


# Cosmological evolution of scalar field dark matter in the CLASS code: Accuracy and precision of numerical solutions

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We present a numerical analysis of the cosmological evolution of scalar field dark matter (SFDM) in the Boltzmann code CLASS, based on a dynamical system analysis of previous works. We show a detailed study of the evolution of the different dynamical variables and in particular of the energy density and its corresponding linear perturbations. The numerical results are in good agreement with those of the original SFDM equations of motion and have better accuracy than other approaches. In addition, we calculate the temperature and matter power spectra and discuss the reliability of their numerical results. We also give simple examples in which we can put constraints on the field mass using recent likelihoods incorporated in the Monte Carlo Markov chain sampler MontePython.

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## I. INTRODUCTION

In the modern era of cosmology [1,2], it is mandatory to develop theoretical models capable of describing the Universe at large scales, with the precision that current data demand [2]. One key theoretical ingredient, particularly important for the formation of cosmic structure, is dark matter (DM), which must capture the physics of such a process in a mathematically consistent way [3]. For the cosmological setting, this always starts with properly solving the Einstein-Boltzmann system describing the cosmological evolution for the initial perturbations of both the matter components and the metric tensor at the linear level [4].

Different Boltzmann solvers have been programmed for the linearized form of Einstein equations in a Friedmann-Robertson-Walker-Lemaître (FRWL) universe, such as CMBFAST [5], CMBEASY [6], CAMB [7], and CLASS [8]. Only the latter two, CAMB and CLASS, have been kept up to date, and both are used by the cosmology community. The aforementioned codes consider the cold dark matter (CDM) model as the main matter component, and they are very well suited to explore most of its properties, although they have been amended to study alternative dark matter models in recent years.

One of these alternative models to CDM, which is currently one of the compelling proposals that has been explored for the last two decades, is that based on a scalar

field. It is found in the literature under several names: scalar field dark matter (SFDM), ultralight axions, fuzzy dark matter, axionlike particle, Bose-Einstein condensate, and wave dark matter (some initial works on this model are [9–18], whereas more recent work can be found at [19–28]). All of these names reflect the particular properties of this DM particle. Its mathematical description is given by a scalar field  $\phi$  (which can be real or complex), with a fiducial ultralight mass of  $m_\phi c^2 \sim 10^{-22}$  eV that can be produced by the Peccei-Quinn mechanism for pseudo-Goldstone bosons (as is the case for QCD axions) and whose quantum nature manifests itself at cosmological scales through its imprint on the structure formation at small scales. Therefore, we shall refer to this model as SFDM hereafter.

To solve for the cosmological dynamics of linear perturbations of the SFDM, there exists AxionCAMB [20],<sup>1</sup> an amended version of CAMB for a scalar field endowed with a quadratic potential  $V(\phi) \propto \phi^2$ , although a new version was recently presented for cases similar to axions [29]. On the other hand, one of the authors of this present work modified CLASS with the aim of including the SFDM model with the same quadratic potential: CLASS.FreeSF [19].<sup>2</sup> Both aforementioned codes deal with the background dynamics and linear perturbations of the SFDM, and it is possible to obtain the anisotropies of the cosmic microwave background

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<sup>1</sup><https://github.com/dgrin1/axionCAMB>.

<sup>2</sup><https://github.com/lurena-lopez/class.FreeSF>.

radiation (CMB) as well as the matter power spectrum (MPS) for several mass values of the SFDM particle.

We have made other amendments to CLASS to study potentials with self-interaction in the scalar field potential. For example, the full axionlike potential  $V(\phi) \propto 1 - \cos \phi$  was fully implemented for the first time in CLASS in the work [30], where it was shown that this model presents an excess of power on small scales in the MPS with respect to CDM,<sup>3</sup> with a greater discussion of the cosmological signatures of this SFDM model in [32].<sup>4</sup> The other SFDM potential we have considered in CLASS is that of a hyperbolic function  $V(\phi) \propto \cosh \phi - 1$  [15,16,37,38], where the self-interacting term has the opposite sign to that of axions.

As we will show later, the mathematical treatment of the SFDM cosmological evolution used in the modified versions of CLASS makes use of new dynamical variables and the tools of dynamical systems. The output from the codes (initial conditions in the radiation-dominated era, physical effects on observables such as CMB, MPS, the halo mass function, and others) turns out to be crucial for the subsequent realization of realistic numerical simulations of structure formation in the nonlinear regime; see, for example, Refs. [39–43]. We have even shown that this is useful for a unified description of different scalar potentials in a one-parametric way [44].

Our main goal in the present work is to give more details of the mathematical approach we have used in previously amended versions of CLASS and in turn to show its robustness in describing the physical processes of SFDM up to the level of linear density perturbations. This is done here for SFDM endowed with a quadratic potential,

$$V(\phi) = \frac{1}{2} m_\phi^2 \phi^2, \quad (1)$$

where  $m_\phi$  is the mass of the scalar field particle and the only free parameter in the model. Note that we are using natural units with  $\hbar = c = 1$ , and then  $m_\phi$  is given in units of eV. Whenever appropriate, we will compare our method to the common fluid approximation to the SFDM dynamics used in other works, following the description in Refs. [20,45,46].

This paper is organized as follows. In Sec. II, we develop the mathematical formulation for the evolution of the background and linear density perturbations, in terms of new dynamical variables that are appropriate to handle the particularities of SFDM. We also establish the appropriate

<sup>3</sup>Recently, these kinds of features in the MPS have also been reported for a scalar field with a nonperiodic potential [31].

<sup>4</sup>Separately, the authors in Refs. [33–36] developed AXICLASS (the corresponding repository can be found at <https://github.com/PoulinV/AxiCLASS>). The potential of the scalar field in this code is of the form  $V(\phi) \propto (1 - \cos \phi)^n$ . The interest in such a generalization arose mostly for the so-called early dark energy models.

initial conditions for the scalar field to behave as the DM component at late times, for both background and linear quantities.

Section III is dedicated to the description of the typical regime of rapid oscillations of the SFDM at late times in its evolution and to the way in which we deal with them for their reliable numerical computation. In particular, we show a detailed study of the evolution of the (barotropic) equation of state, the energy density, and finally the linear density perturbations. As an application of our method in the Boltzmann code CLASS, we present the temperature and matter power spectra and discuss the reliability of our numerical results. Moreover, we also give simple examples in which we can put constraints on the field mass  $m_\phi$  using recent likelihoods incorporated in the Monte Carlo Markov chain (MCMC) sampler MontePython [47,48], taking advantage of its close interoperability with CLASS.

The comparison and equivalence between our approach and the original formulation in terms of the scalar field itself are presented in Sec. IV. We explicitly show the transformation between our variables and the original field ones ( $\phi, \dot{\phi}$ ) and that the numerical results of our method are completely equivalent to the standard field approach. Finally, in Sec. V, we summarize and discuss our results, highlighting the advantages of our method.

## II. MATHEMATICAL BACKGROUND

In this section, we present the equations of motion for the SFDM model in the context of an expanding universe and the subsequent transformations we use to make them more suitable for numerical computations. The original motivations and some extra details of the method described here can be found in Refs. [19,30,49].

### A. Background evolution

Let us consider a spatially flat FRWL line element,

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)], \quad (2)$$

where  $a(t)$  is the scale factor. The background equations for ordinary matter, which is represented by perfect fluids with density  $\rho_j$  and pressure  $p_j$ , as well as for SFDM field  $\phi$  endowed with the potential (1), are given by

$$H^2 = \frac{\kappa^2}{3} \left( \sum_j \rho_j + \rho_\phi \right), \quad (3a)$$

$$\dot{H} = -\frac{\kappa^2}{2} \left[ \sum_j (\rho_j + p_j) + (\rho_\phi + p_\phi) \right], \quad (3b)$$

$$\dot{\rho}_j = -3H(\rho_j + p_j), \quad (3c)$$

$$\ddot{\phi} = -3H\dot{\phi} - m_\phi^2 \phi, \quad (3d)$$

where  $\kappa^2 = 8\pi G$ . The dot denotes the derivative with respect to cosmic time  $t$ , and  $H = \dot{a}/a$  is the Hubble parameter. In the equations above, the scalar field density  $\rho_\phi$  and pressure  $p_\phi$  are defined, respectively, as

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m_\phi^2\phi^2, \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m_\phi^2\phi^2. \quad (4)$$

For the SFDM part, it is convenient to use the following change of variables [19,30,32,49–51]:

$$\frac{\kappa\dot{\phi}}{\sqrt{6}H} = e^\beta \sin(\theta/2), \quad -\frac{\kappa m_\phi \phi}{\sqrt{6}H} = e^\beta \cos(\theta/2), \quad (5a)$$

$$y_1 \equiv \frac{2m_\phi}{H}. \quad (5b)$$

To understand the meaning of the new variables, we write here the SFDM density parameter  $\Omega_\phi$  and equation of state (EOS)  $w_\phi$ ,

$$\Omega_\phi = \frac{8\pi G\rho_\phi}{3H^2} = e^{2\beta}, \quad (6a)$$

$$w_\phi = \frac{\dot{\phi}^2 - m_\phi^2\phi^2}{\dot{\phi}^2 + m_\phi^2\phi^2} = -\cos\theta. \quad (6b)$$

Thus,  $\beta$  is the logarithm of the energy density parameter, and  $\theta$ , which is an internal polar angle, is directly related to  $w_\phi$ . Lastly,  $y_1$  is simply the ratio of the field mass to the Hubble parameter (i.e., dimensionless by definition), which is a ubiquitous quantity in all methods for solutions of the SFDM equations of motion.

Using the new variables, the Klein-Gordon equation (3d) transforms into the following set of first-order differential equations [19]:

$$\theta' = -3\sin\theta + y_1, \quad (7a)$$

$$y_1' = \frac{3}{2}(1 + w_{\text{tot}})y_1, \quad (7b)$$

$$\beta' = \frac{3}{2}(w_{\text{tot}} + \cos\theta). \quad (7c)$$

Here, a prime denotes derivatives with respect to the number of  $e$ -folds of expansion  $N = \ln a$ .<sup>5</sup> The total EOS  $w_{\text{tot}}$ , which is used in the foregoing equations, can be calculated from the ratio of the total pressure  $p_{\text{tot}}$  to the total density

$\rho_{\text{tot}}$  in the Universe, and its explicit expression is given in terms of the EOS of the different components of matter as

$$w_{\text{tot}} = \frac{p_{\text{tot}}}{\rho_{\text{tot}}} = \frac{\sum_j w_j \rho_j + w_\phi \rho_\phi}{\sum_j \rho_j + \rho_\phi}. \quad (8)$$

Although the new dynamical variables are convenient for numerical purposes, we need to recover the standard quantities used in Boltzmann and cosmological codes in general, for instance, the density and pressure of the SFDM component. It can be shown that such quantities can be recovered in the form

$$\rho_\phi = \frac{e^{2\beta}}{1 - e^{2\beta}} \sum_j \rho_j, \quad p_\phi = -\cos\theta \rho_\phi, \quad (9)$$

where the sum takes into account only the density components other than the SFDM one.

## B. Linear density perturbations

When considering linear perturbations, the line element in the synchronous gauge and the perturbed scalar field are given by

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + \bar{h}_{ij})dx^i dx^j, \quad (10)$$

$$\phi(\vec{x}, t) = \phi(t) + \varphi(\vec{x}, t), \quad (11)$$

where  $\bar{h}_{ij}$  and  $\varphi$  are the spatial part of the metric perturbation and scalar field fluctuation, respectively. The linearized Klein-Gordon equation for the scalar field perturbation is written (in Fourier space) as [52–55]

$$\ddot{\varphi}(\vec{k}, t) = -3H\dot{\varphi}(\vec{k}, t) - \left(m_\phi^2 + \frac{k^2}{a^2}\right)\varphi(\vec{k}, t) - \frac{1}{2}\dot{\phi}\dot{h}. \quad (12)$$

In a way similar to the procedure we have used for the Klein-Gordon equation (3b), we propose the following change of variables for the scalar field perturbation  $\varphi$  and its derivative  $\dot{\varphi}$  [19,30]:

$$\sqrt{\frac{2\kappa\dot{\varphi}}{3H}} = -e^{\alpha+\beta} \cos(\vartheta/2), \quad \frac{\kappa y_1 \varphi}{\sqrt{6}} = -e^{\alpha+\beta} \sin(\vartheta/2). \quad (13)$$

Field perturbations are then represented by the new variables  $\alpha$  and  $\vartheta$ . However, it is more convenient to further define two new variables in the form of

$$\delta_0 = -e^\alpha \sin\left(\frac{\theta - \vartheta}{2}\right), \quad \delta_1 = -e^\alpha \cos\left(\frac{\theta - \vartheta}{2}\right). \quad (14)$$

<sup>5</sup>It is important to be aware that, since version 2.10.2 of CLASS, differential equations are integrated in  $\ln a$  rather than conformal time. This makes the integration of our formalism into CLASS simpler and more natural, as  $\ln a$  is the preferred choice when studying dynamical systems such as Eq. (7) [19,50].

