

Polarization modes of gravitational waves in generalized Proca theory

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In this paper, we study polarization modes of gravitational waves in generalized Proca theory in the homogeneous and isotropic Minkowski background. The results show that the polarizations of gravitational waves depend on the parameter space of this gravity theory and can be divided into quite rich cases by parameters. In some parameter space, it only allows two tensor modes, i.e., the $+$ and \times modes. In some parameter space, besides tensor modes, it also allows one scalar mode, or two vector (vector- x and vector- y) modes, or both one scalar mode and two vector modes. The scalar mode is a mixture mode of a breathing mode and a longitudinal mode, or just a pure breathing mode. Interestingly, it is found that the amplitude of the vector modes is related to the speed of the tensor modes. This allows us to give the upper bound of the amplitude of the vector modes by detecting the speed of the tensor modes. Specifically, if the speed of tensor modes is strictly equal to the speed of light, then the amplitude of vector modes is zero.

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I. INTRODUCTION

The successful detection of gravitational waves [1–5] implies the arrival of the gravitational wave astronomy era. In addition to using electromagnetic waves, we can also develop astronomy and cosmology by detecting gravitational waves [6–10]. Not only that, gravitational waves can also help us test various modified gravities and deepen our understanding of gravity [11,12].

Polarization is an important property of gravitational waves. In 1973, Eardley *et al.* pointed out that there are up to six independent polarization modes of gravitational waves in the general four-dimensional metric theory [13]. For a specific modified gravity, due to the field equations constraining the possible value of the Riemann tensor, it generally does not allow all six polarization modes, but rather allows a subset of them. For example, for general relativity, the vacuum Einstein field equation requires the Ricci tensor is zero, which results in the theory having only two polarization modes of gravitational waves, namely the $+$ mode and the \times mode. Different theories predict different classes of polarization modes, so it can be expected that some of them will be excluded from

the detection of polarization modes of gravitational waves in the future.

In many modified gravities, the polarization modes of gravitational waves have been studied, like $f(R)$, Horndeski, Palatini generalized Brans-Dicke, Palatini-Horndeski, teleparallel Horndeski, Einstein-aether, tensor-vector-scalar, Horava, scalar-tensor-vector, $f(T)$, dCS, EdGB, and Bumblebee theories [14–24]. Additionally, there are also some studies on the generation of gravitational waves in modified gravities, for example, Refs. [25–27]. The ground-based gravitational wave detectors of LIGO, VIRGO, and KAGRA are located at different locations on the Earth. They can detect gravitational waves in different two-dimensional planes. Therefore, different detectors can cooperate to detect the polarization modes of gravitational waves. In this regard, some related work has already begun [4,5,28–40]. Polarization information of gravitational waves can also be detected using Pulsar Timing Arrays (PTAs) [41]. For example, in 2021, Huang *et al.* discovered a tentative indication for scalar-transverse gravitational waves in NANOGrav 12.5 year data set [42]. Furthermore, one can also expect to use space gravitational wave detectors such as Lisa, Taiji, and TianQin [43–46] to detect polarization modes of gravitational waves in the future.

Adding additional fields is one way to modify gravity. The additional degrees of freedom may explain inflation and the accelerated expansion of the Universe [47–50]. If the additional field is a scalar field, then such a theory is

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called scalar-tensor theory. In the metric formalism, the most general scalar-tensor theory that can derive second-order field equations is Horndeski theory [51]. This theory was first discovered by Horndeski in 1974. Later, it was realized that extending the scalar Galileon theory [52] to curved spacetime would lead to the rediscovery of Horndeski theory [53,54].

The observation of the anisotropy of the cosmic microwave background (CMB) may indicate the existence of a preferred direction in the Universe [55]. This may be naturally generated by an additional vector field [50]. Therefore, in addition to adding a scalar field, we can also add an additional vector field to modify gravity. Such a theory is called vector-tensor theory. Similar to the scalar Galileon theory, in 2014, Tasinato constructed a vector Galileon theory [56]. Heisenberg further gave generalization of the Proca action in curved spacetime [57]. This covariant vector-tensor theory constitutes the Galileon-type generalization of the Proca action, and it is called generalized Proca theory. When one considers the Lagrangian of a vector field with derivative self-interactions with *a priori* arbitrary coefficients, which does not have ghostlike pathologies, and extends it to the case of curved spacetime, generalized Proca theory will be obtained [57]. The field equations of generalized Proca theory are second order, so there is no the Ostrogradsky instability in this theory [58]. There has been a lot of research on this theory, such as cosmology [59–63] and black holes [64–68].

In this paper, we will study polarization modes of gravitational waves in generalized Proca theory in the range of a linear analysis. We consider a homogeneous and isotropic Minkowski background and our goal is to obtain the linearized field equations and then analyze the polarization modes of gravitational waves allowed by these

equations. Therefore, in Sec. II, we review the action of generalized Proca theory. In Sec. III, we obtain the background equations and linear perturbation equations. In Sec. IV, we combine the perturbations into some gauge invariants. These gauge invariants can help us analyze the linear perturbation equations. In Sec. V, we analyze the polarization modes of gravitational waves in generalized Proca theory. In fact, in Refs. [59,60], the authors analyzed all the permutations and propagating degree of freedom of generalized Proca theory on top of a Friedmann-Lemaître-Robertson-Walker background, and the perturbation equations and wave speed expressions we need can be directly obtained from these results by taking the Minkowski background limit.

We will use the natural system of units $c = G = 1$ and consider four-dimensional spacetime in this paper. We set the metric signature as $(-, +, +, +)$. The Latin alphabet indices (a, b, c, d, e) range over spacetime indices $(0, 1, 2, 3)$, and the Latin alphabet indices (i, j, k) range over space indices $(1, 2, 3)$ which point to $(+x, +y, +z)$ directions, respectively.

II. GENERALIZED PROCA THEORY

Generalized Proca theory is a vector-tensor theory. In addition to being related to the metric g_{ab} , the action also depends on a vector field A^a .

The action of generalized Proca theory is [57,59,69]

$$S(g_{ab}, A^a) = \int d^4x \sqrt{-g} (\mathcal{L}_F + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5), \quad (1)$$

where

$$\mathcal{L}_F = -\frac{1}{4} F_{ab} F^{ab}, \quad (2)$$

$$\mathcal{L}_2 = G_2(X), \quad (3)$$

$$\mathcal{L}_3 = G_3(X) \nabla_a A^a, \quad (4)$$

$$\mathcal{L}_4 = G_4(X) R + G_{4,X}(X) [(\nabla_a A^a)^2 + c_2 \nabla_a A_b \nabla^a A^b - (1 + c_2) \nabla_a A_b \nabla^b A^a], \quad (5)$$

$$\begin{aligned} \mathcal{L}_5 = & G_5(X) \left(R_{ab} - \frac{1}{2} g_{ab} R \right) \nabla^a A^b \\ & - \frac{1}{6} G_{5,X}(X) [(\nabla_a A^a)^3 - 3d_2 \nabla_a A^a \nabla_b A_c \nabla^b A^c - 3(1 - d_2) \nabla_a A^a \nabla_b A_c \nabla^c A^b \\ & + (2 - 3d_2) \nabla_a A_b \nabla^c A^a \nabla^b A_c + 3d_2 \nabla_a A_b \nabla^c A^a \nabla_c A^b]. \end{aligned} \quad (6)$$

