From the Weyl-Schrödinger connection to the accelerating Universe: Extending Einstein's gravity via a length preserving nonmetricity

Lei Ming,[1,*] Shi-Dong Liang,[1,†] Hong-Hao Zhang,[1,‡] and Tiberiu Harko[2,3,4,§]

[1]*School of Physics, Sun Yat-Sen University, Guangzhou 510275, People's Republic of China*
[2]*Department of Theoretical Physics, National Institute of Physics and Nuclear Engineering (IFIN-HH), Bucharest, 077125 Romania*
[3]*Department of Physics, Babes-Bolyai University, Kogalniceanu Street, Cluj-Napoca, 400084, Romania*
[4]*Astronomical Observatory, 19 Ciresilor Street, Cluj-Napoca 400487, Romania*

(Received 10 October 2023; accepted 3 December 2023; published 2 January 2024)

One of the important extensions of Riemann geometry is Weyl geometry, which is essentially based on the ideas of conformal invariance and nonmetricity. A similar non-Riemannian geometry was proposed by Erwin Schrödinger in the late 1940s, in a geometry which is simpler, and (probably) more elegant than the Weyl geometry. Even if it contains nonmetricity, the Schrödinger connection preserves the length of vectors under parallel transport, and thus seems to be more physical than the Weyl connection. Interestingly enough, Schrödinger's approach did not attract much interest in the field of gravitational physics. It is the goal of the present paper to reconsider the Schrödinger geometry as a potential candidate for a gravitational theory extending standard general relativity. We consider a gravitational action constructed from a length preserving nonmetricity, in the absence of torsion, and investigate its variation in both Palatini and metric formalisms. While the Palatini variation leads to standard general relativity, the metric version of the theory adds some nonmetricity dependent extra terms in the gravitational Einstein equations, which can be interpreted as representing a geometric type dark energy. After obtaining the generalized Friedmann equations, we analyze in detail the cosmological implications of the theory, by considering two distinct models, corresponding to a dark energy satisfying a linear equation of state, and to conserved matter energy, respectively. In both cases we compare the predictions of the Weyl-Schrödinger cosmology with a set of observational data for the Hubble function, and with the results of the ΛCDM standard paradigm. Our results show that the Weyl-Schrödinger cosmological models give a good description of the observational data, and, for certain values of the model parameters, they can reproduce almost exactly the predictions of the ΛCDM model. Hence, the Weyl-Schrödinger theory represents a simple, and viable alternative to standard general relativity, in which dark energy is of purely geometric origin.

DOI: 10.1103/PhysRevD.109.024003

I. INTRODUCTION

The creation of the theory of general relativity, as realized in the essential contributions by Einstein and Hilbert [1–3] had an overwhelming impact not only on the various branches of the gravitational physics, including cosmology, but also on mathematics. In their theoretical approaches, Einstein and Hilbert extensively applied the Riemannian geometry [4], in which on a manifold one can introduce an additional structure, the metric, determined by a metric tensor $g_{\mu\nu}$, and a symmetric connection $\Gamma^{\alpha}_{\mu\nu}$, respectively. The metric tensor allows us to define distances and angles, while with the help of the connection one can define the covariant derivative ∇_λ of a vector V_μ as $\nabla_\lambda V_\mu = \partial_\lambda V_\mu - \Gamma^{\sigma}_{\lambda\mu} V_\sigma$. The geometric properties of the space time manifold are characterized by the curvature tensor $R^{\mu}_{\nu\sigma\lambda}$, constructed from the connection, and its contractions $R_{\nu\lambda} = R^{\sigma}_{\nu\sigma\lambda}$, and $R = R^{\nu}_{\nu}$, from which the Einstein tensor $G_{\mu\nu}$ is obtained. The gravitational field equations can be also derived from the Einstein-Hilbert variational principle, with the help of the action $S = \int R\sqrt{-g}d^4x$. The Einstein-Hilbert action can be generalized to the $f(R)$ modified gravity theory action, given by $S = \int f(R)\sqrt{-g}d^4x$, where f is an arbitrary analytical function of the Ricci scalar [5,6] For a detailed review of $f(R)$ gravity see [7]. Extensions of $f(R)$ gravity theories by including matter-geometry couplings were considered in Refs. [8,9,10].

In 1918, a few years after the birth of general relativity, Weyl [11,12] did propose a generalization of Riemannian geometry, which was inspired by the idea of developing the

*minglei@mail.sysu.edu.cn
†stslsd@mail.sysu.edu.cn
‡zhh98@mail.sysu.edu.cn
§tiberiu.harko@aira.astro.ro

first unified theory of gravity and electromagnetism. In generalizing Riemann geometry, Weyl abandoned the metric condition $\nabla_\lambda g_{\mu\nu} = 0$, by generalizing it to $\nabla_\lambda g_{\mu\nu} = Q_{\lambda\mu\nu}$, where $Q_{\lambda\mu\nu}$ is the nonmetricity of the spacetime. In the initial formulation by Weyl, the nonmetricity has the form $Q_{\lambda\mu\nu} = \omega_\lambda g_{\mu\nu}$, where ω_λ is the Weyl vector. Weyl suggested that the nonmetricity of the spacetime is the source of the electromagnetic field. Weyl's unified theory was severely criticized by Einstein, leading essentially to its abandonment for more than a half century. Einstein's criticism can be summarized as follows. Under a rescaling of the metric tensor $g_{\mu\nu} \to (1+\epsilon\omega) g_{\mu\nu}$, the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is rescaled according to $ds \to \exp^{(\omega/2)} ds$. Einstein asserted that since ds represents the ticking of a clock, or the spacings of atomic spectral lines, if it is not absolutely invariant, the basic physical quantities (Compton wavelength, electron mass, etc.) would vary in space and time, an effect which is not observed experimentally. This pathological behavior is called the second clock effect. For a recent discussion of it see [13] For early discussions of the Weyl geometry and of its applications see [14,15], respectively.

Soon after the publication of Weyl's work, another fundamental advance occurred in differential geometry, namely, the definition of the concept of torsion [16]. Theories based on torsion represent another interesting generalization of Einstein's general relativity [17–19], presently called the Einstein-Cartan theory [20]. In the Einstein-Cartan theory, the torsion field $T^\mu_{\sigma\lambda} \neq 0$ is assumed to be proportional to the spin density of the matter [20].

It is also worth mentioning, for the sake of completeness, a third mathematical and physical enlargement of the gravitational field theories. This extension was initiated by the work of Weitzenböck [21], who introduced a class of new geometrical structures, known as the Weitzenböck spaces. A Weitzenböck space is characterized by the basic mathematical properties $\nabla_\mu g_{\sigma\lambda} = 0$, $T^\mu_{\sigma\lambda} \neq 0$, and $R^\mu_{\nu\sigma\lambda} = 0$, respectively, and when $T^\mu_{\sigma\lambda} = 0$, reduces to a Euclidean manifold. Moreover, in a Weitzenböck manifold $T^\mu_{\sigma\lambda}$ has values that depend on the regions of the manifold. Due to the fact that the Riemann curvature tensor identically vanishes, the Weitzenböck geometries have the property of distant parallelism, known also as absolute parallelism teleparallelism. Einstein was the first to apply teleparallelism in physics by proposing a unified teleparallel theory of electromagnetism and gravitation [22]. Weitzenböck geometries are extensively used in teleparallel equivalent of general relativity (TEGR) type theories, proposed initially in [23–25], also known as the $f(\mathbb{T})$ gravity theory, where $\mathbb{T}$ is the torsion scalar. These theories can explain the late-time acceleration of the Universe, without introducing the dark energy, or the cosmological constant [26–29]. For a review of teleparallel gravity see [30].

With a few notable exceptions, in the physics community Weyl's geometry was almost totally ignored in the first 50 years of its existence. But this situation began to change especially after 1970, when the interest for the physical and mathematical applications of Weyl geometry at both macroscopic and microscopic levels significantly increased. For a detailed description of the fascinating history of Weyl geometry, and of its applications in physics see [31].

An important development related to Weyl geometry can be related to the investigations by Dirac [32,33]. In proposing an extension of Weyl's theory, and geometry, Dirac introduced the Lagrangian

$$L = -\beta^2 R + kD^\mu\beta D_\mu\beta + c\beta^4 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

which contains a real scalar field β of weight $w(\beta) = -1$, and the electromagnetic field tensor $F_{\mu\nu}$ coming from the Weyl curvature. Moreover, Dirac adopted for the constant k the value $k = 6$. The Lagrangian (1) has the important property of conformal invariance. In [34] the cosmological implications of a Dirac type model were investigated. The Weyl-Dirac type Lagrangian

$$\begin{aligned} L = {} & W^{\lambda\rho}W_{\lambda\rho} - \beta^2 R + \sigma\beta^2 w^\lambda w_\lambda + 2\sigma\beta w^\lambda \beta_{,\lambda} \\ & + (\sigma+6)\beta_{,\rho}\beta_{,\lambda}g^{\rho\lambda} + 2\Lambda\beta^4 + L_m, \end{aligned} \tag{2}$$

was considered in [35], where β is the Dirac scalar field, while σ and Λ are constants. $W_{\mu\nu}$ is the Weyl length curvature tensor, obtained from the Weyl vector w_μ. In the cosmological applications of this model it was shown that ordinary matter is created by the Dirac's gauge function in the very early Universe. On the other hand, at late times, Dirac's gauge function generates to dark energy that accelerates the present day Universe.

Weyl's geometry can be naturally generalized to include torsion, thus leading to the Weyl-Cartan geometry, which was intensively studied from both mathematical and physical points of view [36–41]. For the physical applications of the Riemann-Cartan and Weyl-Cartan spacetimes see the review [42]. A class of teleparallel gravity models, called Weyl-Cartan-Weitzenböck gravity, was proposed in [43], with the action formulated with the help of the dynamical variables $(g_{\mu\nu}, w_\mu, T^\lambda{}_{\mu\nu})$. The teleparallel gravity and the Weyl-Cartan-Weitzenböck theory was generalized in [44], by inserting the Weitzenböck condition into the Weyl-Cartan gravitational action via a Lagrange multiplier. The cosmological analysis of the theory shows that both accelerating and decelerating cosmological models can be obtained.

The theoretical investigations performed by using Riemannian, Cartan and teleparallel geometries indicate that general relativity, or more generally, geometric theories of gravity, can be formulated in (at least) two formalisms, which are mathematically equivalent: the curvature representation (with the nonmetricity and torsion vanishing identically), and the teleparallel representation, in which the nonmetricity and the curvature vanish identically.

A third, mathematically equivalent geometric representation of general relativity has also been formulated. The properties of the gravitational field can be described geometrically by the nonmetricity Q of the metric. From a geometric point of view the nonmetricity describes the change of the length of a vector when parallelly transported around a closed loop. The gravitational theory describing gravity via nonmetricity is called the symmetric teleparallel theory, and it was initially introduced in [45]. The connection describing the geometry can be decomposed generally into the Levi-Civita connection of the Riemannian geometry, and a deformation one form, $A^\alpha{}_\beta$, so that $\Gamma^\alpha{}_\beta = \Gamma^{\{\}\alpha}{}_\beta - A^\alpha{}_\beta$. The deformation one form is generally given by $A_{\alpha\beta} = K_{\alpha\beta} - Q_{\alpha\beta}/2 - Q_{\gamma[\alpha\beta]}\theta^\gamma$, where $K_{\alpha\beta}$ is the contorsion, while $Q_{\alpha\beta}$ denotes the nonmetricity, which is generally defined according to $Q_{\alpha\beta} = -Dg_{\alpha\beta}$.

In a teleparallel frame, in which the condition $\Gamma \equiv 0$ is satisfied, and after also requiring the condition of the vanishing of the torsion, it follows that $Q_{\mu\nu\lambda} = -g_{\mu\nu,\lambda}$. Hence, the deformation tensor becomes the Christoffel symbol $\gamma^\alpha_{\beta\gamma}$, so that $A^\alpha{}_{\beta\gamma} = \gamma^\alpha_{\beta\gamma}$. Then the gravitational action can be represented as $L_g = \sqrt{-g}g^{\mu\nu}(\gamma^\alpha_{\beta\mu}\gamma^\beta_{\nu\alpha} - \gamma^\alpha_{\beta\alpha}\gamma^\beta_{\mu\nu})$, which is exactly the Einstein-Hilbert Lagrangian. In symmetric teleparallel gravity the associated energy-momentum density is the Einstein pseudotensor, which now becomes a true tensor.

The symmetric teleparallel gravity approach was generalized to the $f(Q)$ gravity theory (or the coincident general relativity) in [46]. As a first step in constructing the theory one introduces the quadratic nonmetricity scalar

$$Q = -\frac{1}{4}Q_{\alpha\beta\mu}Q^{\alpha\beta\mu} + \frac{1}{2}Q_{\alpha\beta\mu}Q^{\beta\mu\alpha} + \frac{1}{4}Q_\alpha Q^\alpha - \frac{1}{2}Q_\alpha\bar{Q}^\alpha, \quad (3)$$

where $Q_\mu = Q_\mu{}^\alpha{}_\alpha$, and $\tilde{Q}^\mu = Q_\alpha{}^{\mu\alpha}$. Then, the nonmetricity conjugate $P^\alpha{}_{\mu\nu}$ is defined as

$$P^\alpha{}_{\mu\nu} = c_1 Q^{\alpha\mu\nu} + c_2 Q_{(\mu}{}^\alpha{}_{\nu)} + c_3 Q^\alpha g_{\mu\nu} + c_4 \delta^\alpha_{(\mu}\bar{Q}_{\nu)} + \frac{c_5}{2}(\tilde{Q}^\alpha g_{\mu\nu} + \delta^\alpha_{(\mu}Q_{\nu)}). \quad (4)$$

Finally, after introducing the general quadratic form Q as $\mathrm{Q} = Q^{\mu\nu}_\alpha P^\alpha_{\mu\nu}$, one can write down the gravitational action of the $f(Q)$ theory as [46]

$$S = \int d^n x\left[-\frac{1}{2}\sqrt{-g}\mathrm{Q} + \lambda_\alpha{}^{\beta\mu\nu}R^\alpha{}_{\beta\mu\nu} + \lambda_\alpha{}^{\mu\nu}T^\alpha{}_{\mu\nu}\right]. \quad (5)$$

The physical, cosmological and geometrical properties of the $f(Q)$ gravity have been extensively investigated recently [47–53]. For a review of the $f(Q)$ theory see [54]. An extension of the $f(Q)$ theory was considered in [55] by assuming that the nonmetricity Q could nonminimally couple to the matter Lagrangian. The action of the theory is thus given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}f_1(Q) + f_2(Q)L_m\right], \quad (6)$$

where L_m is the matter Lagrangian, and f_1 and f_2 are arbitrary analytical functions of Q. The existence of a nonminimal coupling between geometry and matter leads to the nonconservation of the matter energy-momentum tensor, and to the presence of an extra force in the geodesic equation of motion of massive particles. The cosmological solutions obtained in the framework of this model can describe the accelerating evolution of the Universe.

The most general extension of the $f(Q)$ gravity, with the gravitational Lagrangian L constructed from an arbitrary function f of the nonmetricity Q and of the trace T of the matter-energy-momentum tensor, was introduced in [56]. The action of the theory is

$$S = \int\left[\frac{1}{16\pi}f(Q,T) + L_m\right]\sqrt{-g}d^4x, \quad (7)$$

Within the framework of $f(Q,T)$ gravity one can construct cosmological models by assuming some simple functional forms of the function $f(Q,T)$. In these models the Universe enters in an accelerating phase, which usually ends with a de Sitter type expansion.