After some straightforward algebra, the linearized Klein-Gordon equation (12) is represented by the following set of first-order differential equations:

$$\delta'_0 = \left[ -3 \sin \theta - \frac{k^2}{k_J^2} (1 - \cos \theta) \right] \delta_1 + \frac{k^2}{k_J^2} \sin \theta \delta_0 - \frac{\bar{h}'}{2} (1 - \cos \theta), \quad (15a)$$

$$\delta'_1 = \left[ -3 \cos \theta - \frac{k^2}{k_J^2} \sin \theta \right] \delta_1 + \frac{k^2}{k_J^2} (1 + \cos \theta) \delta_0 - \frac{\bar{h}'}{2} \sin \theta. \quad (15b)$$

In writing Eq. (15), there appears a natural definition of a Jeans wave number given by  $k_J^2 = a^2 H^2 y_1$ , which also acts as a normalization factor of the wave number  $k$ .

Notice that for scales larger than the Jeans one,  $k^2/k_J^2 \ll 1$ , the scale-dependent terms in Eq. (15) can be neglected, whereas in the opposite case,  $k^2/k_J^2 \gg 1$ , the evolution of the perturbations are different for each scale. Our definition of the Jeans wave number coincides with those given in the literature if we write it as  $k_J^2 = 2a^2 m_\phi H$  using  $y_1 = 2m_\phi/H$  [see Eq. (5b)].

To understand the physical meaning of the new dynamical variables  $\delta_0, \delta_1$ , we first write their expressions in terms of the original field variables,

$$\delta_0 = \frac{m_\phi^2}{\kappa^2 \rho_\phi} \left[ \frac{\kappa \dot{\phi}}{m_\phi} \frac{\kappa \dot{\phi}}{m_\phi} + (\kappa \phi)(\kappa \phi) \right], \quad (16a)$$

$$\delta_1 = \frac{m_\phi^2}{\kappa^2 \rho_\phi} \left[ \frac{\kappa \dot{\phi}}{m_\phi} (\kappa \phi) - \frac{\kappa \dot{\phi}}{m_\phi} (\kappa \phi) \right]. \quad (16b)$$

Now, we recall the expressions for the scalar field density contrast  $\delta_\phi$  and the velocity divergence  $\Theta_\phi$  in the form

$$\delta_\phi = \frac{\dot{\phi} \dot{\phi} + m_\phi \phi \phi}{\rho_\phi} = \delta_0, \quad (17a)$$

$$(\rho_\phi + p_\phi) \Theta_\phi = \frac{k^2}{a} \dot{\phi} \phi = \frac{k^2 \rho_\phi}{a H y_1} [(1 - \cos \theta) \delta_1 - \sin \theta \delta_0]. \quad (17b)$$

Hence, the variable  $\delta_0$  is the scalar field density contrast, whereas the velocity divergence is based on a combination of the two variables  $\delta_0$  and  $\delta_1$ . See Appendix A for an extended discussion of the equivalence between our approach and the fluid approximation.

### C. Initial conditions

Calculation of initial conditions implies an approximate solution of the equations of motion starting well within the epoch of radiation domination, with a corresponding initial value of the scale factor of the order of  $a_i = 10^{-14}$  [19].

We start with the initial condition of the auxiliary variable, which reads

$$y_{1i} = \frac{2m_\phi H_0}{H_0 H_i} = 1.85 \times 10^{11} \frac{a_i^2}{\sqrt{\Omega_{r0} h^2}} \left( \frac{m_\phi}{10^{-22} \text{ eV}} \right), \quad (18)$$

where  $H_0$  ( $H_i$ ) is the present (initial) value of the Hubble parameter and  $h$  is its reduced value. Note that for the calculation of  $H_i$  we assume radiation domination at early times. It is clear from Eq. (18) that the auxiliary variable is very small at early times,  $y_{1i} \sim 10^{-14}$ , for the values of  $m_\phi$  that are of interest for SFDM models. On the contrary, its value at present is very large  $y_1 \sim 10^{10}$ , which means that it changes by almost 24 orders of magnitude during its evolution.

As for the polar angle, there is an attractor solution when the equations of motion are solved in the linear regime at early times, from which we obtain the following equation:

$$\theta_i = \frac{1}{5} y_{1i}. \quad (19)$$

The initial condition of the variable  $\beta$  is found by matching the early- and late-time solutions at the beginning of the rapid oscillations of the field. The resultant equation is

$$e^{2\beta_i} = A \times a_i \frac{\Omega_{\phi 0}}{\Omega_{r0}} \left[ \frac{4\theta_i^2}{\pi^2} \left( \frac{1 + \pi^2/36}{1 + \theta_i^2/9} \right) \right]^{3/4}. \quad (20)$$

Here,  $A$  is a constant coefficient that is adjusted by the numerical code, typically with a shooting mechanism, to match the value of the desired density parameter  $\Omega_{\phi 0}$  at the present time.

On the other hand, the initial conditions of the density perturbations is a more involved procedure, but it reveals the existence of an attractor solution for the dynamical variables in the form

$$\delta_{0i} = \frac{2}{7} \bar{h} \sin(\theta_i/2) \sin(\theta_i/12), \quad (21a)$$

$$\delta_{1i} = \frac{2}{7} \bar{h} \sin(\theta_i/2) \cos(\theta_i/12). \quad (21b)$$

The details of the numerical implementation of the polar method of the sections above in the amended version of the Boltzmann code CLASS are presented in Appendix C, from which we obtained the numerical solutions that are presented in the sections below.

### III. STAGE OF RAPID FIELD OSCILLATIONS

For the field to behave as a CDM component, it should enter a phase of rapid oscillations around the minimum of the potential. Under our polar transformation (4), such fast oscillations are equivalent to the following averages during a Hubble time:  $\langle \sin \theta \rangle = 0$  and  $\langle \cos \theta \rangle = 0$ .

However, we must be careful how the oscillations are dealt with, as the solutions at late times depend on the choices made for the averaged dynamical quantities. Here, we explain in detail our method for the cutoff of the rapid oscillations proposed in Refs. [19,30,32].

#### A. Outline of the general method

It is well known that the stage of rapid oscillations is difficult to solve numerically, and then we follow here the prescription in Ref. [19] in that the cosine and sine functions in the equations of motion are replaced by the cutoff trigonometric functions,

$$\cos_{\star} \theta = \frac{1}{2} [1 - \tanh(\theta - \theta_{\star})] \cos \theta, \quad (22a)$$

$$\sin_{\star} \theta = \frac{1}{2} [1 - \tanh(\theta - \theta_{\star})] \sin \theta, \quad (22b)$$

where  $\theta_{\star}$  is a reference value. In this form,  $\cos_{\star} \theta = \cos \theta$  ( $\sin_{\star} \theta = \sin \theta$ ) if  $0 \leq \theta < \theta_{\star}$ , while  $\cos_{\star} \theta \rightarrow 0$  ( $\sin_{\star} \theta \rightarrow 0$ ) if  $\theta \gg \theta_{\star}$ .

In the following, we will refer to  $t_{\star}$  as the time at which we apply the cutoff for the trigonometric functions and then to  $\theta(t_{\star}) = \theta_{\star}$ . We will also refer to  $t_{\text{osc}}$  as the time for the beginning of the rapid oscillations. However, and in contrast to  $t_{\star}$ , the value of  $t_{\text{osc}}$  cannot be precisely determined, and in our formalism, it is just a reference value without a major effect on the numerical solutions.

The general method can be described as follows. We replace all the sine and cosine terms with the cutoff functions (22) in the equations of motion (7) and then solve them numerically. Note that the solutions are then continuous at  $t = t_{\star}$  by construction, and we only need to be sure that the cutoff is applied after the onset of the rapid oscillations so that  $t_{\star} > t_{\text{osc}}$ .

However, it is difficult to determine the time at the start of the oscillations, and this also makes impractical the calculation of  $t_{\star}$ . The reason is that cosmic time is a dimensional quantity calculated from the integration of the Friedmann equation (3a), which depends on all dynamical variables in a cosmological model.<sup>6</sup> As we shall show in the following, it is better to calibrate the cutoff time using the

<sup>6</sup>This is a problem similar to those in other approaches that require to set the value of the ratio  $m_{\phi}/H$  at the start of the oscillations, where  $H$  is also a dimensional output value of the cosmological equations.

polar variable  $\theta$ , which is dimensionless and also a direct dynamical variable in our set of field equations.

#### B. Case of the polar angle $\theta$

To understand the general behavior of the solutions after the cutoff of the trigonometric functions, we start with the equation of motion (7a) of the polar angle  $\theta$ , which for convenience we write in terms of cosmic time  $t$ ,

$$\dot{\theta} = -\frac{3}{2t} \sin \theta + 2m_{\phi}, \quad (23)$$

where we have considered  $H = 1/(2t)$  for Radiation Domination (RD). Notice that we can write Eq. (23) in the form

$$\frac{d\theta}{d(2m_{\phi}t)} = -\frac{3}{2} \frac{\sin \theta}{(2m_{\phi}t)} + 1, \quad (24)$$

which shows that the evolution of  $\theta$  as a function of the dimensionless variable  $2m_{\phi}t$  is the same, regardless of the value of the field mass  $m_{\phi}$ . We will use this feature in the plots below, but we still refer to Eq. (23) to obtain semianalytical expressions.

First, we assume at the beginning that  $0 < \theta \ll 1$ , and then  $\sin \theta \simeq \theta$ . As a consequence, Eq. (23) becomes

$$\dot{\theta} = -\frac{3}{2t} \theta + 2m_{\phi}, \quad (25a)$$

and the solution that satisfies the initial condition  $\theta(0) = 0$  is

$$\theta(t) = \frac{4}{5} m_{\phi} t \quad (\text{early}). \quad (25b)$$

We now use Eq. (25b) on the right-hand side of Eq. (23), from which we obtain the new differential equation,

$$\dot{\theta} = -\frac{3}{2t} \sin_{\star}(4m_{\phi}t/5) + 2m_{\phi}, \quad (26a)$$

whose solution is

$$\theta(t) = 2m_{\phi}t - \frac{3}{2} \text{Si}(4m_{\phi}t/5), \quad (26b)$$

where  $\text{Si}(x)$  is the sine integral. It can be shown that, at early times  $m_{\phi}t \ll 1$ , we recover the solution (25b). Likewise, for the late-time evolution, we can approximate the sine integral by its asymptotic behavior,  $\text{Si}(x) \simeq \pi/2 - \cos(x)/x + \mathcal{O}(1/x^2)$  for  $x \gg 1$ , to obtain

$$\theta(t) = 2m_{\phi}t - \frac{3\pi}{4} \quad (\text{late}). \quad (26c)$$

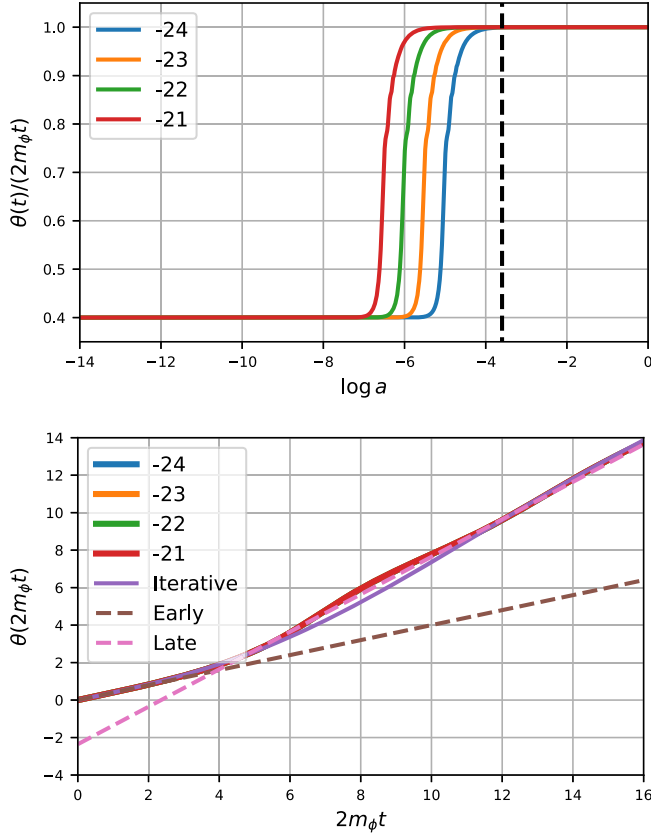


FIG. 1. Numerical solutions of the polar variable  $\theta$  as a function of the dimensionless variable  $2m_\phi t$ . The cases correspond to different values of the field mass  $\log(m_\phi/eV) = -24, -23, -22, -21$ . Also shown are the semianalytical solutions (26b) (iterative, purple), (25b) (early, dashed brown), and (26c) (late, dashed pink). See the text for more details.

