

Towards entanglement entropy of random large-N theories

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A large class of strongly correlated quantum systems can be described in certain large-N limits by quadratic in field actions along with self-consistency equations that determine the two-point functions. We use the replica approach and the notion of shifted Matsubara frequency to compute von Neumann and Rényi entanglement entropies for generic bipartitioning of such systems. We argue that the von Neumann entropy can be computed from equilibrium spectral functions without partitioning, while the Rényi entropy requires recalculating the spectrum in the interacting case. We demonstrate the flexibility of the method by applying it to examples of a two-site problem in presence of decoherence and coupled Sachdev-Ye-Kitaev models.

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I. INTRODUCTION

Entanglement is one of the central concepts of quantum mechanics and a notion based on which many of the modern physical phenomena are understood. The entanglement between the degrees of freedom in a region of space A and the rest of the system \bar{A} , is fully characterized by the so-called entanglement spectrum (ES), i.e., eigenvalues of the reduced density matrix $\rho_A = \text{Tr}_{\bar{A}}[\rho]$, or equivalently, its various moments. Among different measures of the entanglement, Rényi and von Neumann entanglement entropies (EEs),

$$S_A^{\text{R}_M} \equiv \frac{1}{1-M} \log \text{Tr}[\rho_A^M], \quad S_A^{\text{vN}} \equiv -\text{Tr}[\rho_A \log \rho_A], \quad (1)$$

are frequently used, where the latter can also be extracted from the limit $S_A^{\text{vN}} = \lim_{M \rightarrow 1^+} S_A^{\text{R}_M}$.

It is known that the EE of typical pure states depends on the sizes of the Hilbert spaces [1,2], whereas the EE of the ground state scales with the spatial extent of the regions. This is because roughly speaking, EE counts the number of entangled states; for gapped systems with short-range correlation an “area law” and for gapless systems with long-range correlation, a “volume law” is expected [3–5].

Entanglement entropy has many important applications. For example, in $1+1$ -dimensional gapless systems, EE is the natural probe of the central charge of the underlying conformal field theory (CFT) [6]. Furthermore, in $2+1$ -dimensional gapped systems with perimeter L_A , the entropy has the form $S_A^{\text{vN}} = \alpha L_A - \gamma$ [7], where γ is a signature of topological order and can be extracted using a procedure that eliminates the extensive part [8,9].

Moreover, according to eigenstate thermalization hypothesis (ETH) [10–12], the reduced density matrix of a chaotic system in a pure state has the Boltzmann form

$\rho_A \sim e^{-H_A/T_{\text{eff}}}$, where H_A is the Hamiltonian of *detached* A part and the temperature T_{eff} depends on the state’s energy. A somewhat unexpected example is the Laughlin state, whose ES contains the spectrum of gapless edge states that would exist if A and \bar{A} were physically detached [13], as if due to topology and despite the gap, ρ_A shares the same spectrum with H_A . Similar physics is present in other topological systems [14,15] and is understood in terms of the relevance of the coupling between the edge states across A - \bar{A} border [16] in the renormalization group (RG) sense.

There are also connections to holography [17,18]. According to Ryu-Takayanagi conjecture, the EE of CFT_{d+1} is given geometrically $S_A^{\text{vN}} \propto \mathcal{A}_A$ by the extremal area \mathcal{A}_A of the minimal spacelike surface anchored to A region and extending in the AdS_{d+2} bulk. As external parameters are varied, \mathcal{A}_A may switch from isolated surfaces to a joint surface, and this is interpreted as the formation of a wormhole. Hence, certain transitions in EE are holographically topological.

Despite its prevalence and important applications, the class of problems where EE can be computed are limited to noninteracting problems [19–21], 2D CFTs [6,22,23], a number of integrable models [24–26], as well as systems amenable to quantum Monte Carlo simulations [27,28], exact diagonalization [29], or density matrix renormalization. Here, we develop a versatile new technique that allows us to extend this list to problems that can be described by quadratic actions, e.g., models studied using static mean-field [30,31] and dynamical large-N techniques. The latter includes random Sachdev-Ye-Kitaev (SYK) models [32] as well as various tensor models that describe Kondo systems [33–43] and large-N theories of strange metals [44,45]. To the best of our knowledge such a versatile technique that can be applied to all these problems was not available until now.

Previous attempts at calculating EE of these systems [46–50] have been mostly limited to second Rényi entropy and restricted to random SYK model, which thanks to its exact solvability and maximally chaotic behavior [32], have attracted considerable interest. In particular, the thermalization of the SYK and coupled-SYK models [51–56], have been studied due to their holographic equivalence to black holes, connected by traversable wormholes [52,53]. Therefore, we apply the method to study coupled-SYK models, as an example in this paper.

The rest of the paper is organized as follows. Section II is the central part of the paper where we develop our new approach to computing EE as well as comment on the role of topology. The method is then applied to various problems in Sec. III. We conclude in Sec. IV and present some future directions. A number of appendices present the usual approach to non-interaction systems, the equilibrium action of the SYK model and the detailed proof of various statements made in the paper.

II. METHOD

In this section, we develop a formalism to compute EE in large-N theories, which are described by quadratic action, in which the interaction is incorporated self-consistently into self-energies. We also discuss the non-interacting limit of the formalism and its connection to topology and ETH.

A. Replica approach

We consider field theories whose action $\mathcal{S} = \mathcal{S}_Q + \mathcal{S}_C$ can be reduced into a quadratic quantum part $\mathcal{S}_Q[\bar{\psi}, \psi; \Sigma]$ in any dimension, possibly by introducing a number of dynamical constraints, and a Luttinger-Ward functional in the free energy, collected in $\mathcal{S}_C[G, \Sigma]$. The two parts are linked by self-consistency equations $\Sigma[G]$. We imagine dividing the system into A and $B = \bar{A}$ parts [Fig. 1(a)] with (bosonic, fermionic, or mixed) quantum fields ψ_a and ψ_b , each having an arbitrary number of modes which capture the spatial extension of the region. To compute $S^{\text{R}M}$, we introduce $r = 1 \dots M$ replica of quantum fields $\psi^{(r)}(\tau)$, with imaginary-time boundary-conditions [57],

$$\psi_a^{(r)}(\beta) = \tilde{\zeta}^{1/M} \psi_a^{(r+1)}(0), \quad \psi_b^{(r)}(\beta) = \tilde{\zeta} \psi_b^{(r)}(0), \quad (2)$$

for the fields in A and B, respectively [see Fig. 1(b)]. Here, $\tilde{\zeta} = \pm 1$ for bosons/fermions, and we have chosen a gauge in which $\tilde{\zeta}$ is distributed uniformly among ψ_a [58]. In terms of these fields, $\text{Tr}[\rho_A^M] = Z_M/Z_0^M$ where $Z_M = e^{-NS_C} \int D(\bar{\psi}, \psi) e^{-NS_Q}$ has to be computed on the manifold of Fig. 1(b).

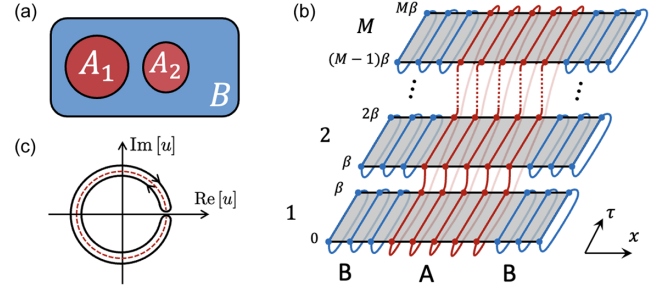


FIG. 1. (a) General bipartite setting considered in this paper. A and B sections do not need to be simply connected. (b) The replica method for computing Rényi entropy. The boundary condition in the imaginary time direction for A and B sections, represented by the red/blue lines, are different. (c) The contour integral used to define von Neumann EE in the fermionic case. Bosonic case is the vertical mirror of this.