Here, $F_{ab} = \nabla_a A_b - \nabla_b A_a$, R_{ab} is the Ricci tensor and R is the Ricci scalar, c_2, d_2 are constants, G_2, G_3, G_4 , and G_5 are arbitrary functions of the variable $X = -\frac{1}{2}g_{ab}A^aA^b$. In addition, $G_{n,X} = dG_n/dX$ ($n = 2, 3, 4, 5$). The remaining derivatives are still represented by this notation. For example, $G_{2,XX}$ represents the second-order derivative of G_2 with respect to the variable X . In the action of generalized Proca theory, L_3, L_4 , and L_5 correspond to the first, second, and third power terms of the derivative interaction of the vector field, respectively. And $G_4(X)R$ and $G_5(X)(R_{ab} - \frac{1}{2}g_{ab}R)\nabla^a A^b$ are the nonminimal couplings between the vector field and curvature that must be introduced in order for the field equation to be second order. It can be seen that unlike Einstein-aether theory [70], the action of generalized Proca theory does not have a Lagrangian multiplier for an *a priori* fixed norm of the vector field.

The first term of \mathcal{L}_4 can be equivalently written as $(16\pi)^{-1}R + [G_4(X) - (16\pi)^{-1}]R$. Therefore, it is easy to see that this theory includes the Einstein-Hilbert action. We hope that generalized Proca theory can retain the contribution of the Einstein-Hilbert term, so it requires

$$G_4(X) \neq 0. \quad (7)$$

III. BACKGROUND EQUATIONS AND LINEAR PERTURBATION EQUATIONS

Now consider a homogeneous and isotropic Minkowski background

$$g_{ab} = \eta_{ab}, \quad A^a = \dot{A}^a = (A, 0, 0, 0), \quad (8)$$

where \dot{A}^a is a constant vector and η_{ab} is the Minkowski metric. Due to spatial isotropy, the vector \dot{A}^a only has one nonvanishing temporal component A , and its spatial components are all zero.

By substituting the assumption (8) into the field equations derived from the variation of the action (1) with respect to the metric tensor g^{ab} and the vector field A^a , respectively, we can determine that \dot{A}^a should satisfy the following background equations:

$$-\frac{1}{2}\dot{G}_2\eta_{ab} + \frac{1}{2}\dot{G}_{2,X}\dot{A}_a\dot{A}_b = 0, \quad (9)$$

$$-\dot{G}_{2,X}\dot{A}_a = 0. \quad (10)$$

Here and below, the notation “ \circ ” above the letter means that the corresponding function takes the value of $X = -\frac{1}{2}\eta_{ab}\dot{A}^a\dot{A}^b$. Using the background equations (9) and (10), we can divide the background solution (8) into two cases:

$$\text{Case A: } \dot{G}_2 = 0, \quad \dot{G}_{2,X} \neq 0, \quad A = 0;$$

$$\text{Case B: } \dot{G}_2 = 0, \quad \dot{G}_{2,X} = 0, \quad A = \text{constant}.$$

To describe gravitational waves, we investigate perturbations of the background

$$g_{ab} = \eta_{ab} + h_{ab}, \quad A^a = \dot{A}^a + B^a; \\ |h_{ab}| \ll 1, \quad |B^a| \ll |\dot{A}^a|. \quad (11)$$

In the following, we use η_{ab} and η^{ab} to lower and raise the spacetime index and define $h = \eta^{ab}h_{ab}$.

By substituting the perturbations (11) into the field equations derived from the variation of the action (1), and writing the first-order terms of the perturbations, we can obtain two linear-perturbation equations that describe gravitational waves.

Substituting the perturbations into the field equation derived from the variation of vector A^a , we obtain the first linear-perturbation equation

$$\mathcal{V}_a = 0. \quad (12)$$

Here, the specific expression of \mathcal{V}_a is very lengthy, and we place it in the Appendix.

Substituting the perturbations into the field equation derived from the variation of tensor g^{ab} , we have the second linear-perturbation equation

$$\mathcal{T}_{ab} = 0. \quad (13)$$

Here, we also list the expression of \mathcal{T}_{ab} in the Appendix.

It should be noted that for the case of $A = 0$, i.e., $\dot{A}^a = 0$, it is easy to analyze polarization modes of gravitational waves. In this case, Eq. (13) becomes

$$-2\dot{G}_2h_{ab} - 2\dot{G}_4(\partial_a\partial_b h - \partial_a\partial_c h^c_b - \partial_b\partial_c h^c_a \\ + \square h_{ab} + \eta_{ab}\partial_c\partial_d h^{cd} - \eta_{ab}\square h) = 0. \quad (14)$$

On the other hand, we always have $\dot{G}_2 = 0$ for the background solution (8). In addition, since $G_4(X) \neq 0$, Eq. (14) can be further written as

$$\overset{(1)}{G}_{ab} = -\frac{1}{2}(\partial_a\partial_b h - \partial_a\partial_c h^c_b - \partial_b\partial_c h^c_a + \square h_{ab} \\ + \eta_{ab}\partial_c\partial_d h^{cd} - \eta_{ab}\square h) = 0. \quad (15)$$

Here, G_{ab} is the Einstein tensor, and the notation “(1)” above the letter G means that only the first-order term of the perturbations is taken here. It can be seen that under linear approximation, Eq. (15) is no different from Einstein’s field equation. Therefore, just like general relativity, there are only two tensor modes (+ and \times) propagating at the speed

of light for the case of $A = 0$. It should be pointed out that this does not mean that when $A = 0$ general Proca theory will return to general relativity. When we consider not only linear-order but also the full nonlinear field equation, we will find that the field equations of generalized Proca theory contain higher-order terms of vector-field perturbations.

One may ask whether another linear-perturbation equation (12) further constrains the tensor-mode gravitational waves. The answer is no. Note that only the transverse-traceless part h_{ij}^{TT} of the metric perturbation h_{ab} contributes to the tensor-mode gravitational waves, and Eq. (12) is a vector type equation with a single index. We can know that this equation does not constrain the value of h_{ij}^{TT} , and therefore does not constrain the tensor mode gravitational waves.

Through the above analysis, we find that when the background solution (8) belongs to Case A, it only allows two tensor (+ and \times) modes propagating at the speed of light. The result is the same for Case B with $A = 0$. Therefore, in the following sections, we only need to analyze the case where the background solution (8) meets the following conditions:

$$\dot{G}_2 = 0, \quad \dot{G}_{2,x} = 0, \quad A \neq 0. \quad (16)$$

IV. GAUGE INVARIANTS

In this section, we will introduce a method for simplifying linear perturbation equations. After using this method, the linear perturbation equations can be decomposed into independent tensor, vector, and scalar equations. These equations can be represented by some gauge invariants combined with perturbations.

