Irreversibility and gravitational radiation: A proof of Bondi's conjecture

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It is shown that the evolution of an axially and reflection symmetric fluid distribution, satisfying the Tolman condition for thermal equilibrium, is not accompanied by the emission of gravitational radiation. This result, which was conjectured by Bondi many years ago, expresses the irreversibility associated to the emission of gravitational waves. The observational consequences emerging from this result are commented. The resulting models are not only nondissipative and vorticity free, but also shear-free and geodesic, furthermore all their complexity factors vanish.

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I. INTRODUCTION

In his seminal paper on gravitational radiation [1] Bondi wrote (Section 6): "If the distinction between radiative and non-radiative motions is locally significant then the clearest self-consistent distinction appears to be between cases where the equations of state do not involve the time explicitly and are time reversible (no dissipation), and others."

In other words, the irreversibility of the process of emission of gravitational waves must be reflected in the equation of state of the source through an entropy increasing (dissipative) factor.

The rationale supporting this conjecture is very clear: radiation is an irreversible process, this fact emerges at once if absorption is taken into account and/or Sommerfeld type conditions, which eliminate inward traveling waves, are imposed. Therefore, it is obvious that an entropy generator factor should be present in the description of the source.

However, since the Bondi's work deals exclusively with the space-time outside the source (more so, far from the source), the above mentioned relationship between gravitational radiation and dissipative processes within the source, remained so far a conjecture (a very reasonable one though).

It is the purpose of this work to provide a definitive proof of the Bondi's conjecture.

For doing that we shall resort to a general formalism to describe the evolution of dissipative axially and reflection symmetric fluid distribution presented in [2], based in the $1+3$ formalism developed in [3–6].

Our proof develops in two steps. We shall first prove that assuming the Tolman condition [7] to be satisfied (implying the absence of dissipative flux), the fluid is necessarily vorticity free. Next, using this last condition and the absence of dissipation we shall prove that the magnetic part of the Weyl tensor vanishes. This last result closes the proof of the Bondi's conjecture, since it implies the vanishing of the super-Poynting vector. Indeed, in the theory of the super-Poynting vector, a state of gravitational radiation is associated to a nonvanishing component of the latter (see [8–10]). This in turn is in agreement with the established link between the super-Poynting vector and the news functions [11], in the context of the Bondi-Sachs approach [1,12].

Besides we shall see that the fluid is necessarily shear-free, geodesic and all their complexity factors vanish.

In the next section we shall briefly summarize the main equations required for our proof. Then we shall proceed with the proof following the steps outlined before. Finally we discuss about the physical relevance of our results. Some basic definitions and intermediate formulae are given in the Appendix.

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II. THE METRIC AND THE SOURCE: BASIC EQUATIONS AND NOTATION

As mentioned before, we shall resort to the general approach fully deployed in [2] in order to achieve our goal. In this section we shall present very briefly the most general properties of the space-time under consideration and the matter content of the source. The reader is referred to [2] and the Appendix for any specific detail of calculation.

We shall consider axially (and reflection) symmetric sources. For such a system the most general line element may be written in "Weyl spherical coordinates" as

$$ds^2 = -A^2dt^2 + B^2(dr^2 + r^2d\theta^2) + C^2d\phi^2 + 2Gd\theta dt, \quad (1)$$

where A, B, C, G are functions of t, r, and θ, of class C^ω, with A, B, C positive defined. We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$. At this point it is important to stress that due to the reflection symmetry [no $dtd\phi$ terms in (1)], rotations around the symmetry axe are excluded, and vorticity is associated with motion along the θ direction.

The inverse components of the metric are given by

$$g^{\alpha\beta} = \begin{pmatrix} -\frac{B^2r^2}{A^2B^2r^2+G^2} & 0 & \frac{G}{A^2B^2r^2+G^2} & 0 \\ 0 & \frac{1}{B^2} & 0 & 0 \\ \frac{G}{A^2B^2r^2+G^2} & 0 & \frac{A^2}{A^2B^2r^2+G^2} & 0 \\ 0 & 0 & 0 & \frac{1}{C^2} \end{pmatrix}. \quad (2)$$

We shall assume that our source is filled with an anisotropic and dissipative fluid. The energy momentum tensor may be written in the "canonical" form as

$$T_{\alpha\beta} = (\mu + P)V_\alpha V_\beta + Pg_{\alpha\beta} + \Pi_{\alpha\beta} + q_\alpha V_\beta + q_\beta V_\alpha. \quad (3)$$

The above is the canonical, algebraic decomposition of a second order symmetric tensor with respect to unit timelike vector, which has the standard physical meaning where $T_{\alpha\beta}$ is the energy-momentum tensor describing some energy distribution and V^μ the four-velocity assigned by certain observer. In our case we are considering an Eckart frame where fluid elements are at rest.

With the above definitions it is clear that μ is the energy density (the eigenvalue of $T_{\alpha\beta}$ for eigenvector V^α), q_α is the heat flux, whereas P is the isotropic pressure, and $\Pi_{\alpha\beta}$ is the anisotropic tensor.