B. Replica momentum

Despite the quadratic form of the action, computing Z_M is highly nontrivial due to the boundary condition (2). Following [59], we transform both fields to the so-called *replica-momentum* space,

$$\forall p = 0 \dots M-1, \quad \psi^{(p)}(\tau) \equiv \frac{1}{\sqrt{M}} \sum_{r=1}^M \Omega^{pr} \psi^{(r)}(\tau). \quad (3)$$

In this space, the ψ_a have the boundary condition $\psi_a^{(p)}(\beta) = u_p \psi_a^{(p)}(0)$ with $u_p \equiv \tilde{\zeta}^{1/M} \Omega^{-p}$ in terms $\Omega \equiv e^{2\pi i/M}$, whereas ψ_b have the usual $\psi_b^{(p)}(\beta) = \tilde{\zeta} \psi_b^{(p)}(0)$ periodicity. For a field with a periodicity $\psi(\beta) = u\psi(0)$, the Matsubara frequencies are shifted according to $u = e^{i\beta\bar{\omega}_n}$. Such shifted Matsubara frequency have been encountered in the perturbative calculation of the second Rényi entropy before [60], which are generalized here to arbitrary Rényi entropies. The summation over shifted frequencies $\bar{\omega}_n$, can be done using contour integration with $n_u(z) \equiv [ue^{\beta z} - 1]^{-1}$, and such a field has the partition sum $Z_e[u] \equiv [1 - u^{-1}e^{\beta\epsilon}]^{-\zeta}$. Note that $\tilde{\zeta} n_{\tilde{\zeta}}(\omega)$ is Bose-Einstein and Fermi-Dirac distributions for $\tilde{\zeta} = \pm 1$, respectively.

Quite generically, the quadratic action \mathcal{S}_Q on the manifold of Fig. 1(b) decouples into different p sectors and using Einstein summation can be expressed as

$$\mathcal{S}_Q = \sum_{p=0}^{M-1} (\bar{\psi}_{a,n} \bar{\psi}_{b,m})^{(p)} [-\mathcal{G}_{u_p}^{-1}]^{ab,a'b'} \begin{pmatrix} \psi_{a',n'} \\ \psi_{b',m'} \end{pmatrix}^{(p)}. \quad (4)$$

Here, n, n' indices refer to shifted Matsubara frequencies $\bar{\omega}_n[u] = 2\pi nT - iT \log u$, whereas m, m' refer to regular bosonic/fermionic Matsubara frequencies $i\omega_m$.

C. Time-translational invariance assumption

The Rényi entropy, proportional to $\log Z_M/Z_0^M = \sum_p \log Z[u_p]/Z_0$ can be expressed as a contour integral in the complex u plane [Fig. 1(c)],

$$\log \frac{Z_M}{Z_0^M} = \oint \frac{du}{2\pi i} \log \left(\frac{Z[u]}{Z_0} \right) \partial_u \log(u^M - \tilde{\zeta}). \quad (5)$$

This enables us to extend S_A^{RM} to nonintegers values of M , justifying the $S_A^{\text{vN}} = \lim_{\epsilon \rightarrow 0} S_A^{\text{R}^{1+\epsilon}}$ limit. See Ref. [20] for a discussion of uniqueness. Although for $u \neq \tilde{\zeta}$ the (imaginary) time-translational symmetry is broken [61], we expect it to be recovered in the $\epsilon \rightarrow 0$ limit, and thus, $\mathcal{G}_u(\tau_1, \tau_2) = G(\tau_1 - \tau_2) + \epsilon \delta \mathcal{G}_u(\tau_1, \tau_2)$ for the Green's function. For interacting systems, this feeds into the self-energy $\Sigma = \Sigma[G]$, giving $\Sigma_u(\tau_1, \tau_2) = \Sigma(\tau_1 - \tau_2) + \epsilon \delta \Sigma_u(\tau_1, \tau_2)$ (see the Appendix A 2). The first observation of our paper is that since the $\epsilon = M - 1 \rightarrow 0$ limit of Eq. (5) is explicitly proportional to ϵ , the ϵ -correction to the self-energy is not needed to compute the von Neumann EE. Therefore, we assume that self-energy has time-translational symmetry. For noninteracting problems, this is an exact statement, but for interacting large-N problems, this approximation is only valid for the von Neumann entropy.

D. Entanglement entropy formula

Absorbing the Hamiltonian into the self-energy, the inverse Green's function in (4) can be written as

$$[\mathcal{G}_u^{-1}]_{nm, n'm'}^{ab, a'b'} = \begin{pmatrix} [i\bar{\omega}_n \delta^{aa'} - \Sigma_n^{aa'}] \delta_{nn'} & \frac{1 - \tilde{\zeta} u}{\beta} \frac{\Sigma_m^{ab'}}{i\bar{\omega}_n - i\omega_m} \\ \frac{1 - \tilde{\zeta} u^{-1}}{\beta} \frac{\Sigma_m^{ba'}}{i\omega_m - i\bar{\omega}_n} & [i\omega_m \delta^{bb'} - \Sigma_m^{bb'}] \delta_{mm'} \end{pmatrix}, \quad (6)$$

where we have taken advantage of time-translational symmetry of self-energies. The off diagonal elements in frequency originate from the mismatch in Matsubara frequencies of ψ_a and ψ_b fields. However, a knowledge of equilibrium Green's function G alone, is sufficient to build the \mathcal{G}_u^{-1} . See Appendix A 3 for a derivation of Eq. (6).

The u -sector partition function of action (4), (6) is

$$Z[u] = \det^{-\zeta} [(-\mathcal{G}_u^{-1})^{aa'}] \det^{-\zeta} [(-\mathcal{G}_u^{BB'})^{-1}]. \quad (7)$$

We use $\zeta = 1$ for bosons, $\zeta = -1$ and $\zeta = -1/2$ for complex/real fermions and notice that $\tilde{\zeta} = \text{sign}(\zeta)$. After summation over shifted frequencies $\bar{\omega}_n$ and expressing the Green's function of A by its spectral representation $A^{aa'}(\omega) \equiv iG^{aa'}(z)_{\omega-i\eta}^{\omega+i\eta}$, the $\mathcal{G}_u^{BB'}$ can be written as

$$(\mathcal{G}_u^{BB'})^{-1} = [G_m^{-1}]^{BB'} \delta_{mm'} - \int \frac{dx}{2\pi} \frac{K_u(x) \Sigma_m^{ba} A^{aa'}(x) \Sigma_m^{a'b'}}{\beta (\omega_m - x)(i\omega_m' - x)},$$

where $K_u(\omega) \equiv (\tilde{\zeta} u - 1)n_u(\omega)/n_{\tilde{\zeta}}(\omega)$. Here, $G^{BB'}(z)$ with uppercase B and $G^{aa'}(z)$ with lowercase a are the equilibrium Green's function of the *attached* B part and *detached* A part (possibly modified due to self-consistency equations), respectively. In other words, $G^{aa'}(z)$ is the inverse of the first block of $\mathcal{G}_{u=\tilde{\zeta}}^{-1}$, but $G^{BB'}$ is the last block of the inverted matrix $\mathcal{G}_{u=\tilde{\zeta}}$.