Under spatial rotation transformation, h_{ij} is transformed like a second-order tensor, h_{0i} and B^i are transformed like vectors, and h_{00} and B^0 are transformed like scalars. We can uniquely decompose the perturbations as follows [24,71,72]:

$$\begin{aligned} B^0 &= B^0, \\ B^i &= \partial^i \omega + \mu^i, \\ h_{00} &= h_{00}, \\ h_{0i} &= \partial_i \gamma + \beta_i, \\ h_{ij} &= h_{ij}^{TT} + \partial_i \epsilon_j + \partial_j \epsilon_i + \frac{1}{3} \delta_{ij} H + \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \Delta \right) \zeta. \end{aligned} \quad (17)$$

Here,

$$\partial_i \mu^i = \partial_i \beta^i = \partial_i \epsilon^i = 0, \quad (18)$$

$$\delta^{ij} h_{ij}^{TT} = \delta^i h_{ij}^{TT} = 0. \quad (19)$$

Here and below, we use δ_{ij} and δ^{ij} to lower and raise the space index. As can be seen, Eq. (17) uniquely decomposes a spatial vector into a scalar part and a transverse vector part, and uniquely decomposes a spatial tensor into two scalar parts, a transverse vector part, and a transverse-traceless tensor part.

Considering that the left sides of Eqs. (12) and (13) are actually a vector and a tensor, respectively, we can also apply this decomposition method to the linear-perturbation equations. Due to the uniqueness of the decomposition, requiring the equations to be zero is equivalent to requiring that each decomposed part of the linear-perturbation equations be zero, so we can obtain a series of independent tensor, vector, and scalar equations. Due to the spatial homogeneous and isotropic of the background solution (8), by substituting (17) into Eqs. (12) and (13), we can see that these decomposed scalar, vector, and tensor equations only rely on tensor, vector, and scalar perturbations, respectively [73]. Therefore, the linear perturbation equations are decoupled.

We can also continue to combine the perturbations in these equations into gauge invariants. Under linear approximation, the gauge transformations of the metric and the vector field are [74]

$$h_{ab} \rightarrow h_{ab} - \partial_a \xi_b - \partial_b \xi_a, \quad (20)$$

$$B^a \rightarrow B^a + \dot{A}^b \partial_b \xi^a. \quad (21)$$

Here, ξ^a is an arbitrary function of spacetime coordinates.

Under the above gauge transformations, the spatial tensor

$$h_{ij}^{TT}, \quad (22)$$

the two spatial vectors

$$\begin{aligned} \Xi_i &= \beta_i - \partial_0 \epsilon_i, \\ \Sigma_i &= \mu_i + A \partial_0 \epsilon_i, \end{aligned} \quad (23)$$

and the four spatial scalars

$$\begin{aligned} \phi &= -\frac{1}{2} h_{00} + \partial_0 \gamma - \frac{1}{2} \partial_0 \partial_0 \zeta, \\ \Theta &= \frac{1}{3} (H - \Delta \zeta), \\ \Omega &= B^0 - A \partial_0 \gamma + \frac{1}{2} A \partial_0 \partial_0 \zeta, \\ \Psi &= \omega + \frac{1}{2} A \partial_0 \zeta, \end{aligned} \quad (24)$$

are gauge invariant [24,72]. In the next section, we will check that the scalar, vector, and tensor equations decomposed from

Eqs. (12) and (13) can be represented by the gauge invariants (22)–(24) combined with perturbations.

V. POLARIZATION MODES OF GRAVITATIONAL WAVES

In this section, we analyze the polarization modes of gravitational waves in generalized Proca theory in Case B.

We detect gravitational waves by detecting the relative displacement of free particles [75]. In this paper, similar to Ref. [13], we assume that the matter field only has minimal coupling with the metric g_{ab} , while the vector field A^a is not directly coupled with the matter field. So, the action describing the motion of a free particle is

$$S = \int ds = \int \sqrt{|g_{ab} dx^a dx^b|}, \quad (25)$$

the relative motion between two adjacent free-test particles used to detect the polarization modes of gravitational waves satisfies the equation of geodesic deviation [75],

$$\frac{d^2 \eta_i}{dt^2} = -{}^{(1)}R_{i0j0} \eta^j. \quad (26)$$

Here, η_i is the relative displacement of the two test particles and R_{i0j0} is the $(i0j0)$ component of the Riemannian tensor. The notation “(1)” above the letter R means that only the first-order term of the perturbations is taken here. It can be seen from Eq. (26) that R_{i0j0} will completely determine the motion behavior of the free test particles under the gravitational waves, and the polarization modes of gravitational waves are defined by different values of R_{i0j0} [75].

Using Eqs. (17), and (22)–(24), we can write R_{i0j0} as

$$\begin{aligned} {}^{(1)}R_{i0j0} = & -\frac{1}{2} \partial_0 \partial_0 h_{ij}^{TT} + \frac{1}{2} \partial_0 \partial_i \Xi_j + \frac{1}{2} \partial_0 \partial_j \Xi_i \\ & + \partial_i \partial_j \phi - \frac{1}{2} \delta_{ij} \partial_0 \partial_0 \Theta. \end{aligned} \quad (27)$$

It can be seen that among several gauge invariants, only $h_{ij}^{TT}, \Xi_i, \phi, \Theta$ contribute to gravitational waves, while Σ_i, Ω, Ψ do not cause the relative displacement of the test particles.

Now, taking the propagation direction of gravitational waves as $+z$ direction we mark the components of R_{i0j0} as [13,21]

$$R_{i0j0} = \begin{pmatrix} P_4 + P_6 & P_5 & P_2 \\ P_5 & -P_4 + P_6 & P_3 \\ P_2 & P_3 & P_1 \end{pmatrix}. \quad (28)$$

We can define six independent polarization modes of gravitational waves P_1, \dots, P_6 . Any plane gravitational wave can be written as a linear combination of these six modes. We show the relative motions of the test particles under these six polarization modes in Fig. 1.

Combining Eqs. (27) and (28), we can obtain the expressions of the six polarization modes

$$\begin{aligned} P_1 &= \partial_3 \partial_3 \phi - \frac{1}{2} \partial_0 \partial_0 \Theta, & P_2 &= \frac{1}{2} \partial_0 \partial_3 \Xi_1, \\ P_3 &= \frac{1}{2} \partial_0 \partial_3 \Xi_2, & P_4 &= -\frac{1}{2} \partial_0 \partial_0 h_{11}^{TT}, \\ P_5 &= -\frac{1}{2} \partial_0 \partial_0 h_{12}^{TT}, & P_6 &= -\frac{1}{2} \partial_0 \partial_0 \Theta. \end{aligned} \quad (29)$$