In the 1940s Erwin Schrödinger, who was mostly interested in metric-affine theories, tried to find the most general possible symmetric connection [57,58]. From general considerations he arrived at the result that such a connection is given by $\Gamma^\lambda_{\mu\nu} = \gamma^\lambda_{\mu\nu} + g^{\lambda\rho}S_{\rho\mu\nu}$, where $S_{\rho\mu\nu}$ is a geometric quantity consisting of the combination of two antisymmetric connections. The geometry based on Schrödinger's connection represents essentially a new geometry, which is distinct from that of Weyl. A systematic investigation of the Schrödinger connection was performed in [59], where an action of the form

$$S = \frac{1}{2\kappa^2}\int d^3x\left(\sqrt{-g}f(R) + \frac{1}{2\mu}\epsilon^{\mu\nu\rho}Q_\rho\hat{R}_{\nu\mu}\right) + \int d^3x\epsilon^{\mu\nu\rho}\zeta_{\nu\sigma}T_{\rho\mu}{}^\sigma, \quad (8)$$

was considered, where $f(R)$ denotes an arbitrary function of the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}(\Gamma)$, $\hat{R}_{\mu\nu} \coloneqq R^\lambda{}_{\lambda\mu\nu} = \partial_{[\mu}Q_{\nu]}$ is the homothetic curvature tensor, μ is a Chern-Simons coupling constant, and $\zeta_{\nu\sigma}$ a Lagrange multiplier, respectively. Moreover, $\epsilon^{\mu\nu\rho} = \sqrt{-g}\varepsilon^{\mu\nu\rho}$ the Levi-Civita symbol, $\varepsilon^{\mu\nu\rho}$ is the Levi-Civita tensor, while Γ denotes an arbitrary connection. Solutions with constant scalar curvature were obtained in the framework of this model, leading to a self-duality relation for the nonmetricity vector. This relation gives a Proca type equation, which may be an indication of the inhomogeneous Maxwell equations as originating from affine geometry.

It is the main goal of the present investigation to consider the possibility of the Schrödinger geometry as being an important and viable candidate for the geometric extension of standard general relativity. To implement this idea we begin by considering a gravitational action, which is formulated in terms of a length preserving nonmetricity, in the absence of torsion. The variation of this action is considered in both Palatini and metric formalisms. It turns out that the Palatini variation leads to standard general relativity, and hence the two theories coincide in this formulation. However, the metric variation of the Schrödinger action leads to the presence of nonmetricity dependent extra terms in the gravitational Einstein equations. We interpret these terms as representing a geometric type dark energy.

In order to investigate the physical implications, and the viability of the Weyl-Schrödinger theory we consider the field equations in the FLRW cosmological metric. After deriving the generalized Friedmann equations, we analyze in detail the cosmological implications of the Weyl-Schrödinger theory. In the generalized Friedmann equations the presence of nonmetricity generates two new terms, which can be interpreted as an effective geometric energy density of the dark matter, and an effective pressure. To test viability of the theory we consider two distinct cosmological models. In the first model, we assume that dark energy satisfies a linear equation of state, that is, the effective geometric pressure is proportional to the dark matter energy density, with the parameter of the equation of state assumed to be a redshift dependent function. In the second model we assume that the matter energy density is conserved. For both models we perform a comparison of the predictions of the Weyl-Schrödinger cosmology with a set of observational data for the Hubble function, and with the similar results obtained within the framework of the ΛCDM standard cosmological paradigm. Our results show that the Weyl-Schrödinger cosmological models can give a good description of the observational data for the Hubble function. Moreover, for specific values of the model parameters, they can reproduce almost exactly the predictions of the ΛCDM model. Therefore, the Weyl-Schrödinger theory in its nonmetricity representation could provide a simple, and viable alternative to standard general relativity, in which dark energy is of purely geometric origin.

The present paper is organized as follows. We introduce the fundamentals of the Weyl and Schrödinger geometries in Sec. II. The action of the Weyl-Schrödinger theory is introduced in Sec. III, where the gravitational field equations are derived in both Palatini and metric formalisms. The cosmological implications of the theory are investigated in Sec. IV, where the predictions of two distinct cosmological models are compared with the observational data, and the similar predictions of the ΛCDM model. We discuss our results, and we conclude our work in Sec. V. The calculational details of the variation of the action in the Palatini formalism are presented in Appendix A 1. The technical details of the calculation of the variation of the action with respect to the metric tensor are given in Appendix A 2. Finally, the derivation of the generalized Friedmann equations for the FLRW metric is presented in Appendix A 3.

II. FROM WEYL GEOMETRY TO THE SCHRÖDINGER CONNECTION

In his book [58], Schrödinger wished to find the most general class of an affine connection to be in accordance with the affine measure of distance along every geodesic, i.e., a relationship between $g_{\mu\nu}$ and $\Gamma^{\lambda}{}_{\mu\nu}$. While allowing the existence of a nonzero nonmetricity, a Schrödinger connection is supposed to preserve the length of vectors under parallel transport, which in general does not hold in Weyl geometry. To find such a form of connection, one can start with a sufficient condition with vanishing nonmetricity that $Q_{\alpha\mu\nu} = -\nabla_{\alpha} g_{\mu\nu} = 0$, then the circling of it gives

$$0 = \nabla_{\rho} g_{\mu\nu} = \partial_{\rho} g_{\mu\nu} - g_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\rho} - g_{\nu\alpha}\Gamma^{\alpha}{}_{\mu\rho}, \tag{9}$$

$$0 = \nabla_{\mu} g_{\nu\rho} = \partial_{\mu} g_{\nu\rho} - g_{\nu\alpha}\Gamma^{\alpha}{}_{\rho\mu} - g_{\rho\alpha}\Gamma^{\alpha}{}_{\nu\mu}, \tag{10}$$

$$0 = \nabla_{\nu} g_{\rho\mu} = \partial_{\nu} g_{\rho\mu} - g_{\rho\alpha}\Gamma^{\alpha}{}_{\mu\nu} - g_{\mu\alpha}\Gamma^{\alpha}{}_{\rho\nu}. \tag{11}$$

Adding the later two equations and minus the first, and contracting with $\frac{1}{2}g^{\rho\lambda}$ yields

$$\begin{aligned} 0 &= \frac{1}{2}g^{\rho\lambda}(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\rho\mu} - \partial_{\rho}g_{\mu\nu}) - \frac{1}{2}g^{\rho\lambda}g_{\rho\alpha}(\Gamma^{\alpha}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\nu\mu}) \\ &\quad + \frac{1}{2}g^{\rho\lambda}g_{\nu\alpha}(\Gamma^{\alpha}{}_{\mu\rho} - \Gamma^{\alpha}{}_{\rho\mu}) + \frac{1}{2}g^{\rho\lambda}g_{\mu\alpha}(\Gamma^{\alpha}{}_{\nu\rho} - \Gamma^{\alpha}{}_{\rho\nu}) \\ &= \gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}{}_{(\mu\nu)} + g^{\rho\lambda}(g_{\nu\alpha}\Gamma^{\alpha}{}_{[\mu\rho]} + g_{\mu\alpha}\Gamma^{\alpha}{}_{[\nu\rho]}), \end{aligned} \tag{12}$$

or

$$\Gamma^{\lambda}{}_{\mu\nu} = \gamma^{\lambda}_{\mu\nu} + g^{\rho\lambda}(g_{\nu\alpha}\Gamma^{\alpha}{}_{[\mu\rho]} + g_{\mu\alpha}\Gamma^{\alpha}{}_{[\nu\rho]}) + \Gamma^{\lambda}{}_{[\mu\nu]}, \tag{13}$$

where $\gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\rho\mu} - \partial_{\rho}g_{\mu\nu})$ is the Christoffel symbol, $\Gamma^{\lambda}{}_{(\mu\nu)} \equiv \frac{1}{2}(\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}{}_{\nu\mu})$ and $\Gamma^{\lambda}{}_{[\mu\nu]} \equiv \frac{1}{2}(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu})$ are the symmetric and antisymmetric parts of $\Gamma^{\lambda}{}_{\mu\nu}$, respectively. Considering that in the equations of geodesic, antisymmetry cancels in the overall connection, we can drop the last term in (13) and are led to

$$\Gamma^{\lambda}{}_{\mu\nu} = \gamma^{\lambda}_{\mu\nu} + g^{\rho\lambda}(g_{\nu\alpha}\Gamma^{\alpha}{}_{[\mu\rho]} + g_{\mu\alpha}\Gamma^{\alpha}{}_{[\nu\rho]}). \tag{14}$$

From this relation Schrödinger concluded that the connections $\Gamma^{\lambda}{}_{\mu\nu}$, which are compatible with the metric $g_{\mu\nu}$ in a general meaning, while the condition $Q_{\rho\mu\nu} = 0$ is not necessary satisfied, should have the form

$$\Gamma^\lambda{}_{\mu\nu} = \gamma^\lambda_{\mu\nu} + g^{\rho\lambda} S_{\rho\mu\nu}, \tag{15}$$

where the Schrödinger tensor $S_{\rho\mu\nu}$ is symmetric in its later indices

$$S_{\rho\mu\nu} = S_{\rho\nu\mu}. \tag{16}$$

He also concluded that with this form of connections, the necessary condition that a vector preserves its length under parallel transport is

$$S_{(\rho\mu\nu)} = 0. \tag{17}$$

To see this, notice that for a tangent vector of a geodesic, $\xi^\mu \propto \frac{dx^\mu}{d\lambda}$, we have $\xi^\rho \nabla_\rho \xi^\mu = 0$, then its length being constant gives

$$\begin{aligned}
0 &= \xi^\rho \nabla_\rho (g_{\mu\nu}\xi^\mu\xi^\nu) \\
&= \nabla_\rho g_{\mu\nu}\xi^\rho\xi^\mu\xi^\nu + g_{\mu\nu}\xi^\nu\xi^\rho\nabla_\rho\xi^\mu + g_{\mu\nu}\xi^\mu\xi^\rho\nabla_\rho\xi^\nu \\
&= (\partial_\rho g_{\mu\nu} - g_{\mu\alpha}\Gamma^\alpha{}_{\nu\rho} - g_{\nu\alpha}\Gamma^\alpha{}_{\mu\rho})\xi^\rho\xi^\mu\xi^\nu \\
&= (\partial_\rho g_{\mu\nu} - g_{\mu\alpha}\gamma^\alpha_{\nu\rho} - g_{\nu\alpha}\gamma^\alpha_{\mu\rho} - g_{\mu\alpha}g^{\lambda\alpha}S_{\lambda\nu\rho} - g_{\nu\alpha}g^{\lambda\alpha}S_{\lambda\mu\rho})\xi^\rho\xi^\mu\xi^\nu \\
&= -(S_{\mu\nu\rho} + S_{\nu\mu\rho})\xi^\rho\xi^\mu\xi^\nu \\
&= -2S_{\mu\nu\rho}\xi^\rho\xi^\mu\xi^\nu,
\end{aligned} \tag{18}$$

and thus we arrive at the result $S_{(\mu\nu\rho)} = 0$.

To summarize, a Schrödinger connection, which preserves the length of vectors under parallel transport, although involving nonzero nonmetricity, has the form (15), while fulfilling the conditions (16) and (17), respectively.

As is well known, the generic decomposition of an affine connection is given by

$$\begin{aligned}
\Gamma^\lambda{}_{\mu\nu} &= \gamma^\lambda{}_{\mu\nu} + N^\lambda{}_{\mu\nu} \\
&= \gamma^\lambda{}_{\mu\nu} + L^\lambda{}_{\mu\nu} + C^\lambda{}_{\mu\nu} \\
&= \gamma^\lambda{}_{\mu\nu} + \frac{1}{2} g^{\rho\lambda}(Q_{\mu\nu\rho} + Q_{\nu\rho\mu} - Q_{\rho\mu\nu}) \\
&\quad + g^{\rho\lambda}(T_{\rho\mu\nu} + T_{\mu\nu\rho} - T_{\nu\rho\mu}),
\end{aligned} \tag{19}$$

where $T^\lambda{}_{\mu\nu} \coloneqq \Gamma^\lambda{}_{[\mu\nu]}$ is the torsion tensor, $N^\lambda{}_{\mu\nu}$, $L^\lambda{}_{\mu\nu}$ and $C^\lambda{}_{\mu\nu}$ are the distortion, deflection and contorsion tensor, respectively. In the case of symmetric connection (i.e., vanishing torsion) and $N_{(\rho\mu\nu)} = 0$, (19) reduces to

$$\begin{aligned}
\Gamma^\lambda{}_{\mu\nu} &= \gamma^\lambda_{\mu\nu} + \frac{1}{2} g^{\rho\lambda}(Q_{\mu\nu\rho} + Q_{\nu\rho\mu} - Q_{\rho\mu\nu}) \\
&= \gamma^\lambda{}_{\mu\nu} + \frac{1}{2} g^{\rho\lambda}(-Q_{\rho\mu\nu} - Q_{\rho\mu\nu}) \\
&= \gamma^\lambda_{\mu\nu} - g^{\rho\lambda} Q_{\rho\mu\nu}
\end{aligned} \tag{20}$$

with $Q_{(\rho\mu\nu)} = 0$. This corresponds to a Schrödinger connection for which the Schrödinger tensor $S_{\rho\mu\nu} = -Q_{\rho\mu\nu}$. It was also discussed that the Schrödinger connection can be written only in terms of torsion with vanishing nonmetricity [59]. We will focus on the torsion free case and consider a length preserving nonmetricity.

Having the affine connection (20) at hand, we can then define the Riemann curvature tensor $R^\mu{}_{\nu\alpha\beta}$, which describes how parallel transport modifies the orientation of a vector, by acting the commutator of two covariant derivatives on a vector v^μ,

$$[\nabla_\alpha, \nabla_\beta] v^\mu = 2\nabla_{[\alpha}\nabla_{\beta]} v^\mu = R^\mu{}_{\nu\alpha\beta} v^\nu, \tag{21}$$

where $R^\mu{}_{\nu\alpha\beta}$ is related to the Schrödinger connection $\Gamma^\lambda{}_{\mu\nu}$ via

$$R^\mu{}_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^\mu{}_{\nu\beta} - \partial_\beta \Gamma^\mu{}_{\nu\alpha} + \Gamma^\mu{}_{\rho\alpha}\Gamma^\rho{}_{\nu\beta} - \Gamma^\mu{}_{\rho\beta}\Gamma^\rho{}_{\nu\alpha}. \tag{22}$$

Without the help of a metric, there exist two possible independent contractions of $R^\mu{}_{\nu\alpha\beta}$, namely the Ricci tensor $R_{\mu\nu}$

$$R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha} + \Gamma^\alpha{}_{\rho\alpha}\Gamma^\rho{}_{\mu\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\mu\alpha}, \tag{23}$$

and the homothetic curvature tensor $\hat{R}_{\mu\nu}$

$$\hat{R}_{\mu\nu} \equiv R^\alpha{}_{\alpha\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\alpha\nu} - \partial_\nu \Gamma^\alpha{}_{\alpha\mu}. \tag{24}$$

It is known that when nonmetricity is present ($Q_{\rho\mu\nu} \neq 0$), the homothetic curvature $\hat{R}_{\mu\nu}$ is nonvanishing and can be expressed as $\hat{R}_{\mu\nu} = \partial_{[\mu} Q_{\nu]}$. However, in the next section we will see that only the symmetric part of Ricci tensor appears in the equations of motion, since $\hat{R}_{\mu\nu}$ is anti-symmetric in the indices μ and ν.