Using determinant shuffling technique (see the Appendix A 4) and defining $\mathbb{1} \equiv 2\pi\delta(\omega - \omega')\delta_{aa'}$, Eq. (7) becomes

$$Z[u] = Z_a[u] Z_B \det^{-\zeta} [\mathbb{1} + \tilde{\zeta} K_u(\omega) \mathbb{A}_a(\omega) \mathbb{J}_A(\omega, \omega')] \quad (8)$$

(see Appendix A 5 for details) written in terms of

$$\mathbb{J}_A(\omega, \omega') \equiv \frac{1}{\beta} \sum_m \frac{\mathbb{R}_A(i\omega_m)}{(i\omega_m - \omega)(i\omega_m - \omega')}, \quad (9)$$

where $R^{aa'} \equiv \Sigma^{ab} G^{BB'} \Sigma^{ba'}$. Alternatively in terms of the attached/detached A correlators, $\mathbb{R}_A = \mathbb{G}_a^{-1} \mathbb{G}_A \mathbb{G}_a^{-1} - \mathbb{G}_a^{-1}$. The boundary condition in imaginary time u appears in Eq. (8) only via $K_u(x)$. We can write the determinant term as $\det^{-\zeta} [n_u/n_{\tilde{\zeta}}(\mathbb{C} - u\mathbb{D})]$, where [62]

$$\mathbb{C}(\omega, \omega') = \tilde{\zeta} n_{\tilde{\zeta}}(\omega) \mathbb{1} + \sqrt{\mathbb{A}_a(\omega)} \mathbb{J}_A(\omega, \omega') \sqrt{\mathbb{A}_a(\omega')}, \quad (10)$$

and $\mathbb{D} = \mathbb{1} + \tilde{\zeta}\mathbb{C}$. Considering that $\mathbb{J}_A \rightarrow 0$ for a reference \mathbb{C}_0 with detached A and B parts, the system-independent thermal prefactor can be eliminated by taking the ratio of the two determinants. Using $\text{Tr}[\rho^M] = Z^{-M} \prod_p Z[u_p]$ and $Z = Z_a Z_B$, we finally have

$$\text{Tr}[\rho_A^M] = \prod_p \frac{Z_a[u_p]}{Z_a} \frac{\det^{-\zeta} [\mathbb{D} - u_p^{-1}\mathbb{C}]}{\det^{-\zeta} [\mathbb{D}_0 - u_p^{-1}\mathbb{C}_0]}. \quad (11)$$

Equation (11) is the central result of our paper. We have succeeded to single-out the parameter u , characterizing the boundary condition in each sector, and express the rest in terms of equilibrium Green's functions of region A. This enables us to evaluate the p -product using the identity $\prod_p \det[\mathbb{D} - u_p^{-1}\mathbb{C}] = \det[\mathbb{D}^M - \tilde{\zeta}\mathbb{C}^M]$.

E. Thermal part of EE

Rényi entropies can be written as a sum of two terms $S_A^{\text{RM}} = S_a^{\text{RM}} + \Delta S_A^{\text{RM}}$. The first term is the (thermal) Rényi entropy of the detached A system,

$$S_a^{\text{RM}} = \frac{1}{1-M} [\log Z_a(M\beta) - M \log Z_a(\beta)], \quad (12)$$

where $Z_a(\beta) = e^{-\beta F_a(\beta)}$ is the partition function of the detached A system at inverse temperature β . In the $M \rightarrow 1^+$ limit, $S_a^{\text{vN}} = -dF_a/dT$ becomes the thermodynamical

entropy of the detached A system. Note that $S_a(T \rightarrow 0)$, vanishes for all gapped systems, as well as most gapless systems that lack a residual $T = 0$ entropy.

F. Quantum corrections

The quantum correction to EE ΔS_A , requires a diagonalization of $\mathbb{C}(\omega, \omega')$ matrix. The eigenvalues of \mathbb{C} are real and positive ($c \leq 1$ for fermions). We define the entanglement density of states (DoS) $\Delta\rho$ as the difference $\Delta\rho(c) \equiv \rho(c) - \rho_0(c) \equiv \sum_j \delta(c - c_j) - \sum_j \delta(c - c_{j0})$ in \mathbb{C} and \mathbb{C}_0 DoSs. $\Delta\rho(c)$ vanishes for physically detached A and B. Defining $c^+ \equiv c + i\eta$, $\Delta\rho$ can be expressed as

$$\Delta\rho(c) = -\frac{1}{\pi} \partial_c \text{Im} \log \{ \det[(c^+ \mathbb{1} - \mathbb{C}_0)^{-1} (c^+ \mathbb{1} - \mathbb{C})] \}. \quad (13)$$

in terms of which, $\Delta S_A^{\text{R}_M} = \int dc \Delta\rho(c) g^{\text{R}_M}(c)$, where

$$g^{\text{R}_M}(c) \equiv \frac{-\zeta}{1-M} \log[(1 + \zeta c)^M - \zeta c^M], \quad (14)$$

and $g^{\text{vN}}(c) = g^{\text{R}_+}(c) = \zeta[(1 + \zeta c) \log(1 + \zeta c) - \zeta c \log(c)]$. Generally $g \geq 0$, and for fermions $g \leq g(1/2) = \log(2)$. The matrix \mathbb{C} has to be discretized and diagonalized numerically. Assuming \mathcal{N} frequency points, ρ and ρ_0 are each $\mathcal{O}(\mathcal{N})$, but $\Delta\rho$ is an $\mathcal{O}(1)$ zero-mean function, independent of frequency discretization [63]. The form of Eq. (14) is familiar from Luttinger's theorem [64]. $\Delta\rho(c)$ consists of unit-area resonances located at c values where the phase of the determinant winds, corresponding to excess or deficit of an eigenvalue on top of a continuum.

Equations (13)–(14) indicate that the \mathbb{C} part of the ES can be emulated by an infinite set of auxiliary particles $\tilde{\psi}_{a,\omega}$ in an extra dimension [57] in thermal equilibrium with occupations $\langle \tilde{\psi}_{\omega}^\dagger \tilde{\psi}_{\omega'} \rangle = \mathbb{C}(\omega, \omega')$ [65]. The relation $\mathbb{C} = (e^{\beta \tilde{\mathbb{H}}} - \zeta)^{-1}$ defines *entanglement Hamiltonian* $\tilde{\mathbb{H}}$ [66].

G. Noninteracting limit and topology

In the noninteracting limit, the spectral function $\mathbb{A}^a(\omega)$ consists of a series of delta functions, which reduce the dimension of \mathbb{C} to the number of modes. More importantly, $\Delta\rho < 0$ contribution by \mathbb{C}_0 exactly cancels the thermal contribution to EE, S_a . In this limit $\tilde{\psi}_{a,\omega} \rightarrow \psi_a \delta(\omega)$, the matrix \mathbb{C} represents occupation of physical particles ψ_a , and our formalism reduces to known results [20]. See Appendix A 6 for an explicit proof.

Generally, when A-B coupling is weaker than temperature, Eq. (13) offers a perturbative expansion without the need to diagonalize \mathbb{C} (See Appendix A 7). If the A-B coupling is irrelevant in a renormalization group sense, $\Sigma^{ab} \rightarrow 0$ and $\mathbb{J}_A \rightarrow 0$ and ΔS vanishes. On the other hand, if A-B coupling is relevant, for example in presence of edge modes in the energy spectrum of detached systems, $\Sigma^{ab} \rightarrow \infty$. In this case, it is justified to *flatten the spectrum* [15] by neglecting the k dependence of Green's functions

involved in computing $\mathbb{J}_A(\omega, \omega')$. Writing $V^2 \delta^{aa'} = \Sigma^{ab} \Sigma^{ba'}$, for each mode in A, $R^{aa'}(z) \rightarrow \delta^{aa'} V^2 / (z - V^2/z)$ will have the same form as a two-site fermion problem with a coupling V . The latter has a zero mode in the ES and an EE of $\log(2)$. The original model has a highly degenerate zero mode, whose degeneracy is lifted by $A^a(\omega)$, resulting in a gapless mode in ES, in apparent agreement with ETH [16].