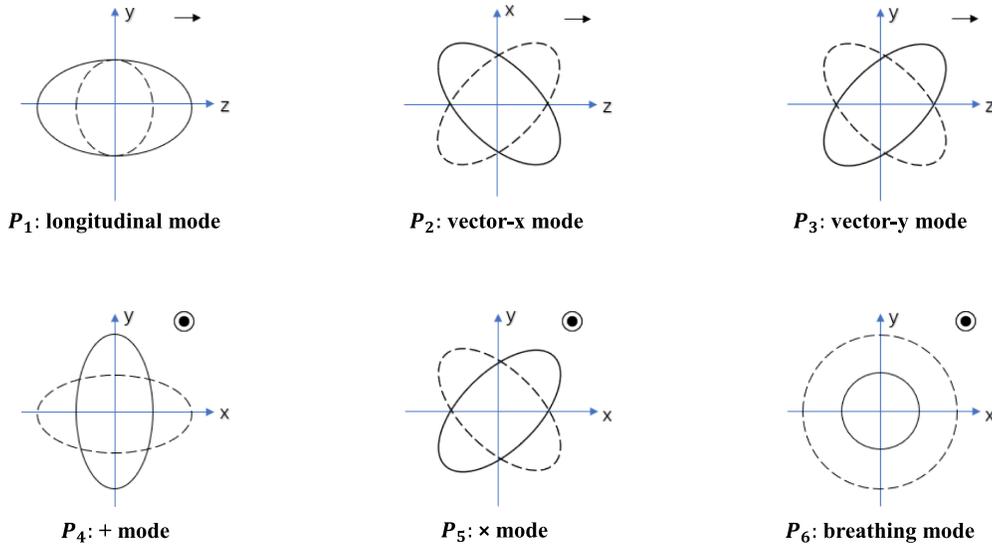


FIG. 1. Six polarization modes of gravitational waves [13]. We have assumed that the gravitational waves propagate in the $+z$ direction. The solid and dotted lines are the cases of a circle of test particles when the phases of waves are 0 and π , respectively. There is no relative motion between test particles in the direction of the third axis that is not drawn.

It should be pointed out that in this calculation, we assume that the propagation direction of gravitational waves is in the $+z$ direction. Therefore, when we consider wave solutions, $h_{ij}^{TT}, \Xi_i, \phi, \Theta$ are functions of only t and z . This will result in, for example, $h_{22}^{TT} = -h_{11}^{TT}$ and $\partial_1 \partial_1 \phi = 0$. It can be seen that h_{ij}^{TT} only contributes to tensor (+ and \times) modes, Ξ_i only contributes to vector (vector-x and vector-y) modes, ϕ only contributes to longitudinal mode, and Θ contributes to both breathing mode and longitudinal mode.

Now, we will use the method introduced in Sec. IV to simplify the linear perturbation equations (12) and (13) and analyze the polarization modes of gravitational waves in generalized Proca theory in case (16).

A. Tensor mode

The equation describing the tensor mode gravitational waves can be derived from the transverse traceless tensor part of the ij component in Eq. (13), which is

$$(\dot{G}_4 - \dot{G}_{4,X} A^2) \partial_0 \partial_0 h_{ij}^{TT} - \dot{G}_4 \Delta h_{ij}^{TT} = 0. \quad (30)$$

This equation is consistent with the result of taking Minkowski limit in Refs. [59,60]. In order for generalized Proca theory to have tensor mode (+ and \times) gravitational waves, combining Eqs. (7) and (30), we should have the condition

$$\dot{G}_4 - \dot{G}_{4,X} A^2 \neq 0. \quad (31)$$

If the condition (31) is not satisfied, then Eq. (30) will become $\Delta h_{ij}^{TT} = 0$. It indicates that h_{ij}^{TT} has no wave solution, so this is not what we expected.

With the condition (31), we can obtain from Eq. (30) that the speed of the tensor mode gravitational waves c_T is

$$c_T^2 = \frac{\dot{G}_4}{\dot{G}_4 - \dot{G}_{4,X} A^2}. \quad (32)$$

If $c_T^2 < 0$, the solution will exponentially diverge, and then it is linearly unstable. To ensure the linear stability of the solution, we require $c_T^2 > 0$.

GW170817 and GRB170817A require the speed of tensor mode gravitational waves c_T to meet [76,77]

$$-3 \times 10^{-15} \leq c_T - 1 \leq 7 \times 10^{-16}. \quad (33)$$

This result will constrain the values of the theoretical parameters. Considering that Eq. (33) requires c_T not to deviate significantly from the speed of light. Using Eq. (32), it shows that

$$|\dot{G}_{4,X} A^2| \ll |\dot{G}_4|. \quad (34)$$

Therefore, we can expand c_T^2 as

$$c_T^2 \approx 1 + \frac{\dot{G}_{4,X} A^2}{\dot{G}_4}. \quad (35)$$

Accordingly, c_T has an approximation of

$$c_T \approx 1 + \frac{\dot{G}_{4,X} A^2}{2\dot{G}_4}. \quad (36)$$

Therefore, it can be seen that the constrain (33) requires

$$\left| \frac{\dot{G}_{4,X} A^2}{\dot{G}_4} \right| \lesssim 10^{-15}. \quad (37)$$

In particular, when the speed of tensor mode gravitational waves is strictly equal to the speed of light, we have $\dot{G}_{4,X} = 0$.

B. Vector mode

We can derive three equations describing vector mode gravitational waves from the linear perturbation equations (12) and (13).

The first equation can be derived from the transverse vector part of the i component of Eq. (12), which is

$$(1 - 2c_2 \dot{G}_{4,X}) \square (\Sigma_i + A \Xi_i) - \dot{G}_{4,X} A \Delta \Xi_i = 0. \quad (38)$$

The second and third equations can be derived respectively from the transverse vector parts of the ij and $0i$ components of Eq. (13),

$$\dot{G}_4 \Xi_i + \dot{G}_{4,X} A \Sigma_i = 0, \quad (39)$$

$$(1 - 2c_2 \dot{G}_{4,X}) A \square (\Sigma_i + A \Xi_i) - \dot{G}_4 \Delta \Xi_i - \dot{G}_{4,X} A \Delta (\Sigma_i + A \Xi_i) = 0. \quad (40)$$

These equations are consistent with the results of taking Minkowski limit in Refs. [59,60]. Perhaps one would ask why there are only two variables Ξ_i, Σ_i but three equations here. In fact, these three equations are not independent of each other. We can easily find that Eq. (40) can be obtained using Eqs. (38) and (39). Therefore, we only need to consider the first two.

Using Eq. (39), we have

$$\Xi_i = -\frac{\dot{G}_{4,X} A}{\dot{G}_4} \Sigma_i. \quad (41)$$

It can be seen that when $\dot{G}_{4,X} = 0$, we have $\Xi_i = 0$. However, from Eq. (27), it can be seen that the gauge invariant vector that contributes to gravitational waves is

only Ξ_i . Therefore, in this case, generalized Proca theory does not allow vector mode gravitational waves.

When $\dot{G}_{4,X} \neq 0$, we can substitute Eq. (41) into Eq. (38), and obtain

$$\mathcal{M}\partial_0\partial_0\Sigma_i - \mathcal{N}\Delta\Sigma_i = 0, \quad (42)$$

where

$$\mathcal{M} = (1 - 2c_2\dot{G}_{4,X})(\dot{G}_4 - \dot{G}_{4,X}A^2), \quad (43)$$

$$\mathcal{N} = \mathcal{M} + \dot{G}_{4,X}^2A^2. \quad (44)$$

This is an equation for the variable Σ_i . When we solve the wave solution of Σ_i from Eq. (42), we can obtain Ξ_i with Eq. (41), and then obtain the wave solution of vector mode gravitational waves. We already have the condition (31), so when considering $\dot{G}_{4,X} \neq 0$, we can study this equation in the following three cases:

Case 1: When $1 - 2c_2\dot{G}_{4,X} = 0$, Eq. (42) becomes $\Delta\Sigma_i = 0$. It indicates that Σ_i has no wave solution, so we have $\Xi_i = \Sigma_i = 0$. This case does not allow vector mode gravitational waves.