Since we choose the fluid to be comoving in our coordinates, then

$$V^\alpha = \left(\frac{1}{A}, 0, 0, 0\right), \qquad V_\alpha = \left(-A, 0, \frac{G}{A}, 0\right). \quad (4)$$

Next, let us introduce the unit, spacelike vectors $\mathbf{K}$, $\mathbf{L}$, $\mathbf{S}$, with components

$$K_\alpha = (0, B, 0, 0), \qquad K^\alpha = \left(0, \frac{1}{B}, 0, 0\right), \quad (5)$$

$$L^\alpha = \left(\frac{G}{A\sqrt{A^2B^2r^2 + G^2}}, 0, \frac{A}{\sqrt{A^2B^2r^2 + G^2}}, 0\right), \quad (6)$$

$$L_\alpha = \left(0, 0, \frac{\sqrt{A^2B^2r^2 + G^2}}{A}, 0\right), \quad (7)$$

$$S_\alpha = (0, 0, 0, C), \qquad S^\alpha = \left(0, 0, 0, \frac{1}{C}\right), \quad (8)$$

satisfying the following relations:

$$V_\alpha V^\alpha = -K^\alpha K_\alpha = -L^\alpha L_\alpha = -S^\alpha S_\alpha = -1, \quad (9)$$

$$V_\alpha K^\alpha = V^\alpha L_\alpha = V^\alpha S_\alpha = K^\alpha L_\alpha = K^\alpha S_\alpha = S^\alpha L_\alpha = 0. \quad (10)$$

The unitary vectors V^α, L^α, S^α, K^α form a canonical orthonormal tetrad (say $e^{(a)}_\alpha$), such that

$$e^{(0)}_\alpha = V_\alpha, \qquad e^{(1)}_\alpha = K_\alpha, \qquad e^{(2)}_\alpha = L_\alpha, \qquad e^{(3)}_\alpha = S_\alpha,$$

with $a = 0, 1, 2, 3$ (latin indices labeling different vectors of the tetrad). The dual vector tetrad $e^\alpha_{(a)}$ is easily computed from the condition

$$\eta_{(a)(b)} = g_{\alpha\beta}e^\alpha_{(a)}e^\beta_{(b)}.$$

We shall express all kinematical and physical variables, as well as the equations relating them, in terms of their tetrad components. These expressions are explicitly deployed in the Appendix.

III. PROVING THAT NO DISSIPATION IMPLIES NO GRAVITATIONAL RADIATION

In order to ensure the absence of dissipation we have to impose the Tolman conditions for thermodynamic equilibrium [7]. Such conditions emerge from the fact that, according to special relativity, all forms of energy have inertia, and therefore this should also apply to heat. Then, because of the equivalence principle, there should also be some weight associated to heat, and one should expect that thermal energy tends to displace to regions of lower gravitational potential. This in turn implies that the condition of thermal equilibrium in the presence of a gravitational field must change with respect to its form in the absence of gravity. Thus, a temperature gradient is necessary in thermal equilibrium in order to prevent the flow of heat from regions of higher to lower gravitational potential. Tolman deduced such conditions without any reference to any specific transport equation, however, as expected, for any consistent transport equation, the absence of dissipation should lead to Tolman conditions.

Thus, for example, in the Müller-Israel-Stewart second order phenomenological theory for dissipative fluids [13–16]), the transport equation reads

$$\tau h^{\mu}_{\nu} q^{\nu}_{;\beta} V^{\beta} + q^{\mu} = -\kappa h^{\mu\nu}(T_{,\nu} + T a_{\nu}) - \frac{1}{2}\kappa T^2 \left(\frac{\tau V^{\alpha}}{\kappa T^2}\right)_{;\alpha} q^{\mu}, \quad (11)$$

where τ, κ, T denote the relaxation time, the thermal conductivity, and the temperature, respectively.

From (11) we see that the absence of dissipative flux implies at once

$$h^{\mu\nu}(T_{,\nu} + T a_{\nu}) = 0, \quad (12)$$

which are the Tolman conditions.

We have now all the ingredients required for our proof. Some relevant equations are written down in the Appendix.

We shall assume that the system is in thermodynamic equilibrium, implying that the Tolman conditions (12) are satisfied, i.e.,

$$a_{\mu} = -h^{\nu}_{\mu}\Gamma_{,v} \qquad \Gamma \equiv \ln T. \quad (13)$$

From the above equation it follows that

$$a_1 = -\Gamma', \qquad a_2 = -\frac{G\dot{\Gamma}}{A^2} - \Gamma_{,\theta}. \quad (14)$$

Using (14) in (A40) produces

$$K^{[\mu} L^{\nu]} a_{\mu;\nu} = V^{\mu}\Gamma_{,\mu}\Omega, \quad (15)$$

which combined with (A39) produces

$$\Omega_{,\delta} V^{\delta} + \frac{1}{3}(2\Theta + \sigma_I + \sigma_{II} + 3V^{\mu}\Gamma_{,\mu})\Omega = 0. \quad (16)$$

The consequences derived from the above equation are far reaching. Indeed, if we assume that the system is initially static (at $t = 0$ say), and assume that it starts to evolve afterward, keeping the thermodynamic equilibrium, then the evolving fluid would be vorticity-free. This result is in full agreement with earlier works indicating that vorticity generation is sourced by entropy gradients [17–21]. At the same time this result reinforces further the Bondi's conjecture about the absence of radiation for nondissipative systems, if we recall the radiation-vorticity link discussed in [11,22]. However, we have not yet a formal proof of the conjecture. For that we need to prove that a system evolving without dissipation and vorticity cannot radiate gravitational radiation, i.e., we have to show that during the evolution regime, after leaving the dynamic equilibrium, $H_1 = H_2 = 0$ all along the evolution.