At $T \rightarrow 0$, the negative part of $\Delta\rho$ can be ignored, and resonances can be represented by their entanglement “energies” $\varepsilon = \log(1/c + \zeta)$. An ES gap closing and reopening with a zero mode then indicates a topological transition in the bulk and formation of edge states. Indeed the quantum EE $\Delta S / \log(2) = \text{nullity}(\tilde{\mathbb{H}})$ is related to the number of zero modes of $\tilde{\mathbb{H}}$, a topological invariant.

III. EXAMPLES

In this section, we show that the formalism developed above can be used to compute EE in large-N theories. For simplicity, we limit ourselves to two-site fermionic problems.

A. Models with self-energy

The simplest example is a system in which integrating out some internal degrees of freedom has led to a self-energy. Consider the four-site problem in a U geometry [inset of Fig. 2(a)] where A and B are coupled by V but each are coupled by W to a single-site bath, resulting in $\Sigma^{aa}(z) = \Sigma^{bb}(z) = W^2/z$. Figure 2(b) shows EE in perfect agreement with exact diagonalization. At $W \rightarrow 0^+$, the bath sites are forced to be entangled with each other, as can be seen by a Shrieffer-Wolff produced coupling, thus $S \rightarrow 2 \log(2)$. Although the EE is constant for $W \ll V$, there is a crossover from quantum to thermal contributions as W/T is varied [Fig. 2(c)]. Figure 2(a) shows that at $W < V$ effectively two of the eigenvalues of \mathbb{C} move to $c \rightarrow 1/2$, forming zero modes that increase EE to $2 \log(2)$, but they are canceled at $T < W$ by the spectral migration of \mathbb{C}_0 eigenvalues to zero. The EE decreases with increasing W , due to the entanglement monogamy.

Our technique readily generalizes to the case where A and B are *decohered* [67] by coupling to a fermionic bath [Fig. 2(d)], an example for which many other methods fail. In the wideband limit, the self-energy can be taken to be independent of frequency, i.e., $\Sigma(\omega + i\eta) = -i\Gamma$ for both sites. The resulting EE shows no interbath entanglement, but an overshoot at $\Gamma \sim V$ remains.

B. Models with self-consistency

As an example of problems with self-consistency, we look at coupled-SYK models, defined as $H_0 + H_{\text{int}}$, where

$$H_0 = \frac{1}{4!} \sum_{\mu=A,B} \sum_{ijkl=1}^{N_\mu} J_{ijkl}^\mu \chi_i^\mu \chi_j^\mu \chi_k^\mu \chi_l^\mu. \quad (15)$$

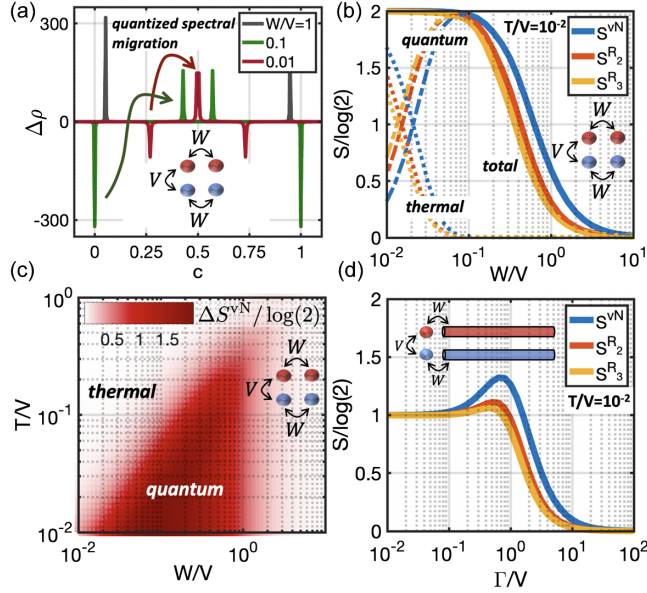


FIG. 2. Free fermions coupled to (a)–(c) single-site and (d) wideband bath geometries indicated in the insets. (a) Entanglement DoS $\Delta\rho(c)$ at $T/V = 10^{-2}$ for various W/V ratios, show a quantized migration of positive spectrum toward $c \rightarrow 1/2$ for $T < W$, followed by a negative spectrum migration at $W < T$. Each peak has a unit area. (b) von Neumann and Rényi EE as function of W resolved into thermal part and quantum correction. (c) The quantum part of S^{vN} show its W/T scaling. (d) von Neumann and Rényi EE as a function of Γ/V for $\Gamma = \pi W^2/\Lambda$ in the large bandwidth Λ limit.

H_0 describes two copies of SYK dots. Here, χ_j^μ are Majorana fermions and J_{ijkl}^μ are random numbers taken from a zero mean Gaussian distribution (ZMGD) with the variance J^2/N . After disorder averaging and in the large- $N_{A,B}$ limit, this model reduces to a quadratic action with the two-point function that is determined self-consistently by the self-energy $\Sigma_\mu(\tau) = J^2 G_\mu^3(\tau)$ and the Dyson equation $G_\mu^{-1}(z) = z - \Sigma_\mu(z)$. Readers are referred to [32,68] for important omitted aspects as well as Appendix A 8 for a review of the equilibrium case. Without coupling, $\Delta S_A = 0$ and thus $S_A(T) = S_{\text{th}}^{\text{SYK}}(T)$, which at $T \rightarrow 0$ is given by the residual entropy of a single SYK.

We now assume that the two SYK dots are connected [69] by four-fermion couplings $H = H_0 + \sum_{ijkl} V_{ijkl} \chi_i^A \chi_j^A \chi_k^B \chi_l^B$, where V_{ijkl} are again ZMGD with variance V^2/N . In this latter case, the self-consistency equations becomes ($p = N_B/N_A$),

$$\Sigma_{aa}(\tau) = J^2 G_{AA}^3(\tau) + 2V^2 \sqrt{p} G_{BB}^2(\tau) G_{AA}(\tau), \quad (16)$$

and a similar equation for Σ_{bb} with $A \leftrightarrow B$ and $p \rightarrow 1/p$. For $V = J$, this is a single composite SYK dot with the total number of $N = N_A + N_B$ fermions.

A common feature of all these four-fermion coupling models is that the coupling is irrelevant in an RG sense near

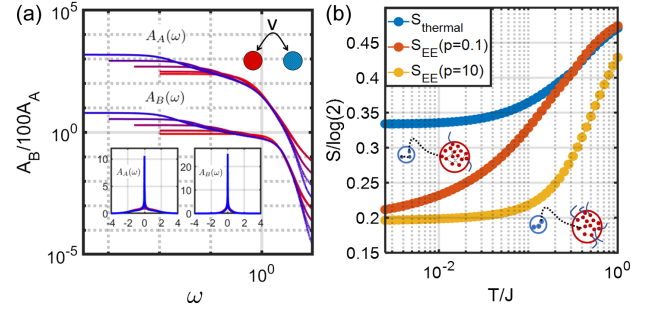


FIG. 3. Coupled-SYK model with $V/J = 1$. (a) Spectral function A_A and A_B for various temperatures (T/J from 0 to 1), (b) von Neumann EE S^{vN} for different components.

infrared fixed point. Furthermore, $\Sigma^{ab} = \Sigma^{ba} = 0$, and thus, there are no quantum corrections $\Delta S_A = 0$. The EE is given entirely by the thermal part $S_A^{vN} = S_a$, which also includes the classical part of the action.