Case 2: When $(1 - 2c_2\dot{G}_{4,X})(\dot{G}_4 - \dot{G}_{4,X}A^2) + \dot{G}_{4,X}^2A^2 = 0$, Eq. (42) becomes $\partial_0\partial_0\Sigma_i = 0$. It also indicates that Σ_i has no wave solution, so we have $\Xi_i = \Sigma_i = 0$. This case also does not allow vector mode gravitational waves.

Case 3: When $1 - 2c_2\dot{G}_{4,X} \neq 0$ and $(1 - 2c_2\dot{G}_{4,X}) \times (\dot{G}_4 - \dot{G}_{4,X}A^2) + \dot{G}_{4,X}^2A^2 \neq 0$, Eq. (42) will have wave solutions of Σ_i . In this case, generalized Proca theory allows two vector (vector-x and vector-y) modes. Using Eq. (42), we obtain the speed of vector mode gravitational waves c_V :

$$c_V^2 = 1 + \frac{\dot{G}_{4,X}^2A^2}{(1 - 2c_2\dot{G}_{4,X})(\dot{G}_4 - \dot{G}_{4,X}A^2)}. \quad (45)$$

Similar to the previous analysis of tensor modes, in order for the solution to have linear stability, we require $c_V^2 > 0$. It can be seen that in Case 3, vector mode gravitational waves do not propagate at the speed of light, i.e., $c_V \neq 1$.

Based on the above research, we find that the existence of vector mode gravitational waves in generalized Proca theory depends on the parameter space. We summarize the results in Table I.

Finally, note that Ξ_i in Eq. (41) depends on $\dot{G}_{4,X}$, which allows us to use the speed of tensor mode gravitational waves to constrain the upper limit of the amplitude of vector-mode gravitational waves.

First, all the perturbations are weak relative to the background,

$$|h_{ab}| \ll 1, \quad |B^a| \ll |\dot{A}^a|. \quad (46)$$

Then with Eqs. (17) and (23), we have

$$|\Sigma_i| \ll |A|. \quad (47)$$

Therefore, using Eqs. (35) and (41), the condition (47) implies

$$|\Xi_i| = \left| \frac{\dot{G}_{4,X}A}{\dot{G}_4} \Sigma_i \right| \ll \left| \frac{\dot{G}_{4,X}A^2}{\dot{G}_4} \right| \approx |c_T^2 - 1|. \quad (48)$$

The above inequality gives the relationship between the vector mode amplitude and the tensor mode speed. Then, considering GW170817 and combining with the condition (37), we have from Eq. (48) that

$$|\Xi_i| \ll 10^{-15}. \quad (49)$$

Using Eqs. (17) and (23), the above constrain indicates that the metric perturbation (abbreviated as h) corresponding to the vector gravitational waves satisfy $|h| \ll 10^{-15}$. If we further require the speed of tensor modes to be the speed of light, i.e., $c_T = 1$, then according to Eq. (48), we have $\Xi_i = 0$. At this point, the amplitude of the vector modes is zero, and we cannot detect the vector gravitational waves.

C. Scalar mode

We can derive six equations describing scalar mode gravitational waves from the linear perturbation equations (12) and (13).

TABLE I. The number of vector mode gravitational waves under various cases. The numbers in the rightmost column of the table represent the degrees of freedom of the vector mode gravitational waves in the corresponding case.

Cases	Conditions	Vector degrees of freedom
Case 0	$\dot{G}_{4,X} = 0$.	0
Case 1	$\dot{G}_{4,X} \neq 0, 1 - 2c_2\dot{G}_{4,X} = 0$.	0
Case 2	$\dot{G}_{4,X} \neq 0, (1 - 2c_2\dot{G}_{4,X})(\dot{G}_4 - \dot{G}_{4,X}A^2) + \dot{G}_{4,X}^2A^2 = 0$.	0
Case 3	$\dot{G}_{4,X} \neq 0, 1 - 2c_2\dot{G}_{4,X} \neq 0, (1 - 2c_2\dot{G}_{4,X})(\dot{G}_4 - \dot{G}_{4,X}A^2) + \dot{G}_{4,X}^2A^2 \neq 0$.	2

The first and second equations can be derived from the 0 component and the scalar part of the i component of Eq. (12), respectively, which are given by

$$\begin{aligned} & -2(1-2c_2\dot{G}_{4,X})\Delta(\Omega+2A\phi+\partial_0\Psi) \\ & +\dot{G}_{3,X}A(2\Delta\Psi+3A\partial_0\Theta) \\ & -4\dot{G}_{4,X}A\Delta\Theta+2\dot{G}_{2,XX}A^2(\Omega+A\phi)=0, \end{aligned} \quad (50)$$

$$\begin{aligned} & -(1-2c_2\dot{G}_{4,X})\partial_0(\Omega+2A\phi+\partial_0\Psi) \\ & -\dot{G}_{3,X}A(\Omega+A\phi)-2\dot{G}_{4,X}A\partial_0\Theta=0. \end{aligned} \quad (51)$$

The third and fourth ones can be obtained respectively from the trace part and the $\partial_i\partial_j$ part of the ij components of Eq. (13),

$$\begin{aligned} & \dot{G}_4(-4\Delta\phi-2\Delta\Theta+6\partial_0\partial_0\Theta)-3\dot{G}_{3,X}A^2\partial_0(\Omega+A\phi) \\ & -4\dot{G}_{4,X}A\Delta(\Omega+A\phi+\partial_0\Psi)-6\dot{G}_{4,X}A^2\partial_0\partial_0\Theta=0, \end{aligned} \quad (52)$$

$$\dot{G}_4(2\phi+\Theta)+2\dot{G}_{4,X}A(\Omega+A\phi+\partial_0\Psi)=0. \quad (53)$$

Finally, the fifth and sixth equations can be derived respectively from the scalar part of the $0i$ component and the 00 component of Eq. (13):

$$(1-2c_2\dot{G}_{4,X})A(\Omega+2A\phi+\partial_0\Psi)+2\dot{G}_4\Theta=0, \quad (54)$$

$$\begin{aligned} & 4(1-2c_2\dot{G}_{4,X})A\Delta(\Omega+2A\phi+\partial_0\Psi) \\ & +4\dot{G}_4\Delta\Theta+4\dot{G}_{4,X}A^2\Delta\Theta-\dot{G}_{3,X}A^2 \\ & \times(2\Delta\Psi+3A\partial_0\Theta)-2\dot{G}_{2,XX}A^3(\Omega+A\phi)=0. \end{aligned} \quad (55)$$

These equations are consistent with the results of taking Minkowski limit in Refs. [59,60]. The above six equations are not independent of each other. Equation (55) can be obtained from Eqs. (50) and (54), and Eq. (51) from Eqs. (52)–(54). Therefore, only Eqs. (50) and (52)–(54) are independent. Now, we have four equations and four variables $\phi, \Theta, \Psi, \Omega$.