The Ricci tensor $R_{\mu\nu}$ can be decomposited into two parts, the pure Riemannian part $\mathring{R}_{\mu\nu}$ computed for the Levi-Civita connection $\gamma^\lambda_{\mu\nu}$ and the part containing the contribution of nonmetricity,

$$\begin{aligned}
R_{\mu\nu} &\equiv \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha} + \Gamma^\alpha{}_{\rho\alpha}\Gamma^\rho{}_{\mu\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\mu\alpha} \\
&= \partial_\alpha \gamma^\alpha_{\mu\nu} - \partial_\alpha Q^\alpha{}_{\mu\nu} - \partial_\nu \gamma^\alpha_{\mu\alpha} + \partial_\nu \tilde{Q}_\mu \\
&\quad + \gamma^\alpha_{\rho\alpha}\gamma^\rho_{\mu\nu} - \tilde{Q}_\rho \gamma^\rho_{\mu\nu} - \gamma^\alpha_{\rho\alpha} Q^\rho{}_{\mu\nu} + \tilde{Q}_\rho Q^\rho{}_{\mu\nu} \\
&\quad - \gamma^\alpha_{\rho\nu}\gamma^\rho_{\mu\alpha} + Q^\alpha{}_{\rho\nu}\gamma^\rho_{\mu\alpha} + \gamma^\alpha_{\rho\nu} Q^\rho{}_{\mu\alpha} - Q^\alpha{}_{\rho\nu} Q^\rho{}_{\mu\alpha} \\
&= \mathring{R}_{\mu\nu} - \mathring{\nabla}_\alpha Q^\alpha_{\mu\nu} + \mathring{\nabla}_\nu \tilde{Q}_\mu + \tilde{Q}_\rho Q^\rho{}_{\mu\nu} - Q^\alpha{}_{\rho\nu} Q^\rho{}_{\mu\alpha},
\end{aligned} \tag{25}$$

where $\mathring{\nabla}_\alpha$ is the covariant derivative corresponding to $\gamma^\rho_{\mu\nu}$, i.e., $\mathring{\nabla}_\alpha g_{\mu\nu} = 0$, and we use

$$\mathring{R}_{\mu\nu} \coloneqq \partial_\alpha \gamma^\alpha_{\mu\nu} - \partial_\nu \gamma^\alpha_{\mu\alpha} + \gamma^\alpha_{\rho\alpha}\gamma^\rho_{\mu\nu} - \gamma^\alpha_{\rho\nu}\gamma^\rho_{\mu\alpha}, \tag{26}$$

$$\mathring{\nabla}_\alpha Q^\alpha{}_{\mu\nu} = \partial_\alpha Q^\alpha{}_{\mu\nu} + \gamma^\alpha_{\rho\alpha} Q^\rho{}_{\mu\nu} - \gamma^\rho_{\mu\alpha} Q^\alpha{}_{\rho\nu} - \gamma^\rho_{\nu\alpha} Q^\alpha{}_{\mu\rho}, \tag{27}$$

$$\mathring{\nabla}_\nu \tilde{Q}_\mu = \partial_\nu \tilde{Q}_\mu - \gamma^\rho_{\mu\nu} \tilde{Q}_\rho. \tag{28}$$

Before going further into the Weyl-Schrödinger geometry, we shall discuss more about the conditions of the nonmetricity. With the two independent vectors of nonmetricity, Q_μ and $\tilde{Q}_\mu$, the nonmetricity tensor $Q_{\lambda\mu\nu}$ can be decomposed in n dimension as [60]

$$Q_{\lambda\mu\nu} = \frac{n+1}{(n+2)(n-1)} Q_\lambda g_{\mu\nu} - \frac{2}{(n+2)(n-1)} Q_{(\mu} g_{\nu)\lambda} - \frac{2}{(n+2)(n-1)} \tilde{Q}_\lambda g_{\mu\nu} + \frac{2n}{(n+2)(n-1)} \tilde{Q}_{(\mu} g_{\nu)\lambda} + \Omega_{\lambda\mu\nu}, \tag{29}$$

where $\Omega_{\lambda\mu\nu}$ is the traceless part of $Q_{\lambda\mu\nu}$: $g^{\mu\nu}\Omega_{\lambda\mu\nu} = g^{\lambda\mu}\Omega_{\lambda\mu\nu} = 0$.

If we assume $\Omega_{\lambda\mu\nu} = 0$ and $\tilde{Q}_\mu = mQ_\mu$ with some constant m, then

$$Q_{\lambda\mu\nu} = \frac{n+1-2m}{(n+2)(n-1)} Q_\lambda g_{\mu\nu} + \frac{2mn-2}{(n+2)(n-1)} Q_{(\mu} g_{\nu)\lambda}, \tag{30}$$

and thus

$$Q_{(\lambda\mu\nu)} = \frac{n+1-2m}{(n+2)(n-1)} Q_{(\lambda} g_{\mu\nu)} + \frac{2mn-2}{(n+2)(n-1)} Q_{(\mu} g_{\nu\lambda)} = \frac{2m+1}{n+2} Q_{(\lambda} g_{\mu\nu)}, \tag{31}$$

therefore the condition $Q_{(\lambda\mu\nu)} = 0$ leads to $m = -\frac{1}{2}$, i.e.,

$$\tilde{Q}_\mu = -\frac{1}{2} Q_\mu. \tag{32}$$

We are going to find a solution of the Schrödinger connection that satisfies the above condition, together with the constraint $\Omega_{\lambda\mu\nu} = 0$.

III. GRAVITATIONAL FIELD EQUATIONS IN THE WEYL-SCHRÖDINGER GEOMETRY

In this paper we first work in the Palatini formalism in which the affine connection and metric are considered to be two independent variables and the matter part of the action does not depend on the connection. The gravitational action we will study is

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(R + \frac{5}{24} Q_\rho Q^\rho + \frac{1}{6} \tilde{Q}_\rho \tilde{Q}^\rho + 2T_\rho Q^\rho + \zeta^{\rho\sigma}{}_\alpha T^\alpha{}_{\rho\sigma} \right) + \int d^4x \sqrt{-g} L_m, \tag{33}$$

here $R \coloneqq g^{\mu\nu} R_{\mu\nu}(\Gamma)$, $T_\rho \coloneqq T^\sigma{}_{\rho\sigma}$, and $\zeta^{\rho\sigma}{}_\alpha$ is a Lagrange multiplier.

The variation of (33) with respect to $\zeta^{\mu\nu}{}_\lambda$ leads to a vanishing torsion,

$$T^\lambda{}_{\mu\nu} = 0. \tag{34}$$

The variation with respect to the metric $g^{\mu\nu}$ gives the modified Einstein equation,

$$R_{(\mu\nu)} - \frac{1}{2} R g_{\mu\nu} + \frac{5}{24} \left(\frac{1}{2} g_{\mu\nu} Q_\alpha Q^\alpha + Q_\mu Q_\nu - 2 g_{\mu\nu} Q_\alpha \tilde{Q}^\alpha - 2 g_{\mu\nu} g^{\alpha\beta} \nabla_\beta Q_\alpha \right) + \frac{1}{6} \left(-\frac{1}{2} g_{\mu\nu} \tilde{Q}_\alpha \tilde{Q}^\alpha - \tilde{Q}_\mu \tilde{Q}_\nu + Q_\mu \tilde{Q}_\nu - 2\nabla_{(\mu} \tilde{Q}_{\nu)} \right) = 8\pi T_{\mu\nu}, \tag{35}$$

where we define the energy-momentum tensor as usual, according to the relation

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}}. \tag{36}$$

Varying (33) with respect to $\Gamma^\lambda{}_{\mu\nu}$ results in

$$-\frac{\nabla_\lambda(\sqrt{-g} g^{\mu\nu})}{\sqrt{-g}} + \frac{\nabla_\rho(\sqrt{-g} g^{\mu\rho})\delta^\nu_\lambda}{\sqrt{-g}} + \frac{1}{3} \tilde{Q}_\lambda g^{\mu\nu} + \frac{1}{3} \tilde{Q}^\mu \delta^\nu_\lambda + Q^\mu \delta^\nu_\lambda - \frac{1}{6} Q^\nu \delta^\mu_\lambda + \zeta^{[\mu\nu]}{}_\lambda = 0. \tag{37}$$

The detailed calculations for these equations of motion can be found in Appendix A 1.

Noticing that

$$\frac{\nabla_\lambda \sqrt{-g}}{\sqrt{-g}} = \frac{1}{2g} \nabla_\lambda g = \frac{1}{2g} g g^{\alpha\beta} \nabla_\lambda g_{\alpha\beta} = -\frac{1}{2} Q_\lambda, \tag{38}$$

and

$$\nabla_\lambda g^{\mu\nu} = g^{\alpha\mu} g_{\alpha\beta} \nabla_\lambda g^{\beta\nu} = -g^{\alpha\mu} g^{\beta\nu} \nabla_\lambda g_{\alpha\beta} = Q^{\mu\nu}_\lambda, \tag{39}$$

respectively, and using the decomposition of the nonmetricity (29) in 4 dimensions,

$$Q_{\lambda\mu\nu} = \frac{5Q_\alpha - 2\tilde{Q}_\alpha}{18} g_{\mu\nu} + \frac{4\tilde{Q}_{(\mu} g_{\nu)\alpha} - Q_{(\mu} g_{\nu)\alpha}}{9} + \Omega_{\lambda\mu\nu}, \tag{40}$$

one can simplify (37) to

$$0=\frac{4}{9}\left(\tilde{Q}_{\lambda}+\frac{1}{2}Q_{\lambda}\right)g^{\mu\nu}+\frac{10}{9}\left(\tilde{Q}^{\mu}+\frac{1}{2}Q^{\mu}\right)\delta^{\nu}_{\lambda}-\frac{2}{9}\left(\tilde{Q}^{\nu}+\frac{1}{2}Q^{\nu}\right)\delta^{\mu}_{\lambda}-\Omega_{\lambda}{}^{\mu\nu}+\zeta^{[\mu\nu]}{}_{\lambda}. \tag{41}$$

Contracting the above equation with $g_{\mu\nu}$ one gets

$$\frac{8}{3}\left(\tilde{Q}_{\lambda}+\frac{1}{2}Q_{\lambda}\right)=0. \tag{42}$$

Thus we have

$$\tilde{Q}_{\lambda}=-\frac{1}{2}Q_{\lambda}. \tag{43}$$

Using (43) and contracting (41) with any nonzero antisymmetric tensor $\Sigma_{\rho[\mu\nu]}$, we find

$$\zeta^{[\mu\nu]}{}_{\lambda}=0, \tag{44}$$

and

$$\Omega_{\lambda}{}^{\mu\nu}=0, \tag{45}$$

respectively. With the use of Eqs. (43) and (45), we obtain a Schrödinger connection (20), as discussed in the previous section.

Furthermore, after a straightforward calculation by inserting (25) and (A23) into (35), the modified Einstein equation in the Palatini formalism can be largely simplified to obtain

$$\mathring{R}_{\mu\nu}-\frac{1}{2}\mathring{R}g_{\mu\nu}=8\pi T_{\mu\nu}, \tag{46}$$

where $\mathring{R}_{\mu\nu}$ is the Ricci tensor constructed from the Levi-Civita connection $\gamma^{\lambda}_{\mu\nu}$ which is metric compatible, i.e., $\mathring{\nabla}_{\alpha}g_{\mu\nu}=0$. This means our model in its Palatini formalism is equivalent to the general relativity.

Now, we consider the metric formalism, in which the connection has to be assumed to depend on the metric in *a priori* way. If we adopt the Schrödinger connection (20), then the variation of (33) with respect to $g_{\mu\nu}$ gives the modified Einstein equation in the metric formalism as

$$\mathring{R}_{\mu\nu}-\frac{1}{2}\mathring{R}g_{\mu\nu}-\frac{2}{9}Q_{\rho}Q^{\rho}g_{\mu\nu}-\frac{11}{18}Q_{\mu}Q_{\nu}+\frac{2}{3}g_{\mu\nu}\nabla_{\rho}Q^{\rho}+\frac{1}{6}g_{\rho\mu}\nabla_{\nu}Q^{\rho}+\frac{1}{6}g_{\rho\nu}\nabla_{\mu}Q^{\rho}=8\pi T_{\mu\nu}. \tag{47}$$

IV. COSMOLOGICAL APPLICATIONS

In the present section we will consider the cosmological applications of the Weyl-Schrödinger gravity theory, as we have introduced it in Sec. III. As a first step in ours study, we will obtain the generalized Friedmann equations of the theory, by assuming a flat, isotropic and homogeneous Universe. We also point out the presence of extra terms, of geometric nature, in the generalized Friedmann equations, which can be interpreted as a dark energy, and which trigger the accelerated expansion of the Universe. Then, we will reformulate the basic equations in a dimensionless form, and in the redshift space. The existence of a de Sitter type solution will be investigated in detail. Two cosmological models, obtained by obtained various conditions on the dark energy terms, are obtained, and studied in detail. In each case a comparison with the standard ΛCDM model and a small set of observational data is also performed.

A. Generalized Friedmann equations in Weyl-Schrödinger gravity

We assume first that the Universe is described by the isotropic, homogeneous and spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, given by

$$\mathrm{d}s^2=-\mathrm{d}t^2+a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{48}$$

where $a(t)$ is the scale factor. We also assume that, due to spatial symmetry, the first Weyl vector can be taken to be of the form

$$Q_{\rho}=[\omega(t),0,0,0]. \tag{49}$$

Moreover, we consider that the matter content of the Universe can be described as a perfect fluid, characterized by only two thermodynamic parameters, the energy density ρ, and the thermodynamic pressure p. Hence, the ordinary matter energy-momentum tensor is given by

$$T_{\mu\nu}=\rho u_{\mu}u_{\nu}+p(u_{\mu}u_{\nu}+g_{\mu\nu}), \tag{50}$$

where u^{μ} is the normalized four-velocity of the fluid, satisfying the condition $u_{\mu}u^{\mu}=-1$. Then, the field equations (47) give the two generalized Friedmann equations of the Weyl-Schrödinger theory as (see Appendix A 3 for their derivation)

$$\frac{3\dot{a}^2}{a^2}+\frac{2\dot{a}}{a}\omega-\frac{1}{2}\omega^2+\dot{\omega}=8\pi\rho \tag{51}$$

and

$$-\frac{2\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}-\frac{7\dot{a}}{3a}\omega-\frac{1}{6}\omega^2-\frac{2}{3}\dot{\omega}=8\pi p. \tag{52}$$

By introducing the Hubble function H, defined as $H = \dot{a}/a$, we can reformulate the generalized Friedmann equations as

$$3H^2 = 8\pi(\rho + \rho_{\rm DE}) = 8\pi\rho_{\rm eff}, \tag{53}$$

and

$$2\dot{H} + 3H^2 = -8\pi(p + p_{\rm DE}) = -8\pi p_{\rm eff}, \tag{54}$$

where we have denoted

$$\rho_{\rm DE} = \frac{1}{8\pi}\left(-\dot{\omega} - 2H\omega + \frac{1}{2}\omega^2\right), \tag{55}$$

and

$$p_{\rm DE} = \frac{1}{8\pi}\left(\frac{2}{3}\dot{\omega} + \frac{1}{6}\omega^2 + \frac{7}{3}H\omega\right), \tag{56}$$

respectively. From the generalized Friedman equations we obtain the global energy balance equation, as given by

$$\dot{\rho}_{\rm eff} + 3H(\rho_{\rm eff} + p_{\rm eff}) = 0, \tag{57}$$

which can be explicitly written as

$$\dot{\rho} + 3H(\rho + p) + \frac{1}{8\pi}\frac{d}{dt}\left(-\dot{\omega} - 2H\omega + \frac{1}{2}\omega^2\right) + \frac{3}{8\pi}H\left(-\frac{1}{3}\dot{\omega} + \frac{1}{3}H\omega + \frac{2}{3}\omega^2\right) = 0. \tag{58}$$

As an indicator of the accelerated/decelerated expansion, we introduce the deceleration parameter q, defined as

$$q = \frac{d}{dt}\frac{1}{H} - 1 = -\frac{\dot{H}}{H^2} - 1. \tag{59}$$

With the use of the generalized Friedmann equations we obtain for the deceleration parameter the expression

$$q = \frac{1}{2} + \frac{3}{2}\frac{p_{\rm eff}}{\rho_{\rm eff}} = \frac{1}{2} + \frac{3}{2}\frac{p + \frac{1}{8\pi}\left(\frac{2}{3}\dot{\omega} + \frac{1}{6}\omega^2 + \frac{7}{3}H\omega\right)}{\rho + \frac{1}{8\pi}\left(-\dot{\omega} - 2H\omega + \frac{1}{2}\omega^2\right)}. \tag{60}$$

Once the condition $q < 0$ is satisfied, the Universe will enter into an accelerated phase of expansion. Thus a transition can be triggered in the present model by the dynamical evolution of the Weyl field ω.