Figure 3 shows the spectra of A and B as well as both thermodynamical and entanglement entropy of the two systems. The same residual entropy per particle for A and B shown in Fig. 3(b) indicates that the larger part of the coupled system is still entangled to the outside at $T = 0$. In addition to that, there are some intersubsystem entanglement as indicated by the EEs.

IV. CONCLUSION

In summary, we have provided a Green's function formalism to compute ES of theories with a quadratic action through diagonalization of a single matrix built out of equilibrium functions. In this sense, our approach is different from the Z_M gauge theory approach taken in [70–72]. This Green's function approach already simplifies the computation of entanglement entropy in specific noninteracting scenarios.

Interactions can be treated perturbatively within this formalism. However, we argued that this formalism provides access to the von Neumann EE of large- N theories described by a Luttinger-Ward functional of two-point Green's functions. The latter includes contributions from both quantum and classical parts of the action. The focus of this work has been on the quantum part and the examples chosen are large- N models which have simple classical parts. Generalization to other examples with more complicated classical actions, e.g., [52] is left for future. More importantly, further work is needed to investigate other systems and verify the time-translational invariant assumption that enables such an extension.

We have applied our method to a noninteracting problem with self-energy as well as the coupled SYK model. The method can be in principle applied to Kondo lattices [30,34,35,41] where changes in the pattern of entanglement are shown to be playing major role in Kondo breakdown transition [36,73,74]. Extension to nonequilibrium

steady-state as well as quench dynamics is an interesting future direction.

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APPENDIX

This appendix contains further details and extended proof of various statements in the paper.

1. Noninteracting case

For the sake of completeness, we remind ourselves of the known noninteracting results [20,21].

Bosons—The reduced density matrix is

$$\rho_A = \frac{e^{-H_A}}{Z_A},$$

where $H_A = \sum_{mn} \epsilon_{mn} a_m^\dagger a_n$ is the entanglement Hamiltonian and $Z_A = \prod_k (1 - e^{\epsilon_k})^{-1}$. We define the correlators,

$$C_{mn} \equiv \langle a_m^\dagger a_n \rangle.$$

The spectrum of \mathbb{C} and H_A have the following relation:

$$\nu_k = \frac{1}{e^{\epsilon_k} - 1},$$

where $\{\nu_k\}$ is the spectrum of \mathbb{C} and $\{\epsilon_k\}$ is spectrum of H_A . The von Neumann entropy and Rényi entropy are

$$\begin{aligned} S_A^{\text{vN}} &= -\text{Tr}[\rho_A \ln \rho_A] \\ &= \sum_k \left[\frac{\epsilon_k e^{-\epsilon_k}}{1 - e^{-\epsilon_k}} - \ln(1 - e^{-\epsilon_k}) \right], \end{aligned} \quad (\text{A1})$$

and

$$S_A^{\text{R}_M} = \frac{\text{Tr}[\ln \rho_A^M]}{1 - M}.$$

Instead of using the spectrum $\{\epsilon\}$, one can use the spectrum $\{\nu_k\}$ to calculate von Neumann entropy and Rényi entropy as

$$S_A^{\text{vN}} = \sum_k (1 + \nu_k) \ln(1 + \nu_k) - \nu_k \ln \nu_k$$

and

$$S_A^{\text{R}_M} = \text{Tr} \ln[(1 + \nu_k)^M - \nu_k^M].$$

or equivalently, in terms of the \mathbb{C} matrix as

$$S_A^{\text{vN}} = -\text{Tr}[(1 + \mathbb{C}) \ln(1 + \mathbb{C}) - \mathbb{C} \ln \mathbb{C}]$$

and

$$S_A^{\text{R}_M} = -\frac{1}{1 - M} \ln \det[(1 + \mathbb{C})^M - \mathbb{C}^M].$$

Fermions—The entanglement Hamiltonian is

$$\rho_A = \frac{e^{-H_A}}{Z_A},$$

where $H_A = \sum_{mn} \epsilon_{mn} a_m^\dagger a_n$ and $Z_A = \prod_k (1 + e^{-\epsilon_k})$. Again, we define the correlator,

$$C_{mn} \equiv \langle a_m^\dagger a_n \rangle.$$

The spectrum of \mathbb{C} and H_A are related according to

$$\nu_k = \frac{1}{e^{\epsilon_k} + 1},$$

where $\{\nu_k\}$ is the spectrum of \mathbb{C} and $\{\epsilon_k\}$ is spectrum of H_A . Therefore, von Neumann entropy and Rényi entropy are

$$S_A^{\text{vN}} = -\text{Tr}[\rho_A \ln \rho_A] = \sum_k \left(\frac{\epsilon_k e^{-\epsilon_k}}{1 + e^{-\epsilon_k}} + \ln(1 + e^{-\epsilon_k}) \right),$$

and

$$S_A^{\text{R}_M} = \frac{1}{1 - M} \ln \text{Tr}[\rho_A^M].$$

Instead of using the spectrum $\{\epsilon\}$, one can use the spectrum $\{\nu_k\}$ to calculate von Neumann entropy and Rényi entropy as

$$S_A^{\text{vN}} = -\sum_k (1 - \nu_k) \ln(1 - \nu_k) + \nu_k \ln \nu_k$$

and

$$S_A^{\text{R}_M} = \text{tr} \ln[(1 + \nu_k)^M - \nu_k^M],$$

which in terms of the \mathbb{C} matrix are given by or

$$S_A^{\text{vN}} = -\text{Tr}[(1 - \mathbb{C}) \ln(1 - \mathbb{C}) + \mathbb{C} \ln \mathbb{C}]$$

and

$$S_A^{R_M} = \frac{1}{1-M} \ln \det[(\mathbb{1} - \mathbb{C})^M + \mathbb{C}^M].$$

We can unify the entropies of Bosons and Fermions into the following:

$$S_A^{vN} = \zeta \text{Tr}[(\mathbb{1} + \tilde{\zeta} \mathbb{C}) \ln(\mathbb{1} + \tilde{\zeta} \mathbb{C}) - \tilde{\zeta} \mathbb{C} \ln \mathbb{C}]$$

and

$$S_A^{R_M} = \frac{-\zeta}{1-M} \ln \det[(\mathbb{1} + \tilde{\zeta} \mathbb{C})^M - \tilde{\zeta} \mathbb{C}^M].$$

