}^\dagger_{ab}}\delta_0\mathscr{H}^\dagger_{ab} &= 2\frac{\partial L}{\partial \mathscr{H}^\dagger_{ab}}\left[H^\dagger_{\mu\nu}h_b{}^\nu\delta_0 h_a{}^\mu + h_b{}^\nu\mathscr{D}^\dagger_a\left(\delta_0 B_\nu - \frac{1}{3}\delta_0 T_\nu\right)\right],\\ &= 2\frac{\partial L}{\partial \mathscr{H}^\dagger_{ab}}H^\dagger_{\mu\nu}h_b{}^\nu\delta_0 h_a{}^\mu + \frac{2}{3}b^a{}_{[\mu}\mathscr{D}^\dagger_{c]}\left[(\mathscr{T}^{\dagger c}{}_{pq} + 2\delta^c_p\mathscr{D}^\dagger_q)\left(\frac{\partial L}{\partial \mathscr{H}^\dagger_{pq}}\right)\right]\delta_0 h_a{}^\mu\\ &\quad + \frac{2}{3}\eta_{c[a}h_{b]}{}^\mu\left(\delta^c_p\mathscr{D}^\dagger_q + \frac{1}{2}\mathscr{T}^{\dagger c}{}_{pq}\right)\left(\frac{\partial L}{\partial \mathscr{H}^\dagger_{pq}}\right)\delta_0 A^{\dagger ab}{}_\mu\\ &\quad -\frac{2}{3}\mathscr{D}^\dagger_c\left\{\left[\frac{\partial L}{\partial \mathscr{H}^\dagger_{pq}}\mathscr{T}^{\dagger[c}{}_{pq}b^{a]}{}_\mu + 2\delta^{[c}_p\mathscr{D}^\dagger_q\left(\frac{\partial L}{\partial \mathscr{H}^\dagger_{pq}}\right)b^{a]}{}_\mu\right]\delta_0 h_a{}^\mu\right\}.\end{aligned} \tag{71}$$

In the above expressions it is again assumed that the appropriate antisymmetrizations, arising from the symmetries of the field strength tensors, are performed when the rhs are evaluated. It is also easily shown that the quantity in brackets in each of the last terms in (69)–(71) has Weyl weight $w = -3$, so according to (67) each such term contributes a surface term to the variation of the action (9).

Following an analogous approach to that adopted for WGT, one may then substitute the expressions (68)–(71) into (66), which may itself subsequently be substituted into (9) to obtain an expression of the general form (12) for Noether's first theorem. This may be written as

$$\begin{aligned}\delta S = \int [&v^A\delta_0\varphi_A + \tau^{\dagger a}{}_\mu\delta_0 h_a{}^\mu + \sigma_{ab}{}^\mu\delta_0 A^{\dagger ab}{}_\mu\\ &+ h^{-1}\mathscr{D}^\dagger_p(h\mathscr{J}^p)]d^4x = 0,\end{aligned} \tag{72}$$

where the current $h\mathscr{J}^p$ is given by

$$\begin{aligned}h\mathscr{J}^p &= \frac{\partial L}{\partial(\mathscr{D}^\dagger_p\varphi_A)}\delta_0\varphi_A\\ &\quad + 2\left[\frac{1}{3}\frac{\partial L}{\partial(\mathscr{D}^\dagger_{[p}\varphi_A)}w_A\varphi_A b^{b]}{}_\mu + \frac{\partial L}{\partial \mathscr{T}^\dagger_{abp}}b_{a\mu} - \frac{2}{3}\eta_{rs}\frac{\partial L}{\partial \mathscr{T}^\dagger_{rs[p}}b^{b]}{}_\mu - \frac{1}{3}\frac{\partial L}{\partial \mathscr{H}^\dagger_{rs}}\mathscr{T}^{\dagger[p}{}_{rs}b^{b]}{}_\mu - \frac{2}{3}\delta^{[p}_r\mathscr{D}^\dagger_s\left(\frac{\partial L}{\partial \mathscr{H}^\dagger_{rs}}\right)b^{b]}{}_\mu\right]\delta_0 h_b{}^\mu\\ &\quad - 2\frac{\partial L}{\partial \mathscr{R}^\dagger_{abcp}}h_c{}^\mu\delta_0 A^\dagger_{ab\mu} + b^p{}_\mu\xi^\mu L,\end{aligned} \tag{73}$$

and we have defined the variational derivative[13] $v^A \equiv (\delta\mathscr{L}/\delta\varphi_A)_\dagger = \delta\mathscr{L}/\delta\varphi_A$ with respect to the matter field φ_A, and the total modified dynamical energy-momentum $\tau^{\dagger a}{}_\mu \equiv (\delta\mathscr{L}/\delta h_a{}^\mu)_\dagger$ and spin-angular-momentum

[13] We denote the variational derivative of $\mathscr{L}$ with respect to any one of the fields χ in the new set of variables by $(\delta\mathscr{L}/\delta\chi)_\dagger$ to distinguish it from the variational derivative $\delta\mathscr{L}/\delta\chi$ in the original set.

$\sigma_{ab}{}^{\mu} \equiv (\delta\mathscr{L}/\delta A^{\dagger ab}{}_{\mu})_{\dagger} = \delta\mathscr{L}/\delta A^{ab}{}_{\mu}$ of both the matter and gravitational gauge fields. It is also worth noting that the (identically vanishing) dilation current $\zeta^{\dagger\mu} \equiv (\delta\mathscr{L}/\delta B_{\mu})_{\dagger}$ in the new set of variables is related to that in the original set by $\zeta^{\dagger\mu} = \zeta^{\mu} - 2h_a{}^{\mu}\sigma^{ab}{}_{b}$, so that the latter is given simply by $\zeta^{\mu} = 2h_a{}^{\mu}\sigma^{ab}{}_{b}$. Manifestly covariant forms for the variational derivatives may then be read off from the expressions (68)–(71). Converting all Greek indices to roman and defining the quantities $\tau^{\dagger a}{}_{b} \equiv \tau^{\dagger a}{}_{\mu}h_b{}^{\mu}$ and $\sigma_{ab}{}^{c} \equiv \sigma_{ab}{}^{\mu}b^{c}{}_{\mu}$, one then makes the following identifications:

$$h\upsilon^{A} = \frac{\bar{\partial}L}{\partial\varphi_A} - \mathscr{D}^{\dagger}_{a}\frac{\partial L}{\partial(\mathscr{D}^{\dagger}_{a}\varphi_A)}, \tag{74a}$$

$$h\tau^{\dagger a}{}_{b} = \frac{\partial L}{\partial(\mathscr{D}^{\dagger}_{a}\varphi_A)}\mathscr{D}^{\dagger}_{b}\varphi_A + 2\frac{\partial L}{\partial\mathscr{R}^{\dagger}_{pqra}}\mathscr{R}^{\dagger}_{pqrb} + 2\frac{\partial L}{\partial\mathscr{H}^{\dagger}_{pa}}\mathscr{H}^{\dagger}_{pb} + 2\frac{\partial L}{\partial\mathscr{T}^{\dagger}_{pqa}}\mathscr{T}^{\dagger}_{pqb} - (\mathscr{T}^{\dagger a}{}_{qr} + 2\delta^{a}_{q}\mathscr{D}^{\dagger}_{r})\frac{\partial L}{\partial\mathscr{T}^{\dagger b}{}_{qr}}$$
$$- \delta^{b}_{a}L - 2\mathscr{D}^{\dagger}_{c}(h\hat{\sigma}^{ca}{}_{b}), \tag{74b}$$