To simplify the mathematical formalism we introduce a set of dimensionless variables (τ, h, r, P, Ω), defined according to the transformations

$$\tau = H_0 t, \qquad H = H_0 h, \qquad \rho = \frac{3H_0^2}{8\pi} r, \qquad p = \frac{3H_0^2}{8\pi} P, \qquad \omega = H_0 \Omega, \tag{61}$$

where H_0 is the present-day value of the Hubble function. Then the system of the generalized Friedmann equations takes the following dimensionless form

$$h^2 = r - \frac{2}{3}h\Omega + \frac{1}{6}\Omega^2 - \frac{1}{3}\frac{d\Omega}{d\tau}, \tag{62}$$

$$2\frac{dh}{d\tau} + 3h^2 = -3P - \frac{7}{3}h\Omega - \frac{1}{6}\Omega^2 - \frac{2}{3}\frac{d\Omega}{d\tau}. \tag{63}$$

To facilitate the comparison with the observational data we reformulate the cosmological evolution equations in the redshift space, with the redshift variable defined according to

$$1 + z = \frac{1}{a}, \tag{64}$$

giving

$$\frac{d}{d\tau} = -(1+z)h(z)\frac{d}{dz}. \tag{65}$$

Then in the redshift space the generalized Friedman equations are given by

$$h^2(z) = r(z) - \frac{2}{3}h(z)\Omega(z) + \frac{1}{6}\Omega^2(z) + \frac{1}{3}(1+z)h(z)\frac{d\Omega}{dz}, \tag{66}$$

$$-2(1+z)h(z)\frac{dh(z)}{dz} + 3h^2(z) = -3P(z) - \frac{7}{3}h(z)\Omega(z) - \frac{1}{6}\Omega^2(z) + \frac{2}{3}(1+z)h(z) \times \frac{d\Omega}{dz}. \tag{67}$$

To test the relevance, and the viability of the cosmological predictions of the Weyl-Schrödinger gravity theory, we will perform a detailed comparison of it with the standard ΛCDM cosmology, as well as with a small sample of observational data points, obtained for the Hubble function.

In the ΛCDM model the Hubble function is given by

$$H = H_0\sqrt{\frac{\Omega_m}{a^3} + \Omega_\Lambda} = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}, \tag{68}$$

where $\Omega_m = \Omega_b + \Omega_{\rm DM}$, with $\Omega_b = \rho_b/\rho_{cr}$, $\Omega_{\rm DM} = \rho_{\rm DM}/\rho_{cr}$ and $\Omega_\Lambda = \Lambda/\rho_{cr}$, where ρ_{cr} is the critical density of the Universe. Ω_b, $\Omega_{\rm DM}$ and $\Omega_{\rm DE}$ represent the density parameters of the baryonic matter, dark matter, and dark energy, respectively. The deceleration parameter can be obtained from the relation

$$q(z) = \frac{3(1+z)^3\Omega_m}{2[\Omega_\Lambda + (1+z)^3\Omega_m]} - 1. \tag{69}$$

In the following analysis for the matter and dark energy density parameters of the ΛCDM model we will use the numerical values $\Omega_{\rm DM} = 0.2589$, $\Omega_b = 0.0486$, and $\Omega_\Lambda = 0.6911$, respectively [61,62]. Hence, the total matter density parameter $\Omega_m = \Omega_{\rm DM} + \Omega_b = 0.3075$, where we have neglected the contribution of the radiation to the total matter energy balance in the late Universe. The present day value of q, as predicted by the ΛCDM model, is thus $q(0) = -0.5912$, indicating that the recent Universe is in an accelerating expansionary stage. For the observational data we use the values of the Hubble functions from the compilation presented in [63].

B. The de Sitter solution: $h = \text{constant}$

We look first for exact vacuum solutions of the Weyl-Schrödinger cosmological models, with $P = 0$, under the condition of a constant expansion rate, with $h = h_0 =$ constant. Equation (63) then becomes

$$\frac{d\Omega}{d\tau} + \frac{7}{2}h_0\Omega + \frac{1}{4}\Omega^2 + \frac{9}{2}h_0^2 = 0, \tag{70}$$

with the general solution given by

$$\Omega(\tau) = h_0\left\{\sqrt{31}\tanh\left[\frac{1}{4}\sqrt{31}h_0(\tau - 4c_1)\right] - 7\right\}, \tag{71}$$

where c_1 is an arbitrary constant of integration. In the limit of large times we have $\lim_{\tau\to\infty}\Omega\tau = (\sqrt{31} - 7)h_0$, that is, the Weyl vector takes negative values when τ is very large. Then, from Eq. (62) we obtain the variation of the matter density during the de Sitter type era as

$$r(\tau) = \frac{1}{4}h_0^2\left\{12\sqrt{31}\tanh\left[\frac{1}{4}\sqrt{31}h_0(\tau - 4c_1)\right] + 31\text{sech}^2\left[\frac{1}{4}\sqrt{31}h_0(\tau - 4c_1)\right] - 68\right\}. \tag{72}$$

In the large time limit the matter density tends to $\lim_{r\to\infty} r(\tau) = (3\sqrt{31} - 17)h_0^2$, indicating a slight violation of the energy condition $r > 0$ at large time intervals.

C. Model I: Dark energy models with a linear EOS

We will consider now dark energy models that do not satisfy anymore the condition of the constancy of the Hubble function. As a first dark energy model in the Weyl-Schrödinger gravity theory we assume that the effective pressure and energy density of the dark energy are related by a linear equation of state, given by

$$p_{\rm DE}(z) = \sigma(z)\rho_{\rm DE} - \frac{\lambda}{8\pi}, \tag{73}$$

where λ is a constant. For the parameter $\sigma(z)$ of the dark energy equation of state we adopt the Chevallier-Polarski-Linder (CPL) parametrization [64,65], so that

$$\sigma(z) = \sigma_0 + \sigma_a\frac{z}{1+z}. \tag{74}$$

This form allows to extend the dark energy EOS to very high redshifts, since $\lim_{z\to\infty} = \sigma_0 + \sigma_a$. Hence, the dynamical cosmological evolution equations describing the expansion of the dust Universe, with $P = 0$, take the form

$$-\frac{2}{3}\left[1 + \frac{3}{2}\sigma(z)\right](1+z)h(z)\frac{d\Omega(z)}{dz} + \frac{7}{3}\left[1 + \frac{6\sigma(z)}{7}\right] \times h(z)\Omega(z) + \frac{1}{6}[1 - 3\sigma(z)]\Omega^2(z) + \lambda = 0, \tag{75}$$

and

$$-2(1+z)h(z)\frac{dh(z)}{dz} + 3h^2(z) - \lambda + \sigma(z)\left[-2h(z)\Omega(z) + \frac{1}{2}\Omega^2(z) + (1+z)h(z)\frac{d\Omega(z)}{dz}\right] = 0, \tag{76}$$

respectively.

The system of equations (75) and (76) must be integrated with the initial conditions $h(0) = 1$, and $\Omega(0) = \Omega_0$.

Once the functions $h(z)$ and $\Omega(z)$ are known as solutions of the evolution equations, the matter density can be obtained as

$$r(z) = h^2(z) + \frac{2}{3}h(z)\Omega(z) - \frac{1}{6}\Omega^2(z) - \frac{1}{3}(1+z)h(z)\frac{d\Omega(z)}{dz}. \tag{77}$$

The variations as functions of the redshift of the Hubble function and of the deceleration parameter are represented, for different values of λ, in Fig. 1. The Weyl-Schrödinger model, closed with an effective equation of state of the dark energy, gives a good description of the observational data, and, for a certain range of the model parameters, can reproduce almost exactly the predictions of the ΛCDM model. However, some differences do appear in the behavior of the deceleration parameter. Similarly to the standard cosmological models, the Weyl-Schrödinger models predicts a decelerating expansion of the Universe at redshifts

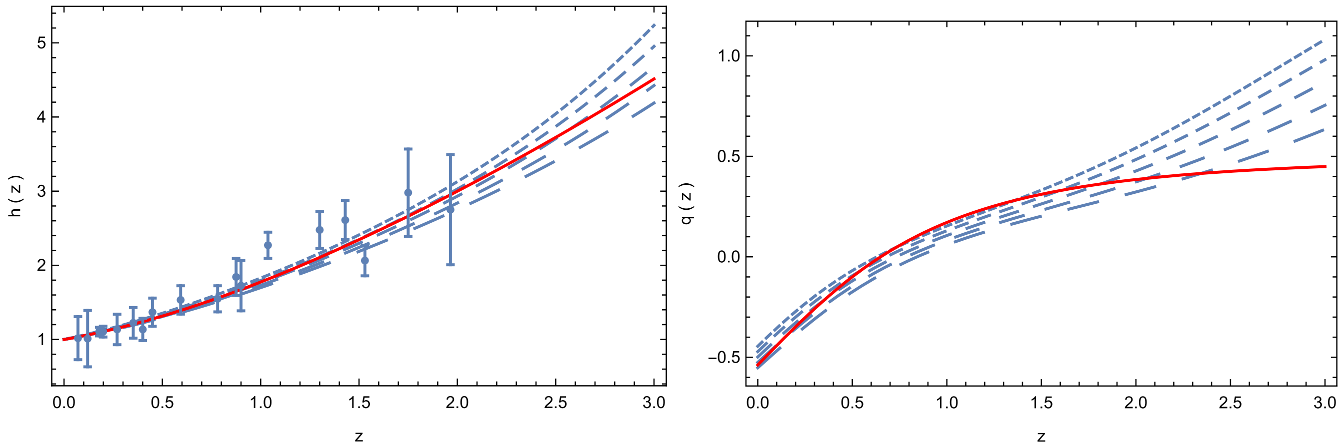

FIG. 1. Variations as functions of the cosmological redshift of the dimensionless Hubble function h (left panel) and of the deceleration parameter (right) panel in the Weyl-Schrödinger cosmological model with a linear equation of state for the dark energy for $\lambda = 3.1$ (dotted curve), $\lambda = 3.2$ (short dashed curve), $\lambda = 3.3$ (dashed curve), $\lambda = 3.4$ (long-dashed curve), and $\lambda = 3.5$ (ultralong dashed curve). The initial conditions used to integrate the cosmological evolution equations are $\Omega(0) = -9.7 \times 10^{-1}$, and $h(0) = 1$, respectively. For the numerical values of the coefficients of the parameter of the dark energy equation of state we have adopted the values $\sigma_0 = 0.58$ and $\sigma_a = 0.0018$, respectively. The observational data are represented with their error bars, while the red curve depicts the predictions of the ΛCDM model.

higher than $z \approx 1$, and an accelerating expansion at lower redshifts.

The variations of the temporal component of the Weyl-Schrödinger vector Ω, and of the matter energy density $r(z)$ are represented, as a functions of the redshift, in Fig. 2. The cosmological Weyl-Schrödinger vector is a monotonically increasing function of the redshift (a monotonically decreasing function of the cosmological time), and its evolution is strongly dependent, at high redshifts, by the adopted values of the model parameters. Up to a redshift of around $z \approx 0.5$, the cosmological dynamics of the Weyl-Schrödinger vector is relatively independent on the numerical values of the model parameter, including the choice of the initial conditions. The matter energy density of the Weyl-Schrödinger model coincides, up to a redshift of around $z \approx 2$, with the predictions of the ΛCDM model. However, at larger redshifts, there are significant differences between the predictions of the two models. Generally, the increase in the matter density occurs faster in the ΛCDM model, and thus, standard cosmology predicts the existence of a much higher amount of cosmic matter in the early Universe, as compared with the predictions of the Weyl-Schrödinger model.

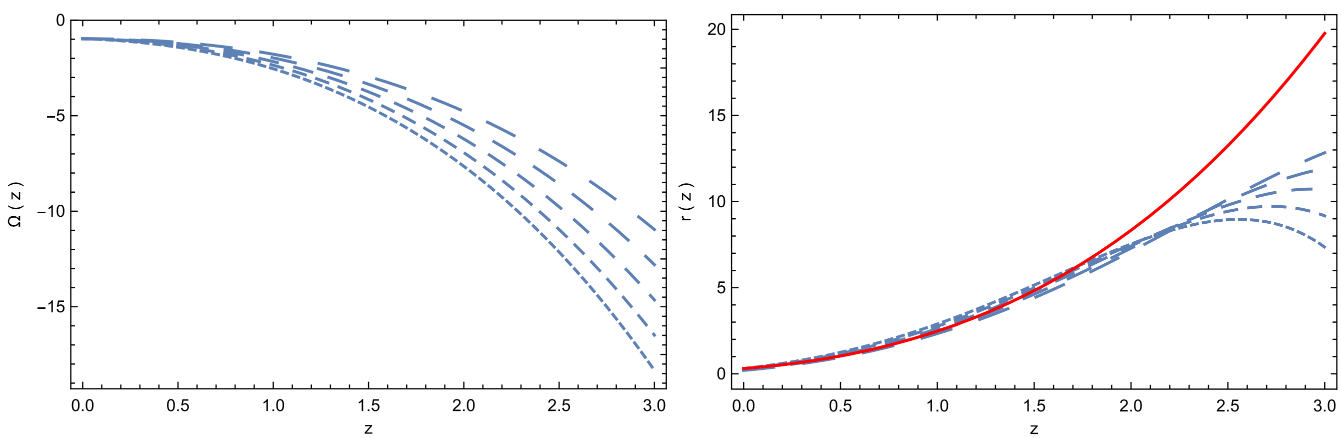

FIG. 2. Variation as a function of the redshift of the dimensionless Weyl-Schrödinger vector Ω (left panel) and of the dimensionless matter energy density r in the Weyl-Schrödinger model with a linear equation of state for the dark energy for $\lambda = 3.1$ (dotted curve), $\lambda = 3.2$ (short dashed curve), $\lambda = 3.3$ (dashed curve), $\lambda = 3.4$ (long-dashed curve), and $\lambda = 3.5$ (ultralong dashed curve). The initial conditions used to integrate the cosmological evolution equations are $\Omega(0) = -9.7 \times 10^{-1}$, and $h(0) = 1$, respectively. For the numerical values of the coefficients of the parameter of the dark energy equation of state we have adopted the values $\sigma_0 = 0.58$ and $\sigma_a = 0.0018$, respectively. The red curve depicts the prediction of the ΛCDM model for the matter energy density, $r(z) = 0.3075(1 + z)^3$.

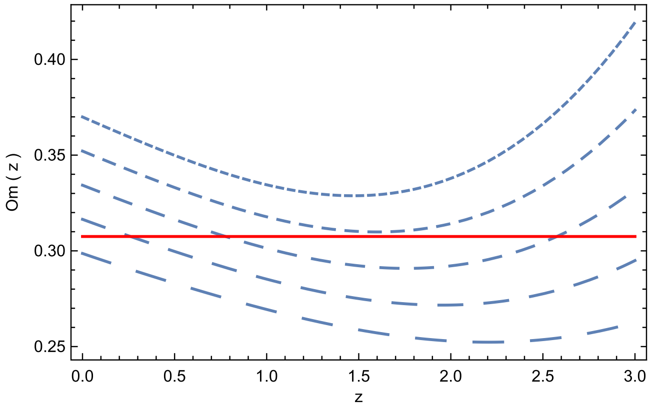

FIG. 3. Variation of the function $Om(z)$ for the Weyl-Schrödinger cosmological model with a linear equation of state of the dark energy for $\lambda = 3.1$ (dotted curve), $\lambda = 3.2$ (short dashed curve), $\lambda = 3.3$ (dashed curve), $\lambda = 3.4$ (long-dashed curve), and $\lambda = 3.5$ (ultralong dashed curve). The initial conditions used to integrate the cosmological evolution equations are $\Omega(0) = -9.7 \times 10^{-1}$, and $h(0) = 1$, respectively. The red curve corresponds to the prediction of the ΛCDM model for the $Om(z)$ function.