We can use Eqs. (52) and (53) to obtain the following equation:

$$2(\dot{G}_4-\dot{G}_{4,X}A^2)\partial_0\Theta-\dot{G}_{3,X}A^2(\Omega+A\phi)=0. \quad (56)$$

This equation can replace Eq. (52), and we can solve scalar-mode gravitational waves with Eqs. (50), (53), (54), and (56).

Now, we study scalar mode gravitational waves in the following four cases:

Case 1: $\dot{G}_{3,X}=1-2c_2\dot{G}_{4,X}=0$. In this case, using Eq. (31), we find that Eq. (56) requires $\Theta=0$. Thus, Eq. (54) becomes

$$(1-2c_2\dot{G}_{4,X})A(\Omega+2A\phi+\partial_0\Psi)=0. \quad (57)$$

It can be seen that the equation is an identity. We now have only two equations [(50) and (53)] left to constrain the three variables ϕ, Ψ, Ω , which indicates that these variables cannot be determined. So we do not consider this case.

Case 2: $\dot{G}_{3,X}=0, 1-2c_2\dot{G}_{4,X}\neq 0$. In this case, using Eqs. (31) and (56) also requires $\Theta=0$, while Eq. (54) requires

$$\Omega+2A\phi+\partial_0\Psi=0. \quad (58)$$

Then, from (53) we have

$$(\dot{G}_4-\dot{G}_{4,X}A^2)\phi=0. \quad (59)$$

Using Eq. (31), requires $\phi=0$. Since $\Theta=\phi=0$, Eq. (27) indicates that scalar mode gravitational waves are not allowed in this case.

Case 3: $\dot{G}_{3,X}\neq 0, 1-2c_2\dot{G}_{4,X}=0$. In this case, using Eq. (54), we have $\Theta=0$. Then, Eq. (56) requires $\Omega+A\phi=0$. Using the above conditions, Eq. (50) will require $\Psi=0$. Finally, from Eq. (53), it can be seen that $\phi=0$. Since $\Theta=\phi=0$, scalar mode gravitational waves are not allowed in this case.

Case 4: $\dot{G}_{3,X}\neq 0, 1-2c_2\dot{G}_{4,X}\neq 0$. In this case, from Eqs. (56) and (54), we have

$$\Omega+A\phi=\frac{2(\dot{G}_4-\dot{G}_{4,X}A^2)}{\dot{G}_{3,X}A^2}\partial_0\Theta, \quad (60)$$

$$\Omega+2A\phi+\partial_0\Psi=-\frac{2\dot{G}_4}{A(1-2c_2\dot{G}_{4,X})}\Theta. \quad (61)$$

Substituting Eq. (60) into Eq. (61) yields

$$A\phi+\partial_0\Psi=-\frac{2\dot{G}_4}{A(1-2c_2\dot{G}_{4,X})}\Theta-\frac{2(\dot{G}_4-\dot{G}_{4,X}A^2)}{\dot{G}_{3,X}A^2}\partial_0\Theta. \quad (62)$$

By substituting Eqs. (60) and (61) into Eq. (50), we obtain

$$\Delta\Psi = -\frac{2(\dot{G}_4 - \dot{G}_{4,X}A^2)}{\dot{G}_{3,X}A^2}\Delta\Theta - \frac{3\dot{G}_{3,X}^2A^2 + 4\dot{G}_{2,XX}(\dot{G}_4 - \dot{G}_{4,X}A^2)}{2\dot{G}_{3,X}^2A}\partial_0\Theta. \quad (63)$$

From Eqs. (62) and (63), it can be derived that

$$\Delta\phi = \frac{3\dot{G}_{3,X}^2A^2 + 4\dot{G}_{2,XX}(\dot{G}_4 - \dot{G}_{4,X}A^2)}{2\dot{G}_{3,X}^2A^2}\partial_0\partial_0\Theta - \frac{2\dot{G}_4}{A^2(1 - 2c_2\dot{G}_{4,X})}\Delta\Theta, \quad (64)$$

which gives the relationship between the amplitudes of the two scalar perturbations Θ and ϕ that affect scalar-mode gravitational waves. Finally, with Eqs. (61) and (64), we obtain the equation for the variable Θ from Eq. (53),

$$\mathcal{P}\partial_0\partial_0\Theta - \mathcal{Q}\Delta\Theta = 0, \quad (65)$$

where

$$\mathcal{P} = (1 - 2c_2\dot{G}_{4,X})(\dot{G}_4 - \dot{G}_{4,X}A^2)\mathcal{C}_1, \quad (66)$$

$$\mathcal{Q} = \dot{G}_{3,X}^2\dot{G}_4\mathcal{C}_2, \quad (67)$$

$$\mathcal{C}_1 = 3\dot{G}_{3,X}^2A^2 + 4\dot{G}_{2,XX}(\dot{G}_4 - \dot{G}_{4,X}A^2), \quad (68)$$

$$\mathcal{C}_2 = 4\dot{G}_4 - (1 - 2c_2\dot{G}_{4,X})A^2. \quad (69)$$

We can further divide Case 4 into following cases:

Case 4.1: $\mathcal{C}_1 = 0, \mathcal{C}_2 = 0$. In this case, we have

$$\Psi = -\frac{2(\dot{G}_4 - \dot{G}_{4,X}A^2)}{\dot{G}_{3,X}A^2}\Theta, \quad (70)$$

$$\phi = \frac{1}{2}\Theta. \quad (71)$$

We also have $\mathcal{P} = \mathcal{Q} = 0$ and so we cannot solve the function Θ with Eq. (65). Therefore, we do not consider this nonphysical case.

Case 4.2: $\mathcal{C}_1 \neq 0, \mathcal{C}_2 = 0$. In this case, Eq. (65) becomes $\partial_0\partial_0\Theta = 0$. This requires $\Theta = 0$, and then by Eq. (64), ϕ is also zero. Thus, generalized Proca theory does not allow scalar-mode gravitational waves in this case.

Case 4.3: $\mathcal{C}_1 = 0, \mathcal{C}_2 \neq 0$. In this case, we have $\mathcal{P} = 0$ and $\mathcal{Q} \neq 0$, which indicates that Eq. (65) becomes $\Delta\Theta = 0$. This requires $\Theta = 0$, and then by Eq. (64), ϕ is also zero. Therefore, generalized Proca theory does not allow scalar-mode gravitational waves in this case either.

Case 4.4: $\mathcal{C}_1 \neq 0, \mathcal{C}_2 \neq 0$. In this case, Eq. (65) is a wave equation and generalized Proca theory allows scalar-mode gravitational waves with one degree of freedom. The speed of the scalar mode gravitational waves c_S is

$$c_S^2 = \frac{\mathcal{Q}}{\mathcal{P}}. \quad (72)$$

To ensure the linear stability of the solution, we require $c_S^2 > 0$.