$$h\sigma_{ab}{}^{c} = \frac{1}{2}\frac{\partial L}{\partial(\mathscr{D}^{\dagger}_{c}\varphi_A)}\Sigma_{ab}\varphi_A + (\mathscr{T}^{\dagger c}{}_{rs} + 2\delta^{c}_{r}\mathscr{D}^{\dagger}_{s})\frac{\partial L}{\partial\mathscr{R}^{\dagger ab}{}_{rs}} - 2\frac{\partial L}{\partial\mathscr{T}^{\dagger[ab]}{}_{c}} + h\hat{\sigma}_{ab}{}^{c}, \tag{74c}$$

where for convenience we have also defined the quantity

$$h\hat{\sigma}_{ab}{}^{c} = \frac{1}{3}\delta^{c}_{[a}\eta_{b]r}\frac{\partial L}{\partial(\mathscr{D}^{\dagger}_{r}\varphi_A)}w_A\varphi_A + \frac{2}{3}\eta_{pr}\delta^{c}_{[a}\eta_{b]q}\frac{\partial L}{\partial\mathscr{T}^{\dagger}_{pqr}} - \frac{1}{3}\delta^{c}_{[a}\eta_{b]r}(\mathscr{T}^{\dagger r}{}_{pq} + 2\delta^{r}_{p}\mathscr{D}^{\dagger}_{q})\frac{\partial L}{\partial\mathscr{H}^{\dagger}_{pq}}. \tag{75}$$

Once again, it is assumed that the appropriate antisymmetrizations, arising from the symmetries of the field strength tensors, are performed when the rhs are evaluated. As mentioned above, $\zeta^{\dagger a} \equiv 0$ since $\mathscr{L}$ does not explicitly depend on B_{μ} in the new set of variables; the dilation current in the original set of variables is thus given by $\zeta^{a} = 2\sigma^{ab}{}_{b}$. As anticipated, the expressions (74a)–(74c) are manifestly covariant (and hence so too are the equations of motion obtained by setting each rhs to zero) and straightforward to evaluate, requiring one only to differentiate the Lagrangian L with respect to the matter fields, their covariant derivatives and the field strengths. One may easily confirm that the above expressions lead to precisely the same variational derivatives as those obtained by using the standard (but much longer) approach of evaluating (5a) for each field.

It is worth comparing the expressions (74a)–(74c) with their counterparts (35a)–(35c) in WGT. One sees that the eWGT expression for $h\upsilon_A$ is obtained simply by 'replacing asterisks with daggers' and recalling that $\mathscr{T}^{\dagger}_{a} \equiv 0$, but the expressions in eWGT for $h\tau^{\dagger a}{}_{b}$ and $h\sigma_{ab}{}^{c}$ each contain an additional final term beyond those obtained by performing the same process on their WGT counterparts (35b)–(35c). In particular, one sees that the final terms in (74b) and (74c) each depend on the quantity (75) and have no analog in WGT. It is a noteworthy feature of eWGT that the additional term in the expression for $h\tau^{\dagger a}{}_{b}$ is given by the covariant derivative of the additional term (with permuted indices) in the expression for $h\sigma_{ab}{}^{c}$, and this has some novel consequences. First, one notes that for a Lagrangian L that does not contain the gauge field strength tensors, but depends only on the matter fields and their covariant derivatives, the variational derivatives with respect to the gauge fields do *not* reduce to the *covariant canonical currents* [11,21] of the matter fields. Indeed, there exist additional terms proportional to the dilational generator $\Delta = w_A I$ for the matter fields φ_A, so that any matter field with nonzero Weyl weight w_A contributes additionally both to the modified energy-momentum tensor and to the spin-angular-momentum tensor, irrespective of its spin. Second, for Lagrangians that do depend on the gauge field strengths, there are additional terms capable of producing a dependence on the covariant derivatives of the field strength tensors, and in each case these terms depend on the covariant derivatives of field strength tensors for different gauge fields than those with respect to which the variational derivative is taken. Moreover, the final term on the rhs of (74b) contains *second* covariant derivatives of $\partial L/\partial\mathscr{H}^{\dagger}_{ab}$.

From (60), it appears at first sight that $\mathscr{H}^{\dagger}_{ab}$ is linear in second-order derivatives of $h_a{}^{\mu}$ and first-order derivatives of $h_a{}^{\mu}$ and $A^{\dagger ab}{}_{\mu}$ (and hence of $A^{ab}{}_{\mu}$ and B_{μ}). In that case, if the Lagrangian contains a term proportional to $\mathscr{H}^{\dagger}_{ab}\mathscr{H}^{\dagger ab}$ (which has the required Weyl weight $w = -4$ to be scale-invariant) it would follow that the final term on the rhs of (74b) is linear in fourth-order derivatives of $h_a{}^{\mu}$ and third-order derivatives of all three gauge fields $h_a{}^{\mu}$, $A^{ab}{}_{\mu}$ and B_{μ}. Similarly, the final term in (74c) would be linear in third-order derivatives of $h_a{}^{\mu}$. Moreover, if the Lagrangian contains a term proportional to $\mathscr{R}^{\dagger}_{[ab]}\mathscr{H}^{\dagger ab}$, the final term on the rhs of (74b) would be linear in third-order derivatives of $h_a{}^{\mu}$, $A^{ab}{}_{\mu}$ and B_{μ}. These considerations would seem to indicate that eWGTs containing either term in the Lagrangian suffer from Ostrogradsky's instability [39,40]. As noted in [20], however, this conclusion is not clear cut, since in applying such theories to particular physical systems or in the general linearized case, one finds that the resulting field equations always organize themselves into combinations of coupled second-order equations in the gauge fields [20]. Specifically, one finds the terms containing higher-order derivatives correspond to the derivative of already known expressions, and so contain no new information. Having now identified the gauge symmetry (56) and obtained the general expressions (74b) and (74c) for the variational derivatives, one may indeed show that this always occurs in the general nonlinear case. First, one may use the gauge transformation (56) to set $T_\mu = 0$, so that $\mathscr{H}^\dagger_{ab}$ is merely linear in first-order derivatives of B_μ. Nonetheless, if the Lagrangian contains a term proportional to $\mathscr{H}^\dagger_{ab}\mathscr{H}^{\dagger ab}$, the final term in (74b), specifically the part that arises from the final term in (75), still contains third-order derivatives of B_μ. This is unproblematic, however, since this term is the covariant derivative of an expression that is already known from the field equation $h\sigma_{ab}{}^c = 0$. Hence, in the final field equations one encounters field derivatives of only second-order or lower, thereby avoiding Ostrogradsky's instability.

It is also worth pointing out that, as for WGT, we have not assumed the equations of motion to be satisfied in deriving (74a)–(74c). Thus, one may calculate the corresponding variational derivatives for *any subset* of terms in L that is a scalar density of weight $w = -4$. Individually, however, such quantities do *not* vanish, in general. Rather, each equation of motion requires only the vanishing of the sum of such quantities, when derived from disjoint subsets that exhaust the total Lagrangian L.

B. Relationship between first- and second-order variational principles in eWGT

As we did for WGT, we now demonstrate how the approach outlined above is well suited to comparing first- and second-order variational derivatives. We again focus on the example of the variational derivatives obtained by setting the (eWGT) torsion to zero *after* the variation is performed (first-order approach) with those obtained by setting the torsion to zero in the action *before* carrying out the variation (second-order approach). As mentioned in the Introduction, however, in eWGT one faces an additional complication relative to WGT, since setting the torsion to zero does not lead to an explicit expression for the rotational gauge field in terms the other gauge fields, but instead an implicit constraint relating all the gauge fields.