Any further iteration to integrate Eq. (23) cannot be expressed in closed form, but as we present below, Eq. (26b) suffices to analyze the main properties in the time evolution of the polar angle  $\theta$ .

The numerical solutions of Eq. (23) for different values of the field mass  $m_\phi$ , and as a function of the scale factor  $a$ , are shown in the top panel of Fig. 1. It can be seen that the polar variable shows two asymptotic behaviors that correspond to the semianalytical solutions: at early times,  $\theta/(2m_\phi t) \rightarrow 5/4$ , while at late times,  $\theta/(2m_\phi t) \rightarrow 1$ , as indicated by Eqs. (25b) and (26c), respectively. These asymptotic limits are the same for any field mass  $m_\phi$ , and the latter only influences the time at which the transition occurs between the two values.

In the bottom panel of Fig. 1, we have the evolution of the polar variable but now in terms of dimensionless cosmic time  $2m_\phi t$ . Although the numerical solutions are shown in different colors, it is clear that the corresponding curves are superimposed on each other because their behavior is the same.

It can be seen that the semianalytical solutions agree well with the numerical ones. In particular, the iterative

solution (26b) gives a reliable description of the early- and late-time trends of the solutions, and it even gives a good approximation to the oscillations of the numerical solutions at intermediate times  $2m_\phi t \simeq 4$ , which is also the time at which  $\theta \simeq \pi/2$ . That is, it also corresponds to the time at which the scalar field EOS first crosses the zero value  $w_\phi \simeq 0$ . As this occurs within radiation domination, we also find  $2m_\phi t = m_\phi/H \simeq 4$ , which is the typical time for the start of the oscillations estimated for these field systems.

Surprisingly, the bottom panel of Fig. 1 also shows that the late-time expression (26c) also seems to work very well from the intermediate times onward, that is, almost from the start of the rapid oscillations. This means that we can safely write

$$\theta(t > t_{\text{osc}}) = 2m_\phi t - \frac{3\pi}{4}. \quad (27a)$$

That Eq. (27a) is also a very good approximation can be understood from the properties of the sine integral  $\text{Si}(x)$ , which rapidly converges to its asymptotic value of  $\pi/2$ , with small oscillations around it that rapidly decay away. In what follows, we will use Eq. (27a) to describe the behavior of the polar angle after the onset of rapid oscillations of the field  $\phi$ .

We can also use Eq. (27a) also to convert the cutoff time  $t_\star$  into a cutoff polar angle  $\theta_\star$ , which is both a dynamical variable and the argument in the modified trigonometric functions (22). Hence, the relation between the cutoff values  $t_\star$  and  $\theta_\star$  is

$$2m_\phi t_\star = \theta_\star + \frac{3\pi}{4}. \quad (27b)$$

Equation (27) are a central result in the description of our method. First, Eq. (27a) shows that the polar angle evolves linearly with cosmic time  $t$  after the cutoff time. Second, Eq. (27b) allows us to determine the cutoff point of rapid oscillations by means of the polar angle  $\theta_\star$ , which is more convenient from the numerical point of view and justifies the use of the cutoff expressions (22).

To finish this section, in Fig. 2, we show the numerical evolution of the scalar field EOS  $w_\phi$  as a function again of the dimensionless variable  $2m_\phi t$ , and we see that it first passes through zero (for  $\theta = \pi/2$ ) at around the time  $2m_\phi t_{\text{osc}} \simeq 3.47$ , which we use to mark the time  $t_{\text{osc}}$  for the start of the rapid oscillations. Notice that in terms of the usual mass-to-Hubble ratio this is equivalent to  $m_\phi/H_{\text{osc}} \simeq 3.47$ , a value used as a reference in other studies of field models.

For comparison, we also plot in Fig. 2 the result of the expression  $\cos(2m_\phi t - 3\pi/4)$ . Notice that there is very good agreement of this curve with the original EOS  $w_\phi$  almost from the start of the rapid oscillations, which means that we can use the following expression for the field EOS:

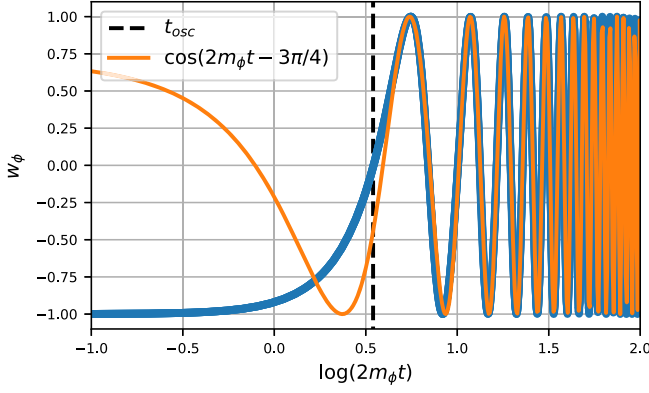


FIG. 2. The numerical solution of the scalar field EOS  $w_\phi$  as a function of the variable  $2m_\phi t$ , under which all the different cases collapse into a single curve (blue curve). The dashed vertical lines mark the beginning of the rapid field oscillations at  $t_{\text{osc}}$ . The curve of the semianalytical approximation (28) (orange curve), which is well matched to the numerical solutions at  $t > t_{\text{osc}}$ , is also plotted. See the text for more details.

$$w_\phi(t > t_{\text{osc}}) = -\cos(2m_\phi t - 3\pi/4). \quad (28)$$

Equation (28) agrees with the common wisdom that the EOS oscillates with a frequency directly related to the field mass via  $2m_\phi$ . The phase of  $3\pi/4$  becomes negligible at very late times, but as we shall see, it must be taken into account for a correct description of the dynamics at intermediate times of other variables after the onset of the rapid oscillations.

### C. Case of the energy density $\rho_\phi$

We have found semianalytical results to follow the evolution of the polar variable  $\theta(t)$ , which are in good agreement with the numerical results. However, one should worry about the numerical accuracy as the scalar field equations of motion must be solved together with other matter components in Boltzmann codes, covering an ample time interval for a complete description of diverse cosmological phenomena.

Here, we perform some accuracy tests using the amended version of the Boltzmann code CLASS, taking some guidelines from our semianalytical results. Our main concern is the choice of the cutoff value  $\theta_\star$ . As we shall see, the cutoff procedure leaves a residual difference with respect to the expected late-time evolution of a given variable that can be minimized if  $\theta_\star \gg 1$ .

An example of the effects of the cutoff on the evolution of different quantities is the energy density  $\rho_\phi$ , which obeys the equation

$$\dot{\rho}_\phi = -3H(1 + w_\phi)\rho_\phi = -3H(1 - \cos\theta)\rho_\phi, \quad (29)$$

whose formal solution after the onset of the rapid oscillations can be written as

$$\rho_\phi a^3 = \rho_{\phi, \text{osc}} a_{\text{osc}}^3 \exp[F(t) - F(t_{\text{osc}})], \quad (30a)$$

with

$$F(t) - F(t_{\text{osc}}) = 3 \int_{t_{\text{osc}}}^t H(x) \cos(\theta(x)) dx. \quad (30b)$$

It suffices to understand the behavior of the density before the time of radiation-matter equality, and for that, we proceed as follows. Equation (30b) can be written in a more convenient form if we use Eq. (27a) for the evolution of the polar angle, and then it can be shown that

$$F(t) = \frac{3\sqrt{2}}{4} [\text{Si}(2m_\phi t) - \text{Ci}(2m_\phi t)], \quad (31a)$$

where  $\text{Si}(x)$  and  $\text{Ci}(x)$  are the sine and cosine integrals, respectively.

We are interested in the evolution of the density at late times, that is,  $2m_\phi t \gg 1$ . Given the asymptotic properties of the sine and cosine integrals for  $x \gg 1$ ,  $\text{Si}(x) \simeq \pi/2 - \cos(x)/x$ , and  $\text{Ci}(x) \simeq \sin(x)/x$ , and substituting the polar angle  $\theta$  using Eq. (27a), we find that

$$F(t \gg t_{\text{osc}}) \simeq \frac{3\sqrt{2}\pi}{8} + \frac{3 \sin \theta}{2(\theta + 3\pi/4)}. \quad (31b)$$

The last term in Eq. (31b) will be responsible for a residual oscillatory term in the density, which will decay away. In fact, if we define

$$\rho_{\phi 0} \equiv \rho_{\phi, \text{osc}} a_{\text{osc}}^3 \exp\left[\frac{3\sqrt{2}\pi}{8} - F(t_{\text{osc}})\right], \quad (32)$$

we can also write Eq. (30a) in a neater form as

$$\rho_\phi(t \gg t_{\text{osc}}) = (\rho_{\phi 0}/a^3) \exp\left(\frac{3 \sin \theta}{2(\theta + 3\pi/4)}\right), \quad (33)$$

which is correct for  $\theta \gg 1$ . There are two parts in the rhs of Eq. (33): one that evolves steadily at the same rate as a pressureless component ( $\sim a^{-3}$ ) and another one that contributes with a decaying oscillating term around unity provided by the exponential function. Moreover,  $\rho_{\phi 0}$  represents the correct asymptotic value of the field density at late times.

The following question arises: can we be assured that our cutoff procedure of the rapid oscillations recovers the right evolution of the density at late times? First, notice that, in principle,  $\rho_{\phi 0}$  in Eq. (32) is fixed at the onset of the rapid oscillations, but we do not need to be very specific about the values of the parameters at this time. In our method, unlike others in the literature, we do not require knowing the precise value of  $t_{\text{osc}}$ , and we can be completely oblivious to it as long as we ensure  $t_\star > t_{\text{osc}}$ . The reason is simple: the

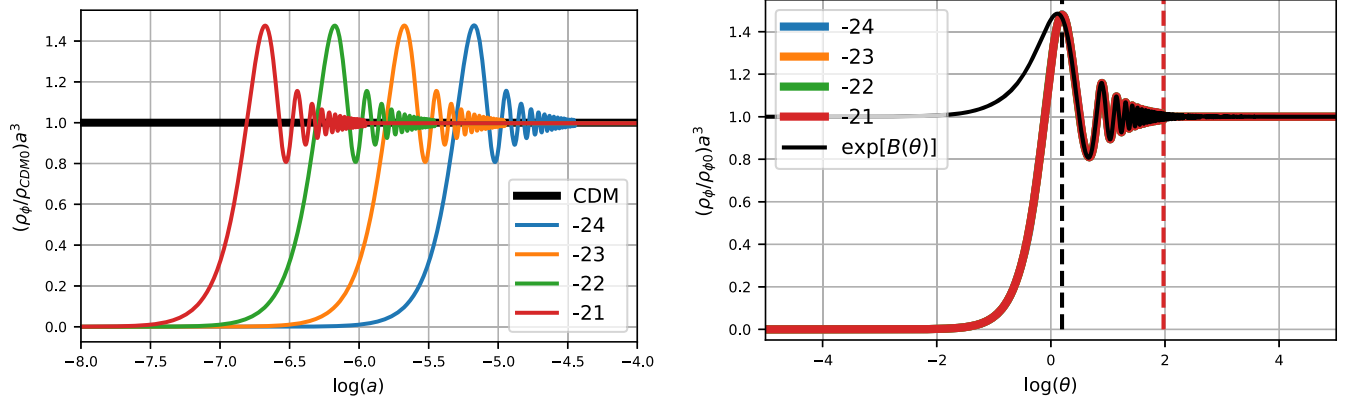


FIG. 3. Numerical solutions of the scalar field energy density  $\rho_\phi$ , in a combination that highlights the asymptotic, nonoscillatory value at late times; the curves represent the same cases as in Fig. 1. Left: the energy density, as a function of the scale factor  $a$ , behaves as a pressureless component from the onset of rapid oscillations, which occur at times that depend on the field mass  $m_\phi$ , but the late-time behavior is the same: decaying oscillations around a fixed value. The cutoff value was set at  $\theta_\star = 30\pi$  for the four cases. Right: if plotted as a function of the polar angle  $\theta$ , all curves collapse into a single one. Also shown is the semianalytical formula (33), with  $B(\theta) \equiv 3 \sin \theta / (\theta + 3\pi/4)$  (black curve), which is in good agreement with the numerical solutions from the start of the rapid oscillations at  $\theta_{\text{osc}} = \pi/2$  (black dashed line). The cutoff value  $\theta_\star = 30\pi$  (red dashed line) is also shown as a reference. See the text for more details.

cutoff of the rapid oscillations is made smoothly at the level of the equations of motion, and then there is no loss of continuity in the numerical variables.