Thus we consider a system that during its evolution satisfies the conditions

$$q_I = q_{II} = 0 \Rightarrow \Omega = 0 \Rightarrow \sigma_I = \sigma_{II} = \sigma. \quad (17)$$

Then from (A42) we obtain

$$\frac{(2\Theta - \sigma)'}{3} = \frac{\sigma C'}{C}, \quad (18)$$

whereas (A43) reads

$$\frac{(2\Theta - \sigma)_{,\theta}}{3} = \frac{\sigma C_{,\theta}}{C}. \quad (19)$$

Also, (A44)–(A46) become

$$H_1 = -\frac{(\sigma C)_{,\theta}}{2rBC}, \quad (20)$$

$$H_2 = \frac{(\sigma C)'}{2BC}, \quad (21)$$

and

$$\frac{H_1'}{B} + \frac{H_{2,\theta}}{\mathrm{Br}} + \frac{H_1}{B}\left[\frac{2C'}{C} + \frac{(\mathrm{Br})'}{\mathrm{Br}}\right] + \frac{H_2}{\mathrm{Br}}\left[\frac{2C_{,\theta}}{C} + \frac{(\mathrm{Br})_{,\theta}}{\mathrm{Br}}\right] = 0, \quad (22)$$

respectively.

Using (20) and (21) in (22) we obtain

$$H_1 C' + H_2 \frac{C_{,\theta}}{r} = 0 \quad (23)$$

$$\frac{H_1'}{B} + \frac{H_{2,\theta}}{\mathrm{Br}} + \frac{H_1}{B}\frac{(\mathrm{Br})'}{\mathrm{Br}} + \frac{H_2}{\mathrm{Br}}\frac{(\mathrm{Br})_{,\theta}}{\mathrm{Br}} = 0, \quad (24)$$

from which it is obvious that the vanishing of either one of the scalars (H_1 or H_2) implies the vanishing of the other.

Finally let us notice that using (18) and (19) in (20) and (21) we may write

$$H_1 = -\frac{1}{rB}\left(\frac{\dot{B}}{AB}\right)_{,\theta}, \quad (25)$$

$$H_2 = \frac{1}{B}\left(\frac{\dot{B}}{AB}\right)'. \quad (26)$$

Let us now proceed to the second part of the proof.

We start from an initially static situation, meaning that at $t = 0$, we have $\dot{A} = \dot{B} = \dot{C} = \sigma = \Theta = H_1 = H_2 = 0$. Besides, conditions (17) are satisfied for all t.

Let us take the first time derivatives of (25), (26), (A42), and (A43) evaluated at $t = 0$, we obtain, respectively,

$$\dot{H}_1 = -\frac{1}{rB}\left(\frac{\ddot{B}}{AB}\right)_{,\theta}, \tag{27}$$

$$\dot{H}_2 = \frac{1}{B}\left(\frac{\ddot{B}}{AB}\right)', \tag{28}$$

$$8\pi\dot{q}_I B = \left[\frac{1}{A}\left(\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}\right)\right]' - \left(\frac{\ddot{B}}{AB}-\frac{\ddot{C}}{AC}\right)\frac{C'}{C} = 0, \tag{29}$$

and

$$8\pi\dot{q}_{II}\mathrm{Br} = \left[\frac{1}{A}\left(\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}\right)\right]_{,\theta} - \left(\frac{\ddot{B}}{AB}-\frac{\ddot{C}}{AC}\right)\frac{C_{,\theta}}{C} = 0. \tag{30}$$

From regularity conditions (A19) and (A20) at $r \approx 0$, and from the fact that A, B, C and their derivatives are regular at $r \approx 0$ we may write at $r \approx 0$, using (29)

$$W \equiv \frac{\ddot{B}}{AB} - \frac{\ddot{C}}{AC} \approx r \approx 0. \tag{31}$$

Taking successive r derivatives of (29) it is a simple matter to check that all r derivatives (of any order) of W vanish at $r \approx 0$, implying that $W = 0$ for all values of r within the fluid distribution.

Thus at $t \approx 0$ we have

$$\frac{\ddot{B}}{AB} - \frac{\ddot{C}}{AC} = 0, \tag{32}$$

for all values of r within the fluid distribution.

Feeding back (32) into (30), we obtain

$$\left(\frac{\ddot{B}}{AB}\right)_{,\theta} = \left(\frac{\ddot{C}}{AC}\right)_{,\theta} = 0, \tag{33}$$

which combined with (27) produces $\dot{H}_1 = 0$, and by virtue of (23), $\dot{H}_2 = 0$ as well.

Next, feeding back (31) into (29) produces

$$\left(\frac{\ddot{B}}{AB}\right)' = \left(\frac{\ddot{C}}{AC}\right)' = 0, \tag{34}$$

implying because A and B and C are independent functions, that the only admissible solution to (34) (which is an identity) is $A' = 0$, and B and C are separable functions.