Finally, in Fig. 3 we present the $Om(z)$ diagnostic of the Weyl-Schrödinger cosmological model. The $Om(z)$ diagnostic [66] is an important theoretical tool which can be used to differentiate alternative cosmological models from the ΛCDM paradigm. The $Om(z)$ function is defined as

$$Om(z) = \frac{H^2(z)/H_0^2 - 1}{(1+z)^3 - 1} = \frac{h^2(z) - 1}{(1+z)^3 - 1}. \tag{78}$$

In the case of the ΛCDM model, $Om(z)$ is a constant, and it is equal to the present day matter density $r(0) = 0.3075$. For cosmological models satisfying an equation of state with a constant equation of state parameter $w =$ constant, the existence of a positive slope of $Om(z)$ is evidence for a phantomlike evolution, while a negative slope indicates a quintessence-like dynamics. The function $Om(z)$ is represented for the present particular Weyl-Schrödinger type cosmological model in Fig. 3.

D. Model II: Models with conserved matter energy density

As a second example of a cosmological model in Weyl-Schrödinger theory, we consider the case in which both the matter and the Weyl-Schrödinger energy-momentum tensors are conserved independently. Hence, we split the total conservation equation (58) as

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{79}$$

and

$$\frac{d}{dt}\left(-\dot{\omega} - 2H\omega + \frac{1}{2}\omega^2\right) + 3H\left(-\frac{1}{3}\dot{\omega} + \frac{1}{3}H\omega + \frac{2}{3}\omega^2\right) = 0, \tag{80}$$

respectively. For a pressureless dust, Eq. (79) can be immediately integrated to give

$$r(z) = r_0(1+z)^3, \tag{81}$$

where $r_0 = r(0)$ is the present day matter density. After introducing the dimensionless coordinates as defined in Eq. (61), and introducing the new variable $u = d\Omega/d\tau$, the conservation equation of the effective energy of the Weyl-Schrödinger field can be reformulated as

$$-\frac{du}{d\tau} - 2\frac{dh}{d\tau}\Omega - 3hu + \Omega u + h^2\Omega + 2h\Omega^2 = 0. \tag{82}$$

Hence, in the redfshift space, the cosmological evolution equations of the Weyl-Schrödinger theory with conserved matter and vector field effective energy can be formulated as

$$(1+z)h(z)\frac{d\Omega}{dz} + u(z) = 0, \tag{83}$$

$$(1+z)h(z)\frac{du(z)}{dz} + 2(1+z)h(z)\frac{dh(z)}{dz}\Omega(z) - 3h(z)u(z) + \Omega(z)u(z) + h^2(z)\Omega(z) + 2h(z)\Omega^2(z) = 0, \tag{84}$$

$$-2(1+z)h(z)\frac{dh(z)}{dz} + 3h^2(z) + \frac{7}{3}h(z)\Omega(z) + \frac{1}{6}\Omega^2(z) + \frac{2}{3}u(z) = 0. \tag{85}$$

The system of differential equations (83)–(85) must be integrated with the initial conditions $h(0) = 1$, $\Omega(0) = \Omega_0$, and $u(0) = u_0$, respectively. However, these initial conditions are not arbitrary, since they must satisfy the constraint, following from the first Friedmann equation (66), which gives

$$1 = r_0 - \frac{2}{3}\Omega_0 + \frac{1}{6}\Omega_0^2 - \frac{1}{3}u_0. \tag{86}$$

The variations with respect to the redshift z of the Hubble function and of the deceleration parameter for the Weyl-Schrödinger cosmological model with conserved quantities are represented, for different values of Ω_0, in Fig. 4. The model gives a good description of the observational data up to a redshift of $z = 2$, and can reproduce almost exactly, for specific values of the initial condition Ω_0, $\Omega_0 \approx -1$, the predictions of the ΛCDM model. At redshifts higher than $z = 2$, there are some important deviations with respect to

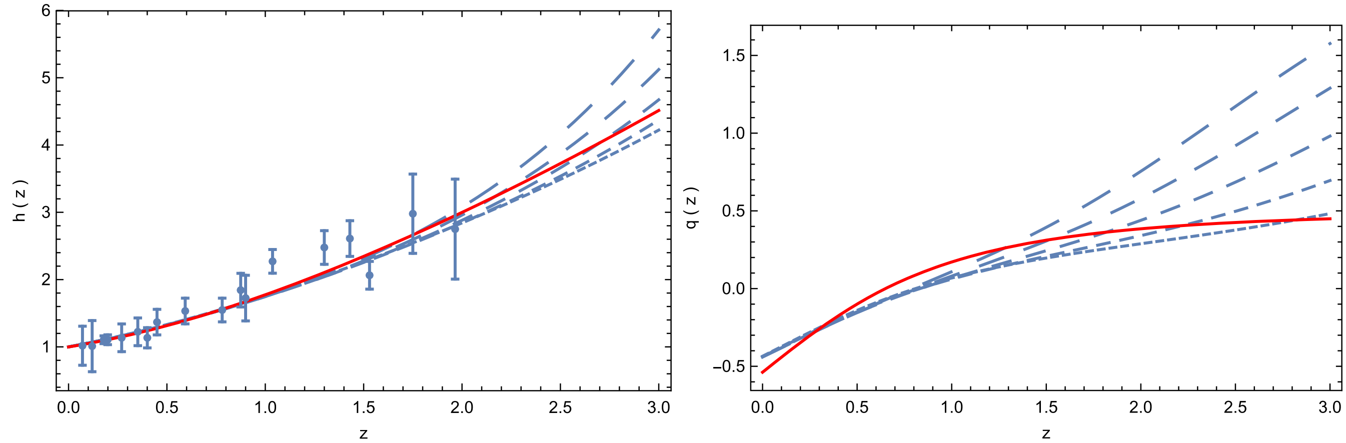

FIG. 4. Variations as functions of the cosmological redshift of the dimensionless Hubble function h (left panel) and of the deceleration parameter (right) panel in the Weyl-Schrödinger cosmological model with conserved matter density, for different values of the initial condition of the Weyl-Schrödinger vector field: $\Omega_0 = -0.86$ (dotted curve), $\Omega_0 = -0.90$ (short dashed curve), $\Omega_0 = -0.94$ (dashed curve), $\Omega_0 = -0.98$ (long-dashed curve), and $\Omega_0 = -1.02$ (ultralong dashed curve). The values of u_0 are obtained by using Eq. (86). The observational data for the Hubble function are represented with their error bars, while the red curve show the theoretical predictions of the ΛCDM model.

the predictions of ΛCDM. Moreover, in the case of this particular Weyl-Schrödinger cosmological model, significant differences do appear in the behavior of the deceleration parameter, which at high redshifts has a very different behavior, as compared with the ΛCDM predictions.

The redshift variations of the Weyl vector Ω, and of its derivative with respect to the redshift u are presented in Fig. 5. The Weyl-Schrödinger vector field takes negative values, and it is a decreasing function of the redshift. Its behavior at higher redshifts show a strong dependence on the initial condition used to numerically integrate the cosmological evolution equation. The derivative of the Weyl-Schrödinger field has only positive values, and it is monotonically increasing function of the redshift. While at low redshifts, in the range $0 < z < 1.5$, the behavior of u is basically independent on the initial condition for Ω_0, at higher redshifts the behavior of u essentially depends on the initial condition for the Weyl-Schrödinger vector field.

The variation of the $Om(z)$ function for the Weyl-Schrödinger cosmological model with conserved matter energy density is represented in Fig. 6.

V. DISCUSSIONS AND FINAL REMARKS

In the present paper we have considered a gravitational theory based on a geometry that goes beyond the standard Riemannian one. More exactly, we have investigated the physical implications of a geometry proposed a long time ago by Erwin Schrödinger [58], and which, interestingly enough, despite of its many remarkable features, did not

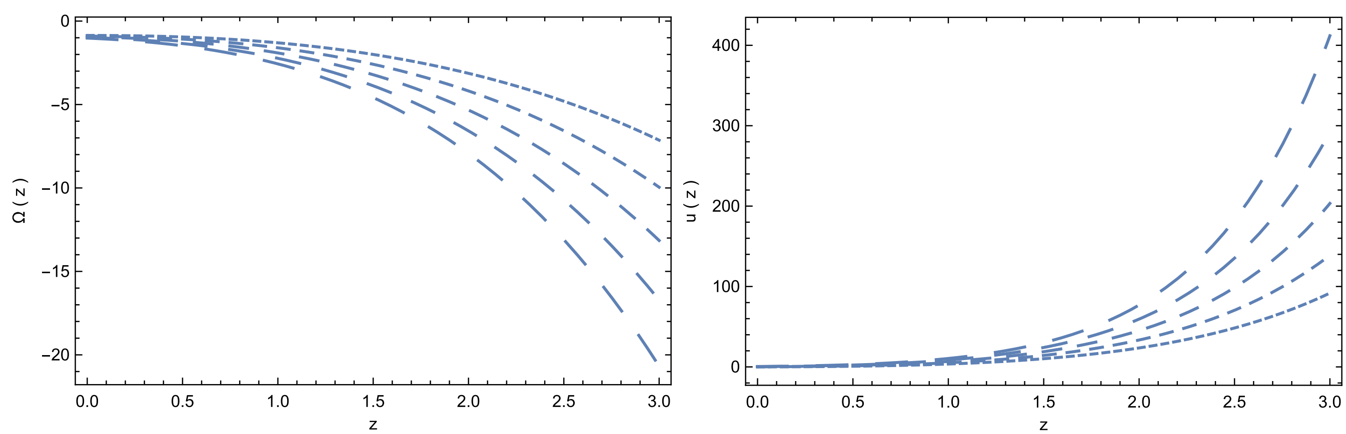

FIG. 5. Variations as functions of the cosmological redshift of the dimensionless Weyl-Schrödinger vector Ω (left panel) and of its derivative u (right) panel in the Weyl-Schrödinger cosmological model with conserved matter density, for different values of the initial condition of the Weyl-Schrödinger vector field: $\Omega_0 = -0.86$ (dotted curve), $\Omega_0 = -0.90$ (short dashed curve), $\Omega_0 = -0.94$ (dashed curve), $\Omega_0 = -0.98$ (long-dashed curve), and $\Omega_0 = -1.02$ (ultralong dashed curve). The values of u_0 are obtained by using Eq. (86).

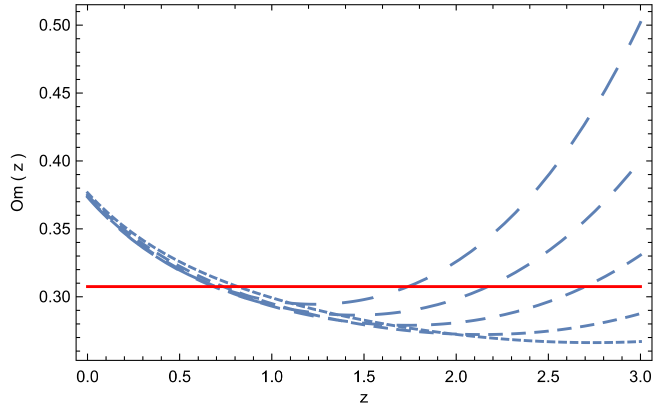

FIG. 6. Variation of the function $Om(z)$ in the Weyl-Schrödinger cosmological model with conserved matter density, for different values of the initial condition of the Weyl-Schrödinger vector field: $\Omega_0 = -0.86$ (dotted curve), $\Omega_0 = -0.90$ (short dashed curve), $\Omega_0 = -0.94$ (dashed curve), $\Omega_0 = -0.98$ (long-dashed curve), and $\Omega_0 = -1.02$ (ultralong dashed curve). The values of u_0 are obtained by using Eq. (86). The prediction of the ΛCDM model for the $Om(z)$ function is represented by the red solid curve.

attract much attention in the scientific community. The starting point of Schrödinger's theory is Weyl geometry. In its initial formulation, Weyl, in an attempt to unify the gravitational and the electromagnetic fields, introduced a connection who adds some new terms to the standard Levi-Civita connection of the Riemannian geometry. These extra terms are known generally as the nonmetricity $Q_{\mu\nu\lambda}$. In Weyl's theory under parallel transport not only the direction, but also the length of vectors vary. The trace of the nonmetricity (the Weyl vector) was identified by Weyl as the electromagnetic field potential. However, Einstein severely criticized Weyl's theory, and this criticism led to its long time abandonment [31]. Weyl's theory is based on the principle of conformal invariance, which has many attractive features, and it is assumed to be a fundamental symmetry of nature [67–70], unifying the Standard Model of the elementary particles, and gravitation.

On the other hand, Schrödinger [58], tried to overcome Einstein's criticism of the Weyl theory by considering a symmetric connection in which the length of vectors is not changed under parallel transport, even in the presence of nonmetricity. The Schrödinger connection $\Gamma^{\lambda}{}_{\mu\nu}$ can be defined generally as [59]

$$\Gamma^{\lambda}{}_{\mu\nu} = \gamma^{\lambda}{}_{\mu\nu} + g^{\lambda\rho} S_{\rho\mu\nu}, \tag{87}$$

where $S_{\mu\nu\rho}$ is a tensor having the properties

$$S_{\lambda\mu\nu} = S_{\lambda\nu\mu}, \qquad S_{(\lambda\mu\nu)} = 0. \tag{88}$$

If

$$S_{\lambda\mu\nu} = -Q_{\lambda\mu\nu}, \tag{89}$$

the length of the vectors is invariant during parallel transport [59]. But, similarly to the standard Riemannian case, the angle between vectors changes due to the parallel transport [58]. It is interesting to note that one could consider geometries with vanishing nonmetricity, and nonzero torsion, and then symmetrize the connection in μ, ν [59]. Thus, we find

$$\Gamma^{\lambda}{}_{(\mu\nu)} \coloneqq \check{\Gamma}^{\lambda}{}_{(\mu\nu)} = \gamma^{\lambda}{}_{\mu\nu} - 2g^{\lambda\rho} T_{(\mu|\rho|\nu)}. \tag{90}$$

Then, if

$$S_{\lambda\mu\nu} = -2T_{(\mu|\lambda|\nu)}, \tag{91}$$

$\check{\Gamma}^{\lambda}{}_{(\mu\nu)}$ is identical with (87) [59]. Hence, it turns out that (87) can be written down either with regard to torsion, by using nonmetricity only, or as relating to both nonmetricity and torsion.

We have seen that a general geometry of spacetime can be characterized by three geometric variables: curvature, torsion and nonmetricity. Thus, in different theories of gravity one could decide whether to include any of them. This gives us eight possible choices, as shown in Fig. 7. The most general gravity theory that contains all of the three variables is based on a non-Riemannian geometry and is called Metric-Affine gravity, while the most trivial case with all three being zero leads to a Minkowski spacetime. The Schrödinger geometry that we have introduced, and studied in detail, as well as the Weyl geometry belong to the same

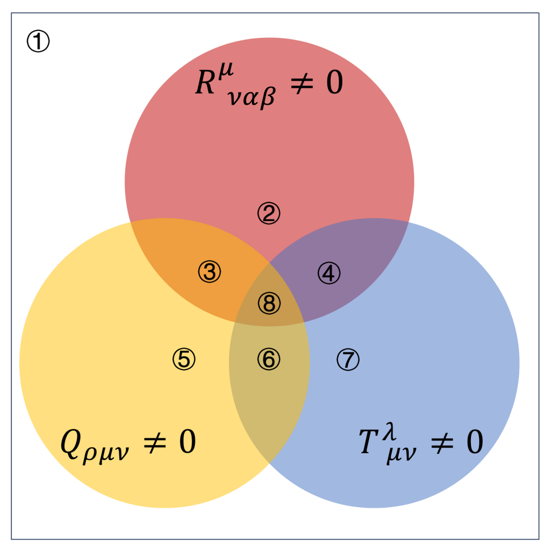

FIG. 7. Eight possibilities of gravity theories (geometries): (1) Minkowski, (2) Riemann, (3) Weyl and Schrödinger, (4) Cartan, (5) symmetric teleparallel, (6) generic teleparallel, (7) metric teleparallel (Weitzenböck) and (8) metric-affine.

category, in which both curvature and nonmetricity are present, but at different levels. The classification of the geometries based on their relationship with the fundamental geometrical quantities (curvature, torsion, nonmetricity), and of the corresponding gravitational theories, is presented in Fig. 7.