To answer our question above, Eq. (33) should be compared with the truncated case. After the cutoff, the equation of motion for the density is

$$\dot{\rho}_\phi = -3H\rho_\phi, \quad (34a)$$

whose solution simply is  $\rho(t > t_\star) = \rho_\star/a^3$ , where  $\rho_\star$  is the density value at  $t = t_\star$ . By the continuity of the solutions at  $t_\star$  for Eqs. (29) and (34a), we finally get

$$\rho(t > t_\star) = (\rho_{\phi 0}/a^3) \exp\left(\frac{3 \sin \theta_\star}{2(\theta_\star + 3\pi/4)}\right). \quad (34b)$$

The result is quite direct: the cutoff introduces a small mismatch, and the correct asymptotic value of the density is not recovered from the solution (34b). But the discrepancy depends on the cutoff value  $\theta_\star$ , and in principle, it can be made as small as required if  $\theta_\star \gg 1$ .

However, Eq. (34b) itself suggests a faster route, which is to choose  $\theta_\star = n\pi$ , where  $n$  is an integer number, although a large enough one so that still  $\theta_\star \gg 1$ , as this also allows us to neglect other oscillatory terms in the sine and cosine integrals in Eq. (31a) that are of order  $\mathcal{O}(1/\theta_\star^2)$  and smaller.

Numerical examples of the evolution of the field density, in the combination  $(\rho_\phi / \rho_{\phi 0}) a^3$ , are shown in Fig. 3, for different values of the field mass  $m_\phi$  but with a fixed value  $\theta_\star = 30\pi$ . Here,  $\rho_0$  is the last value in each of the numerical solutions. Note that the asymptotic value is always unity. In the left panel, we see that the density makes a transition to a

pressureless behavior once the rapid oscillations start, but as before, the transition time depends on the field mass  $m_\phi$ .

If the density is plotted as a function of the variable  $\theta$ , as in the right panel of Fig. 3, we find that all curves collapse again into a single one, and there is a common evolution for all cases. Moreover, we also show the curve arising from Eq. (33) (denoted by  $\exp[B(\theta)]$ ), and it can be seen that it matches quite well the numerical curves after the onset of the rapid oscillations.

Now, in Fig. 4, we show the effects arising from different choices of the cutoff value  $\theta_\star$  and with a fixed mass  $m_\phi = 10^{-24}$  eV. In the left panel, we take  $\theta_\star = 10\pi, 20\pi, 30\pi$ , and we see that the late-time evolution is the same for all cases (the curves are superimposed on each other) even though the cutoff of the oscillations appears at different times. Also, the correct asymptotic value of the density is recovered, and in all cases, it corresponds to the expected average of the density oscillations.

For the right panel in Fig. 4, we take the cutoff values  $\theta_\star = 10\pi, 10.5\pi$ . To have a good matching of the first density oscillations in the two cases and to facilitate the comparison of their asymptotic values, we applied the correction (exponential) factor that appears in Eq. (33) to the case  $\theta_\star = 10.5\pi$ .<sup>7</sup> It can be seen that the cutoff occurs at the maximum of the last oscillation, and hence the asymptotic value is larger than the correct one.

<sup>7</sup>It must be noticed that the value of the present density  $\rho_0$  is the right one in all the numerical cases, as CLASS adjusts accordingly the initial conditions of the dynamical variables to make the numerical solution recover the same final result. This is the reason why we needed to match the early oscillations for a fair comparison at late times between the two cases  $\theta_\star = 10\pi, 10.5\pi$ .



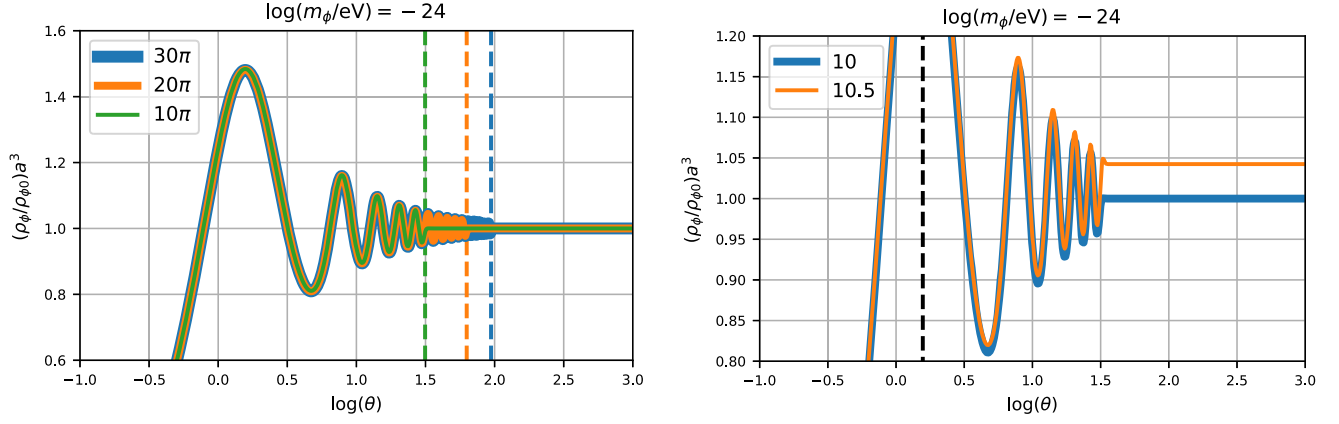


FIG. 4. The behavior of the normalized background density for different choices of the cutoff angle  $\theta_*$ . The density at late times matches the average value well if  $\theta_*$  is an integer multiple of  $\pi$  (left panel), while in any other cases, there is a noticeable mismatch with respect to the average value (right panel). The dashed vertical lines mark the corresponding cutoff values  $\theta_*$  of the numerical solutions. The field mass was fixed at  $m_\phi = 10^{-24}$  eV in the numerical examples. See the text for more details.

We can give an estimate of the error between the two asymptotic values, which according to Eq. (34b) is

$$\frac{\Delta\rho_{\phi 0}}{\rho_{\phi 0}} = \exp\left(\frac{3 \sin \theta_*}{2(\theta_* + 3\pi/4)}\right) - 1. \quad (35a)$$

For the particular case with  $\theta_* = 10.5\pi$ , we get

$$100 \times \left[ \exp\left(\frac{3 \sin(10.5\pi)}{22.5\pi}\right) - 1 \right] = 4.3\%. \quad (35b)$$

This difference is not negligible if one desires high precision of the solution, and it clearly illustrates the necessity to choose the cutoff value  $\theta_*$  wisely.

#### D. Linear density perturbations

In contrast to the background quantities in the sections above, the analysis of Eq. (15) is much more involved because the evolution of the quantities  $\delta_0$  and  $\delta_1$  is coupled to that of the so-called metric continuity  $\bar{h}'/2$  through the perturbed Einstein equations. It is not possible to make a clear separation of the oscillatory and nonoscillatory terms in the formal solution, and a wise decision on the cutoff value  $\theta_*$  cannot easily be decided.

In Fig. 5, we show the evolution of the density contrast  $\delta_0$  for a scale much larger than the Jeans wave number, so that  $k^2 \ll k_J^2$ . In the upper panels, we see the cases with the same cutoff values  $\theta_*$  used previously in Fig. 4. We can see that the numerical solutions have a larger stage of rapid oscillations for larger values of  $\theta_*$ , as in the background case. Also, the choice  $\theta_* = n\pi$ , see Eq. (34b) and the text below, does not make the numerical solution coincide with the nonoscillatory solutions at  $t_*$ . There is a small, but visible, mismatch between the solutions for the different values of  $\theta_*$  considered in the graphs.

Nevertheless, the numerical solution is able to catch up with the nonoscillatory solution after the cutoff of the rapid oscillations, and this is because of the structure of the system (15): its solution is driven by the nonhomogeneous term involving the metric continuity  $\bar{h}'$ , which acts as an attractor solution even at early times. At late times, the value of  $|\delta_0|$  oscillates around  $\bar{h}/2$  with an amplitude that does not decay. It is only on average that the scalar field density contrast can be identified with the CDM one,  $\langle \delta_0 \rangle = \delta_{\text{CDM}}$ .

Other intrinsic oscillations, which we refer to as scale oscillations, are noticeable for scales smaller than the Jeans scale,  $k^2/k_J^2 \gtrsim 1$ , which appear even after the cutoff of the rapid oscillations. This is because the term  $k^2/k_J^2$  plays the role of a frequency in terms of the number of  $e$ -folds  $N$  in Eq. (15) and not in cosmic time  $t$ . For such small scales, from the very beginning, there may be a combination of rapid oscillations with scale oscillations, and the choices of  $\theta_*$  at the transition time give different results for  $\delta_0(t_*)$  and  $\delta_1(t_*)$ .

Our numerical results for the density contrast  $\delta_0$  for small scales are shown in the right panels of Fig. 5, where we see noticeable differences in the late-time behavior of the solutions. It is clear that it is necessary to follow the numerical solutions for longer before cutting off the rapid oscillations and to achieve some convergence of the solutions. However, the evolution of metric continuity  $\bar{h}'/2$  is always smooth and the same regardless of the cutoff value  $\theta_*$ , although its amplitude is also highly suppressed with respect to the CDM case.

We also present the cases  $\theta_* = 10\pi, 10.5\pi$  in Fig. 6. Both cases show that at the cutoff time none of the numerical solutions agrees with the CDM solution, but the solutions quickly become the same because of the driving term  $\bar{h}'/2$  in Eq. (14).

In summary, the numerical solutions clearly show that the scalar field density contrast behaves on average like the

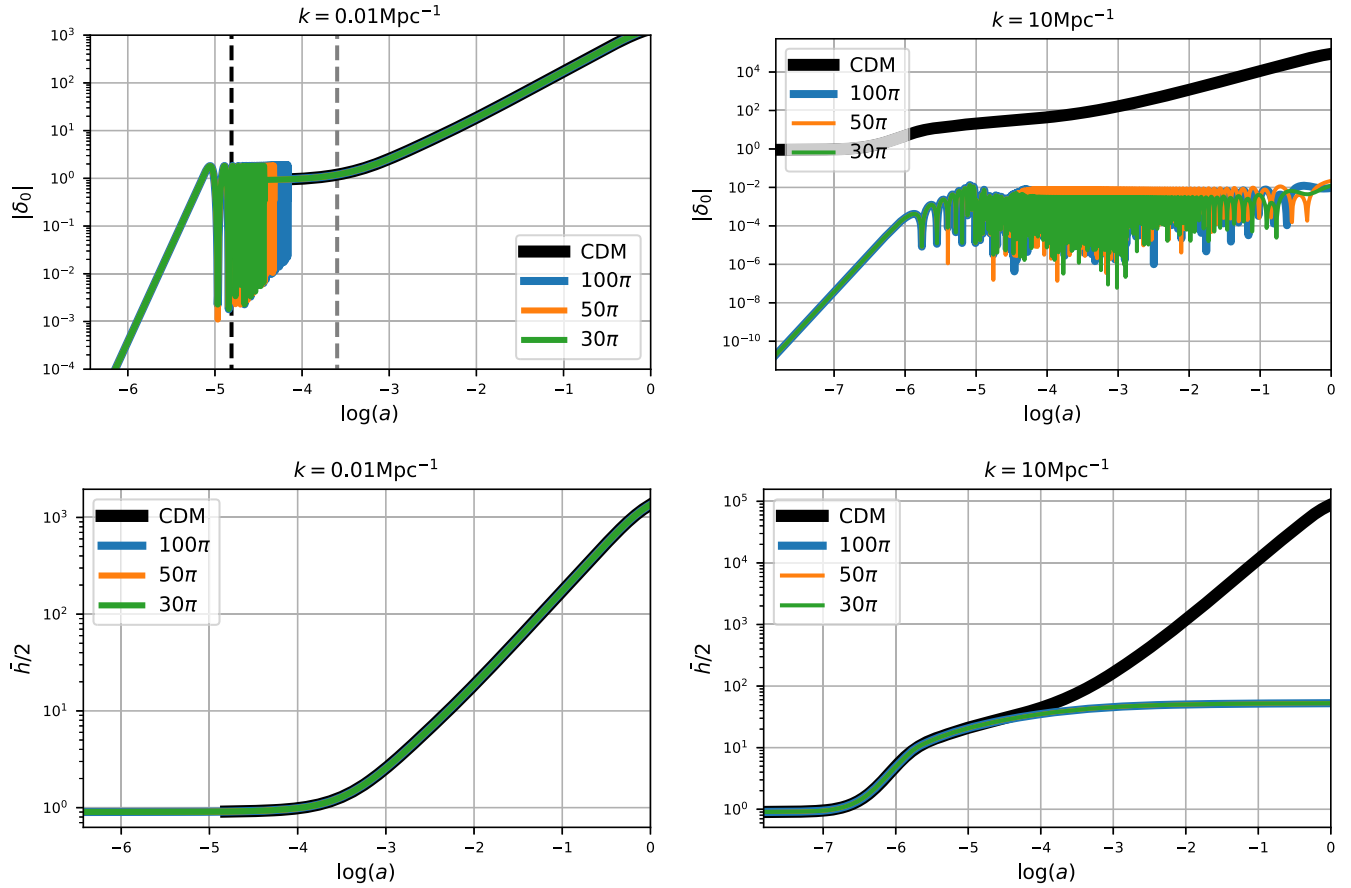


FIG. 5. Behavior of the field density contrast  $\delta_0$  (top panels) and of the metric continuity  $\bar{h}/2$  (bottom panels), for the different wave numbers  $k$  in the plots and for a selection of values for  $\theta_*$ . It is clear that the behavior of both quantities is the same as that of their counterparts in the CDM for large scales (here represented by  $k = 0.01/\text{Mpc}$ ), whereas for small scales ( $k = 10/\text{Mpc}$ ), they are considerably smaller than those of CDM at late times. The field mass was fixed at  $m_\phi = 10^{-24}$  eV in the numerical examples. See the text for more details.