2. Replica symmetry and self-energy

In order to compute the Rényi entropy, one need to solve the large-N path-integral problem on an extended manifold shown in Fig. 1(c). The Rényi entropy is given by

$$\text{Tr}[\rho_A^M] = \frac{1}{Z^M} \int D[G, \Sigma] D\chi e^{-NS[\chi, G, \Sigma]}, \quad (\text{A2})$$

here, χ represents the fermions and $G - \Sigma$ are used to decouple the interaction. Here, we show how this problem reduces to the action (4) and (6) of the paper. In order to be concrete and without loss of generality, we consider the coupled SYK model [69]. This equilibrium path integral description of this model is reviewed in Sec. A 8 of the present Appendix. The replica action is

$$S = \sum_{\mu=A,B} \sum_r \int d\tau \left\{ \frac{1}{2N} \sum_{i=1}^{N_\mu} \chi_{i,\mu}^{(r)}(\tau) \partial_\tau \chi_{i,\mu}^{(r)}(\tau) + \frac{J_{i,j,k,l}^\mu}{4!} \sum_{i,j,k,l=1}^{N_\mu} \chi_{i,\mu}^{(r)} \chi_{j,\mu}^{(r)} \chi_{k,\mu}^{(r)} \chi_{l,\mu}^{(r)} + \frac{V_{i,j,k,l}}{4!} \sum_{i,j=1}^{N_A} \sum_{k,l=1}^{N_B} \chi_{i,A}^{(r)} \chi_{j,A}^{(r)} \chi_{k,B}^{(r)} \chi_{l,B}^{(r)} \right\}, \quad (\text{A3})$$

which is diagonal in replica and needs to be supplemented with the boundary condition (2). The random variables J_{ijkl}^μ and V_{ijkl}^μ have zero mean and the variance,

$$J_{ijkl}^\mu J_{i'j'k'l'}^\nu = \delta^{\mu\nu} \delta_{ii'} \delta_{jj'} \delta_{kk'} \delta_{ll'} \frac{J^2}{4N_\mu^3}, \quad \langle V_{ijkl}^2 \rangle = \delta_{ii'} \delta_{jj'} \delta_{kk'} \delta_{ll'} \frac{V^2}{(N_A N_B)^{\frac{3}{2}}}. \quad (\text{A4})$$

After disorder averaging, the action develops off-diagonal-in-replica contributions and after $G - \Sigma$ decoupling becomes

$$S = \sum_{r,r'} \sum_{\mu=A,B} \int d\tau d\tau' \left\{ \frac{1}{2N} \sum_i \chi_{i,\mu}^{(r)}(\tau) [\partial_\tau \delta(\tau, \tau') \delta^{rr'} + \Sigma_\mu^{(rr')}(\tau, \tau')] \chi_{i,\mu}^{(r')}(\tau') \right. \\ \left. - \frac{1}{2} [\Sigma_\mu^{(rr')}(\tau, \tau') G_\mu^{(r'r)}(\tau', \tau) + \frac{J^2}{4} [G_\mu^{(rr')}(\tau, \tau')]^4] - \frac{V^2 \sqrt{N_A N_B}}{2} G_A^{(rr')}(\tau, \tau')^2 G_B^{(rr')}(\tau, \tau')^2 \right\}, \quad (\text{A5})$$

where μ and ν are a or b for self-energy Σ_μ and A or B for Green's function G_μ . Note that the interacting part of the action contains inter-replica interaction and such four-fermion terms are decoupled by the $G^{(rr')}$ Green's function, leading to off diagonal replica self-energy $\Sigma^{(rr')}$. Transforming from the replica sector n , to replica momentum space p , we find

$$S = \frac{1}{2} \int d\tau d\tau' \left\{ \frac{1}{N} \sum_{p,p'} \sum_{\mu=A,B} \sum_i \tilde{\chi}_{i,\mu}^{(p)}(\tau) (\partial_\tau \delta(\tau, \tau') \delta^{pp'} + \Sigma_\mu^{(pp')}(\tau, \tau')) \chi_{i,\mu}^{(p')}(\tau') - \sum_{p,p'} \sum_\mu \Sigma_\mu^{(pp')}(\tau', \tau) G_\mu^{(pp')}(\tau, \tau') \right. \\ \left. - \sum_{p_1, p_2, p_3, p'_1, p'_2, p'_3} \left[\frac{V^2 \sqrt{N_A N_B}}{2M^2} G_A^{(p_1 p'_1)}(\tau, \tau') G_A^{(p_2 p'_2)}(\tau, \tau') G_B^{(p_3 p'_3)}(\tau, \tau') G_B^{(-p_1 - p_2 - p_3, -p'_1 - p'_2 - p'_3)}(\tau, \tau') \right. \right. \\ \left. \left. + \frac{J^2}{4M^2} \sum_\mu G_\mu^{(p_1 p'_1)}(\tau, \tau') G_\mu^{(p_2 p'_2)}(\tau, \tau') G_\mu^{(p_3 p'_3)}(\tau, \tau') G_\mu^{(-p_1 - p_2 - p_3, -p'_1 - p'_2 - p'_3)}(\tau, \tau') \right] \right\},$$

where we have used

$$\chi^{(r)}(\tau) = \frac{1}{\sqrt{M}} \sum_p \Omega^{-pr} \chi^{(p)}(\tau), \quad G^{(rr')}(\tau, \tau') = \frac{1}{M} \sum_{pp'} \Omega^{-(pr-p'r')} G^{(pp')}(\tau, \tau'), \quad \Sigma^{(rr')}(\tau, \tau') = \frac{1}{M} \sum_{pp'} \Omega^{-(pr-p'r')} \Sigma^{(pp')}(\tau, \tau'),$$

with inverse relations,

$$\chi^{(p)}(\tau) = \frac{1}{\sqrt{M}} \sum_r \Omega^{pr} \chi^{(r)}(\tau), \quad G^{(pp')}(\tau, \tau') = \frac{1}{M} \sum_{rr'} \Omega^{pr-p'r'} G^{(rr')}(\tau, \tau'), \quad \Sigma^{(pp')}(\tau, \tau') = \frac{1}{M} \sum_{rr'} \Omega^{(pr-p'r')} \Sigma^{(rr')}(\tau, \tau').$$

Now in the p space, one set of saddle point solutions are found by varying the Green's function $G_\mu^{(pp')}$,

$$\begin{aligned} \Sigma_\mu^{(pp')}(\tau', \tau) = & - \sum_{p_1, p_2, p'_1, p'_2} \left[\frac{J^2}{M^2} G_\mu^{(p_1 p'_1)}(\tau, \tau') G_\mu^{(p_2 p'_2)}(\tau, \tau') G_\mu^{(-p_1 - p_2 - p, -p'_1 - p'_2 - p')}(\tau, \tau') \right. \\ & \left. + p^{\frac{\tilde{\mu}}{2}} \frac{V^2}{M^2} G_\mu^{(p_1 p'_1)}(\tau, \tau') G_{-\mu}^{(p_2 p'_2)}(\tau, \tau') G_{-\mu}^{(-p_1 - p_2 - p, -p'_1 - p'_2 - p')}(\tau, \tau') \right], \end{aligned}$$

where $\tilde{\mu}$ is ± 1 for $\mu = A$ and B , respectively. The $-\mu$ means the other part besides μ . The variations with respect to $\Sigma_\mu^{(pp')}$ gives the Dyson equation,

$$G^{(p'p)}(\tau, \tau') = \frac{1}{N} \sum_i \langle \chi_i^{(p')}(\tau') \chi_i^{(p)}(\tau) \rangle = \mathcal{G}_{p,p'}(\tau, \tau'). \quad (\text{A6})$$

We note that while these equations do generally support replica off diagonal solutions $G^{(pp')}$ and $\Sigma^{(pp')}$, However, replica symmetry is also preserved by these equations. This means if we assume $G^{(pp')} \propto \delta^{pp'}$, we find $\Sigma^{(pp')} \propto \delta^{pp'}$ which means the action decouples into different p sectors, leading to $G^{(pp')} \propto \delta^{pp'}$. Considering that at UV, $G \sim J \delta^{pp'}$ is replica symmetric, we conclude that the replica symmetry is preserved. Therefore, the Green's functions and self-energies can be represented by online diagonal p indices, $G^{(p)}(\tau, \tau') = \mathcal{G}_p(\tau, \tau')$. Likewise, the self consistency equations become the same for different p sectors,

$$\Sigma_\mu^{(p)}(\tau', \tau) = - \sum_{p_1 p_2} \left[\frac{J^2}{M^2} G_\mu^{(p_1)}(\tau, \tau') G_\mu^{(p_2)}(\tau, \tau') G_\mu^{(-p - p_1 - p_2)}(\tau, \tau') + p^{\frac{\tilde{\mu}}{2}} \frac{V^2}{M^2} G_\mu^{(p_1)}(\tau, \tau') G_{-\mu}^{(p_2)}(\tau, \tau') G_{-\mu}^{(-p - p_1 - p_2)}(\tau, \tau') \right].$$