In order to formulate the gravitational theory based on the Weyl-Schrödinger geometry we have introduced the gravitational action (33), which has essentially a very simple mathematical structure. In the absence of torsion, the action is constructed additively from the Weyl scalar R plus the squares of the two contractions of the nonmetricity Q_ρ and $\tilde{Q}_\rho$, respectively. In order to recover the Schrödinger connection one must impose the condition $Q_\rho = -2\tilde{Q}_\rho$, which gives finally the field equations (47), which are the basic equations of the present Weyl-Schrödinger theory.

It would be interesting to perform a comparison between the present Weyl-Schrödinger theory, standard general relativity, and another important modified gravity theory, the $f(R)$ gravity theory [5–7]. The field equations of $f(R)$ gravity are given by

$$\frac{1}{2}g_{\mu\nu}f(R) - R_{\mu\nu}f'(R) + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f'(R) = -\frac{\kappa^2}{2}T_{\mu\nu}. \tag{92}$$

As compared to the $f(R)$ gravity field equations, the field equations of the Weyl-Schrödinger theory, as well as the corresponding connection, have a very interesting mathematical feature, in the sense that no free arbitrary parameters are introduced in the theory, and all the coefficients in the action, and field equations, are purely numerical. Thus, except the standard gravitational coupling constant of general relativity, no new parameter does appear in the field equations. In the $f(R)$ theory, the action depends on the analytical form of the function f, whose mathematical form is not determined *a priori* by the theory. Even if a functional form of f is obtained empirically from the comparison with observations, and a viable theoretical model is obtained, generally one cannot find any theoretical justification for it in the framework of a basic gravity theory. Moreover, it is rather difficult to obtain a unified $f(R)$ theory describing gravitational dynamics from the Solar System level to galactic, extra-galactic and cosmological scales. More importantly, the field equations of the $f(R)$ in the metric formulation are fourth order strongly nonlinear differential equations, whose general solution requires a significant extension of the space of initial conditions. However, in the Palatini formulation the $f(R)$ gravity field equations are of second order [7]. Moreover, if one uses a fluid representation for the cosmological equations of the $f(R)$ theory, the generalized gravitational fluid contains higher-derivative curvature invariants, including the third order time derivative of the Hubble function [7]. The effective cosmological fluid representation of the Weyl-Schrödinger gravity contains only the Hubble function, the square of the Weyl vector, and of its derivative. Moreover, the generalized Friedmann equations of the present model are much simpler mathematically then the corresponding Friedmann equations of the $f(R)$ theory. In the $f(R)$ theory dark energy originates from curvature, while in the Weyl-Schrödinger theory the source of the dark energy is a special type of nonmetricity. Thus, Weyl-Schrödinger gravity significantly extends the geometric space, and this may be interpreted as a disadvantage with respect to theories describing gravity in terms of a single geometry.

Standard general relativity is a beautiful theory in its simplicity, and it has a powerful descriptive and predictive potential. However, to describe the cosmological observational data one must resort to the cosmological constant Λ, whose physical or geometrical interpretation is still unknown [71,72]. Hence, the search for finding the nature of Λ is still going one, and alternative models of dark energy can offer a solution to this problem. The Weyl-Schrödinger theory offers such a dark energy approach, which is relatively simple, and uniquely fixes the cosmological dynamics by introducing a geometric equivalent description of the cosmological constant. In this theory dark energy has a purely geometric origin, and it is intimately related to the space-time structure. Of course, from a theoretical point of view, it would be preferable to have a clear physical/geometrical interpretation of Λ, but at the present moment this seems to be unlikely. Moreover, it turns out that due to the significant increase of the precision of the cosmological observations, the ΛCDM model must also face some important challenges. Perhaps the most important present of these cosmological problems is the significant difference between the Hubble expansion rates of the Universe as obtained from the cosmic microwave background (CMB) experiments by the Planck satellite, and the low redshift (local) measurements using type IA supernovae. The differences in the determinations of the present-day value of the Hubble constant H_0 led to what is called the Hubble tension problem. For a detailed presentation and discussion of the Hubble tension, and of its possible solutions, see [73], and references therein. The Hubble constant H_0, as determined from the Planck satellite data, has the value of $H_0 = 66.93 \pm 0.62$ km/s/Mpc, while the SHOES collaboration did find the value $H_0 = 73.24 \pm 1.74$ km/s/Mpc [73]. If the Hubble tension does indeed exist, it strongly indicates the necessity of considering new gravitational theories, and of the replacing the ΛCDM model with alternative cosmological approaches. In [74] it was argued that a promising way forward to solve the Hubble tension would ultimately involve a combination of early- and late-time new physics, as well as a local at $z = 0$), new physics. Even that detailed investigation in this problem are still necessary, at least in principle, the Weyl-Schrödinger theory has the potential of addressing the Hubble tension since it leads to both early and late time cosmological evolutions.

In order to consider the physical implications of the Weyl-Schrödinger gravity, and its viability, we have analyzed in detail the cosmological models that follow from the theory. As a first step, after adopting the homogeneous, isotropic and flat FLRW metric, and adopting a specific form for the nonmetricity vector, we have obtained the generalized Friedmann equations, in which two new terms do appear. These two terms, representing some extra contributions coming from nonmetricity, can correspond to a dark energy type fluid, whose energy density $\rho_{\rm DE}$ and pressure $p_{\rm DE}$ are completely determined by the temporal component of Q_ρ. In the present approach the two generalized Friedmann equations contain four unknowns (H, ω, ρ, p), and even after imposing an equation of state the system is still over-determined. But this allows to construct various cosmological scenarios, by imposing some physically reasonable conditions on the effective dark energy and pressure. In this context we have considered two distinct cosmological models. In the first model we have imposed a linear equation of state relating the dark energy pressure and density, the equation of state being parameterized by redshift dependent parameter, defined according to the CPL prescription. The model thus obtained, depending on four parameters $(\Omega_0, \sigma_0, \sigma_a, \lambda)$, can be studied numerically in the redshift space. Once the numerical solution is known, a comparison with a small set of observational data of the Hubble function, and with the ΛCDM model can be performed. The model describes well the observational data for the Hubble function, and for some specific values of the model parameters the ΛCDM model can be recovered almost exactly. The matter density as predicted by this Weyl-Schrödinger cosmological model coincides with the ΛCDM predictions up to a redshift of $z \approx 1.5$, but at higher redshifts the predictions of the two models are rather different.

A second simple cosmological model can be obtained by imposing the condition of the conservation of the matter energy density, which is required to satisfy the standard equation $\dot{\rho} + 3H\rho = 0$. The conservation equation determines the matter energy density as having the same form as in the ΛCDM model. The cosmological evolution is thus determined by the initial condition $\Omega(0)$ of the Weyl vector, and of its derivative $u(0)$. But the first Friedmann equation gives a constraint at $z = 0$, which allows to express $u(0)$ in terms of $r(0)$ and $\Omega(0)$. Hence, in this cosmological model the dynamical evolution depends on a single parameter only, the present day value of the dimensionless Weyl vector $\Omega(0)$. It is interesting that the value $\Omega(0) = -1$ reproduces (almost) exactly the predictions of the ΛCDM model for the Hubble function. This gives for the present day value of the temporal component of the Weyl vector $\omega(0) = -H_0$. Therefore, this two parameters model, depending on the present day values of the matter density, and with $\omega(0) = -H_0$, represents an intriguing, but effective alternative of the ΛCDM paradigm.

The ΛCDM model gives a very good fit of the observational data. Hence, if a given cosmological model can reproduce ΛCDM at least in some redshift range, it can also give a good description of the data, which are represented, in our present approach, by a set of Hubble function values, obtained at various redshifts. However, even if the considered Weyl-Schrödinger type cosmological models give a good concordance with ΛCDM, and the observational data, there are several important differences with respect to the predictions of the two approaches for other important cosmological parameters. One such parameter is the deceleration parameter q, which determines the expansion rate of the Hubble sphere, and the dynamics of the observable galaxy number variation. There are some significant differences on the behavior of q in Weyl-Schrödinger models, and ΛCDM, which do appear already at low redshifts. These differences become extremely important in both considered models at higher redshifts, where the Weyl-Schrödinger models predict much larger values of q. This shows that at high redshifts the Universe expanded much slower than in ΛCDM, and the transition to an accelerated phase occurred much more rapidly than expected from standard cosmology. Determinations of the values of q for $z > 2$ would thus allow to discriminate between the various Weyl-Schrödinger models, and ΛCDM.

The total, baryonic plus dark matter density also generally behaves differently in the Weyl-Schrödinger and standard general relativistic models. In the first cosmological model the matter energy density basically coincides with the ΛCDM prediction, but for $z > 1.5$ the evolution is very different, with the ΛCDM model predicting a much faster increase of the total matter density. However, some parameter values of the Weyl-Schrödinger model predict a decrease of the matter density at high redshifts, and thus those parameter values can be ruled out as unphysical. In the second cosmological model we have considered, with matter satisfying the usual conservation equation, the matter evolution fully coincides with the ΛCDM evolution.

Finally, we would like to point out that important differences between the present theoretical approach and ΛCDM do appear in the $Om(z)$ diagnostic. If the value of the function $Om(z)$ is a constant at any redshift, dark energy corresponds exactly to a cosmological constant. An evolving $Om(z)$ corresponds to other dynamical dark energy models. In both our considered cosmological models $Om(z)$ is an evolving function, and hence our dark energy models do not exactly correspond to a cosmological constant. This represents an important difference between the considered Weyl-Schrödinger cosmologies, describing an evolving dark energy, and the ΛCDM model. Moreover, the slope of the $Om(z)$ function could distinguish between two different types of dark energy models. A positive slope of $Om(z)$ indicates phantomlike dark energy, with $w < -1$, while a negative slope represents quintessencelike behavior, with $w > -1$ [66]. As one can see from Figs. 3 and 6, for both considered Weyl-Schrödinger models, the slope of the function $Om(z)$ is negative up to a redshift of around $z \approx 1.5$, indicating a quintessencelike behavior, while, for $z > 1.5$, the slope is positive, and thus the cosmological evolution becomes phantom-like. Hence, a transition from quintessence to phantom evolution does occur in the presence of nonmetricity, a feature which does not exist in the ΛCDM models.

To summarize: in the present work we have proposed and analyzed in detail, from the point of view of the theoretical consistency, and of the concordance with observations, a geometrical dark energy model, based on the Weyl-Schrödinger theory, which has its origins in the Weyl geometry. In this theory, an effective fluid type dark energy component can be generated from the non-Riemannian geometric structures that determine the properties of the space-time. The Weyl-Schrödinger type model have a close relationship with the standard general relativistic Friedmann cosmological evolution equations, with the Weyl-Schrödinger models exactly reproducing in some particular case the ΛCDM dynamics. The Weyl-Schrödinger models permit to introduce in a simple and intuitive way a geometric dark energy term, of fluid type, for the description of the cosmological evolution. The considered Weyl-Schrödinger models also give a good description of the cosmological observational data, generally in terms of very few free parameters. They can also (almost) exactly reproduce the predictions of the ΛCDM standard cosmological model. However, one should emphasize that important differences with standard cosmology do appear at high redshifts, and in the numerical values of some cosmographic quantities. Despite these shortcomings, the Weyl-Schrödinger type FLRW cosmological model may become an important and attractive alternative to the ΛCDM model, in terms of theoretical foundations, explanations of the observational data, and predictive power. These models may also yield some new perspectives, and a better understanding of the intricate relation existing between the physical reality and abstract mathematical structures.

ACKNOWLEDGMENTS

The work of T. H. is supported by a grant from the Romanian Ministry of Education and Research, CNCS-UEFISCDI, Project Number PN-III-P4-ID-PCE2020-2255 (PNCDI III). L. M. acknowledges the Project funded by China Postdoctoral Science Foundation (2022M723677). H. H. Z. is supported in part by the National Natural Science Foundation of China (NSFC) under Grants No. 12275367 and No. 11875327, the Fundamental Research Funds for the Central Universities, and the Sun Yat-Sen University Science Foundation. S. D. L. is supported by the Natural Science Foundation of Guangdong Province.

APPENDIX

In this appendix we present explicitly the calculational details of the main mathematical results of our approach.

1. The variation of the action with respect to $g^{\mu\nu}$ and $\Gamma^{\lambda}{}_{\mu\nu}$ in Palatini formalism

Firstly, let us start with the variation of Q_ρ and $\tilde{Q}_\rho$ with respect to $g^{\mu\nu}$. By the definitions

$$Q_{\rho\mu\nu} \equiv -\nabla_\rho g_{\mu\nu}, \qquad Q_\rho \equiv g^{\mu\nu} Q_{\rho\mu\nu}, \tag{A1}$$

and

$$\tilde{Q}_\rho = g^{\mu\nu} Q_{\mu\nu\rho}, \tag{A2}$$

we have

$$\delta_g Q_{\rho\mu\nu} = -\nabla_\rho \delta g_{\mu\nu}. \tag{A3}$$

Thus

$$\begin{aligned}\delta_g Q_\rho &= \delta_g(g^{\mu\nu} Q_{\rho\mu\nu}) = Q_{\rho\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta_g Q_{\rho\mu\nu} \\ &= Q_{\rho\mu\nu}\delta g^{\mu\nu} - g^{\alpha\beta}\nabla_\rho \delta g_{\alpha\beta} \\ &= Q_{\rho\mu\nu}\delta g^{\mu\nu} + g^{\alpha\beta}\nabla_\rho (g_{\alpha\mu} g_{\beta\nu}\delta g^{\mu\nu}),\end{aligned} \tag{A4}$$

and

$$\begin{aligned}\delta_g \tilde{Q}_\rho &= \delta_g(g^{\mu\nu} Q_{\mu\nu\rho}) = Q_{\mu\nu\rho}\delta g^{\mu\nu} + g^{\mu\nu}\delta_g Q_{\mu\nu\rho} \\ &= Q_{\mu\nu\rho}\delta g^{\mu\nu} - g^{\alpha\beta}\nabla_\alpha \delta g_{\beta\rho} \\ &= Q_{\mu\nu\rho}\delta g^{\mu\nu} + g^{\alpha\beta}\nabla_\alpha (g_{\beta\mu} g_{\rho\nu}\delta g^{\mu\nu}),\end{aligned} \tag{A5}$$

where $\delta g_{\alpha\beta} = -g_{\alpha\mu} g_{\beta\nu}\delta g^{\mu\nu}$ is used. With the use of another useful relation, $\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}$, one gets

$$\delta\sqrt{-g} = -\frac{\delta g}{2\sqrt{-g}} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}. \tag{A6}$$