The next step consists in proving that time derivatives of any order of H_1 and H_2 evaluated at $t \approx 0$ also vanish, i.e.,

$$\overset{(m)}{H}_1 = \overset{(m)}{H}_0 = 0 \quad (\text{for any } m \geq 1), \quad \text{where } \overset{(m)}{X} \equiv \frac{\partial^m X}{\partial t^m}, \tag{35}$$

this would imply that $H_1 = H_2 = 0$ for any t.

For doing that we shall retrace the same steps above, using (17) and the results obtained so far.

Thus, taking the m-time derivative (with $m \geq 1$) of (A42) and (A43) we may write

$$\frac{\overset{(m+1)}{B}}{AB} = \frac{\overset{(m+1)}{C}}{AC}. \tag{36}$$

Also taking the m-time derivative of (25) and (26) produces

$$\overset{(m)}{H}_1 = -\frac{1}{\mathrm{Br}}\left[\frac{\overset{(m+1)}{B}}{AB}\right]_{,\theta}, \tag{37}$$

$$\overset{(m)}{H}_2 = \frac{1}{B}\left[\frac{\overset{(m+1)}{B}}{AB}\right]'. \tag{38}$$

Using the separability of B and the fact that $A' = 0$, in (38) it follows at once that

$$\overset{(m)}{H}_2 = 0, \tag{39}$$

which by virtue of (23) implies $\overset{(m)}{H}_1 = 0$.

Thus if the system is initially static and evolves without vorticity and without dissipation then all time derivatives of any order of H_1 and H_2 vanish for all values of r, implying that $H_1 = H_2 = 0$ at all times.

There is yet another, perhaps more simple, way to prove the above mentioned statement. Indeed, it is a simple matter to see that the m-time derivative of σ evaluated at $t = 0$, for any $m \geq 1$ reads

$$\overset{(m)}{\sigma} = \frac{1}{B}\left[\frac{\overset{(m+1)}{B}}{AB} - \frac{\overset{(m+1)}{C}}{AC}\right], \tag{40}$$

implying because of (36) that time derivatives of σ of any order vanish at $t = 0$, implying in its turn that the fluid is shear-free. But as shown in [23], for a shear-free fluid (not necessarily perfect fluid), the necessary and sufficient condition to be irrotational is that the Weyl tensor be purely electric; this generalizes a result by Barnes [24,25] and Glass [26]. Besides, it is worth noticing that $H_1 = H_2 = 0$ implies that the fluid is also geodesic $A' = A_{,\theta} = 0$.

Thus according to (A38) the two components of the super-Poynting vector vanish, meaning that no gravitational radiation is produced during the evolution of the system.

IV. CONCLUSIONS

The purpose of this work was to prove the correctness of the Bondi conjecture about the irreversibility associated with gravitational radiation. In other words, a reversible flow ($q = 0$) implies no gravitational radiation. We proved that, by showing that the absence of dissipative flux during the evolution (fulfillment of Tolman conditions), implies that magnetic parts of the Weyl tensor vanish, thereby implying the vanishing of the super-Poynting vector.

At this point, it is worth stressing the fact that a reversible flow, e.g., a perfect isotropic fluid, does not necessarily imply noncrossing of flow lines (geodesic), unless we assume that pressure gradients vanish. Indeed, for a perfect (isotropic and nondissipative) fluid the equation of motion reads

$$(\mu + P)a^\alpha = h^{\alpha\nu} P_{,\nu}, \tag{41}$$

from where is clear that the geodesic condition ($a^\alpha = 0$), automatically implies vanishing of pressure gradients. In this latter case if the fluid is bounded, and we impose matching conditions on the boundary surface, then the pressure vanishes and we have geodesic dust.

As a byproduct of our proof it appears that the vorticity of the fluid also vanishes under the condition mentioned above, bringing out, on the one hand the link of vorticity with dissipative processes already established in [18–21], and on the other hand the link between vorticity and gravitational radiation discussed in [11,22] (and references therein).

It is worth mentioning that the fluid configuration emerging from our restrictions, not only is nonradiative (gravitationally), shear-free, nondissipative and vorticity free, but is also geodesic, as a consequence of which the Tolman conditions imply a homogeneous temperature. These kind of solutions have been investigated in detail in [27,28]. Such solutions are in general nonconformally flat, with the electric Weyl tensor (A22) being defined through the scalars

$$\mathcal{E}_I = \mathcal{E}_I(0)\exp\left[-\frac{2}{3}\int \Theta dt\right],$$
$$\mathcal{E}_{II} = \mathcal{E}_{II}(0)\exp\left[-\frac{2}{3}\int \Theta dt\right],$$
$$\mathcal{E}_{KL} = \mathcal{E}_{KL}(0)\exp\left[-\frac{2}{3}\int \Theta dt\right], \tag{42}$$

with $\Theta = \Theta(t)$.

Also, these models are characterized by the vanishing of the trace-free part of the tensor $Y_{\alpha\beta}$ (see [23] for details), i.e.,

$$Y_I = Y_{KL} = Y_{II} = 0. \tag{43}$$

Parenthetically, these three scalars haven been proposed to describe the degree of complexity of a fluid distribution [29,30]. Thus according to the criterium assumed in these references, the resulting models are the simplest among those belonging to the family of space-times described by (1).