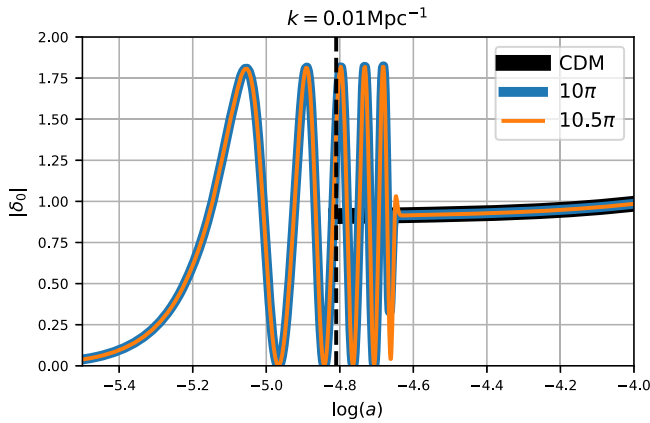


FIG. 6. Illustration of the behavior of the density contrast  $\delta_0$ , for one large scale and for different choices of the cutoff angle  $\theta_*$ . Although there is a mismatch between the solutions at the cutoff time, both follow exactly the CDM solution at late times. The dashed vertical line marks  $\log(a_{eq})$ . The field mass was fixed to  $m_\phi = 10^{-24}$  eV in the numerical examples. See the text for more details.

CDM one on large scales and that the choice  $\theta_* = n\pi$  only helps a little for the numerical solution to have a smooth transition at the cutoff time  $t_*$ . There are no further consequences because the attractor character of the equations of motion for the density perturbations eventually leads to the right numerical result.<sup>8</sup>

### E. Mass power spectrum and temperature anisotropies

To study the issue of convergence in the solution of density perturbations for all scales, we plot the resultant MPS in the upper panel of Fig. 7, the resultant MPS for different choices of  $\theta_*$ , and also the relative differences in the numerical solutions. The field mass in these examples was fixed at  $m_\phi = 10^{-24}$  eV. The first thing to notice is that

<sup>8</sup>See also Appendix B for the equivalent fluid equations of motion of the density perturbations after the cutoff of the rapid oscillations. There, the fluid prescription shows more clearly that SFDM density perturbations grow similarly to CDM under the condition  $k < k_J$ .

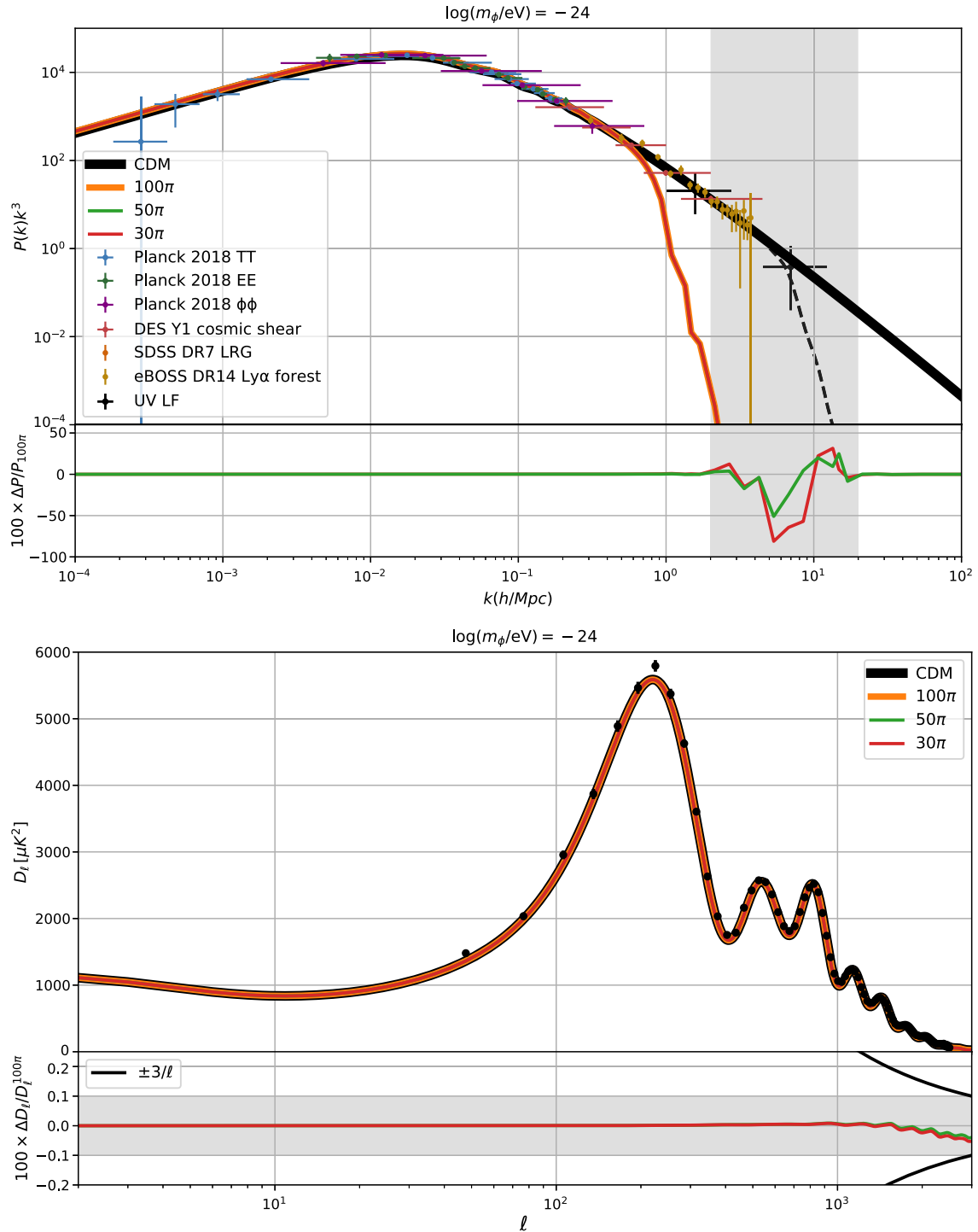


FIG. 7. MPS  $P(k)$  (top panel) and temperature spectrum  $D_\ell$  (bottom panel) calculated for the field mass of  $m_\phi = 10^{-24}$  eV, together with their relative differences with respect to MPS  $P_{100\pi}(k)$  and the CDM temperature spectrum  $D_\ell^{100\pi}$ . The labels of the different curves refer to the cutoff variable  $\theta_*$ . Note that the difference for MPS given the choices of  $\theta_*$  can be as large as 100% in the range  $k = 2\text{--}20$  h/Mpc (vertical gray-shaded region), while for the temperature spectrum, it is less than 0.1% (horizontal gray-shaded region). For reference, we include in the temperature spectrum the (black) curves given by  $\pm 3/\ell$ , which represents an estimated precision threshold beyond which the parameter biases may be significant. The field mass was fixed to  $m_\phi = 10^{-24}$  eV in the numerical examples, but the black dashed line is for  $m_\phi = 10^{-22}$  eV. The data points for the MPS are measurements from the Planck 2018 CMB [1], DES cosmic shear [56], SDSS galaxy clustering [57], SDSS Lyman- $\alpha$  [58], and UV Luminosity Function (LF) [59] datasets. See the text for more details.

there is complete agreement with the MPS of CDM for large scales, represented by the wave numbers  $k < 0.5 h/\text{Mpc}$ .

Regarding the convergence of numerical solutions for different choices of the cutoff value  $\theta_*$ , it can be seen that there is complete agreement for almost all scales, except for the interval  $k = 2\text{--}20 h/\text{Mpc}$ , where the difference can be as large as 100%. However, these discrepancies appear once the MPS is greatly suppressed compared to the CDM, although it should be noticed that the agreement is recovered once the MPS reaches a steady stage at small scales of the form  $P(k) \sim k^{-3}$ . Although one could use the solution with the highest cutoff value  $\theta_*$ , the overall conclusion is that one can safely take  $\theta_* = 30\pi$  for reliable results with the additional advantage of saving computational time.

We repeat the comparison between different cutoff values  $\theta_*$  for the case of the temperature spectrum in terms of the variable  $D_\ell = \ell(\ell + 1)C_\ell/2\pi$  in the lower panel of Fig. 7. This time, the solutions are more similar, among themselves, with different resolutions and also with the corresponding spectrum of CDM. This can be verified in the lower graph with the relative differences of the solutions with respect to choice  $\theta_* = 100\pi$ , which are very small and below 0.1%. For reference, we also show the estimated rule of thumb for bias-free parameter inference, which is given by the curves  $\pm 3/\ell$  at large  $\ell$ . We can see that our numerical results are consistent with such a constraint for the cutoff values chosen for  $\theta_*$  (see also Refs. [46,60–62]).

## F. Constraints on $m_\phi$ from the matter power spectrum

Here, we describe possible constraints of SFDM models from the matter power spectrum, according to recent estimates of the UV galaxy luminous function [59,63] with the package `Gallumi`<sup>9</sup> and the effective field theory of the large scale structure (EFTofLSST) as in Refs. [64,65] with the package `PyBird`.<sup>10</sup> These two are likelihoods of a recent addition to the MCMC software `MontePython` [47,48], which are capable of exploring the power spectrum at semilinear scales and can be as competitive as those of Lyman- $\alpha$  observations.

Two notes of caution are, in turn, on these new likelihoods. In the case of the EFTofLSST, its possible applicability to SFDM is explained well in Ref. [66]: certainly, the standard approach has been designed for a cold and collisionless dark matter particle, but considering

the previous work in Ref. [67], it is concluded that EFTofLSST can be used on SFDM models, taking into account the modified background evolution and linear MPS as input for the formalism. However, the authors in Refs. [66,67] were only interested in cases with very light masses and low contributions of SFDM to dark matter.<sup>11</sup> On the contrary, our aim is to extend the study to values of the field mass to larger values, and this certainly would require a more careful treatment. For now, our results should be considered conservative in that any lower bounds on the field mass  $m_\phi$  would certainly be higher.

Regarding the package `Gallumi`, its main components are clearly explained in Ref. [63] (see also Ref. [59]): the halo mass function (HMF) and the halo-galaxy connection. The HMF depends on cosmological parameters, in particular the comoving matter density, the mass variance on a given mass scale, and the (linear) critical density contrast for gravitational collapse. Many previous works have discussed changes in the HMF for SFDM, especially the necessary adjustments to avoid overcounting of small (ghost) halos [32,73–75]. We have not made these adjustments in the likelihood `Gallumi`. However, the calculation of the HMF in `Gallumi` takes a Sheth-Tormen mass function, which was also considered in Ref. [32] for the semi-analytical estimation of the HMF of the SFDM models. There, it was concluded that the Sheth-Tormen mass function, together with the standard value of the critical overdensity  $\delta_{\text{crit}} = 1.686$ , captures relatively well the lack of low-mass halos in SFDM simulations from the modified linear MPS alone.

Likewise, the galaxy-halo connection depends on astrophysics, which we also keep intact, as it is reasonable to think that the UV emission calculated from a given halo mass is independent of the dark matter model. For the reasons mentioned above, our results from the use of `Gallumi` should be considered reliable but conservative, and again, most likely, any lower bounds on the field mass  $m_\phi$  would certainly be higher.