The only thing different between various sectors is the boundary condition in imaginary-time direction. So, a full solution to the problem requires simultaneous solution to all p sectors. Eventually, the $\text{Tr}[\rho_A^M]$ can be written as

$$\text{Tr}[\rho_A^M] = \frac{1}{Z_0^M} \int D\chi e^{-NS}, \quad S = S_Q + S_C, \quad (\text{A7})$$

where using the saddle-point equations, the classical part is

$$\begin{aligned} S_C = & -\frac{1}{2} \int d\tau d\tau' \left[\sum_p \sum_\mu \Sigma_\mu^{(p)}(\tau', \tau) G_\mu^{(p)}(\tau, \tau') + \frac{V^2 \sqrt{N_A N_B}}{2M^2} \sum_{p_1, p_2, p_3} G_A^{(p_1)}(\tau, \tau') G_A^{(p_2)}(\tau, \tau') G_B^{(p_3)}(\tau, \tau') G_B^{(-p_1 - p_2 - p_3)}(\tau, \tau') \right. \\ & \left. + \frac{J^2}{4M^2} \sum_\mu \sum_{p_1, p_2, p_3} G_\mu^{(p_1)}(\tau, \tau') G_\mu^{(p_2)}(\tau, \tau') G_\mu^{(p_3)}(\tau, \tau') G_\mu^{(-p_1 - p_2 - p_3)}(\tau, \tau') \right], \end{aligned}$$

and the quantum part of the action is

$$S_Q = \sum_p \sum_{\mu=A,B} \int d\tau d\tau' \frac{1}{2N} \sum_i \bar{\chi}_{i,\mu}^{(p)}(\tau) (\partial_\tau \delta(\tau, \tau') + \Sigma_\mu^{(p)}(\tau, \tau')) \chi_{i,\mu}^{(p)}(\tau'). \quad (\text{A8})$$

As we have argued in the paper, however, a full self-consistent solution to all p -sectors is not needed if we are only interested in the von Neumann entanglement entropy. In this case, we could assume that the self-energy $\Sigma^{(p)}(\tau, \tau') = \Sigma(\tau_1 - \tau_2)$ has the same expression as the time-translational invariant $p = 0$ sector. Going to frequency space and using time-translational invariance assumption, the quantum action becomes

$$S_Q = \frac{1}{2N} \sum_p \sum_i \sum_{n,m,n',m'} \left[\bar{\chi}_{A,i}^{(p)}(i\bar{\omega}_n) \quad \bar{\chi}_{B,i}^{(p)}(i\omega_m) \right] \begin{pmatrix} -i\bar{\omega}_n + \Sigma_A^{nn} & 0 \\ 0 & -i\omega_m + \Sigma_B^{mm} \end{pmatrix} \begin{bmatrix} \chi_{A,i}^{(p)}(i\bar{\omega}_n) \\ \chi_{B,i}^{(p)}(i\omega_m) \end{bmatrix}, \quad (\text{A9})$$

where n and indices for shifted Matsubara frequencies, different in each p sector, and m are indices for normal Matsubara frequencies. The classical part becomes

$$\mathcal{S}_C = -\frac{3}{8} \sum_{p,n,m} [\Sigma_A(i\bar{\omega}_n)G_A(i\bar{\omega}_n) + \Sigma_B(i\omega_m)G_B(i\omega_m)]. \quad (\text{A10})$$

The elements of self-energies are worked out in Sec. A3. Then G and Σ will be just the equilibrium Green's functions and self-energies with $p = 0$.

3. Construction of the action in the replica-momentum space

In this section, we construct the elements of the matrix \mathcal{G}_u^{-1} appearing in action (4) of the paper. The diagonal elements are quite straight forward, so we focus on off diagonal elements for both bosons and fermions. We use the following identities:

$$\Sigma(\tau) = \frac{1}{\beta} \sum_n \Sigma(i\omega_n) e^{-i\omega_n \tau}, \quad \Sigma(z) = \int \frac{dx}{\pi} \frac{\Sigma''(x)}{x-z}.$$

Fermions—For the case of fermions, we can write

$$\begin{aligned} \Sigma(\tau) &= -\oint \frac{dz}{2\pi i} [f(z) - \theta_{\tau>0}] e^{-z\tau} \int \frac{dx}{\pi} \frac{\Sigma''(x)}{x-z} \\ &= \int \frac{dx}{\pi} [f(x) - \theta_{\tau>0}] \Sigma''(x) e^{-x\tau}. \end{aligned} \quad (\text{A11})$$

Note that $\theta_{\tau>0}$ is important for convergence, but also necessary to make sure that the Green's function is β antiperiodic. The two-point version is simple but note that $\Sigma(\tau_1, \tau) = T\Sigma(\tau_1 - \tau_2)$. Then,

$$\Sigma_{nm'} = T \int_0^\beta d\tau_1 d\tau_2 e^{i(\bar{\omega}_n \tau_1 - \omega_{m'} \tau_2)} \Sigma(\tau_1 - \tau_2). \quad (\text{A12})$$

The result is

$$\begin{aligned} \Sigma_{nm'} &= T \frac{1+u}{i\bar{\omega}_n - i\omega_{m'}} \int \frac{dx}{\pi} \frac{\Sigma''(x)}{i\omega_{m'} - x} \\ &= -T \frac{1+u}{i\bar{\omega}_n - i\omega_{m'}} \Sigma(i\omega_{m'}), \end{aligned} \quad (\text{A13})$$

where $u = e^{i\bar{\omega}_n \beta}$. Note that choosing $\Sigma(\tau_1, \tau_2) = V\delta(\tau_1, \tau_2)$ or $\Sigma(i\omega_{m'}) = V$ reproduces the known result,

$$V \int_0^\beta d\tau e^{i(\bar{\omega}_n - \omega_{m'})\tau} = -\frac{1}{\beta} V \frac{u+1}{i\bar{\omega}_n - i\omega_{m'}}. \quad (\text{A14})$$

Similarly, we can show that

$$\Sigma_{mn'} = -T \frac{1+u^{-1}}{i\omega_m - i\bar{\omega}_{n'}} \Sigma(i\omega_m). \quad (\text{A15})$$

This is correct, because using $\bar{\omega}_n = \nu_n - iT \log u$, we find

$$\begin{aligned} \lim_{u \rightarrow -1} \Sigma_{nm'} &= \lim_{\epsilon \rightarrow 0} \frac{-T\epsilon \delta_{nm'} \Sigma(i\omega_{m'})}{T \log(-1+\epsilon) - i\pi T} = \delta_{nm'} \Sigma(i\omega_{m'}), \\ \lim_{u \rightarrow -1} \Sigma_{mn'} &= \lim_{\epsilon \rightarrow 0} \frac{-T[(-1+\epsilon)^{-1} + 1] \delta_{mn'} \Sigma(i\omega_m)}{i\pi T - T \log(-1+\epsilon)}, \\ &= \delta_{mn'} \Sigma(i\omega_m). \end{aligned}$$

Bosons—In this case, we have

$$\begin{aligned} \Sigma(\tau) &= \frac{1}{\beta} \sum_n e^{-i\nu_n \tau} \Sigma(i\nu_n) \\ &= \oint \frac{dz}{2\pi i} [n(z) + \theta_\tau] e^{-z\tau} \Sigma(z) \\ &= \int \frac{dx}{\pi} \Sigma''(x) [n(x) + \theta_\tau] e^{-x\tau}. \end{aligned} \quad (\text{A16})$$