Finally, we would like to conclude with two remarks:

(1) The fact that the emission of gravitational radiation requires the presence of dissipative flux within the source to account by the irreversibility of the process, implies that any detected burst of gravitational waves should be accompanied by a burst of thermal radiation, which in principle could be observed too.

(2) An alternative way of proving the Bondi's conjecture could be provided by assuming a perfect fluid (so that $T = (mu + p)VV + pg)$ with an equation of state $p = p(\mu, s)$ where μ is the energy density and s the specific entropy. Then, from the generalized Gibbs equation and the Bianchi identities, it follows that (see for example [31])

$$TS^\alpha_{;\alpha} = -q^\alpha\left[h^\mu_\alpha(\ln T)_{,\mu} + V_{\alpha;\mu}V^\mu + \beta_1 q_{\alpha;\mu}V^\mu + \frac{T}{2}\left(\frac{\beta_1}{T}V^\mu\right)_{;\mu} q_\alpha\right], \tag{44}$$

where S^α is the entropy four current and $\beta_1 = \frac{\tau}{\kappa T}$. If we assume from the beginning that the matter content of the source is a perfect fluid (no heat flux vector) then $S^\alpha_{;\alpha} = 0$, implying that the entropy is constant and Tolman conditions are satisfied. From this point, there are different ways to prove that no gravitational radiation is produced, one of which is the one we have chosen in this manuscript, though it is not the only one.

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APPENDIX: SUMMARY OF SCALAR VARIABLES AND EQUATIONS

The required equations for our proof are given explicitly in [2]. Here for self-consistency we present a brief summary of them, including only those equations explicitly required for our proof. The reader is referred to [2] for details of calculations.

The anisotropic tensor may be expressed in the form

$$\Pi_{\alpha\beta} = \frac{1}{3}(2\Pi_I + \Pi_{II})\left(K_\alpha K_\beta - \frac{h_{\alpha\beta}}{3}\right) + \frac{1}{3}(2\Pi_{II} + \Pi_I)\left(L_\alpha L_\beta - \frac{h_{\alpha\beta}}{3}\right) + 2\Pi_{KL}K_{(\alpha}L_{\beta)}, \tag{A1}$$

with $h_{\mu\nu} = g_{\mu\nu} + V_\nu V_\mu$,

$$\Pi_{KL} = K^\alpha L^\beta T_{\alpha\beta}, \tag{A2}$$

$$\Pi_I = (2K^\alpha K^\beta - L^\alpha L^\beta - S^\alpha S^\beta) T_{\alpha\beta}, \tag{A3}$$

$$\Pi_{II} = (2L^\alpha L^\beta - S^\alpha S^\beta - K^\alpha K^\beta) T_{\alpha\beta}. \tag{A4}$$

The heat flux vector may be written as

$$q_\mu = q_I K_\mu + q_{II} L_\mu, \tag{A5}$$

or, in coordinate components

$$q^\mu = \left(\frac{q_{II} G}{A\sqrt{A^2B^2r^2 + G^2}}, \frac{q_I}{B}, \frac{A q_{II}}{\sqrt{A^2B^2r^2 + G^2}}, 0\right), \tag{A6}$$

$$q_\mu = \left(0, B q_I, \frac{\sqrt{A^2B^2r^2 + G^2}\, q_{II}}{A}, 0\right). \tag{A7}$$

Of course, all the above quantities depend, in general, on t, r, θ.

The kinematical variables (four acceleration, expansion scalar, shear tensor, and vorticity) are

$$a_\alpha = V^\beta V_{\alpha;\beta} = a_I K_\alpha + a_{II} L_\alpha,$$
$$= \left(0, \frac{A'}{A}, \frac{G}{A^2}\left[-\frac{\dot{A}}{A} + \frac{\dot{G}}{G}\right] + \frac{A_{,\theta}}{A}, 0\right), \tag{A8}$$

$$\Theta = V^\alpha_{;\alpha},$$
$$= \frac{AB^2}{r^2A^2B^2 + G^2}\left[r^2\left(2\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{G^2}{A^2B^2}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A} + \frac{\dot{G}}{G} + \frac{\dot{C}}{C}\right)\right], \tag{A9}$$

$$\sigma_{\alpha\beta} = \frac{1}{3}(2\sigma_I + \sigma_{II})\left(K_\alpha K_\beta - \frac{1}{3}h_{\alpha\beta}\right) + \frac{1}{3}(2\sigma_{II} + \sigma_I)\left(L_\alpha L_\beta - \frac{1}{3}h_{\alpha\beta}\right), \tag{A10}$$

where

$$2\sigma_I + \sigma_{II} = \frac{3}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right), \tag{A11}$$

$$2\sigma_{II} + \sigma_I = \frac{3}{A^2B^2r^2 + G^2}\left[AB^2r^2\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) + \frac{G^2}{A}\left(-\frac{\dot{A}}{A} + \frac{\dot{G}}{G} - \frac{\dot{C}}{C}\right)\right], \tag{A12}$$

in the above dots and primes denote derivatives with respect to t and r, respectively.