We replicated the studies in Refs. [59,64] with some shortcuts, as our aim was to focus our attention on the constraints on the field mass  $m_\phi$ . For the analysis, we used Gaussian priors on the following parameters: the angular scale for the sound horizon with  $100\theta_s = 1.0411$  and  $\sigma_{\theta_s} = 0.0003$ <sup>12</sup> and the physical density of baryons

<sup>9</sup>[https://github.com/NNSSA/GALLUMI\\_public](https://github.com/NNSSA/GALLUMI_public).

<sup>10</sup><https://github.com/pierrexyz/pybird>. The code `PyBird` has been calibrated to work on CDM models, which do not predict a cutoff in the MPS. We then assume that its EFTofLSS formalism can be extended to SFDM models in a first approximation; see [66–68], and also [69–71] for other examples beyond  $\Lambda\text{CDM}$ .

<sup>11</sup>Another difference is that previous studies [66,67] were using the `CLASS-PT` pipeline, instead of the `PyBird` one here. For a discussion about the two EFTofLSS pipelines, see Ref. [72].

<sup>12</sup>The original study in Ref. [59] fixed the value of  $100\theta_s = 1.0411$ , as the shooting method that code `CLASS` uses to calculate it by iterations of the Hubble parameter is very efficient for the  $\Lambda\text{CDM}$  model. However, the same method usually does not converge in other alternative models. We decided to use a Gaussian prior to avoid failed attempts of the `CLASS` code under the command of `MontePython`.

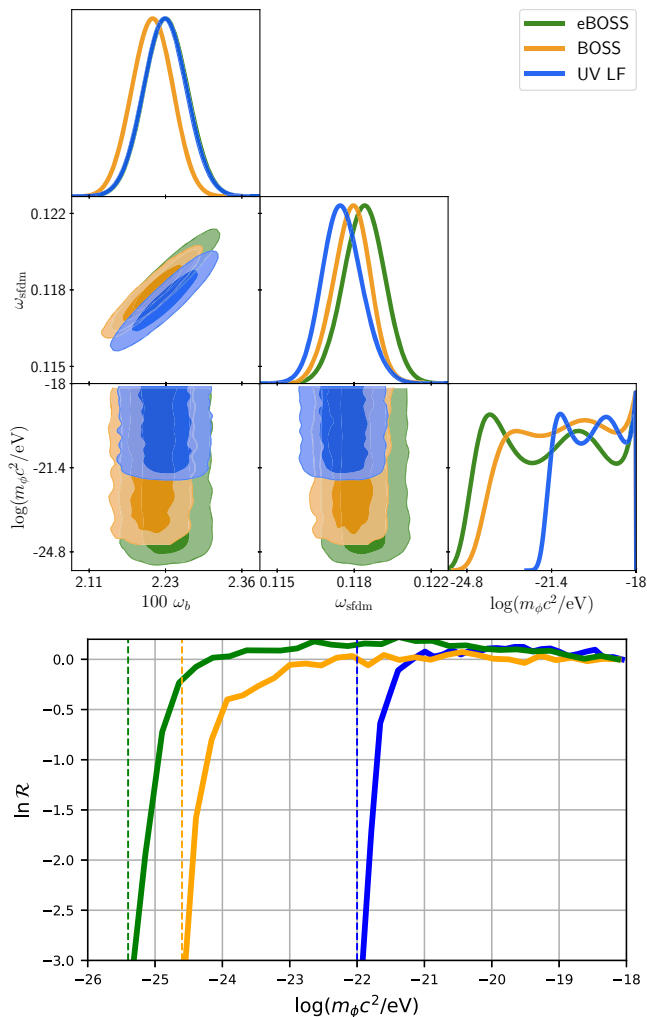


FIG. 8. Top panel: triangle plots obtained from the MCMC code Montepython using the likelihood of the UV galaxy luminous function, for parameters  $\omega_b$ ,  $\omega_{\text{sfdm}}$ , and  $\log(m_\phi c^2 / \text{eV})$ . Bottom panel: plot of the shape distortion function  $\mathcal{R}$  obtained from the posterior distribution obtained for the field mass  $m_\phi$ . See the text for more details.

$\omega_b = 0.02233$  with  $\sigma_b = 0.00036$ .<sup>13</sup> Fixing the sound horizon  $\theta_s$  is known to also fix the combination  $\Omega_m h^{3.4}$ , with  $\Omega_m$  the physical density of total matter and  $h$  the reduced Hubble constant [59,76,77]. This means that our Gaussian prior on  $100\theta_s$ , with  $h = 0.657$ , acts as an indirect prior on the combination  $\Omega_m = \Omega_b + \Omega_\phi$ . For the field mass  $m_\phi$ , we considered a flat prior on the logarithmic scale in the range  $\log(m_\phi / \text{eV}) = [-26, -18]$ . All other cosmological parameters in the models, such as the amplitude of the power spectrum, were fixed to their Planck 2018 CMB values.

<sup>13</sup>This is the same Gaussian prior of Ref. [59] on the physical density of baryons, which is obtained from measurements of the abundance of primordial deuterium.

The likelihood we chose for the UV luminosity function is that of the so-called Model I in Ref. [59], with its corresponding formalities and data. In the case of EFTofLSST, we selected the BOSS and eBOSS datasets as in Ref. [64]. It must be noted that some assumptions in the likelihoods have been made under the CDM paradigm only and would need to be amended for the case of SFDM. Taking into account these caveats, the results reported in the following may be stronger than in the case in which some of the assumptions are corrected for SFDM.

The resulting posterior distributions for the physical densities of baryons  $\omega_b$  and SFDM  $\omega_{\text{sfdm}}$ , and the field mass  $m_\phi$ , after marginalizing over the nuisance parameters of the likelihoods, are shown in Fig. 8. As expected, the separate constraints on the physical densities of baryons and SFDM are practically the same, as they are influenced mostly by the previously assumed priors.

Not surprisingly, the posterior distribution of the field mass shows that each dataset only constrains  $m_\phi$  from below, which means that the likelihoods are insensitive to variations of  $m_\phi$  above a certain threshold value. The dataset with the most constraining power is the UV luminosity function, whereas eBOSS is the less constraining one.

To properly calculate the lower bounds for  $m_\phi$  from the confidence regions in Fig. 8, we use the method in Refs. [78–80] to obtain prior independent constraints by means of the so-called shape distortion function  $\mathcal{R}$ , appropriate for the so-called open likelihoods as in the present case. As explained in Ref. [80], the function  $\mathcal{R}$  will allow us to use the data to define the region below which  $m_\phi$  is disfavored, regardless of the prior assumptions we have chosen.

An advantage of the function  $\mathcal{R}$  is that for its calculation we only need to know the posterior distribution of the field mass  $m_\phi$ , which we obtained from the code Montepython. The resultant shape distortion function is shown in the lower panel of Fig. 8 on the logarithmic scale. Note that  $\mathcal{R} \rightarrow 1$  for large values of the field mass, in this case  $\log(m_\phi c^2 / \text{eV}) \rightarrow -18$ , which is the upper value in our prior range. The sharp decay of  $\mathcal{R}$  at lower values of  $m_\phi$  helps us to calculate an appropriate lower bound. Following the convention in Refs. [78–80], it can be seen that if  $\ln \mathcal{R} = -3$  (moderate level according to Jeffrey’s scale), we can say that the data strongly favor the regions  $\log(m_\phi c^2 / \text{eV}) > -25.4$  for eBOSS,  $\log(m_\phi c^2 / \text{eV}) > -24.6$  for BOSS, and  $\log(m_\phi c^2 / \text{eV}) > -22$  for UV LF.<sup>14</sup>

<sup>14</sup>If we follow the standard wisdom and calculate the lower bound at the 95% confidence level of the histograms, we find instead that  $\log(m_\phi c^2 / \text{eV}) > -24.5$  for eBOSS,  $\log(m_\phi c^2 / \text{eV}) > -23.8$  for BOSS, and  $\log(m_\phi c^2 / \text{eV}) > -21.4$  for UV LF.

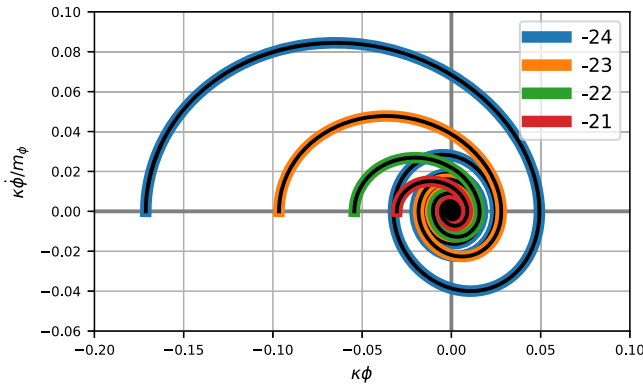


FIG. 9. The phase space of the field variables  $\phi$  and  $\dot{\phi}$ , written in their dimensionless form as in Eq. (36). The colored curves are numerical solutions from the original field variables, whereas the black curves are those obtained from the polar method. The curve labels represent the value of the field mass  $m_\phi$  in units of eV. See the text for more details.

#### IV. COMPARISON WITH THE ORIGINAL FIELD VARIABLES ( $\phi, \dot{\phi}$ )

This section is dedicated to the comparison of the numerical solutions obtained from our polar method to those of the original scalar field equations of motion. To do this, we use the same Boltzmann code CLASS to provide the numerical results, so that they are subject to the same numerical methods and limitations of the code in the two cases. In the amended version, the polar variables are solved as a separate dark matter component, while the field variables are solved using the scalar field equations of the quintessence module already implemented in CLASS.

##### A. Background quantities

The first comparison of background quantities is for the variables  $\phi$  and  $\dot{\phi}$ , which are the dynamical ones in the Klein-Gordon (KG) equation (3d) and the field potential. The relationship between the original and polar variables can be found from the transformation equations (5), in the form of

$$\kappa\phi = -\frac{2\sqrt{6}}{y_1}e^\beta \cos(\theta/2), \quad \frac{\kappa\dot{\phi}}{m_\phi} = \frac{2\sqrt{6}}{y_1}e^\beta \sin(\theta/2). \quad (36)$$

We solved the KG equation (3d) separately, but with the same initial conditions as in the polar case through transformation (36). It is not possible to accurately follow the numerical evolution after the onset of the field oscillations, and then we only solved the KG equation (3d) up to the equivalent time to  $\theta_\star = 100\pi$  ( $2m_\phi t_\star \simeq 100\pi$ ). After this time, the equations of motion are set directly to  $\dot{\phi} = 0$  and  $\ddot{\phi} = 0$ , which means that the late-time solutions of the field variables are just  $\phi(t > t_\star) = \phi(t_\star)$  and  $\dot{\phi}(t > t_\star) = \dot{\phi}(t_\star)$  (and the density remains artificially constant afterward).

The two sets of solutions, the original and the polar ones, are plotted in compact form in the phase space shown in Fig. 9. The thick curves correspond to the system  $\phi - \dot{\phi}$ , with the different colors representing the field mass  $m_\phi$ , while the solutions of the polar system, all in black lines, are superimposed. We see that the agreement between the corresponding curves is exact up to  $t = t_\star$ .

The same comparison exercise for field density  $\rho_\phi$  is shown in Fig. 10, using the same colors for the different curves as in Fig. 9. Furthermore, we normalize the density to the present value of the CDM density  $\rho_{CDM0}$ , to highlight that the final value of the field densities coincides with the equivalent CDM case. The upper panel shows that the two sets of solutions coincide exactly, including the oscillatory phase, which is also confirmed by the comparison in the lower panel: the discrepancies appear at late times in the oscillatory phase and the cutoff in the solution of the field variables [see the explanation below Eq. (36)].

As a final example, we show in Fig. 11 the SFDM EOS calculated directly from the pressure-to-density ratio  $w_\phi = p_\phi/\rho_\phi$ , using the same unit system as in Fig. 2. The variable in the horizontal axis is the dimensionless quantity  $2m_\phi t$ , under which all curves corresponding to a given field mass  $m_\phi$  become the same curve. The EOS oscillates rapidly around the zero value, and we again see that there is excellent agreement between the numerical results of the two approaches.

##### B. Linear density perturbations

We repeated the comparison of the solutions with the case of linear density perturbations. This time, we solved the linearly perturbed KG equation (12), again using the same initial conditions for the two sets of variables, the originals  $\varphi$  and  $\dot{\varphi}$  and the polar ones  $\delta_0$  and  $\delta_1$ . Although the perturbed polar variables are  $\alpha$  and  $\vartheta$ , see Eq. (13), recall that our final perturbed variables are those of Eq. (14) and their corresponding equations of motion (15).

The transformation from the polar variables to the field ones is given by the expressions

$$\kappa\varphi = -\frac{\sqrt{6}}{y_1}e^\beta[\delta_0 \cos(\theta/2) - \delta_1 \sin(\theta/2)], \quad (37a)$$

$$\frac{\kappa\dot{\varphi}}{m_\phi} = \frac{\sqrt{6}}{y_1}e^\beta[\delta_0 \sin(\theta/2) + \delta_1 \cos(\theta/2)]. \quad (37b)$$

We only used Eq. (37) to set the initial conditions of the field variables  $\varphi$  and  $\dot{\varphi}$  in correspondence with those of the polar variables, and then the field equation (12) was solved separately.