Finally, we consider a plane wave solution of Θ satisfying Eq. (65) with a propagation direction of $+z$,

$$\Theta = \Theta_0 e^{ikx}, \quad \frac{k^0}{k^3} = c_S. \quad (73)$$

By substituting (73) into Eq. (64), we obtain

TABLE II. The number of scalar mode gravitational waves in various parameter spaces. The number in the rightmost column of the table represents the degrees of freedom of the scalar mode gravitational waves in the corresponding case. The notation “...” indicates that the parameter space is not physical.

Cases	Conditions	Scalar DoF
Case 1	$\dot{G}_{3,X} = 1 - 2c_2\dot{G}_{4,X} = 0$
Case 2	$\dot{G}_{3,X} = 0, 1 - 2c_2\dot{G}_{4,X} \neq 0$.	0
Case 3	$\dot{G}_{3,X} \neq 0, 1 - 2c_2\dot{G}_{4,X} = 0$.	0
Case 4.1	$\dot{G}_{3,X} \neq 0, 1 - 2c_2\dot{G}_{4,X} \neq 0, \mathcal{C}_1 = 0, \mathcal{C}_2 = 0$
Case 4.2	$\dot{G}_{3,X} \neq 0, 1 - 2c_2\dot{G}_{4,X} \neq 0, \mathcal{C}_1 \neq 0, \mathcal{C}_2 = 0$.	0
Case 4.3	$\dot{G}_{3,X} \neq 0, 1 - 2c_2\dot{G}_{4,X} \neq 0, \mathcal{C}_1 = 0, \mathcal{C}_2 \neq 0$.	0
Case 4.4	$\dot{G}_{3,X} \neq 0, 1 - 2c_2\dot{G}_{4,X} \neq 0, \mathcal{C} \neq 0, \mathcal{C}_2 \neq 0$.	1

$$\Delta\phi = \left[-\frac{C_1}{2\mathring{G}_{3,X}^2 A^2} k_0^2 + \frac{2\mathring{G}_4}{A^2(1-2c_2\mathring{G}_{4,X})} k_3^2 \right] \Theta_0 e^{ikx}. \quad (74)$$

Therefore, ϕ has a plane wave solution $\phi = \phi_0 e^{ikx}$ with

$$\phi_0 = \left[\frac{C_1}{2\mathring{G}_{3,X}^2 A^2} c_S^2 - \frac{2\mathring{G}_4}{A^2(1-2c_2\mathring{G}_{4,X})} \right] \Theta_0. \quad (75)$$

Using Eq. (29), we have

$$P_1 = \left[\frac{\mathring{G}_{3,X}^2 A^2 - C_1}{2\mathring{G}_{3,X}^2 A^2} k_0^2 + \frac{2\mathring{G}_4}{A^2(1-2c_2\mathring{G}_{4,X})} k_3^2 \right] \Theta_0 e^{ikx}; \quad (76)$$

$$P_6 = \frac{1}{2} k_0^2 \Theta_0 e^{ikx}. \quad (77)$$

So the amplitude ratio \mathcal{R} of the longitudinal mode to the breathing mode is

$$\mathcal{R} = \left| \frac{P_1}{P_6} \right| = \left| \frac{\mathring{G}_{3,X}^2 A^2 - C_1}{\mathring{G}_{3,X}^2 A^2} + \frac{4\mathring{G}_4}{A^2(1-2c_2\mathring{G}_{4,X})c_S^2} \right|. \quad (78)$$

It can be seen that the scalar mode of generalized Proca theory is generally a mixture mode of the breathing and longitudinal modes. However, when $\mathcal{R} = 0$, the scalar mode is just a pure breathing mode. It should be pointed out that \mathcal{R} only depends on theoretical parameters. Therefore, once we provide the specific forms of \mathring{G}_2 , \mathring{G}_3 , \mathring{G}_4 , \mathring{G}_5 , c_2 , d_2 in the action and the background vector field, we can uniquely determine the value of \mathcal{R} . Whether it is 0 clearly tells us whether the scalar mode gravitational wave is a breathing mode or a mixture of the longitudinal mode and the breathing mode. Furthermore, it should be pointed out that the condition $\mathcal{R} = 0$ is not equivalent to $c_S = 1$.

Based on the above research, we find that the existence of scalar-mode gravitational waves in generalized Proca theory depends on the parameter spaces. We summarize the results in Table II.

VI. CONCLUSION

In this paper, under the background of the homogeneous and isotropic Minkowski spacetime, we study the polarization modes of gravitational waves in generalized Proca theory under linearized perturbations. We first obtained the background equations and the linear perturbation equations. Then we analyzed the polarization modes of gravitational waves and discussed their linear stability.

We found that the polarization modes of gravitational waves in generalized Proca theory depend on the parameter spaces. There are different vector and scalar polarization modes in different parameter spaces. The specific results have been summarized in Tables I and II, respectively. Generalized Proca theory allows at most two tensor modes, two vector modes and one scalar mode. The scalar mode is

typically a mixture mode of the breathing and longitudinal modes. However, when \mathcal{R} in Eq. (78) is zero, the scalar mode is just a pure breathing mode.

We also found that the amplitude of vector-mode gravitational waves satisfies the condition (48). This allows us to give the upper limit of the amplitude of vector mode gravitational waves by detecting the speed of tensor mode gravitational waves. Combining the results of GW170817, we found that the amplitude of vector gravitational waves has an upper bound of $|h| \ll 10^{-15}$. It should be noted that as the speed of tensor gravitational waves approaches the speed of light, the upper bound of the amplitude of vector gravitational waves decreases. Specifically, if the speed of tensor modes is strictly equal to the speed of light, then the amplitude of vector modes is zero. The relationship between the amplitude of vector modes and the speed of tensor modes is an interesting phenomenon. Does this phenomenon also exist in other vector-tensor gravity? Could we find a physical explanation for this phenomenon? This requires further research.

Finally, we need to point out that we only considered the case of the homogeneous and isotropic Minkowski background. At this point, the background vector \mathring{A}_a has only a nonvanishing temporal component. In the case of a small anisotropic background, the background vector \mathring{A}_a also has nonvanishing small spatial components. In this case, the background vector is timelike. We can always use a Lorentz transformation to make the background vector have only nonvanishing temporal component without losing generality. It can be seen that the polarization modes of gravitational waves in the anisotropic case can be directly obtained by using the Lorentz transformation on the polarization modes of gravitational waves in the isotropic case we have studied. For a general discussion of the transformation law of the polarization modes of gravitational waves under the Lorentz transformation, one can refer to Ref. [78].

The polarization modes of gravitational waves in generalized Proca theory can be divided into quite rich cases by parameter spaces. The appropriate parameter spaces can be expected to be selected in the detection of gravitational wave polarization modes by Lisa, Taiji and TianQin [43,44,46] in the future.