Finally, for the vorticity vector defined as

$$\omega_\alpha = \frac{1}{2}\eta_{\alpha\beta\mu\nu} V^{\beta;\mu} V^\nu = \frac{1}{2}\eta_{\alpha\beta\mu\nu}\Omega^{\beta\mu} V^\nu, \tag{A13}$$

where $\Omega_{\alpha\beta} = V_{[\alpha;\beta]} + a_{[\alpha} V_{\beta]}$ and $\eta_{\alpha\beta\mu\nu}$ denote the vorticity tensor and the Levi-Civita tensor, respectively, we find a single component different from zero, producing

$$\Omega_{\alpha\beta} = \Omega(L_\alpha K_\beta - L_\beta K_\alpha), \tag{A14}$$

and

$$\omega_\alpha = -\Omega S_\alpha, \tag{A15}$$

with the scalar function Ω given by

$$\Omega = \frac{G\left(\frac{G'}{G} - \frac{2A'}{A}\right)}{2B\sqrt{A^2B^2r^2 + G^2}}. \tag{A16}$$

Now, from the regularity conditions, necessary to ensure elementary flatness in the vicinity of the axis of symmetry, and in particular at the center (see [32–34]), we should require that, as $r \approx 0$,

$$\Omega = \sum_{n\geq 1} \Omega^{(n)}(t,\theta) r^n, \tag{A17}$$

implying because of (A16) that in the neighborhood of the center

$$G = \sum_{n\geq 3} G^{(n)}(t,\theta) r^n. \tag{A18}$$

Also, for the length of an orbit at t, θ constant, to be $2\pi r$, close to the origin (elementary flatness), we may write, as $r \to 0$,

$$C \approx r\gamma(t,\theta), \tag{A19}$$

implying

$$C' \approx \gamma(t,\theta), \qquad C_{,\theta} \approx r\gamma_{,\theta}, \tag{A20}$$

where $\gamma(t,\theta)$ is an arbitrary function of its arguments, which as appears evident from the elementary flatness condition, cannot vanish anywhere within the fluid distribution.

Observe that from (A16) and regularity conditions at the centre, it follows that: $G = 0 \Leftrightarrow \Omega = 0$.

Next, for the electric ($E_{\alpha\beta}$) and magnetic ($H_{\alpha\beta}$) parts of the Weyl tensor $C_{\alpha\beta\gamma\delta}$, we have

$$E_{\alpha\beta} = C_{\alpha\nu\beta\delta} V^\nu V^\delta,$$
$$H_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\nu\epsilon\rho} C_{\beta\delta}{}^{\epsilon\rho} V^\nu V^\delta. \tag{A21}$$

The electric part of the Weyl tensor has only three independent nonvanishing components, whereas only two components define the magnetic part. Thus we may also write

$$E_{\alpha\beta} = \frac{1}{3}(2\mathcal{E}_I + \mathcal{E}_{II})\left(K_\alpha K_\beta - \frac{1}{3}h_{\alpha\beta}\right) + \frac{1}{3}(2\mathcal{E}_{II} + \mathcal{E}_I)\left(L_\alpha L_\beta - \frac{1}{3}h_{\alpha\beta}\right) + \mathcal{E}_{KL}(K_\alpha L_\beta + K_\beta L_\alpha), \tag{A22}$$

and

$$H_{\alpha\beta} = H_1(S_\alpha K_\beta + S_\beta K_\alpha) + H_2(S_\alpha L_\beta + S_\beta L_\alpha). \tag{A23}$$

The orthogonal splitting of the Riemann tensor is carried out by means of three tensors $Y_{\alpha\beta}$, $X_{\alpha\beta}$, and $Z_{\alpha\beta}$ defined as

$$Y_{\alpha\beta} = R_{\alpha\nu\beta\delta}V^\nu V^\delta, \tag{A24}$$

$$X_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\nu}{}^{\epsilon\rho}R^\star_{\epsilon\rho\beta\delta}V^\nu V^\delta, \tag{A25}$$

and

$$Z_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\epsilon\rho}R_{\delta\beta}{}^{\epsilon\rho}V^\delta, \tag{A26}$$

where $R^\star_{\alpha\beta\nu\delta} = \frac{1}{2}\eta_{\epsilon\rho\nu\delta}R_{\alpha\beta}{}^{\epsilon\rho}$ and $\epsilon_{\alpha\beta\rho} = \eta_{\nu\alpha\beta\rho}V^\nu$.

The three tensors above may be expressed through the following scalars functions:

$$Y_T = 4\pi(\mu + 3P), \tag{A27}$$

$$Y_I = \mathcal{E}_I - 4\pi\Pi_I, \tag{A28}$$

$$Y_{II} = \mathcal{E}_{II} - 4\pi\Pi_{II}, \tag{A29}$$

$$Y_{KL} = \mathcal{E}_{KL} - 4\pi\Pi_{KL}, \tag{A30}$$

$$X_T = 8\pi\mu, \tag{A31}$$

$$X_I = -\mathcal{E}_I - 4\pi\Pi_I, \tag{A32}$$

$$X_{II} = -\mathcal{E}_{II} - 4\pi\Pi_{II}, \tag{A33}$$

$$X_{KL} = -\mathcal{E}_{KL} - 4\pi\Pi_{KL}, \tag{A34}$$

$$Z_I = (H_1 - 4\pi q_{II}), \qquad Z_{II} = (H_1 + 4\pi q_{II}),$$
$$Z_{III} = (H_2 - 4\pi q_I), \qquad Z_{IV} = (H_2 + 4\pi q_I). \tag{A35}$$

In the above, the scalars Y_T, X_T define the trace of (A24) and (A25), respectively, whereas the scalars Y_I, Y_{II}, Y_{KL}, X_I, X_{II}, X_{KL} define the trace-free part of (A24) and (A25).