We again begin by considering the simpler case of the first-order approach, where one merely sets $\mathscr{T}^{\dagger a}{}_{bc} = 0$ (which is a properly eWGT-covariant condition) in the expressions (74a)–(74c). In eWGT, however, the condition $\mathscr{T}^{\dagger a}{}_{bc} = 0$ results in an *implicit* constraint between the gauge fields $h_a{}^\mu$, $A^{ab}{}_\mu$ and B_μ. Once again, it proves useful in eWGT to work in terms of the modified rotational gauge field, or rather its 'reduced' form in the case $\mathscr{T}^{\dagger a}{}_{bc} = 0$ [20,21]. From (63), this is given by ${}^0A^\dagger_{ab\mu} = b^c{}_\mu{}^0\mathscr{A}^\dagger_{abc}$, where[14]

$$
{}^0\mathscr{A}^\dagger_{abc} = \frac{1}{2}(c_{abc} + c_{bca} - c_{cab}) + \eta_{ac}\left(\mathscr{B}_b - \frac{1}{3}\mathscr{T}_b\right) - \eta_{bc}\left(\mathscr{B}_a - \frac{1}{3}\mathscr{T}_a\right). \tag{76}
$$

In an analogous manner to WGT, under a local extended Weyl transformation, the quantities ${}^0A^{\dagger ab}{}_\mu$ transform in the same way as $A^{\dagger ab}{}_\mu$, and so one may construct the 'reduced' eWGT covariant derivative ${}^0\mathscr{D}^\dagger_a\varphi = h_a{}^{\mu 0}D^*_\mu\varphi = h_a{}^\mu(\partial_\mu + \frac{1}{2}{}^0A^{\dagger ab}{}_\mu\Sigma_{ab} + wB_\mu)\varphi$, which transforms in the same way as $\mathscr{D}^\dagger_a\varphi$. Thus, the corresponding quantities to (74a)–(74c) are obtained simply by evaluating the rhs with $\mathscr{T}^{\dagger a}{}_{bc}$ set to zero, which also implies $\mathscr{D}^\dagger_a \to {}^0\mathscr{D}^\dagger_a$. This yields

$$
h^0 v^A = \frac{\bar{\partial} L}{\partial\varphi_A}\bigg|_0 - {}^0\mathscr{D}^\dagger_a \frac{\partial L}{\partial(\mathscr{D}^\dagger_a\varphi_A)}\bigg|_0, \tag{77a}
$$

$$
h^0\tau^{\dagger a}{}_b = \frac{\partial L}{\partial(\mathscr{D}^\dagger_a\varphi_A)}\bigg|_0 {}^0\mathscr{D}^\dagger_b\varphi_A + 2\frac{\partial L}{\partial\mathscr{R}^\dagger_{pqra}}\bigg|_0 {}^0\mathscr{R}^\dagger_{pqrb} + 2\frac{\partial L}{\partial\mathscr{H}^\dagger_{pa}}\bigg|_0 \mathscr{H}^\dagger_{pb} + 2{}^0\mathscr{D}^\dagger_r\frac{\partial L}{\partial T^{\dagger b}{}_{ar}}\bigg|_0 - \delta^b_a L|_0 - 2{}^0\mathscr{D}^\dagger_c(h^0\hat{\sigma}^{ca}{}_b), \tag{77b}
$$

$$
h^0\sigma_{ab}{}^c = \frac{1}{2}\frac{\partial L}{\partial(\mathscr{D}^\dagger_c\varphi_A)}\bigg|_0 \Sigma_{ab}\varphi_A + 2\delta^{c0}_r\mathscr{D}^\dagger_s\frac{\partial L}{\partial\mathscr{R}^{\dagger ab}{}_{rs}}\bigg|_0 - 2\frac{\partial L}{\partial\mathscr{T}^{\dagger[ab]}{}_c}\bigg|_0 + h^0\hat{\sigma}_{ab}{}^c, \tag{77c}
$$

where by analogy with (75) we have defined the quantity

$$
h^0\hat{\sigma}_{ab}{}^c = \frac{1}{3}\delta^c_{[a}\eta_{b]r}\frac{\partial L}{\partial(\mathscr{D}^\dagger_r\varphi_A)}\bigg|_0 w_A\varphi_A + \frac{2}{3}\delta^c_{[a}\eta_{b]q}\eta_{pr}\frac{\partial L}{\partial\mathscr{T}^\dagger_{pqr}}\bigg|_0 - \frac{2}{3}\delta^c_{[a}\eta_{b]p}{}^0\mathscr{D}^\dagger_q\frac{\partial L}{\partial\mathscr{H}^\dagger_{pq}}\bigg|_0. \tag{78}
$$

[14]It is important to note that there is a fundamental difference with WGT here, since ${}^0A^{\dagger ab}{}_\mu$ depends on the rotational gauge field $A^{ab}{}_\mu$ through the terms containing $\mathscr{T}_a$, and hence cannot be written entirely in terms of the other gauge fields $h_a{}^\mu$ and B_μ.

Once again, it is worth noting that we have not assumed any equations of motion to be satisfied in deriving the quantities (77a)–(77c). Thus, one may derive corresponding quantities for *any subset* of terms in L that are a scalar density with weight $w=-4$, and these quantities do not vanish, in general.

We now consider the second-order approach, where one imposes $\mathscr{T}^{\dagger}_{abc}=0$ at the level of the action, prior to evaluating the variational derivatives. In this case, $A^{\dagger ab}{}_{\mu}$ is again given by (76), in which case one may show that the following constraint must be satisfied while performing the variation:

$$C_{ab\mu}\equiv A^{\dagger}_{ab\mu}-\frac{2}{3}h_d{}^{\nu}b_{[a|\mu}A^{\dagger d}{}_{|b]\nu}-{}^0A_{ab\mu}+\frac{2}{3}h_d{}^{\nu}b_{[a|\mu}{}^0A^d{}_{|b]\nu}=0, \tag{79}$$

where ${}^0A_{ab\mu}=\frac{1}{2}b^c{}_{\mu}(c_{abc}+c_{bca}-c_{cab})$. It is worth noting that $C_{ab\mu}$ depends on all the gauge fields; moreover, since ${}^0A_{ab\mu}$ depends both on the h-field and its derivatives, the expression (79) constitutes a nonholonomic constraint. We therefore consider the *augmented* total Lagrangian density $\hat{\mathscr{L}}\equiv\mathscr{L}+\lambda^{ab\mu}C_{ab\mu}$, where $\lambda^{ab\mu}$ is a field of weight $w=0$ with the same symmetries as $C_{ab\mu}$ that acts as a Lagrange multiplier. Thus, up to terms that are the divergence of a quantity that vanishes on the boundary of the integration region, the integrand in the expression (2) for the variation of the action is given by

$$\left(\frac{\delta\hat{\mathscr{L}}}{\delta\chi_A}\right)_{\dagger}\delta_0\chi_A=v^A\delta_0\varphi_A+\tau^{\dagger a}{}_{\mu}\delta_0h_a{}^{\mu}+\sigma_{ab}{}^{\mu}\delta_0A^{\dagger ab}{}_{\mu}+\lambda^{ab\mu}\delta_0C_{ab\mu}+C_{ab\mu}\delta_0\lambda^{ab\mu}, \tag{80}$$