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APPENDIX: LINEAR PERTURBATION EQUATION

For Eq. (12), the specific expression of \mathcal{V}_a is given by

$$\begin{aligned}
\mathcal{V}_a = & 2\Box B_a - 2\dot{A}^b \partial_a \partial_c h^c_b + 2\dot{A}^b \Box h_{ab} - 2\dot{G}_{2,X} B_a - 2\dot{G}_{2,X} \dot{A}^b h_{ab} \\
& + 2\dot{G}_{3,X} \dot{A}^b \partial_a B_b + \dot{G}_{3,X} \dot{A}^b \dot{A}^c \partial_a h_{bc} - 2\dot{G}_{3,X} \dot{A}_a \partial_b B^b - \dot{G}_{3,X} \dot{A}_a \dot{A}^b \partial_b h \\
& - 4\dot{G}_{4,X} \partial_a \partial_b B^b - 2\dot{G}_{4,X} \dot{A}^b \partial_a \partial_b h - 4c_2 \dot{G}_{4,X} \Box B_a + 2\dot{G}_{4,X} \dot{A}^b \partial_a \partial_c h_b^c \\
& + 4c_2 \dot{G}_{4,X} \dot{A}^b \partial_a \partial_c h_b^c + 2\dot{G}_{4,X} \dot{A}^b \partial_b \partial_c h_a^c - 2\dot{G}_{4,X} \dot{A}_a \partial_b \partial_c h^{bc} - 2\dot{G}_{4,X} \dot{A}^b \Box h_{ab} \\
& - 4c_2 \dot{G}_{4,X} \dot{A}^b \Box h_{ab} + 2\dot{G}_{4,X} \dot{A}_a \Box h + [-2 + 4(1 + c_2) \dot{G}_{4,X}] \partial_a \partial_b B^b \\
& + 2\dot{G}_{2,XX} \dot{A}_a \dot{A}^b B_b + \dot{G}_{2,XX} \dot{A}_a \dot{A}^b \dot{A}^c h_{bc}.
\end{aligned} \tag{A1}$$

For Eq. (13), the specific expression of \mathcal{T}_{ab} is

$$\begin{aligned}
\mathcal{T}_{ab} = & -2\dot{G}_2 h_{ab} - 2\dot{A}_b \Box B_a - 2\dot{A}_a \Box B_b + 2\dot{A}_b \partial_a \partial_c B^c + 2\dot{A}_a \partial_b \partial_c B^c \\
& + 2\dot{A}_b \dot{A}^c \partial_a \partial_d h_c^d + 2\dot{A}_a \dot{A}^c \partial_b \partial_d h_c^d - 2\dot{A}_b \dot{A}^c \Box h_{ac} - 2\dot{A}_a \dot{A}^c \Box h_{bc} \\
& - 2\dot{G}_4 (\partial_a \partial_b h - \partial_a \partial_c h^c_b - \partial_b \partial_c h^c_a + \Box h_{ab} + \eta_{ab} \partial_c \partial_d h^{cd} - \eta_{ab} \Box h) \\
& + 2\dot{G}_{2,X} \dot{A}_b B_a + 2\dot{G}_{2,X} \dot{A}_a B_b + 2\dot{G}_{2,X} \eta_{ab} \dot{A}^c B_c + 2\dot{G}_{2,X} \dot{A}_b \dot{A}^c h_{ac} \\
& + 2\dot{G}_{2,X} \dot{A}_a \dot{A}^c h_{bc} + \dot{G}_{2,XX} \eta_{ab} \dot{A}^c \dot{A}^d h_{cd} + 2\dot{G}_{3,X} \dot{A}_a \dot{A}_b \partial_c B^c \\
& + \dot{G}_{3,X} \dot{A}_a \dot{A}_b \dot{A}^c \partial_c h - 2\dot{G}_{3,X} \eta_{ab} \dot{A}^c \dot{A}^d \partial_d B_c - \dot{G}_{3,X} \eta_{ab} \dot{A}^c \dot{A}^d \dot{A}^e \partial_e h_{cd} \\
& + 2\dot{G}_{4,X} \dot{A}^c \partial_a \partial_b B_c + \dot{G}_{4,X} \dot{A}^c \dot{A}^d \partial_a \partial_b h_{cd} + 2\dot{G}_{4,X} \dot{A}^c \partial_a \partial_b B_c \\
& + \dot{G}_{4,X} \dot{A}^c \dot{A}^d \partial_a \partial_b h_{cd} - 2\dot{G}_{4,X} \dot{A}^c \partial_c \partial_a B_b - 2\dot{G}_{4,X} \dot{A}_b \partial_a \partial_c B^c \\
& - 4c_2 \dot{G}_{4,X} \dot{A}_b \partial_a \partial_c B^c - 2\dot{G}_{4,X} \dot{A}^c \partial_c \partial_b B_a - 2\dot{G}_{4,X} \dot{A}_a \partial_b \partial_c B^c \\
& - 4c_2 \dot{G}_{4,X} \dot{A}_a \partial_b \partial_c B^c + 2\dot{G}_{4,X} \dot{A}_b \Box B_a + 4c_2 \dot{G}_{4,X} \dot{A}_b \Box B_a \\
& + 2\dot{G}_{4,X} \dot{A}_a \Box B_b + 4c_2 \dot{G}_{4,X} \dot{A}_a \Box B_b + 4\dot{G}_{4,X} \eta_{ab} \dot{A}^c \partial_c \partial_d B^d \\
& - 2\dot{G}_{4,X} \dot{A}_b \dot{A}^c \partial_a \partial_d h_c^d - 4c_2 \dot{G}_{4,X} \dot{A}_b \dot{A}^c \partial_a \partial_d h_c^d - 2\dot{G}_{4,X} \dot{A}_a \dot{A}^c \partial_b \partial_d h_c^d \\
& - 4c_2 \dot{G}_{4,X} \dot{A}_a \dot{A}^c \partial_b \partial_d h_c^d - 2\dot{G}_{4,X} \dot{A}^c \dot{A}^d \partial_c \partial_d h_{ab} + 2\dot{G}_{4,X} \dot{A}_a \dot{A}_b \partial_c \partial_d h^{cd} \\
& + 2\dot{G}_{4,X} \eta_{ab} \dot{A}^c \dot{A}^d \partial_c \partial_d h - 4\dot{G}_{4,X} \eta_{ab} \dot{A}^c \Box B_c + 2\dot{G}_{4,X} \dot{A}_b \dot{A}^c \Box h_{ac} \\
& + 4c_2 \dot{G}_{4,X} \dot{A}_b \dot{A}^c \Box h_{ac} + 2\dot{G}_{4,X} \dot{A}_a \dot{A}^c \Box h_{bc} + 4c_2 \dot{G}_{4,X} \dot{A}_a \dot{A}^c \Box h_{bc} \\
& - 2\dot{G}_{4,X} \dot{A}_a \dot{A}_b \Box h - 2\dot{G}_{4,X} \eta_{ab} \dot{A}^c \dot{A}^d \Box h_{cd} - 2\dot{G}_{2,XX} \dot{A}_a \dot{A}_b \dot{A}^c B_c \\
& - \dot{G}_{2,XX} \dot{A}_a \dot{A}_b \dot{A}^c \dot{A}^d h_{cd}.
\end{aligned} \tag{A2}$$

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