The super-Poynting vector defined by

$$P_\alpha = \epsilon_{\alpha\beta\gamma}(Y^\gamma_\delta Z^{\beta\delta} - X^\gamma_\delta Z^{\delta\beta}), \tag{A36}$$

can be written as

$$P_\alpha = P_I K_\alpha + P_{II} L_\alpha, \tag{A37}$$

with

$$P_I = \frac{H_2}{3}(2Y_{II} + Y_I - 2X_{II} - X_I) + H_1(Y_{KL} - X_{KL}) + \frac{4\pi q_I}{3}[2Y_T + 2X_T - X_I - Y_I] - 4\pi q_{II}(X_{KL} + Y_{KL}),$$
$$P_{II} = \frac{H_1}{3}(2X_I + X_{II} - Y_{II} - 2Y_I) + H_2(X_{KL} - Y_{KL}) - 4\pi q_I(Y_{KL} + X_{KL}) + \frac{4\pi q_{II}}{3}[2Y_T + 2X_T - X_{II} - Y_{II}]. \tag{A38}$$

As mentioned before, in the theory of the super-Poynting vector, a state of gravitational radiation is associated to a non–vanishing component of the latter (see [8–10]). Therefore we shall verify the absence of gravitational radiation if the two components of the super-Poynting vector vanish.

From the Ricci identities for the vector V_α, and the Bianchi identities the following set of equations are obtained by contracting with different vectors of the tetrad (see [2] for details). These are as follows: (1) An evolution equation for Ω [Eq. (B5) in [2]]

$$\Omega_{,\delta}V^\delta + \frac{1}{3}(2\Theta + \sigma_I + \sigma_{II})\Omega + K^{[\mu}L^{\nu]}a_{\mu;\nu} = 0, \tag{A39}$$

where

$$(K^\mu L^\nu - L^\mu K^\nu)a_{\nu;\mu} = -\frac{A}{\sqrt{A^2B^2r^2 + G^2}}\left[a_{I,\theta} + \frac{G}{A^2}\dot{a}_I + a_I\left(\frac{B_{,\theta}}{B} + \frac{G}{A^2}\frac{\dot{B}}{B}\right)\right] + \frac{1}{B}\left[a'_{II} + \frac{a_{II}}{A^2B^2r^2 + G^2}\left(A^2B^2r^2\frac{(Br)'}{\mathrm{Br}} + GG' - G^2\frac{A'}{A}\right)\right], \tag{A40}$$

with

$$a_{II} = \frac{Aa_2}{\sqrt{A^2B^2r^2 + G^2}}, \qquad a_I = \frac{a_1}{B}. \tag{A41}$$

(2) Two equations relating q_I and q_{II} with the kinematical variables [Eqs. (B6) and (B7) in [2]]

$$\frac{2}{3B}\Theta_{,r} - \Omega_{;\mu}L^\mu + \Omega(L_{\beta;\mu}K^\mu K^\beta - L^\mu_{;\mu}) + \frac{1}{3}\sigma_I a_I - \Omega a_{II} - \frac{1}{3}\sigma_{I;\mu}K^\mu - \frac{1}{3}(2\sigma_I + \sigma_{II})\left(K^\mu_{;\mu} - \frac{a_I}{3}\right) - \frac{1}{3}(2\sigma_{II} + \sigma_I)\left(L_{\beta;\mu}L^\mu K^\beta - \frac{a_I}{3}\right) = 8\pi q_I, \tag{A42}$$

$$\frac{1}{3\sqrt{A^2B^2r^2 + G^2}}\left(\frac{2G}{A}\Theta_{,t} + 2A\Theta_{,\theta}\right) + \frac{a_{II}\sigma_{II}}{3} + \Omega_{;\mu}K^\mu + \Omega(K^\mu_{;\mu} + L^\mu K^\beta L_{\beta;\mu}) + \Omega a_I - \frac{1}{3}\sigma_{II;\mu}L^\mu + \frac{1}{3}(2\sigma_I + \sigma_{II})\left(L_{\beta;\mu}K^\beta K^\mu + \frac{a_{II}}{3}\right) - \frac{1}{3}(2\sigma_{II} + \sigma_I)\left(L^\mu_{;\mu} - \frac{a_{II}}{3}\right) = 8\pi q_{II}. \tag{A43}$$

(3) Two equations relating the two scalars defining the magnetic part of the Weyl tensor with the kinematical variables [Eqs. (B8) and (B9) in [2]]

$$-\Omega a_I - \frac{1}{2}(K^\mu S_\nu + S^\mu K_\nu)(\sigma_{\mu\delta} + \Omega_{\mu\delta})_{;\gamma}\epsilon^{\nu\gamma\delta} = H_1, \tag{A44}$$

$$-\Omega a_{II} - \frac{1}{2}(L^\mu S_\nu + S^\mu L_\nu)(\sigma_{\mu\delta} + \Omega_{\mu\delta})_{;\gamma}\epsilon^{\nu\gamma\delta} = H_2. \tag{A45}$$