From (79), one finds after some calculation that

$$\begin{aligned}\delta_0C_{ab\mu}={}&\delta_0A^{\dagger}_{ab\mu}-\frac{2}{3}b_{[a|\mu}h_q{}^{\sigma}\delta_0A^{\dagger q}{}_{|b]\sigma}-b^c{}_{\mu}(h_{[c}{}^{\nu 0}\mathscr{D}^{\dagger}_{b]}\delta_0b_{a\nu}\\&+h_{[a}{}^{\nu 0}\mathscr{D}^{\dagger}_{c]}\delta_0b_{b\nu}-h_{[b}{}^{\nu 0}\mathscr{D}^{\dagger}_{a]}\delta_0b_{c\nu})\\&+\frac{2}{3}b^c{}_{\mu}(\eta_{ca}h_{[q}{}^{\sigma 0}\mathscr{D}^{\dagger}_{b]}-\eta_{cb}h_{[q}{}^{\sigma 0}\mathscr{D}^{\dagger}_{a]})\delta_0b^q{}_{\sigma},\end{aligned} \tag{81}$$

from which one may show that (80) becomes (up to a total divergence)

$$\begin{aligned}\left(\frac{\delta\hat{\mathscr{L}}}{\delta\chi_A}\right)_{\dagger}\delta_0\chi_A={}&{}^0v^A\delta_0\varphi_A+{}^0\tilde{\tau}^{\dagger a}{}_{\mu}\delta_0h_a{}^{\mu}\\&+\left({}^0\tilde{\sigma}_{ab}{}^{\mu}+\lambda_{ab}{}^{\mu}-\frac{2}{3}h_a{}^{\mu}\lambda^c{}_{bc}\right)\delta_0A^{\dagger ab}{}_{\mu}\\&+bb^f{}_{\mu}[(\eta_{fa}\delta^e_{[b}{}^0\mathscr{D}^{\dagger}_{c]}+\eta_{fb}\delta^e_{[c}{}^0\mathscr{D}^{\dagger}_{a]}\\&-\eta_{fc}\delta^e_{[a}{}^0\mathscr{D}^{\dagger}_{b]})(h\lambda^{abc})+\frac{4}{3}\delta^e_{[f}{}^0\mathscr{D}^{\dagger}_{b]}(h\lambda^{ab}{}_a)]\\&\times\delta_0h_e{}^{\mu}+C_{ab\mu}\delta_0\lambda^{ab\mu},\end{aligned} \tag{82}$$

$$\equiv v^A\delta_0\varphi_A+t^{\dagger a}{}_{\mu}\delta_0h_a{}^{\mu}+s_{ab}{}^{\mu}\delta_0A^{\dagger ab}{}_{\mu}+C_{ab\mu}\delta_0\lambda^{ab\mu}, \tag{83}$$

where we have again made use of (67) and ${}^0\tilde{\tau}^a{}_{\mu}$ and ${}^0\tilde{\sigma}_{ab}{}^c$ denote quantities analogous to (77b)–(77c), respectively, but *without* the terms containing $\partial L/\partial\mathscr{T}^{\dagger}_{abc}|_0$. In the last line, we have also defined the modified total dynamical energy-momentum $t^{\dagger a}{}_{\mu}$ and spin-angular momentum $s_{ab}{}^{\mu}$ of both the matter and gravitational gauge fields, and the matter field variational derivatives v^A, in the second-order approach.

From (83), one sees immediately that the equation of motion for the Lagrange multiplier field $\lambda^{ab\mu}$ is simply $C_{ab\mu}=0$, which enforces the original constraint (79), as required. By comparing (82) and (83), and converting all indices to Roman, one further finds that the second-order variational derivatives are related to the first-order ones by

$$hv^A=h{}^0v^A, \tag{84}$$

$$\begin{aligned}ht^{\dagger}_{ab}={}&h{}^0\tilde{\tau}^{\dagger}_{ab}+{}^0\mathscr{D}^{\dagger}_c(h\lambda^c{}_{ab}+h\lambda^c{}_{ba}-h\lambda_{ab}{}^c)\\&-\frac{2}{3}\eta_{ab}{}^0\mathscr{D}^{\dagger}_c(h\lambda^{cd}{}_d)+\frac{2}{3}{}^0\mathscr{D}^{\dagger}_b(h\lambda_{ad}{}^d),\end{aligned} \tag{85}$$

$$hs_{abc}=h\left({}^0\tilde{\sigma}_{abc}+\lambda_{abc}+\frac{2}{3}\eta_{c[a}\lambda_{b]d}{}^d\right). \tag{86}$$

To proceed further, one must eliminate the dependence of (85)(86) on the Lagrange multiplier field λ_{abc}. This is achieved by enforcing the A-field equation of motion, so that $hs_{abc}=0$, which now merely determinines λ_{abc} under the constraint $C_{ab\mu}=0$. Using the resulting condition ${}^0\tilde{\sigma}_{abc}+\lambda_{abc}+\frac{2}{3}\eta_{c[a}\lambda_{b]d}{}^d=0$, one may now eliminate the Lagrange multiplier field from (85), and one finally obtains

$$hv^A=h{}^0v^A, \tag{87}$$

$$ht^{\dagger}_{ab}=h{}^0\tilde{\tau}^{\dagger}_{ab}+{}^0\mathscr{D}^{\dagger}_c(h{}^0\tilde{\sigma}_{ab}{}^c-h{}^0\tilde{\sigma}^c{}_{ab}-h{}^0\tilde{\sigma}^c{}_{ba}). \tag{88}$$

As was the case for WGT, the forms of the matter variational derivatives are identical in the first- and second-order approaches, and the form for the modified energy-momentum tensor in the second-order approach is reminiscent of the Belinfante tensor. Since, one has not used the equations of motion for the matter fields and the gauge field $h_a{}^{\mu}$ in deriving the expressions (87)–(88), they remain valid for any subset of the terms in $\mathscr{L}$ that are a scalar density of weight $w=-4$. If one does consider the total Lagrangian L, however, then the second-order equations of motion for the matter and gauge fields are obtained simply by setting the expressions (87)–(88) to zero. In this case, provided the terms of the form $\partial L/\partial\mathscr{T}^{\dagger}_{abc}|_0$ vanish in the first-order equations of motion obtained by setting (37)–(37d) to zero, then this implies that the second-order equations of motion obtained by setting (87)–(88) to zero are also satisfied, but the contrary does not necessarily hold.

C. Manifestly covariant conservation laws in eWGT

We now derive the conservation laws for eWGT in a manner that maintains manifest covariance throughout, by applying the general method outlined in Sec. III in a similar way to that performed in Sec. IV C for WGT. Once again, we begin by considering the general form of the conservations laws given in (15). As in the previous section, we work in the new set of variables φ_A, h_a^μ, ${A^{\dagger ab}}_\mu$, in which the Lagrangian does not depend explicitly on the gauge field B_μ. In this case, under infinitesimal local Weyl transformations consisting of GCTs, rotations of the local Lorentz frames and dilations, parametrized by $\xi^\mu(x)$, $\omega^{ab}(x)$ and $\rho(x)$, the form variations (53) are replaced by

$$\delta_0\varphi = -\xi^\nu\partial_\nu\varphi + \left(\frac{1}{2}\omega^{ab}\Sigma_{ab} + w\rho\right)\varphi, \tag{89a}$$

$$\delta_0 {h_a}^\mu = -\xi^\nu\partial_\nu {h_a}^\mu + {h_a}^\nu\partial_\nu\xi^\mu - ({\omega^b}_a + \rho\delta_a^b){h_b}^\mu, \tag{89b}$$

$$\delta_0 {A^{\dagger ab}}_\mu = -\xi^\nu\partial_\nu {A^{\dagger ab}}_\mu - {A^{\dagger ab}}_\nu\partial_\mu\xi^\nu - 2\omega^{[a}{}_c {A^{\dagger b]c}}_\mu - \partial_\mu\omega^{ab}. \tag{89c}$$

By comparing these transformation laws with the generic form (6), one may read off the functions f_{AC} and f^μ_{AC} in the latter from the coefficients of $\{\lambda^C\} = \{\lambda^1, \lambda^2, \lambda^3\} = \{\xi^\alpha, \omega^{ab}, \rho\}$ and their partial derivatives, respectively. As anticipated, one immediately finds that many of the functions f_{AC} and f^μ_{AC} are not covariant quantities. One therefore again employs the Bessel-Hagen method to obtain new form variations of the fields in which the functions f^μ_{AC} are manifestly covariant, as required, although many of the functions f_{AC} may also be made so. Following the general methodology outlined in Appendix, we consider separately the conservation laws that result from the invariance of the eWGT action under infinitesimal GCTs, local rotations and local dilations, respectively.