We then show in Fig. 12 the evolution of the perturbation variables  $\varphi$  and  $\dot{\varphi}$  as a function of the scale factor and for two values of the wave number  $k$ :  $0.01 \text{ Mpc}^{-1}$  for large

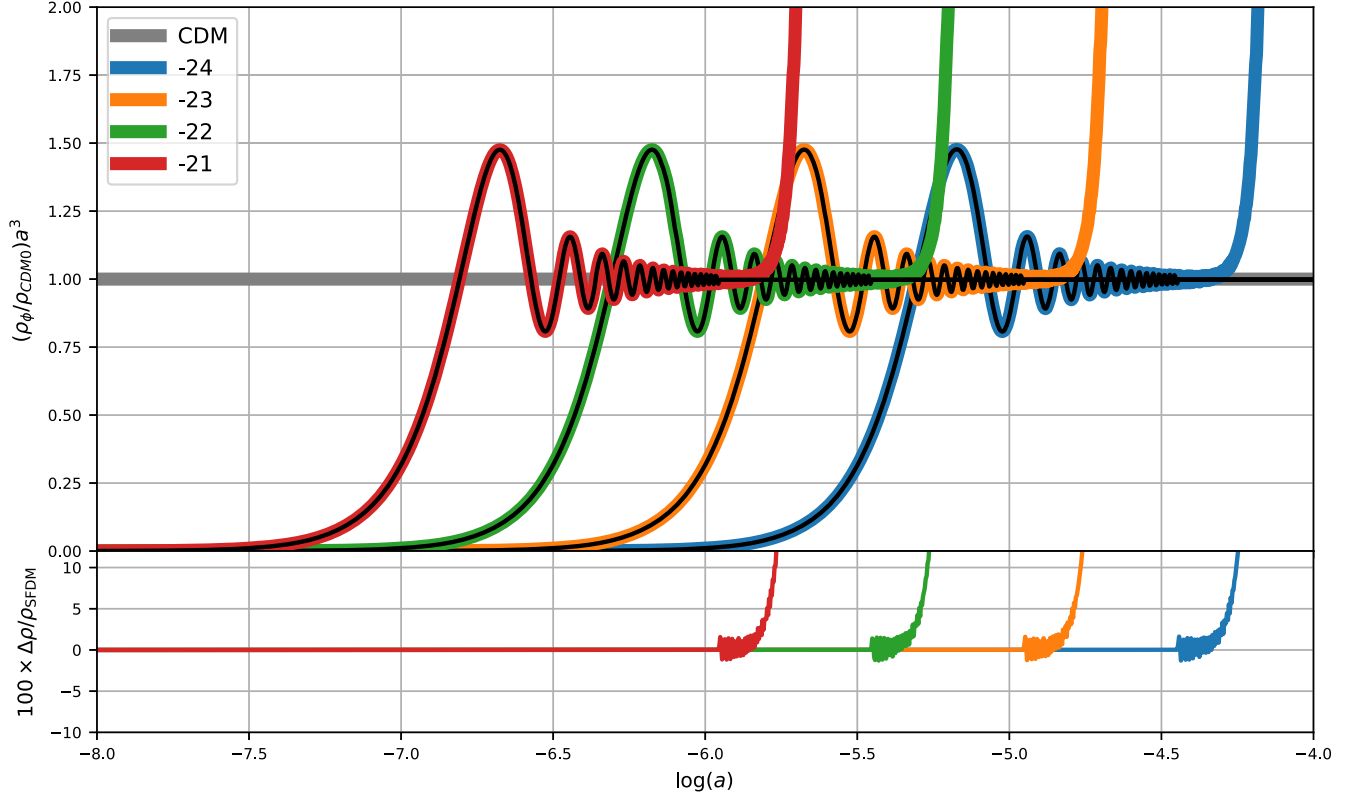


FIG. 10. The evolution of the field density  $\rho_\phi$  for the same cases as in Fig. 3. Top panel: colored curves represent the numerical solution of the original field variables  $\phi$  and  $\dot{\phi}$ , while the corresponding solutions from the polar method are superimposed (black curves). The divergent behavior of the colored curves is due to the cutoff applied to the oscillations of the field solutions at  $\theta_\star = 60\pi$ . Bottom panel: the relative difference between the curves in the upper panel for the same value of the field mass  $m_\phi$ . The difference is close to zero except for the last part, once the oscillations of the field solutions are cut off. See the text for more details.

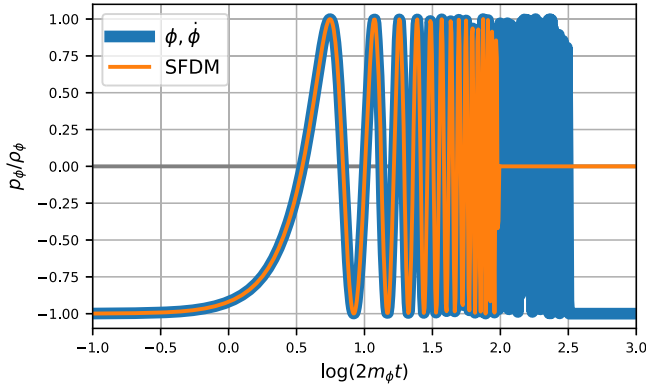


FIG. 11. The numerical solution of the scalar field EOS  $w_\phi$ , calculated from the pressure-to-density ratio, as a function of the variable  $2m_\phi t$ , under which all cases with different field masses  $m_\phi$  collapse into a single curve. The blue curve is the solution of the original variables  $\phi - \dot{\phi}$  (note that  $w_\phi = -1$  after the oscillations are cut off is just an artifact), whereas the orange curve corresponds to the polar method. See the text for more details.

scales (top panels) and  $10 \text{ Mpc}^{-1}$  for small scales (bottom panels).

In the two cases, the solutions from the polar variables via Eq. (37) are superimposed on those of the field variables [obtained directly from Eq. (12)], which are identical up to the time the numerical solutions were followed ( $\theta_\star = 30\pi$  for the polar variables and  $\theta_\star = 100\pi$  for the field ones). Note that the agreement goes beyond the cutoff point of the polar variables, as shown by the bottom panels in each figure, since the relative error is always less than 10%.<sup>15</sup> This means that the cutoff of the trigonometric functions (22) gives from Eq. (15) the expected results of the original field variables.

## V. DISCUSSION

Among the diverse theoretical proposals to describe dark matter, SFDM constitutes a compelling candidate to play

<sup>15</sup>One note is in turn for the calculation of the relative error in Fig. 12. The functions involved oscillate around zero, but their amplitude is of order unity. The lower panels then only show their difference, which we interpret as the relative error between the two numerical results.

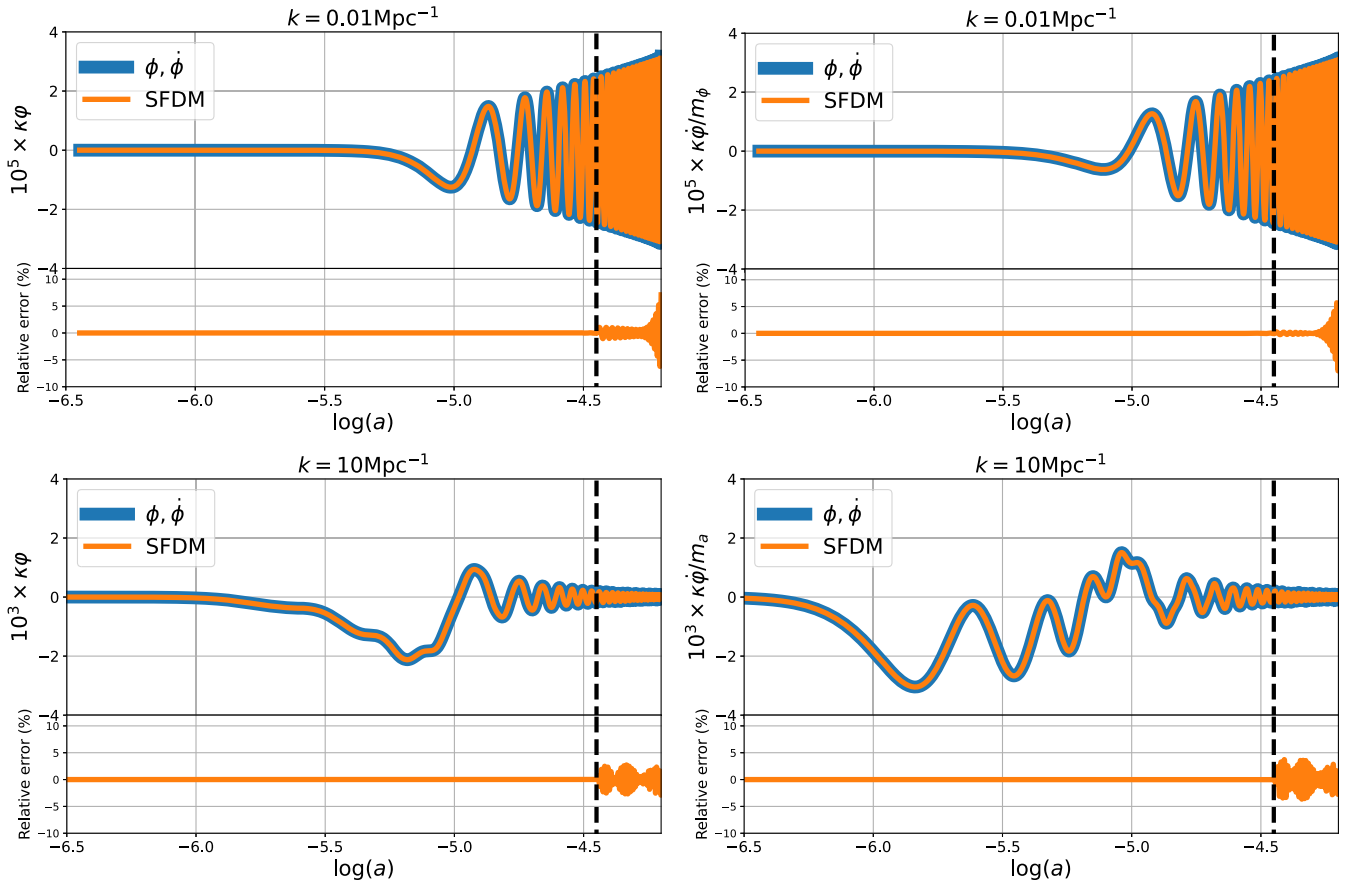


FIG. 12. The evolution of the perturbation variables  $\varphi$  (left column) and  $\dot{\varphi}$  (right column), as obtained from both the original field variables via Eq. (12) and the polar method in Eq. (15); see also Eq. (37). The upper panels show the solutions for large scales, whereas the lower panels show them for small scales. It can be seen that the curves of the two methods agree completely even beyond the cutoff time for the polar variables at  $\theta_* = 30\pi$  (indicated by the black-dashed vertical line). Notice that each panel contains the relative error, in percentage numbers, between the two numerical solutions in each case. See the text for more details.

the role of the CDM component of the Universe. The dynamics of such a particle is modeled through a scalar field endowed with a scalar field potential. In this work, we were particularly interested in the free case of a real scalar field. Given the oscillatory nature of the SFDM when it behaves as CDM, we must be careful to properly handle the differential equations for both background and linear perturbations, in such a way that numerically they were easily solved, allowing us to keep track of the evolution of the scalar field, and of all physical quantities built from it. By posing the Klein-Gordon equations (background and linear perturbations) as a dynamical system once new variables are introduced, we were able to describe the evolution of the SFDM as a system of first-order differential equations.

With this prescription, instead of solving for the original scalar field variables  $(\phi, \dot{\phi}; \varphi, \dot{\varphi})$ , we solved a new set of dynamical variables: the polar variable  $\theta$ , the scalar field energy density  $\beta$ , and the perturbations  $\delta_0$  and  $\delta_1$ . We have also added the variable  $y_1$  which is proportional to the ratio of the mass of the scalar field and the Hubble parameter.

It has been a standard procedure in the literature to deal with the rapid oscillations of the SFDM by averaging the oscillating functions in a Hubble time and then writing down a new set of averaged equations of motion that resemble a standard cosmological fluid. Within our approach, this new set of equations is easier to solve numerically and to include in a standard Boltzmann code. Therefore, there was no need to invoke any approximation or average to cancel out the rapid oscillations of the scalar field. The only consideration of this kind was the introduction of the truncated trigonometric functions that we used [see Eq. (22) in Sec. III A]. In addition, the standard average procedure introduces an undesirable mismatch between the early- and late-time solutions and leaves unanswered the question of the sound speed of the density perturbations of the averaged fluid [45].