And (4) one of equations derived from the Bianchi identities, where one obtains [Eq. (B16) in [2]]

$$-\frac{1}{3}X_{KL}(\sigma_{II} - \sigma_I) + a_I H_1 + a_{II}H_2 - H_{1,\delta}K^\delta - H_{2,\delta}L^\delta - H_1(K^\delta_{;\delta} + K^\nu_{;\delta}S^\delta S_\nu) - H_2(L^\delta_{;\delta} + S^\delta S_\nu L^\nu_{;\delta}) = \left\{8\pi\left[\mu + P - \frac{1}{3}(\Pi_I + \Pi_{II})\right] - Y_I - Y_{II}\right\}\Omega - \frac{4\pi A(q_I B)_{,\theta}}{B\sqrt{A^2B^2r^2 + G^2}} + \frac{4\pi A}{B\sqrt{A^2B^2r^2 + G^2}}\left[\frac{q_{II}\sqrt{(A^2B^2r^2 + G^2)}}{A}\right]_{,r}. \tag{A46}$$

[1] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. R. Soc. A **269**, 21 (1962).
[2] L. Herrera, A. Di Prisco, J. Ibáñez, and J. Ospino, Phys. Rev. D **89**, 084034 (2014).
[3] G. F. R. Ellis, Relativistic cosmology, in *Proceedings of the International School of Physics "Enrico Fermi," Course 47: General Relativity and Cosmology*, edited by R. K. Sachs (Academic Press, New York and London, 1971).
[4] G. F. R. Ellis and H. van Ellst, NATO Sci. Ser. C **541**, 1 (1999).
[5] G. F. R. Ellis, Gen. Relativ. Gravit. **41**, 581 (2009).
[6] G. F. R. Ellis, R. Maartens, and M. A. H. MacCallum, *Relativistic Cosmology* (Cambridge University Press, Cambridge, England, 2012).
[7] R. Tolman, Phys. Rev. **35**, 904 (1930).
[8] L. Bel, C. R. Hebd. Seances Acad. Sci. **247**, 1094 (1958).
[9] L. Bel, Cah. Phys. **16**, 59 (1962); Gen. Relativ. Gravit. **32**, 2047 (2000).
[10] A. García–Parrado Gómez Lobo, Classical Quantum Gravity **25**, 015006 (2008).
[11] L. Herrera, W. Barreto, J. Carot, and A. Di Prisco, Classical Quantum Gravity **24**, 2645 (2007).
[12] R. Sachs, Proc. R. Soc. A **270**, 103 (1962).
[13] I. Müller, Z. Phys. **198**, 329 (1967).
[14] W. Israel, Ann. Phys. (N.Y.) **100**, 310 (1976).
[15] W. Israel and J. Stewart, Phys. Lett. **58A**, 213 (1976).
[16] W. Israel and J. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).
[17] L. Crocco and Z. Agnew, J. Appl. Math. Mech. **17**, 1 (1937).

[18] A. J. Christopherson, K. A. Malik, and D. R. Matravers, Phys. Rev. D **79**, 123523 (2009).
[19] F. Del Sordo and A. Brandenburg, Astron. Astrophys. **528**, A145 (2011).
[20] A. J. Christopherson and K. A. Malik, Classical Quantum Gravity **28**, 114004 (2011).
[21] F. Dosopoulou, F. Del Sordo, C. G. Tsagas, and A. Brandenburg, Phys. Rev. D **85**, 063154 (2012).
[22] L. Herrera, Universe **5**, 164 (2019).
[23] L. Herrera, A. Di Prisco, and J. Ospino, Phys. Rev. D **89**, 127502 (2014).
[24] A. Barnes, Gen. Relativ. Gravit. **4**, 105 (1973).
[25] A. Barnes, in *Classical General Relativity*, edited by W. B. Bonnor, J. N. Islam, and M. A. H. MacCallum (Cambridge University Press, Cambridge, England, 1984).
[26] E. N. Glass, J. Math. Phys. (N.Y.) **16**, 2361 (1975).
[27] A. A. Coley and D. J. McManus, Classical Quantum Gravity **11**, 1261 (1994).
[28] D. J. McManus and A. A. Coley, Classical Quantum Gravity **11**, 2045 (1994).
[29] L. Herrera, Phys. Rev. D **97**, 044010 (2018).
[30] L. Herrera, A. Di Prisco, and J. Ospino, Phys. Rev. D **99**, 044049 (2019).
[31] L. Herrera, A. Di Prisco, and J. Ibáñez, Phys. Rev. D **84**, 064036 (2011).
[32] H. Stephani, D. Kramer, M. MacCallum, C. Honselaers, and E. Herlt, *Exact Solutions to Einstein's Field Equations*, 2nd ed. (Cambridge University Press, Cambridge, England, 2003).
[33] J. Carot, Classical Quantum Gravity **17**, 2675 (2000).
[34] G. T. Carlson, Jr. and J. L. Safko, Ann. Phys. (N.Y.) **128**, 131 (1980).