Considering first the infinitesimal GCTs characterized by $\xi^\alpha(x)$ (which we take to correspond to $C = 1$), one may make use of the invariance of the action under the transformations (89) for arbitrary functions $\omega^{ab}(x)$ and $\rho(x)$ by choosing them in a way that yields covariant forms for the new functions f^μ_{A1} (and also f_{A1} in this case) in the resulting form variations. This is achieved by setting $\omega^{ab} = -{A^{\dagger ab}}_\nu\xi^\nu$ and $\rho = -(B_\nu - \frac{1}{3}T_\mu)\xi^\nu$ (where the minus signs are included for later convenience), which yields transformation laws of a much simpler form than in (89), given by

$$\delta_0\varphi = -\xi^\nu D^\dagger_\nu\varphi, \tag{90a}$$

$$\delta_0 {h_a}^\mu = -\xi^\nu D^\dagger_\nu {h_a}^\mu + {h_a}^\nu\partial_\nu\xi^\mu, \tag{90b}$$

$$\delta_0 {A^{\dagger ab}}_\mu = \xi^\nu {R^{\dagger ab}}_{\mu\nu}. \tag{90c}$$

From these form variations, one may immediately read off the new forms of the functions f_{A1} and f^μ_{A1}, all of which are now manifestly covariant. Inserting these expressions into the general form (15), one directly obtains the manifestly covariant conservation law

$$\mathscr{D}^\dagger_c(h{\tau^{\dagger c}}_\nu) - h({\sigma_{ab}}^\mu {R^{\dagger ab}}_{\mu\nu} - {\tau^{\dagger a}}_\mu D^\dagger_\nu h^\mu_a - \upsilon^A D^\dagger_\nu\varphi_A) = 0, \tag{91}$$

where $h\upsilon^A = (\delta L/\delta\varphi_A)_\dagger = \delta L/\delta\varphi_A$. On multiplying through by ${h_d}^\nu$, one may rewrite the conservation law wholly in term of quantities possessing only Roman indices as

$$\mathscr{D}^\dagger_c(h{\tau^{\dagger c}}_d) - h({\sigma_{ab}}^c {\mathscr{R}^{\dagger ab}}_{cd} - {\tau^{\dagger c}}_b {\mathscr{T}^{\dagger b}}_{cd} - \upsilon^A \mathscr{D}^\dagger_d\varphi_A) = 0. \tag{92}$$

We next consider invariance of the action under infinitesimal local Lorentz rotations characterized by $\omega^{ab}(x)$ (which we take to correspond to $C = 2$). In this case, the functions f^μ_{A2} in the set of transformation laws (89) are already manifestly covariant. One may thus insert the functions f^μ_{A2} and f_{A2} read off from (89) directly into the general form (15), without employing the Bessel-Hagen method. On recalling that $\Gamma^\dagger_\beta {\sigma_{pq}}^\beta = -{A^{\dagger r}}_{p\beta}{\sigma_{rq}}^\beta - {A^{\dagger r}}_{q\beta}{\sigma_{pr}}^\beta$ (since ${\sigma_{ab}}^\mu$ has Weyl weight $w = 0$) one finds that the final set of terms on the lhs of (15) vanish when γ^A corresponds to $h{\sigma_{ab}}^\mu$, and one immediately obtains the manifestly covariant conservation law

$$\mathscr{D}^\dagger_c(h{\sigma_{ab}}^c) + h\tau^\dagger_{[ab]} + \frac{1}{2}h\upsilon^A\Sigma_{ab}\varphi_A = 0. \tag{93}$$

Finally, we consider invariance of the action under infinitesimal local dilations characterized by $\rho(x)$ (which we take to correspond to $C = 3$). Once again, the relevant functions f^μ_{A3} in the set of transformation laws (89) are already manifestly covariant. One may thus insert f^μ_{A3} and f_{A3} read off from (89) directly into the general form (15), which immediately yields the manifestly covariant algebraic conservation law

$$h{\tau^{\dagger c}}_c - h\upsilon^A w_A\varphi_A = 0. \tag{94}$$

It is straightforward to verify that the manifestly covariant conservations WGT laws (92)–(94) have the correct forms [20,21] and match those derived (albeit at considerably greater length) using the standard form of Noether's second theorem (8a).

Before moving on to consider the further condition (8b) arising from Noether's second theorem, in the context of eWGT, we note that the conservation law (93) may be used to simplify the expression (88) for the second-order variational derivative with respect to ${h_a}^\mu$ in terms of first-order variational derivatives. Imposing the condition $\mathscr{T}^{\dagger}_{abc} = 0$, the conservation law (93) becomes

$$ {}^0\mathscr{D}^{\dagger}_c(h^0\tilde{\sigma}_{ab}{}^c) + h^0\tilde{\tau}^{\dagger}_{[ab]} + \frac{1}{2}h^0\tilde{\upsilon}^A\Sigma_{ab}\varphi_A = 0. \quad (95)$$

If one assumes the *matter* equations of motion ${}^0\tilde{\upsilon}^A = 0$ are satisfied (or, equivalently, that the Lagrangian L does not depend on matter fields), the expression (88) can thus be written in the simpler and manifestly symmetric form

$$ ht^{\dagger}_{ab} \stackrel{\mathrm{m}}{\simeq} h^0\tilde{\tau}^{\dagger}_{(ab)} - 2{}^0\mathscr{D}^{\dagger}_c(h^0\tilde{\sigma}^c{}_{(ab)}). \quad (96)$$

D. Relationship between currents in Noether's second theorem in eWGT

We conclude this section by considering the relationship in WGT between the two currents that appear in Noether's second theorem (8b). As discussed in Sec. III C, this equation may be rewritten as $\mathscr{D}^{\dagger}_a[h(\mathscr{J}^a - \mathscr{S}^a)] = 0$, where $h\mathscr{J}^a$ for eWGT is given by (73) and the expression for $h\mathscr{S}^a$ may be obtained from the general form (16), which on using the eWGT field variations (89) yields

$$ h\mathscr{S}^p = h[-\xi^\mu(\tau^{\dagger p}{}_\mu - \sigma_{ab}{}^p A^{\dagger ab}{}_\mu) + \omega^{ab}\sigma_{ab}{}^p]. \quad (97)$$

As was the case for WGT, this expression does not depend on the variational derivatives $\upsilon^A \equiv \delta\mathscr{L}/\delta\psi_A$ with respect to the matter fields since, as expected, the functions f^μ_{AC} vanish in this case, as can be read off from the form variations (89) of the new set of fields. Thus, in order for $h\mathscr{S}^p$ to vanish, it is sufficient that just the equations of motion of the gauge fields are satisfied. Moreover, in eWGT, the current (97) also does not depend on the dilation $\rho(x)$.