Then the variation of (33) with respect to $g^{\mu\nu}$ gives

$$\begin{aligned}\delta_g S = \frac{1}{16\pi}\int d^4x \Big[& R\delta\sqrt{-g} + \sqrt{-g}R_{(\mu\nu)}\delta g^{\mu\nu} + \frac{5}{24}\cdot(Q_\alpha Q^\alpha \delta\sqrt{-g} + \sqrt{-g}Q_\mu Q_\nu \delta g^{\mu\nu} + 2\sqrt{-g}g^{\alpha\beta}Q_\alpha \delta_g Q_\beta)\\
&+\frac{1}{6}(\tilde{Q}_\alpha\tilde{Q}^\alpha\delta\sqrt{-g} + \sqrt{-g}\tilde{Q}_\mu\tilde{Q}_\nu\delta g^{\mu\nu} + 2\sqrt{-g}g^{\alpha\beta}\tilde{Q}_\alpha\delta_g\tilde{Q}_\beta) + 2T^\rho Q_\rho\delta\sqrt{-g} + 2\sqrt{-g}T^\rho\delta_g Q_\rho - 8\pi\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}\Big]\\
=\frac{1}{16\pi}\int d^4x\Big\{& \sqrt{-g}\delta g^{\mu\nu}\left(R_{(\mu\nu)} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{5}{24}\cdot\Big[\sqrt{-g}\delta g^{\mu\nu}\left(-\frac{1}{2}g_{\mu\nu}Q_\alpha Q^\alpha + Q_\mu Q_\nu + 2g^{\alpha\beta}Q_\alpha Q_{\beta\mu\nu}\right)\\
&+2\sqrt{-g}g^{\alpha\beta}Q_\alpha g^{\rho\sigma}\nabla_\beta(g_{\rho\mu}g_{\sigma\nu}\delta g^{\mu\nu})\Big] + \frac{1}{6}\cdot\Big[\sqrt{-g}\delta g^{\mu\nu}\left(-\frac{1}{2}g_{\mu\nu}\tilde{Q}_\alpha\tilde{Q}^\alpha + \tilde{Q}_\mu\tilde{Q}_\nu + 2g^{\alpha\beta}\tilde{Q}_\alpha Q_{\mu\nu\beta}\right)\\
&+2\sqrt{-g}g^{\alpha\beta}\tilde{Q}_\alpha g^{\rho\sigma}\nabla_\rho(g_{\sigma\mu}g_{\beta\nu}\delta g^{\mu\nu})\Big] + 2\sqrt{-g}T^\rho\cdot\Big[\delta g^{\mu\nu}\left(Q_{\rho\mu\nu} - \frac{1}{2}Q_\rho g_{\mu\nu}\right) + g^{\alpha\beta}\nabla_\rho(g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu})\Big] - 8\pi\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}\Big\}\\
=\frac{1}{16\pi}\int d^4x\sqrt{-g}\delta g^{\mu\nu}\Big\{& R_{(\mu\nu)} - \frac{1}{2}Rg_{\mu\nu} + \frac{5}{24}\cdot\Big[-\frac{1}{2}g_{\mu\nu}Q_\alpha Q^\alpha + Q_\mu Q_\nu + 2g^{\alpha\beta}Q_\alpha Q_{\beta\mu\nu} + 4g_{\mu\nu}T_\beta Q^\beta\\
&-\frac{2}{\sqrt{-g}}g_{\rho\mu}g_{\sigma\nu}\nabla_\beta(\sqrt{-g}g^{\alpha\beta}Q_\alpha g^{\rho\sigma})\Big] + \frac{1}{6}\Big[-\frac{1}{2}g_{\mu\nu}\tilde{Q}_\alpha\tilde{Q}^\alpha + \tilde{Q}_\mu\tilde{Q}_\nu + 2g^{\alpha\beta}\tilde{Q}_\alpha Q_{\mu\nu\beta} + 4T_\mu\tilde{Q}_\nu - \frac{2}{\sqrt{-g}}g_{\sigma\mu}g_{\beta\nu}\\
&\cdot\nabla_\rho(\sqrt{-g}g^{\alpha\beta}\tilde{Q}_\alpha g^{\rho\sigma})\Big] + 2\Big[T^\rho\left(Q_{\rho\mu\nu} - \frac{1}{2}Q_\rho g_{\mu\nu} + 2T_\rho g_{\mu\nu}\right) - \frac{1}{\sqrt{-g}}g_{\alpha\mu}g_{\beta\nu}\nabla_\rho(\sqrt{-g}T^\rho g^{\alpha\beta})\Big] - 8\pi T_{\mu\nu}\Big\}. \qquad \text{(A7)}\end{aligned}$$

Here we note that the total derivative terms of the form

$$\int d^4x\nabla_\lambda(\sqrt{-g}X^\lambda) \qquad \text{(A8)}$$

for some vector X^λ cannot be ignored, but rather instead they result in a net contribution given by [75]

$$2\int d^4x\sqrt{-g}T_\lambda X^\lambda. \qquad \text{(A9)}$$

Due to the presence of an extra term in the covariant derivative of a tensor density, and the generally nonsymmetric nature of the connection, we find

$$\begin{aligned}\nabla_\lambda(\sqrt{-g}X^\lambda) &= \partial_\lambda(\sqrt{-g}X^\lambda) + \Gamma^\lambda{}_{\rho\lambda}\sqrt{-g}X^\rho - \Gamma^\rho{}_{\rho\lambda}\sqrt{-g}X^\lambda\\
&= \partial_\lambda(\sqrt{-g}X^\lambda) + \sqrt{-g}(\Gamma^\rho{}_{\lambda\rho} - \Gamma^\rho{}_{\rho\lambda})X^\lambda\\
&= \partial_\lambda(\sqrt{-g}X^\lambda) + 2\sqrt{-g}T_\lambda X^\lambda. \qquad \text{(A10)}\end{aligned}$$

In our model the torsion does not contribute to the final EoM due to (34), as can be seen below.

Noticing that

$$\begin{aligned}&\nabla_\beta(\sqrt{-g}g^{\alpha\beta}Q_\alpha g^{\rho\sigma})\\
&= (\nabla_\beta\sqrt{-g})g^{\alpha\beta}Q_\alpha g^{\rho\sigma} + \sqrt{-g}(\nabla_\beta g^{\alpha\beta})Q_\alpha g^{\rho\sigma}\\
&\quad + \sqrt{-g}g^{\alpha\beta}(\nabla_\beta Q_\alpha)g^{\rho\sigma} + \sqrt{-g}g^{\alpha\beta}Q_\alpha\nabla_\beta g^{\rho\sigma}\\
&= \sqrt{-g}\bigg(-\frac{1}{2}Q_\alpha Q^\alpha g^{\rho\sigma} + \tilde{Q}^\alpha Q_\alpha g^{\rho\sigma} + g^{\rho\sigma}g^{\alpha\beta}\nabla_\beta Q_\alpha\\
&\quad + g^{\alpha\beta}Q_\alpha Q_\beta^{\rho\sigma}\bigg) \qquad \text{(A11)}\end{aligned}$$

and

$$\begin{aligned}&\nabla_\rho(\sqrt{-g}g^{\alpha\beta}\tilde{Q}_\alpha g^{\rho\sigma})\\
&= (\nabla_\rho\sqrt{-g})g^{\alpha\beta}\tilde{Q}_\alpha g^{\rho\sigma} + \sqrt{-g}(\nabla_\rho g^{\alpha\beta})\tilde{Q}_\alpha g^{\rho\sigma}\\
&\quad + \sqrt{-g}g^{\alpha\beta}(\nabla_\rho\tilde{Q}_\alpha)g^{\rho\sigma} + \sqrt{-g}g^{\alpha\beta}\tilde{Q}_\alpha\nabla_\rho g^{\rho\sigma}\\
&= \sqrt{-g}\bigg(-\frac{1}{2}Q_\rho\tilde{Q}_\alpha g^{\alpha\beta}g^{\rho\sigma} + Q_\rho^{\alpha\beta}\tilde{Q}_\alpha g^{\rho\sigma} + g^{\rho\sigma}g^{\alpha\beta}\nabla_\rho\tilde{Q}_\alpha\\
&\quad + g^{\alpha\beta}\tilde{Q}_\alpha\tilde{Q}^\sigma\bigg), \qquad \text{(A12)}\end{aligned}$$

where (38) and (39) are used, we are then able to write down the modified Einstein equation by inserting (A11) and (A12) into (A7) and setting $\delta_g S = 0$:

$$\begin{aligned}&R_{(\mu\nu)} - \frac{1}{2}Rg_{\mu\nu} + \frac{5}{24}\bigg(\frac{1}{2}g_{\mu\nu}Q_\alpha Q^\alpha + Q_\mu Q_\nu - 2g_{\mu\nu}Q_\alpha\tilde{Q}^\alpha\\
&\quad - 2g_{\mu\nu}g^{\alpha\beta}\nabla_\beta Q_\alpha\bigg) + \frac{1}{6}\bigg(-\frac{1}{2}g_{\mu\nu}\tilde{Q}_\alpha\tilde{Q}^\alpha - \tilde{Q}_\mu\tilde{Q}_\nu\\
&\quad + Q_\mu\tilde{Q}_\nu - 2\nabla_\mu\tilde{Q}_\nu\bigg) = 8\pi T_{\mu\nu}. \qquad \text{(A13)}\end{aligned}$$

Now let us turn to the variation of action with respect to $\Gamma^\lambda_{\mu\nu}$. One can easily find first [60]

$$\delta_\Gamma T_\rho = \frac{1}{2}(\delta^\mu_\rho\delta^\nu_\lambda - \delta^\mu_\lambda\delta^\nu_\rho)\delta\Gamma^\lambda_{\mu\nu}, \qquad \text{(A14)}$$

$$\delta_\Gamma R^\alpha{}_{\beta\rho\sigma} = \nabla_\rho\delta\Gamma^\alpha{}_{\beta\sigma} - \nabla_\sigma\delta\Gamma^\alpha{}_{\beta\rho} - 2T^\lambda{}_{\rho\sigma}\delta\Gamma^\alpha{}_{\beta\lambda}, \qquad \text{(A15)}$$

$$\delta_\Gamma Q_\rho = 2\delta^\nu_\rho\delta^\mu_\lambda\delta\Gamma^\lambda_{\mu\nu}, \qquad \text{(A16)}$$

$$\delta_\Gamma \tilde{Q}_\rho = (g^{\mu\nu} g_{\rho\lambda} + \delta^\mu_\rho \delta^\nu_\lambda) \delta\Gamma^\lambda_{\mu\nu}. \tag{A17}$$

With these results the variation of (33) leads to

$$\begin{aligned}
\delta_\Gamma S &= \int d^4x \sqrt{-g} \Big[g^{\alpha\beta} \delta_\Gamma R^\rho{}_{\alpha\rho\beta} + \frac{5}{12} Q^\rho \delta_\Gamma Q_\rho + \frac{1}{3} \tilde{Q}^\rho \delta_\Gamma \tilde{Q}_\rho + 2Q^\rho \delta_\Gamma T_\rho + 2T^\rho \delta_\Gamma Q_\rho + \zeta^{[\mu\nu]}{}_\lambda \delta\Gamma^\lambda{}_{\mu\nu} \Big] \\
&= \int d^4x \sqrt{-g} \Big[g^{\alpha\beta} (\nabla_\rho \delta\Gamma^\rho{}_{\alpha\beta} - \nabla_\beta \delta\Gamma^\rho{}_{\alpha\rho} - 2T^\lambda{}_{\rho\beta} \delta\Gamma^\rho{}_{\alpha\lambda}) + \delta\Gamma^\lambda_{\mu\nu} \cdot \Big(\frac{5}{6} Q^\nu \delta^\mu_\lambda + \frac{1}{3} \tilde{Q}_\lambda g^{\mu\nu} + \frac{1}{3} \tilde{Q}^\mu \delta^\nu_\lambda \\
&\quad + Q^\mu \delta^\nu_\lambda - Q^\nu \delta^\mu_\lambda + 4T^\nu \delta^\mu_\lambda + \zeta^{[\mu\nu]}{}_\lambda \Big) \Big] \\
&= \int d^4x \delta\Gamma^\lambda{}_{\mu\nu} \Big[-\nabla_\lambda (\sqrt{-g} g^{\mu\nu}) + \nabla_\beta (\sqrt{-g} g^{\mu\beta}) \delta^\nu_\lambda + 2\sqrt{-g} (T_\lambda g^{\mu\nu} - T^\mu \delta^\nu_\lambda - g^{\mu\beta} T^\nu{}_{\lambda\beta} + 2T^\nu \delta^\mu_\lambda) \\
&\quad + \sqrt{-g} \Big(\frac{1}{3} \tilde{Q}_\lambda g^{\mu\nu} + \frac{1}{3} \tilde{Q}^\mu \delta^\nu_\lambda + Q^\mu \delta^\nu_\lambda - \frac{1}{6} Q^\nu \delta^\mu_\lambda + \zeta^{[\mu\nu]}{}_\lambda \Big) \Big].
\end{aligned} \tag{A18}$$

Thus $\delta_\Gamma S = 0$ gives

$$-\frac{\nabla_\lambda (\sqrt{-g} g^{\mu\nu})}{\sqrt{-g}} + \frac{\nabla_\rho (\sqrt{-g} g^{\mu\rho}) \delta^\nu_\lambda}{\sqrt{-g}} + \frac{1}{3} \tilde{Q}_\lambda g^{\mu\nu} + \frac{1}{3} \tilde{Q}^\mu \delta^\nu_\lambda + Q^\mu \delta^\nu_\lambda - \frac{1}{6} Q^\nu \delta^\mu_\lambda + \zeta^{[\mu\nu]}_\lambda = 0. \tag{A19}$$

Note that again the torsion terms disappear in the EoM.

2. The variation of the action with respect to $g^{\mu\nu}$ in metric formalism

Since in the metric formalism the connection is assumed to depend on the metric in the way given by Eq. (20), the variation with respect to $g_{\mu\nu}$ now has the extra contribution $\delta_g R_{\mu\nu}$, as compared to Eq. (A7). To calculate this contribution, note that

$$\delta_g Q^\alpha{}_{\mu\nu} = -\delta_g (g^{\alpha\lambda} \nabla_\lambda g_{\mu\nu}) = -g^{\alpha\lambda} \nabla_\lambda \delta g_{\mu\nu} - \delta g^{\alpha\lambda} \nabla_\lambda g_{\mu\nu}, \tag{A20}$$

and

$$\delta_g \tilde{Q}_\rho = -\frac{1}{2} \delta_g Q_\rho = -\frac{1}{2} Q_{\rho\mu\nu} \delta g^{\mu\nu} + \frac{1}{2} g^{\alpha\beta} \nabla_\rho \delta g_{\alpha\beta}. \tag{A21}$$