Our method does not require a separate evolution of the SFDM equations of motion but just a straightforward transformation of the original field equations. Our transformed system of equations remains the same throughout the full evolution, and this applies for both the background



and for the linear density perturbations. Moreover, in the case of linear perturbations, we do not need to define an explicit expression for the sound speed, which is a slippery quantity in the fluid approximation. Nevertheless, in Appendixes A and B, we show the complete equivalence between our equations and the fluid ones at late times, which only required the implicit definition of the sound speed. Our approach also deals successfully with the rapid oscillations in comparison with the original field variables.

The SFDM formulation we present in this work gives accurate predictions on observables such as the CMB anisotropies and the MPS. Moreover, it is possible to put constraints on the main parameter of the model, the SFDM mass  $m_\phi$ . This is of great relevance because, on the one hand, we need numerical solutions with a high precision level to put constraints on the SFDM model in light of new data from present and future surveys. On the other hand, the cosmological dynamics of the SFDM needs to be properly solved in order to set the initial conditions for numerical simulations of the nonlinear process of structure formation. In this sense, our approach offers a mathematical and numerical treatment for the SFDM model, which is advantageous in that it suitably solves the cosmological dynamics of such a scalar field coupled to the Einstein-Boltzmann system.

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## APPENDIX A: FLUID APPROXIMATION

Here, we discuss the possibility of rewriting the equations of motion (15) for the linear density perturbations in their fluid counterparts. Using Eq. (17b), we find that Eq. (15a) becomes

$$\delta'_0 = -\frac{1}{aH}(1+w_\phi)\Theta_\phi - 3\sin\theta\delta_1 - (1+w_\phi)\frac{\bar{h}'}{2}, \quad (\text{A1})$$

where we have also used the relation (6b). From the definition of the adiabatic sound speed  $c_s^2 = \delta p_\phi / \delta \rho_\phi$ , we find that

$$c_s^2\delta_0 = \sin\theta\delta_1 - \cos\theta\delta_0. \quad (\text{A2})$$

The preceding equation can be used to substitute the second term on the right-hand side of Eq. (A1). After some manipulations, we get the result

$$\delta'_0 = -(1+w_\phi)\Theta_\phi - 3\mathcal{H}(c_s^2 - w_\phi)\delta_0 - (1+w_\phi)\frac{\bar{h}'}{2}, \quad (\text{A3a})$$

where now the primes denote the derivatives with respect to the conformal time and  $\mathcal{H}$  is the corresponding Hubble parameter. These calculations show that Eq. (15a) is completely equivalent to the equation of motion for the density contrast in the fluid approximation.

Similarly, we can combine Eqs. (7), (15), and (A2) to write an equation for the momentum density. After some lengthy but otherwise straightforward manipulations, we find that

$$[(1+w_\phi)\Theta_\phi]' = \mathcal{H}(3w_\phi - 1)(1+w_\phi)\Theta_\phi + k^2 c_s^2 \delta_0. \quad (\text{A3b})$$

Equation (A3) are the standard fluid equations for linear density perturbations and then show that the polar equations (15) are their faithful representation *in our alternative approach*.

## APPENDIX B: LINEAR DENSITY PERTURBATIONS AFTER THE CUTOFF POINT

It is illustrative to write Eq. (15) after the cutoff on the rapid field oscillations. For times  $t > t_*$ , we find

$$\begin{pmatrix} \delta_0 \\ \delta_1 \end{pmatrix}' \simeq \frac{k^2}{k_J^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \bar{h}'/2 \\ 0 \end{pmatrix}, \quad (\text{B1})$$

where again a prime denotes derivatives with respect to  $N = \ln a$ .

The system (B1) has the same structure as a forced harmonic oscillator with frequency  $\omega = k^2/k_J^2$ . As explained in Ref. [19], if we assume that  $\omega$  is a constant (as is the case during RD), for large scales, the solution is the same as that of CDM:  $\delta_0 \simeq -\bar{h}/2$  and  $\delta_1 \simeq \text{const}$ . For small scales, the solutions are oscillatory and given by combinations of functions  $\cos(\omega N)$  and  $\sin(\omega N)$ . These behaviors agree with the full numerical results in Fig. 5.

To obtain the corresponding fluid equations, we first notice that the cutoff method applied to Eqs. (12) and (A2) results in the following equations:

$$\Theta_\phi \simeq \frac{k^2}{aHy_1}\delta_1, \quad (c_s^2 - w_\phi)\delta_0 \simeq 0. \quad (\text{B2})$$

The first expression in Eq. (B2) reveals a direct relation between the divergence of the fluid velocity and variable  $\delta_1$ , while the second suggests a nonvarying value of the fluid's EOS (that is,  $dw_\phi/d\rho_\phi = 0$ ; see Ref. [4]).

Repeating the same procedure above for the calculation of Eq. (A3), we find their counterpart for times  $t > t_*$ ,

$$\delta'_0 \simeq -\Theta_\phi - \frac{\bar{h}'}{2}, \quad \Theta'_\phi \simeq -\mathcal{H}\Theta_\phi + \frac{k^2}{4m_\phi^2 a^2} \delta_0, \quad (\text{B3})$$

where a prime now denotes derivatives with respect to conformal time. Finally, the second of Eq. (B3) suggests a scale-dependent sound speed given by

$$c_s^2 \simeq \frac{k^2}{4m_\phi^2 a^2}. \quad (\text{B4})$$

One note on the sound speed of the perturbations is taken in turn. The discussion above gives a clear answer for the sound speed after the cutoff in the field oscillations, but there is none for the right expression of  $c_s^2$  before that. Our formalism for the perturbations does not require an explicit expression of  $c_s^2$ , but the latter is necessary for a correct fluid formulation of the SFDM linear perturbations.

A general expression of  $c_s^2$  was found in Ref. [86] in the comoving gauge of the scalar field, but the same authors argue in Ref. [87] that such an expression cannot be transformed into the synchronous gauge, under which SFDM lacks a definite functional form of  $c_s^2$ . Despite this, the expression for  $c_s^2$  of Ref. [86] has been extensively used in the literature for the fluid approximation of SFDM in the synchronous gauge.<sup>16</sup>

An example is the so-called effective fluid approximation (EFA) discussed in Ref. [46]. The equations of motion of the EFA are

$$\delta'_0 \simeq -\Theta_\phi - 3\mathcal{H}\langle c_s^2 \rangle \delta_0 - \frac{\bar{h}'}{2}, \quad (\text{B5a})$$

$$\Theta'_\phi \simeq -\mathcal{H}\Theta_\phi + k^2 \langle c_s^2 \rangle \delta_0, \quad (\text{B5b})$$

where the cycle-averaged sound speed is

$$\langle c_s^2 \rangle = \frac{k^2 / (4m_\phi^2 a^2)}{1 + k^2 / (4m_\phi^2 a^2)}. \quad (\text{B5c})$$

It can be seen that our Eq. (B3) can be obtained from the EFA if the sound speed is given by the so-called non-relativistic expression  $\langle c_s^2 \rangle \simeq k^2 / (4m_\phi^2 a^2)$ . However, the EFA considers an extra term in the equation of motion of  $\delta_0$ ; see Eq. (B5a).

Noticing this discrepancy, a comparison was made between the fluid approximation and our method in Ref. [46] (see its Appendix B), and no relevant differences

<sup>16</sup>See also Ref. [32] for a suggested expression of  $c_s^2$  in the axion case and Ref. [33] for a generalized one in the case of early dark energy models (for the amended code AXICLASS).

were found in the numerical results. To understand this, we use Eq. (B2) and write the EFA equations (B5a) and (B5b) in terms of our variables  $\delta_0$ ,  $\delta_1$  and  $N = \ln a$ . We find that

$$\begin{pmatrix} \delta_0 \\ \delta_1 \end{pmatrix}' \simeq \frac{k^2}{k_J^2} \begin{pmatrix} -3/y_1 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \bar{h}'/2 \\ 0 \end{pmatrix}. \quad (\text{B6})$$

We now compare Eq. (B6) with Eq. (B1) and spot the extra term  $3/y_1$ . The latter is very small, given that  $y_1 = 2m_\phi/H$  and then  $y_1 \gg 1$  after the start of rapid field oscillations. This is the reason why the comparison in Ref. [46] did not find differences between our approach and the EFA. It can be concluded that our method correctly picks up the only terms in the equations of motion that are valid in the limit  $H/m_\phi \ll 1$ .<sup>17</sup>

### APPENDIX C: NUMERICAL IMPLEMENTATION IN CLASS

Following the design of CLASS, we introduced a new module for the SFDM equations of motion, replicating the same structure as for other dark matter components. In this form, the contribution of the SFDM component could be called with its own parameters in any given parameter file of CLASS (for example, *explanatory.ini*).

The equations of motion for the background variables (12) were included in the file *background.c*, while those of the density perturbations (12) were included in the file *perturbations.c*. Similarly, a shooting routine was incorporated into the file *input.ini* to adjust the initial conditions of the dynamical variables. In each case, the SFDM quantities were added to the matter budget so that they contribute correctly to the right-hand side of the Einstein equations, for both the background and the linearly perturbed ones.

Given the oscillatory nature of all SFDM quantities, we applied the cutoff procedure to the sine and cosine functions that appear in their definition; see Eq. (22). For example, the background pressure was written as  $p_\phi = -\cos_\star \theta \cdot \rho_\phi$ , and then effectively  $p_\phi = 0$  for  $t > t_*$ . Another case would be the combination  $p_\phi + \rho_\phi = (1 - \cos_\star \theta)\rho_\phi$ , so that  $p_\phi + \rho_\phi = \rho_\phi$  for  $t > t_*$ .

The same was applied to the perturbed quantities, as in the density perturbation written in the form  $\delta\rho_\phi = \rho_\phi \cdot \delta_0$ , which is required for the right-hand side of the perturbed Einstein equations. Similarly, the momentum density perturbation  $(p_\phi + \rho_\phi)\Theta_\phi$  was written as in Eq. (17b), that is,

<sup>17</sup>The study in Ref. [46], and others in which the EFA is used, missed neglecting the extra term in Eq. (B5a) because it did not analyze the joint term  $\mathcal{H}\langle c_s^2 \rangle$ ; otherwise the authors would have realized that it is of order  $\mathcal{O}(H/m_\phi)$ . The same applies to the EFA used in Ref. [20] that includes additional terms from the gauge transformation: These terms are also of the order  $\mathcal{O}(H/m_\phi)$  and can be neglected. This effectively leads the EFA to become again Eq. (B3).

$$(\rho_\phi + p_\phi)\Theta_\phi = \frac{k^2\rho_\phi}{aHy_1} [(1 - \cos_\star \theta)\delta_1 - \sin_\star \theta\delta_0]. \quad (\text{C1a})$$

The result for late times  $t > t_\star$  is well defined and is given by

$$(\rho_\phi + p_\phi)\Theta_\phi = \frac{k^2\rho_\phi}{aHy_1} \delta_1. \quad (\text{C1b})$$

It should be noted that in some parts of the file `perturbations.c` one requires calculating the velocity divergence  $\Theta$ , which is one of the fundamental variables in the fluid formalism for linear perturbations of density and momentum. However, the calculation of  $\Theta_\phi$  itself is problematic for SFDM, as can be clearly seen in Eq. (C1a): one needs to divide by the quantity  $(p_\phi + \rho_\phi)$ , which passes through zero during the phase of rapid field oscillations. As explained above, the calculation of  $\Theta_\phi$  is not needed in the perturbed Einstein equations, and then we only used Eq. (C1b) as an additional source of the perturbed momentum density.

A technical aspect not always mentioned for SFDM models and its implementation in a Boltzmann code is that a small quantity of a CDM component is always needed, as stated in Ref. [19]. This is because, to be consistent with the synchronous gauge, the Einstein field equations must be solved in the comoving frame of the CDM fluid, that is,  $\Theta_{\text{cdm}} = 0$ . In fact, an automatic feature in the current versions of CLASS and its amended versions for SFDM, is that  $\Omega_{\text{cdm}} = 10^{-10}$  once the amount of CDM is set to zero in the input file `explanatory.ini`. This ensures that there will be practically a null contribution of CDM, so that SFDM is the dominant nonrelativistic matter component, and also helps us to avoid the known problem that the synchronous gauge is not completely fixed for density perturbations when the SFDM is the only DM component [87].

One last note is that the evolution of the perturbations should start well before the onset of the rapid oscillations. We include an additional condition in the file `perturbations.c` to guarantee that the calculations begin only if  $m_\phi/H < 0.01$ . This gives the system (15) enough time to reach its attractor solution starting from the initial conditions (21).

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