If one substitutes the form variations (89) of the new set of fields into the expression (73) for $h\mathscr{J}^p$, one finds after a long calculation of a similar nature to that required in WGT, which makes careful use of the definition (58) of the field strength tensors, the contracted Bianchi identity (65c) and the manifestly covariant expressions (74b)–(74c) for the variational derivatives with respect to the gravitational gauge fields, that

$$ \begin{aligned} \mathscr{D}^{\dagger}_p(h\mathscr{J}^p) &= \mathscr{D}^{\dagger}_p[-\xi^\mu h(\tau^p{}_q b^q{}_\mu - \sigma_{ab}{}^p A^{ab}{}_\mu) + \omega^{ab}h\sigma_{ab}{}^p] \\ &= \mathscr{D}^{\dagger}_p(h\mathscr{S}^p), \end{aligned} \quad (98)$$

thereby verifying explicitly the relationship between the two currents that is implied by Noether's second theorem (8b), as was the case in WGT. Thus, as expected for an action that is invariant under a set of local symmetries, this relationship contains no further information, but nonetheless provides a useful check of the derivation of the expressions (74b)–(74c). Indeed, in a similar way to WGT, the requirement $\mathscr{D}^{\dagger}_a[h(\mathscr{J}^a - \mathscr{S}^a)] = 0$ from Noether's second theorem can thus be used as an alternative (albeit rather longer) means of deriving the expressions (35b)–(35d) for the variational derivatives with respect to the gravitational gauge fields.

VI. CONCLUSIONS

We have presented a variational principle that maintains manifest covariance throughout when applied to the actions of gauge theories of gravity. In particular, it directly yields field equations and conservation laws that are manifestly covariant under the symmetries to which the action is invariant. This is achieved by deriving explicit manifestly covariant forms for the Euler-Lagrange variational derivatives and Noether's theorems for a generic action of the form typically assumed in gauge theories of gravity.

The manifestly covariant form of Noether's first theorem and the expressions for the variational derivatives derived therefrom not only provide a significant calculational saving relative to the traditional method of evaluation, but also yield useful insights into their general forms. In particular, these expressions enable one easily to establish the relationship between the forms of variational derivatives, and hence the field equations, obtained by applying first- and second-order variational principles, respectively. An interesting case is provided by comparing the variational derivatives obtained by setting the torsion to zero after the variation is performed (first-order approach) with those obtained by setting the torsion to zero in the action before carrying out the variation (second-order approach).

The reexpression of Noether's second theorem in terms of manifestly covariant quantities provides further utility and insights. In particular, one may use it to derive the conservation laws obeyed by the matter and gravitational gauge fields in a manifestly covariant manner. This also relies on being able to express the form variations of these fields such that at least the coefficient functions of the derivatives of the parameters of the symmetry transformations are manifestly covariant. This may be achieved by generalizing the approach introduced by Bessel-Hagen for electromagnetism, which is discussed in Appendix. The reexpression of Noether's second theorem further allows one straightforwardly to verify the relationship between the two currents on which it depends. Indeed, one may use Noether's second theorem as an alternative (albeit somewhat longer) means of deriving manifestly covariant forms for the variational derivatives.

The manifestly covariant variational principle is illustrated by application to the scale-invariant WGT and its recently proposed eWGT version, but can be straightforwardly applied to other gravitational gauge theories with smaller or larger symmetry groups. For WGT and eWGT, the fields in the theory consist of a translational gauge field $h_a{}^\mu$ (with inverse $b^a{}_\mu$), a rotational gauge field $A^{ab}{}_\mu$ and a dilational gauge field B_μ, together with some set of matter fields φ_A, which may include a scalar compensator field. In eWGT, however, it is more natural to work in terms of the alternative set of variables φ_A, h_a^μ, $A^{\dagger ab}{}_\mu$ and B_μ, where the modified rotational gauge field $A^{\dagger ab}{}_\mu \equiv A^{ab}{}_\mu + 2b^{[a}{}_\mu \mathscr{B}^{b]}$ and $\mathscr{B}_a = h_a{}^\mu B_\mu$. Moreover, eWGT may be shown to be invariant under the simultaneous 'torsion-scale' gauge transformations $A^{ab}{}_\mu \to A^{ab}{}_\mu + 2b^{[a}{}_\mu \mathscr{Y}^{b]}$ and $B_\mu \to B_\mu - Y_\mu$, where $\mathscr{Y}_a = h_a{}^\mu Y_\mu$ and Y_μ is an arbitrary vector field; this may be used to set either B_μ or T_μ to zero, which can considerably simplify subsequent calculations. The scale-invariant actions for WGT and eWGT are further assumed to depend only on the matter fields, their covariant derivatives and the field strength tensors of the gravitational gauge fields. In this case, the eWGT action in the alternative set of variables does not depend explicitly on B_μ, hence reducing by one the number of independent variational derivatives. As might be expected from the above considerations, one finds a number of similarities between WGT and eWGT, and also some important and novel differences.

Considering first the manifestly covariant expressions for the variational derivatives in WGT, one finds that these reduce to the corresponding covariant canonical currents of the matter fields if the Lagrangian does not depend on the gravitational gauge field strengths. For Lagrangians that do depend on the gauge field strengths, one finds that the only terms that contain the covariant derivative of a field strength tensor depend on the field strength tensor of the gauge field with respect to which the variational derivative is taken. By contrast, in eWGT one finds that the variational derivatives with respect to the translational and modified rotational gauge fields contain additional terms beyond those obtained by 'replacing asterisks with daggers' in their WGT counterparts. Moreover, the additional terms in the translational variational derivative are given by the covariant derivative of the additional terms (with permuted indices) in the expression for the rotational variational derivative; this has some novel consequences. First, for a Lagrangian that depends only on the matter fields and their covariant derivatives, the variational derivatives with respect to the gauge fields do not reduce to the covariant canonical currents of the matter fields, but comtain additional terms proportional to the dilational generator $\Delta = w_A I$ for the matter fields φ_A. Thus, any matter field with nonzero Weyl weight w_A contributes additionally both to the modified energy-momentum tensor and to the spin-angular-momentum tensor, irrespective of its spin. Second, for Lagrangians L that depend on the gauge field strengths, there are additional terms capable of producing a dependence on the covariant derivatives of the field strength tensors, and in each case these terms depend on the covariant derivatives of field strength tensors for different gauge fields than those with respect to which the variational derivative is taken. Moreover, there exist terms containing covariant derivatives of $\partial L/\partial \mathscr{H}^\dagger_{ab}$. By using the 'torsion-scale' gauge symmetry and the manifestly covariant forms of the variational derivatives, however, one may show that the final eWGT field equations contain field derivatives of only second-order or lower, thereby avoiding Ostrogradsky's instability.

On comparing the variational derivatives obtained by setting the torsion to zero after the variation is performed (first-order approach) with those obtained by setting the torsion to zero in the action before carrying out the variation (second-order approach), one finds important differences between WGT and eWGT. In both cases, the rotational gauge field is no longer an independent field, but in WGT it may be written explicitly in terms of the other gauge fields, whereas in eWGT there exists an implicit constraint relating all the gauge fields. In both cases, however, one may arrive at simple expressions for the variational derivatives in the second-order approach in terms of those from the first-order approach. In particular, the translational variational derivative in the second-order approach for WGT and eWGT is the gauge theory equivalent of the Belinfante tensor. Moreover, in WGT the second-order dilational variational derivative may be considered to define an associated Belinfante dilation current, which is clearly related to the 'field virial' that is relevant to the invariance of an action under special conformal transformations.