Then from Eq. (25) we have

$$\begin{aligned}
\int d^4x \sqrt{-g} g^{\mu\nu} \delta_g R_{\mu\nu} &= \int d^4x \sqrt{-g} g^{\mu\nu} (\delta_g \mathring{R}_{\mu\nu} - \mathring{\nabla}_\alpha \delta_g Q^\alpha{}_{\mu\nu} + \mathring{\nabla}_\nu \delta_g \tilde{Q}_\mu + Q^\rho{}_{\mu\nu} \delta_g \tilde{Q}_\rho + \tilde{Q}_\rho \delta_g Q^\rho{}_{\mu\nu} - Q^\rho{}_{\mu\alpha} \delta_g Q^\alpha{}_{\rho\nu} - Q^\alpha{}_{\rho\nu} \delta_g Q^\rho{}_{\mu\alpha}) \\
&= \int d^4x \sqrt{-g} g^{\mu\nu} \Big[Q^\rho{}_{\mu\nu} \Big(-\frac{1}{2} Q_{\rho\alpha\beta} \delta g^{\alpha\beta} + \frac{1}{2} g^{\alpha\beta} \nabla_\rho \delta g_{\alpha\beta} \Big) - \frac{1}{2} Q_\rho (-g^{\rho\lambda} \nabla_\lambda \delta g_{\mu\nu} - \delta g^{\rho\lambda} \nabla_\lambda g_{\mu\nu}) \\
&\quad - Q^\rho_{\mu\alpha} (-g^{\alpha\lambda} \nabla_\lambda \delta g_{\rho\nu} - \delta g^{\alpha\lambda} \nabla_\lambda g_{\rho\nu}) - Q^\alpha_{\rho\nu} (-g^{\rho\lambda} \nabla_\lambda \delta g_{\mu\alpha} - \delta g^{\rho\lambda} \nabla_\lambda g_{\mu\alpha}) \Big] \\
&= \int d^4x \sqrt{-g} \Big(-\frac{1}{2} Q^\rho Q_{\rho\alpha\beta} \delta g^{\alpha\beta} + \frac{1}{2} Q^\rho g^{\alpha\beta} \nabla_\rho \delta g_{\alpha\beta} + \frac{1}{2} g^{\mu\nu} Q^\lambda \nabla_\lambda \delta g_{\mu\nu} - \frac{1}{2} g^{\mu\nu} Q_\rho \delta g^{\rho\lambda} Q_{\lambda\mu\nu} \\
&\quad + Q^{\rho\nu\lambda} \nabla_\lambda \delta g_{\rho\nu} - Q^{\rho\nu}{}_\alpha \delta g^{\alpha\lambda} Q_{\lambda\rho\nu} + Q^{\alpha\lambda\mu} \nabla_\lambda \delta g_{\mu\alpha} - Q^\alpha{}_\rho{}^\mu \delta g^{\rho\lambda} Q_{\lambda\mu\alpha} \Big) \\
&= \int d^4x \Big[\sqrt{-g} \delta g^{\mu\nu} \Big(-\frac{1}{2} Q^\rho Q_{\rho\mu\nu} - \frac{1}{2} Q_\mu Q_\nu - Q^{\rho\sigma}{}_\mu Q_{\nu\rho\sigma} - Q^\rho{}_\mu{}^\sigma Q_{\nu\sigma\rho} \Big) - \frac{1}{2} \delta g_{\alpha\beta} \nabla_\rho (\sqrt{-g} Q^\rho g^{\alpha\beta})
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\delta g_{\mu\nu}\nabla_\lambda(\sqrt{-g}Q^\lambda g^{\mu\nu}) - \delta g_{\rho\nu}\nabla_\lambda(\sqrt{-g}Q^{\rho\nu\lambda}) - \delta g_{\mu\alpha}\nabla_\lambda(\sqrt{-g}Q^{\alpha\lambda\mu})\Big]\\
&= \int d^4x\sqrt{-g}\delta g^{\mu\nu}\left[-\frac{1}{2}Q^\rho Q_{\rho\mu\nu} - \frac{1}{2}Q_\mu Q_\nu - 2Q^{\rho\sigma}{}_\mu Q_{\nu\rho\sigma} + \frac{1}{\sqrt{-g}}g_{\alpha\mu}g_{\beta\nu}(\nabla_\rho(\sqrt{-g}Q^\rho g^{\alpha\beta}) + 2\nabla_\lambda(\sqrt{-g}Q^{\alpha\beta\lambda}))\right]\\
&= \int d^4x\sqrt{-g}\delta g^{\mu\nu}\left(\frac{1}{2}Q^\rho Q_{\rho\mu\nu} - \frac{1}{2}Q_\mu Q_\nu - 2Q^{\rho\sigma}{}_\mu Q_{\nu\rho\sigma} - \frac{1}{2}Q_\rho Q^\rho g_{\mu\nu} + g_{\mu\nu}\nabla_\rho Q^\rho - Q_\lambda Q_{\mu\nu}{}^\lambda + 2g_{\alpha\mu}g_{\beta\nu}\nabla_\lambda Q^{\alpha\beta\lambda}\right). \qquad \text{(A22)}
\end{aligned}
$$

With the solution (43), we are able to write (40) as

$$
Q_{\lambda\mu\nu} = \frac{1}{3}Q_\lambda g_{\mu\nu} - \frac{1}{6}Q_\mu g_{\nu\lambda} - \frac{1}{6}Q_\nu g_{\lambda\mu}, \qquad \text{(A23)}
$$

and noticing that because of $Q_{(\rho\mu\nu)} = 0$,

$$
\delta g^{\mu\nu}Q_\lambda Q_{\mu\nu}{}^\lambda = \delta g^{\mu\nu}Q^\rho\frac{Q_{\mu\nu\rho} + Q_{\nu\mu\rho}}{2} = -\frac{1}{2}\delta g^{\mu\nu}Q^\rho Q_{\rho\mu\nu} \qquad \text{(A24)}
$$

and

$$
\begin{aligned}
2\delta g^{\mu\nu}g_{\alpha\mu}g_{\beta\nu}\nabla_\lambda Q^{\alpha\beta\lambda} &= 2\delta g^{\mu\nu}g_{\alpha\mu}g_{\beta\nu}\nabla_\lambda\frac{Q^{\alpha\beta\lambda} + Q^{\beta\alpha\lambda}}{2}\\
&= -\delta g^{\mu\nu}g_{\alpha\mu}g_{\beta\nu}\nabla_\lambda Q^{\lambda\alpha\beta}, \qquad \text{(A25)}
\end{aligned}
$$

(A22) can be further simplified to

$$
\begin{aligned}
\int d^4x\sqrt{-g}g^{\mu\nu}\delta_g R_{\mu\nu} = \int d^4x\sqrt{-g}\delta g^{\mu\nu}\Big(&-\frac{2}{9}g_{\mu\nu}Q^\rho Q_\rho\\
&-\frac{11}{18}Q_\mu Q_\nu + \frac{2}{3}g_{\mu\nu}\nabla_\rho Q^\rho\\
&+\frac{1}{6}g_{\rho\mu}\nabla_\nu Q^\rho + \frac{1}{6}g_{\rho\nu}\nabla_\mu Q^\rho\Big). \qquad \text{(A26)}
\end{aligned}
$$

All the other contributions to the Einstein equation will be the same as in the Palatini case. Hence, the field equation in the metric formalism can be obtained by simply adding the above terms into Eq. (46), and thus we finally arrive at Eq. (47).

3. Calculating the Friedmann equations

According to (48) and (49), all the nonzero components of $Q_{\lambda\mu\nu}$ are

$$
Q_{0ii} = \frac{1}{3}a^2 b \quad \text{and} \quad Q_{ii0} = Q_{i0i} = -\frac{1}{6}a^2 b, \qquad \text{(A27)}
$$

where $i = 1$, 2 or 3. Note that $\gamma^\lambda_{\mu\nu}$ has the same structure,

$$
\gamma^0_{ii} = a\dot{a} \quad \text{and} \quad \gamma^i_{i0} = \gamma^i_{0i} = \frac{\dot{a}}{a}, \qquad \text{(A28)}
$$

we can then calculate all the nonzero components of the Schrödinger connection (20),

$$
\Gamma^0{}_{ii} = a\dot{a} + \frac{1}{3}a^2 b \quad \text{and} \quad \Gamma^i{}_{i0} = \Gamma^i{}_{0i} = \frac{\dot{a}}{a} + \frac{1}{6}b, \qquad \text{(A29)}
$$

and get the well known results for $\mathring{R}_{\mu\nu}$

$$
\mathring{R}_{00} = 3\frac{\dot{a}^2}{a^2} \quad \text{and} \quad \mathring{R}_{ii} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}. \qquad \text{(A30)}
$$

On the other hand, since

$$
Q_0 = -2\tilde{Q}_0 = b \quad \text{and} \quad Q_i = \tilde{Q}_i = 0, \qquad \text{(A31)}
$$

one gets

$$
Q_\alpha Q^\alpha = -b^2, \qquad \text{(A32)}
$$

$$
\nabla_0 Q^0 = \partial_0 Q^0 = -\dot{b}, \quad \nabla_i Q^i = \Gamma^i{}_{0i}Q^0 = -\frac{\dot{a}}{a}b - \frac{1}{6}b^2 \qquad \text{(A33)}
$$

and

$$
\nabla_\rho Q^\rho = -\dot{b} - 3\frac{\dot{a}}{a}b - \frac{1}{2}b^2. \qquad \text{(A34)}
$$

Inserting all the above results into (47), and considering $T_{00} = \rho$ and $T_{ii} = p$, we are able to write down the Friedmann equations as (51) and (52).

[1] A. Einstein, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften (Berlin) 844 (1915), https://ui.adsabs.harvard.edu/abs/1915SPAW.......844E/abstract.
[2] D. Hilbert, Nachr. Ges. Wiss. Gött. Math. Phys. Klasse **3**, 395 (1915).
[3] A. Einstein, Ann. Phys. (Berlin) **354**, 769 (1916).
[4] B. Riemann, Abh. K. Ges. Wiss. Gött. **13**, 1 (1867).
[5] H. A. Buchdahl, Mon. Not. R. Astron. Soc. **150**, 1 (1970).
[6] J. D. Barrow and A. C. Otteville, J. Phys. A **16**, 2757 (1983).
[7] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, Phys. Rep. **692**, 1 (2017).
[8] T. Harko and F. S. N. Lobo, Eur. Phys. J. C **70**, 373 (2010).
[9] T. Harko, F. S. N. Lobo, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **84**, 024020 (2011).
[10] T. Harko and F. S. N. Lobo, *Extensions of f(R) Gravity: Curvature-Matter Couplings and Hybrid Metric Palatini Theory* (Cambridge University Press, Cambridge, UK, 2018).
[11] H. Weyl, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften (Berlin) 465 (1918), https://ui.adsabs.harvard.edu/abs/1918SPAW.......465W/abstract.
[12] H. Weyl, *Space, Time, Matter* (Dover, New York, 1952).
[13] C. Pala, O, Sert, and M. Adak, Eur. Phys. J. C **83**, 17 (2023).
[14] A. S. Eddington, *Space, The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, England, 1923).
[15] W. Pauli, *The Theory of Relativity* (Dover Publications, Mineola, New York, 1981).
[16] É. Cartan, C. R. Acad. Sci. (Paris) **174**, 593 (1922).
[17] É. Cartan, Ann. Sci. L'École Norm. **40**, 325 (1923).
[18] É. Cartan, Ann. Sci. L'École Norm. **41**, 1 (1924).
[19] É. Cartan, Ann. Sci. L'École Norm. **42**, 17 (1925).
[20] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976).
[21] R. Weitzenböck, *Invariantentheorie* (Noordhoff, Groningen, 1923).
[22] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Klasse **1928**, 217 (1928).
[23] C. Möller, Mat. Fys. Skr. Dan. Vid. Selsk. **1**, 10 (1961).
[24] C. Pellegrini and J. Plebanski, Mat. Fys. Skr. Dan. Vid. Selsk. **2**, 4 (1963).
[25] K. Hayashi and T. Shirafuji, Phys. Rev. D **19**, 3524 (1979).
[26] J. Levi Said, J. Mifsud, D. Parkinson, E. N. Saridakis, J. Sultana, and K. Zarb Adami, J. Cosmol. Astropart. Phys. 11 (2020) 047.
[27] M. Chakrabortty, N. Sk, and A. Kumar Sanyal, Eur. Phys. J. C **83**, 557 (2023).
[28] S. Bahamonde, G. Trenkler, L. G. Trombetta, and M. Yamaguchi, Phys. Rev. D **107**, 104024 (2023).
[29] R. Briffa, C. Escamilla-Rivera, J. Levi Said, and J. Mifsud, Mon. Not. R. Astron. Soc. **522**, 6024 (2023).
[30] Y. F. Cai, S. Capozziello, M. De Laurentis, and E. N. Saridakis, Rep. Prog. Phys. **79**, 106901 (2016).
[31] E. Scholz, The unexpected resurgence of Weyl geometry in late 20th-century physics, in *Beyond Einstein: Perspectives on Geometry, Gravitation, and Cosmology in the Twentieth Century*, edited by D. E. Rowe, T. Sauer, and S. A. Walter (Springer, New York, NY, 2018), pp. 261–360.
[32] P. A. M. Dirac, Proc. R. Soc. A **333**, 403 (1973).
[33] P. A. M. Dirac, Proc. R. Soc. A **338**, 439 (1974).
[34] N. Rosen, Found. Phys. **12**, 213 (1982).
[35] M. Israelit, Gen. Relativ. Gravit. **43**, 751 (2011).
[36] H.-H. von Borzeszkowski and H.-J. Treder, Gen. Relativ. Gravit. **29**, 455 (1997).
[37] D. Puetzfeld and R. Tresguerres, Classical Quantum Gravity **18**, 677 (2001).
[38] D. Putzfeld, Classical Quantum Gravity **19**, 4463 (2002).
[39] D. Puetzfeld, Classical Quantum Gravity **19**, 3263 (2002).
[40] T. Y. Moon, J. Lee, and P. Oh, Mod. Phys. Lett. A **25**, 3129 (2010).
[41] T. Y. Moon, P. Oh, and J. S. Sohn, J. Cosmol. Astropart. Phys. 11 (2010) 005.
[42] M. Novello and S. E. Perez Bergliaffa, Phys. Rep. **463**, 127 (2008).
[43] Z. Haghani, T. Harko, H. R. Sepangi, and S. Shahidi, J. Cosmol. Astropart. Phys. 10 (2012) 061.
[44] Z. Haghani, T. Harko, H. R. Sepangi, and S. Shahidi, Phys. Rev. D **88**, 044024 (2013).
[45] J. M. Nester and H.-J. Yo, Chin. J. Phys. **37**, 113 (1999).
[46] J. Beltran Jimenez, L. Heisenberg, and T. Koivisto, Phys. Rev. D **98**, 044048 (2018).
[47] J. Beltran Jimenez, L. Heisenberg, and T. S. Koivisto, Universe **5**, 173 (2019).
[48] J. Beltrán Jiménez, L. Heisenberg, T. S. Koivisto, and S. Pekar, Phys. Rev. D **101**, 103507 (2020).
[49] J. A. Nájera, C. Aráoz Alvarado, and C. Escamilla-Rivera, Mon. Not. R. Astron. Soc. **524**, 5280 (2023).
[50] P. Bhar and J. M. Z. Pretel, Phys. Dark Universe **42**, 101322 (2023).
[51] S. A. Narawade, S. P. Singh, and B. Mishra, Phys. Dark Universe **42**, 101282 (2023).
[52] A. Mussatayeva, N. Myrzakulov, and M. Koussour, Phys. Dark Universe **42**, 101276 (2023).
[53] N. Myrzakulov, M. Koussour, and D. J. Gogoi, Phys. Dark Universe **42**, 101268 (2023).
[54] L. Heisenberg, arXiv:2309.15958.
[55] T. Harko, T. S. Koivisto, F. S. N. Lobo, G. J. Olmo, and D. Rubiera-Garcia, Phys. Rev. D **98**, 084043 (2018).
[56] Y. Xu, G. Li, T. Harko, and S.-D. Liang, Eur. Phys. J. C **79**, 708 (2019).
[57] E. Schrödinger, Nature (London) **153**, 572 (1944).
[58] E. Schrödinger, *Space-Time Structure*, Cambridge Science Classics (Cambridge University Press, Cambridge, England, 1985).
[59] S. Klemm and L. Ravera, Phys. Lett. B **817**, 136291 (2021).
[60] D. Iosifidis, arXiv:1902.09643.
[61] Y. Akrami *et al.*, Astron. Astrophys. **641**, A1 (2020).
[62] N. Aghanim *et al.*, Astron. Astrophys. **641**, A6 (2020).
[63] H. Boumaza and K. Nouicer, Phys. Rev. D **100**, 124047 (2019).
[64] M. Chevallier and D. Polarski, Int. J. Mod. Phys. D **10**, 213 (2001).
[65] E. V. Linder, Phys. Rev. Lett. **90**, 091301 (2003).
[66] V. Sahni, A. Shafieloo, and A. A. Starobinsky, Phys. Rev. D **78**, 103502 (2008).
[67] R. Penrose, *Cycles of Time: An Extraordinary New View of the Universe* (Random House, London, UK, 2010).
[68] R. Penrose, Found. Phys. **44**, 873 (2014).

[69] G. 't Hooft, Int. J. Mod. Phys. D **24**, 1543001 (2015).
[70] G. 't Hooft, Singularities, horizons, firewalls, and local conformal symmetry, in *2nd Karl Schwarzschild Meeting on Gravitational Physics*, edited by P. Nicolini, M. Kaminski, J. Mureika, and M. Bleicher (Springer International Publishing, Cham, 2018), pp. 1–12.
[71] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).
[72] S. M. Carroll, Living Rev. Relativity **4**, 1 (2001).
[73] E. Di Valentino, Classical Quantum Gravity **38**, 153001 (2021).
[74] S. Vagnozzi, Universe **9**, 393 (2023).
[75] H. Burton, *On the Palatini Variation and Connection Theories of Gravity* (University of Waterloo, Waterloo, Canada, 1999).