Turning to the re-expression of Noether's second theorem, the resulting derivations of manifestly covariant forms of the conservation laws satisfied by the fields in WGT and eWGT, yield similar forms in both cases for the laws corresponding to invariance under local translations and rotations, respectively. For invariance under local dilations, however, one finds the resulting conservation law is differential in WGT, but algebraic in eWGT. In both WGT and eWGT, one may also use the re-expression of Noether's second theorem to verify the relationship between the two currents on which it depends, although in both cases this verification requires a calculation of considerable length. Alternatively, in each case, one may use Noether's second theorem as an alternative (albeit considerably longer) means of deriving manifestly covariant forms for the variational derivatives.

Whilst this paper has focussed heavily on the *Lagrangian* prescription of field theory, and the associated field equations and conservation laws, we note that the techniques developed here may impart even stronger benefits in the *Hamiltonian* formulation. Hamiltonian gauge field theory is characterized by the presence of field-valued *constraints*, which encode not only the gauge symmetries but also the whole nonlinear dynamics, as elucidated by the consistency algorithm of Dirac and Bergmann [41–43]. The fundamental currency of the consistency algorithm is the Poisson bracket,[15] which is a bilinear in functional variations with respect to dynamical fields. In the context of gravitational gauge fields, the Hamiltonian formulation is typically realized using the

[15]More sophisticated *Dirac* brackets [44] also arise; these are equally relevant to our discussion.

so-called $3+1$ or Arnowitt-Deser-Misner (ADM) technique, whereby manifest diffeomorphism covariance is preserved despite the imposition of a spacelike foliation. Accordingly, the ADM Poisson bracket presents a clear opportunity for manifestly covariant variational methods, such as those expressed in Eqs. (35) and (74). The Hamiltonian demand is, if anything, more pronounced than the Lagrangian demand. In the latter case, a countably small collection of field equations (not including indices) must be obtained (e.g., *one* set of Einstein equations). In the former case and for a gravitational gauge theory, all Poisson brackets between all constraints must be evaluated in order to classify the gauge symmetries: this can in practice correspond to tens or hundreds of brackets [45–49]. Separately, the variations of a constraint can be more challenging than those of an action because: (i) the constraints are typically indexed and always (quasi-) local, necessitating the use of smearing functions; (ii) they may contain more terms in ADM form than the original Lagrangian; and crucially (iii) they are of *unlimited*[16] order in spatial gradients [50] even when the Lagrangian is second order as assumed in (1). The extension of the techniques discussed here to the higher-order, ADM variational derivative, is left to future work.

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APPENDIX: BESSEL-HAGEN METHOD FOR ELECTROMAGNETISM

For classical electromagnetism (EM) in Minkowski spacetime $\mathscr{M}$ labeled using Cartesian inertial coordinates x^μ, the action is given by $S=\int \mathscr{L} d^4x$, where the Lagrangian density $\mathscr{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and the (Faraday) field strength tensor $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, in which A_μ is the electromagnetic 4-potential (which is not to be confused with the rotational gravitational gauge field $A^{ab}{}_\mu$ appearing throughout the main text of the paper). As is well-known, the most general infinitesimal global coordinate transformations under which the EM action is invariant are the conformal transformations[17]; these have the form $x'^\mu=x^\mu+\xi^\mu(x)$, where

$$\xi^\mu(x)=a^\mu+\omega^\mu{}_\nu x^\nu+\rho x^\mu+c^\mu x^2-2c\cdot x x^\mu, \tag{A1}$$

in which the 15 infinitesimal parameters a^μ, $\omega^{\mu\nu}=-\omega^{\nu\mu}$, ρ and c^μ are constants, and we use the shorthand notation $x^2\equiv\eta_{\mu\nu}x^\mu x^\nu$ and $c\cdot x\equiv\eta_{\mu\nu}c^\mu x^\nu$. If the four parameters c^μ defining the so-called special conformal transformation (SCT) vanish, then (A1) reduces to an infinitesimal global Weyl transformation. Moreover, if the parameter ρ defining the dilation (or scale transformation) also vanishes, then (A1) further reduces to an infinitesimal global Poincaré transformation, consisting of a restricted Lorentz rotation defined by the six parameters $\omega^{\mu\nu}$ and a spacetime translation defined by the four parameters a^μ.

Under the action of any infinitesimal coordinate transformation $x'^\mu=x^\mu+\xi^\mu(x)$, the 4-potential has the form variation

$$\delta_0^{(\xi)}A_\mu=\delta^{(\xi)}A_\mu-\xi^\nu\partial_\nu A_\mu=-A_\nu\partial_\mu\xi^\nu-\xi^\nu\partial_\nu A_\mu, \tag{A2}$$

where we have explicitly denoted the form and total variations as being induced by the infinitesimal coordinate transformation. Thus, the corresponding Noether current (5b) has the form

$$\begin{aligned} J^\mu &= \frac{\partial\mathscr{L}}{\partial(\partial_\mu A_\sigma)}\delta_0^{(\xi)}A_\sigma+\xi^\mu\mathscr{L} \\ &= F^{\mu\sigma}(A_\nu\partial_\sigma\xi^\nu+\xi^\nu\partial_\nu A_\sigma)-\frac{1}{4}\xi^\mu F^{\rho\sigma}F_{\rho\sigma}. \end{aligned} \tag{A3}$$

Using the expression (A1) for an infinitesimal global conformal coordinate transformation, one finds that (A3) may be written as

$$J^\mu=-a^\alpha t^\mu{}_\alpha+\frac{1}{2}\omega^{\alpha\beta}M^\mu{}_{\alpha\beta}+\rho D^\mu+c^\alpha K^\mu{}_\alpha, \tag{A4}$$

where the coefficients of the parameters of the conformal transformation are defined by

$$t^\mu{}_\alpha\equiv\frac{\partial\mathscr{L}}{\partial(\partial_\mu A_\sigma)}\partial_\alpha A_\sigma-\delta^\mu_\alpha\mathscr{L}=-F^{\mu\sigma}\partial_\alpha A_\sigma+\frac{1}{4}\delta^\mu_\alpha F^{\rho\sigma}F_{\rho\sigma}, \tag{A5a}$$

$$M^\mu{}_{\alpha\beta}\equiv x_\alpha t^\mu{}_\beta-x_\beta t^\mu{}_\alpha+s^\mu{}_{\alpha\beta}, \tag{A5b}$$

$$D^\mu\equiv-x^\alpha t^\mu{}_\alpha+j^\mu, \tag{A5c}$$

$$K^\mu{}_\alpha\equiv(2x_\alpha x^\beta-\delta^\beta_\alpha x^2)t^\mu{}_\beta+2x^\beta(s^\mu{}_{\alpha\beta}-\eta_{\alpha\beta}j^\mu), \tag{A5d}$$

which are the canonical energy momentum, angular momentum, dilation current and special conformal current, respectively, of the 4-potential A_μ. We have also defined the quantities

$$s^\mu{}_{\alpha\beta}\equiv\frac{\partial\mathscr{L}}{\partial(\partial_\mu A_\sigma)}(\Sigma_{\alpha\beta})_\sigma{}^\rho A_\rho=-2F^\mu{}_{[\alpha}A_{\beta]}, \tag{A6a}$$

$$j^\mu\equiv\frac{\partial\mathscr{L}}{\partial(\partial_\mu A_\sigma)}wA_\sigma=F^{\mu\sigma}A_\sigma, \tag{A6b}$$

which are the canonical spin angular momentum and intrinsic dilation current of the 4-potential; here $(\Sigma_{\alpha\beta})_\sigma{}^\rho=2\eta_{\sigma[\alpha}\delta^\rho_{\beta]}$ are the generators of the vector representation

[16]This is due to cumulative derivatives arising in the course of the Dirac algorithm.

[17]The action is also invariant under finite global conformal coordinate transformations [51–53]; these include conformal inversions $x'^\mu=x^\mu/x^2$ for $x^2\neq 0$, which are not connected to the identity and so are not considered here.

of the Lorentz group and $w = -1$ is the Weyl weight of A_μ.

If the field equations $\delta\mathscr{L}/\delta A_\nu = \partial_\mu F^{\mu\nu} = 0$ are satisfied, then invariance of the action implies the conservation law $\partial_\mu J^\mu \simeq 0$. Since the parameters of the global conformal coordinate transformation in (A4) are constants, one thus obtains separate conservation laws given by

$$\partial_\mu t^\mu{}_\alpha \simeq 0, \tag{A7a}$$

$$\partial_\mu s^\mu{}_{\alpha\beta} + 2t_{[\alpha\beta]} \simeq 0, \tag{A7b}$$

$$\partial_\mu j^\mu - t^\mu{}_\mu \simeq 0, \tag{A7c}$$

$$s^\mu{}_{\alpha\mu} - j_\alpha \simeq 0, \tag{A7d}$$

which hold up to a total divergence of any quantity that vanishes on the boundary of the integration region of the action. It is worth noting that the first condition has been used to derive the second and third conditions, and the first three conditions have all been used to derive the final condition. The conservation laws (A7) may be easily verified directly using the expressions (A5a) and (A6) for $t^\mu{}_\alpha$, $s^\mu{}_{\alpha\beta}$ and j^μ, respectively, and the EM equations of motion. It is worth noting that the conservation law (A6), which results from invariance of the action under special conformal transformations, requires the 'field virial' to vanish [37].

In addition to being invariant under infinitesimal global conformal coordinate transformations of the form (A1), however, the EM action is also well-known to be invariant under the gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu\alpha$, where $\alpha(x)$ is an arbitrary function of spacetime position. Since our considerations thus far have not taken this into account, it is perhaps unsurprising that the canonical quantities $t^\mu{}_\alpha$, $s^\mu{}_{\alpha\beta}$ and j^μ are not invariant under the gauge transformation, as is easily demonstrated. Moreover, it is immediately apparent that the overall Noether current J^μ in (A3) is also not gauge invariant. All these problems originate from the form variation $\delta_0^{(\xi)} A_\sigma$ in (A2) itself not being gauge invariant. The lack of gauge invariance of the canonical expressions is a severe shortcoming, which means that these quantities must be unphysical. The situation is usually remedied, at least for the energy-momentum tensor in electromagnetism, by using the Belinfante method [36] of adding *ad hoc* terms, which do not follow from Noether's theorem, to the canonical energy momentum in order to construct a 'modified' energy-momentum tensor, which is gauge invariant (and symmetric) and can be further 'improved' to be traceless also [54]. One should note, however, that these methods are not guaranteed to yield a gauge-invariant energy-momentum tensor for general gauge field theories when matter fields are coupled to a gauge field [55], although this deficiency is addressed in [56].

An alternative approach, which makes direct use of the gauge invariance of the EM action and Noether's theorem, was first proposed in 1921 by Bessel-Hagen (who acknowledges Noether for suggesting the idea) [33]. This work is not widely known, however, and similar approaches have since been proposed by other authors [57–60], although Bessel-Hagen's original method arguably remains the most straightforward and intuitive [38]. The key to the method is to recognize that the form variations $\delta_0\chi_A$ of the fields appearing in the general expression (5b) for the Noether current J^μ may correspond to *any* transformation that leaves the action invariant. Indeed, it is advantageous to consider the *most general* such transformation. Applying this notion to EM, one should thus replace the form variation (A2) induced solely by the infinitesimal global conformal coordinate transformation by the general form

$$\delta_0 A_\mu = \delta^{(\xi)} A_\mu + \partial_\mu\alpha - \xi^\nu\partial_\nu A_\mu = -A_\nu\partial_\mu\xi^\nu + \partial_\mu\alpha - \xi^\nu\partial_\nu A_\mu, \tag{A8}$$

which also includes the contribution induced by the EM gauge transformation. Since the form variation (A8) leaves the EM action invariant for $\xi^\mu(x)$ given by (A1) and for arbitrary $\alpha(x)$, one may choose the latter to be as convenient as possible. Given that our goal is to arrive at a gauge-invariant form for the Noether current J^μ, one should therefore choose $\alpha(x)$ such that the form variation (A8) is itself gauge-invariant; this is the central idea underlying the Bessel-Hagen method.

One may easily obtain a gauge-invariant form variation by setting $\alpha = A_\nu\xi^\nu$, which immediately yields $\delta_0 A_\mu = \xi^\nu F_{\mu\nu}$. Consequently, the Noether current (A3) is replaced by the new form

$$\begin{aligned} J^\mu &= \frac{\partial\mathscr{L}}{\partial(\partial_\mu A_\sigma)}\delta_0 A_\sigma + \xi^\mu\mathscr{L} = \xi^\nu\left(F^{\mu\sigma}F_{\nu\sigma} - \frac{1}{4}\delta^\mu_\nu F^{\rho\sigma}F_{\rho\sigma}\right) \\ &= -\xi^\nu\tau^\mu{}_\nu, \end{aligned} \tag{A9}$$

where in the final equality we have identified the standard physical energy-momentum tensor $\tau^\mu{}_\nu = -(F^{\mu\sigma}F_{\nu\sigma} - \frac{1}{4}\delta^\mu_\nu F^{\rho\sigma}F_{\rho\sigma})$ of the EM field, which is immediately seen to be gauge invariant, symmetric and traceless. Substituting the form (A1) for ξ^μ into (A9), one finds that the expression (A4) for the Noether current is replaced by the much simpler form

$$\begin{aligned} J^\mu &= -a^\alpha\tau^\mu{}_\alpha + \frac{1}{2}\omega^{\alpha\beta}(x_\alpha\tau^\mu{}_\beta - x_\beta\tau^\mu{}_\alpha) - \rho x^\alpha\tau^\mu{}_\alpha \\ &\quad + c^\alpha(2x_\alpha x^\beta - \delta^\beta_\alpha x^2)\tau^\mu{}_\beta, \end{aligned} \tag{A10}$$

from which one can further identify new forms for the angular momentum, dilation current and special conformal current of the EM field, all of which are gauge invariant. If one again assumes the EM field equations to hold and uses the fact that the parameters of the global conformal coordinate transformation are constants, one obtains separate conservation laws that replace those in (A7) and are given by the succinct forms

$$\partial_\mu \tau^\mu{}_\alpha \simeq 0, \tag{A11a}$$

$$\tau_{[\alpha\beta]} \simeq 0, \tag{A11b}$$

$$\tau^\mu{}_\mu \simeq 0, \tag{A11c}$$

where, in this case, the conservation law derived from the coefficient of the SCT parameters c^μ is satisfied automatically given the other three conservation laws above, all of which may be easily verified directly.

Finally, one should also determine the further conservation law that results solely from invariance of the action under EM gauge transformations. This is easily achieved by setting $\xi^\mu = 0$, which is equivalent to all of the constant parameters in (A1) vanishing. In this case, (A8) becomes simply $\delta_0 A_\mu = \partial_\mu \alpha$ and the Noether current is immediately given by

$$J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\sigma)} \delta_0 A_\sigma = -F^{\mu\sigma} \partial_\sigma \alpha. \tag{A12}$$

Assuming the EM field equations to hold, the resulting conservation law $\partial_\mu J^\mu \simeq 0$ may be written as $F^{\mu\sigma}\partial_\mu\partial_\sigma\alpha \simeq 0$, which is satisfied identically because of the antisymmetry of $F^{\mu\sigma}$.

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