This has the correct half-periodicity, as seen in

$$\begin{aligned} \Sigma(\beta - |\tau|) &= \int \frac{dx}{\pi} \Sigma''(x) [n(x) + 1] e^{-\beta x} e^{+x|\tau|} \\ &= \int \frac{dx}{\pi} \Sigma''(x) n(x) e^{+x|\tau|} = \Sigma(-|\tau|). \end{aligned}$$

Fourier transform is

$$\begin{aligned} \Sigma_{nm'} &= T \int \frac{dx}{\pi} \Sigma''(x) \int_0^\beta d\tau_1 d\tau_2 [n(x) + \theta_{\tau_1 > \tau_2}] \\ &\quad \times e^{i(\bar{\omega}_n - x)\tau_1} e^{-(i\nu_{m'} - x)\tau_2} \\ &= T \frac{u-1}{i\bar{\omega}_n - i\nu_{m'}} \int \frac{dx}{\pi} \frac{\Sigma''(x)}{x - i\nu_{m'}} \\ &= T \frac{u-1}{i\bar{\omega}_n - i\nu_{m'}} \Sigma(i\nu_{m'}) \end{aligned}$$

and

$$\begin{aligned} \Sigma_{mn'} &= T \int \frac{dx}{\pi} \Sigma''(x) \int_0^\beta d\tau_1 d\tau_2 [n(x) + \theta_{\tau_1 > \tau_2}] \\ &\quad \times e^{-i(\bar{\omega}_{n'} - x)\tau_2} e^{(i\nu_m - x)\tau_1} \\ &= T \frac{u^{-1}-1}{i\nu_m - i\bar{\omega}_{n'}} \int \frac{dx}{\pi} \frac{\Sigma''(x)}{x - i\nu_m} = T \frac{u^{-1}-1}{i\nu_m - i\bar{\omega}_{n'}} \Sigma(i\nu_m). \end{aligned}$$

As a check, using $e^{i\beta\bar{\omega}_n} = u$ and $\bar{\omega}_n = \nu_n - iT \log u$, we find

$$\lim_{u \rightarrow -1} \Sigma_{nm'} = \delta_{nm'} \Sigma(i\nu_m), \quad \lim_{u \rightarrow -1} \Sigma_{mn'} = \delta_{mn'} \Sigma(i\nu_{m'}).$$

So, in summary,

$$\Sigma_{nm'} = -\frac{1}{\beta} \frac{\zeta u - 1}{i\bar{\omega}_n - i\omega_{m'}}, \quad \Sigma_{mn'} = -\frac{1}{\beta} \frac{\zeta u^{-1} - 1}{i\omega_m - i\bar{\omega}_{n'}}.$$

4. Useful matrix identities

In this section, we provide some useful matrix identities that are used in the paper. The first is the famous determinant identity,

$$\det \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \det(A) \det(D - BA^{-1}C) = \det(D) \det(A - CD^{-1}B), \quad (\text{A17})$$

which leads to the equation employed in the paper,

$$\det \left[\begin{array}{c|c} I_{mm} & V_{mn} \\ \hline V_{nm} & I_{nn} \end{array} \right] = \det(I_{mm} - V_{mn}V_{nm}) = \det(I_{nn} - V_{nm}V_{mn}). \quad (\text{A18})$$

We also use some matrix inversion identities. If $(A - BD^{-1}C)$ is invertible,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}. \quad (\text{A19})$$

If $(D - CA^{-1}B)$ is invertible,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}. \quad (\text{A20})$$

A consequence of these identities is that

$$\begin{aligned} V_{qk}G_{kk'}^B V_{k'q} &= V_{qk}(G_b^{-1} - V_{kq}G_a V_{qk})^{-1}V_{kq} = \begin{pmatrix} V_{qk} & 0 \\ V_{qk} & G_a^{-1} \end{pmatrix} \begin{pmatrix} G_b^{-1} & V_{kq} \\ V_{qk} & G_a^{-1} \end{pmatrix} \begin{pmatrix} V_{kq} \\ 0 \end{pmatrix} \\ &= V_{qk}[G_b + G_b V_{qk}(G_a^{-1} - V_{qk}G_b V_{kq})^{-1}V_{qk}G_b]V_{kq} \\ &= \Sigma_a + \Sigma_a G_a \Sigma_a = G_a^{-1}G_a G_a^{-1} - G_a^{-1}. \end{aligned} \quad (\text{A21})$$

5. A more detailed proof of Eq. (8)

Here, we provide a more detailed proof of central equation of the paper, Eq. (8). We start from the action (4),

$$\begin{aligned} \mathcal{S} &= \frac{1}{\beta} \sum_n \left[\bar{\psi}_a(i\bar{\omega}_n) \left(-i\bar{\omega}_n + \epsilon_a + \Sigma^{aa}(i\bar{\omega}_n) \right) \psi_a(i\bar{\omega}_n) + \sum_m \bar{\psi}_b(i\omega_m) \left(-i\omega_m + \epsilon_b + \Sigma^{bb}(i\omega_m) \right) \psi_b(i\omega_m) \right. \\ &\quad \left. + \frac{1}{\beta^2} \sum_{n,m} \left\{ \Sigma^{ab}(i\omega_m) \frac{e^{i\bar{\omega}_n\beta} - 1}{i\bar{\omega}_n - i\omega_m} \bar{\psi}_a(i\bar{\omega}_n) \psi_b(i\omega_m) + \Sigma^{ba}(i\omega_m) \frac{e^{-i\bar{\omega}_n\beta} - 1}{i\omega_m - i\bar{\omega}_n} \bar{\psi}_b(i\omega_m) \psi_a(i\bar{\omega}_n) \right\} \right]. \end{aligned}$$

Shift $\psi_a(i\bar{\omega}_n) \rightarrow \psi_a(i\bar{\omega}_n) - \frac{1}{\beta} \sum_m \Sigma^{ba}(i\omega_m) \frac{1}{i\bar{\omega}_n - \epsilon_a - \Sigma^{aa}(i\bar{\omega}_n)} \frac{1-u^{-1}}{i\bar{\omega}_n - i\omega_m} \psi_b(i\omega_m)$, the action becomes

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_a + \frac{1}{\beta^2} \sum_{m,m',n} \bar{\psi}_b(i\omega_m) \left[-[G^{-1}(i\omega_m)]^{bb'} \delta_{m,m'} \right. \\ &\quad \left. + \frac{1}{\beta} \Sigma^{ba}(i\omega_m) \frac{1}{i\bar{\omega}_n - \epsilon_a - \Sigma^{aa}(i\bar{\omega}_n)} \frac{(1-u)(1-u^{-1})}{(i\bar{\omega}_n - i\omega_m)(i\bar{\omega}_n - i\omega_{m'})} \Sigma^{a'b'}(i\omega_{m'}) \right] \psi_b(i\omega_{m'}), \end{aligned}$$

where $[G^{-1}(i\omega_m)]^{bb'} = -i\omega_m + \epsilon_b + \Sigma^{bb'}(i\omega_m)$. Using spectral representation,

$$\mathcal{S} = \mathcal{S}_a + \frac{1}{\beta^2} \sum_{m,m'} \bar{\psi}_b(i\omega_m) \left[-[G^{-1}(i\omega_m)]^{bb'} \delta_{m,m'} + \frac{1}{\beta} \sum_n \Sigma^{ba}(i\omega_m) \int \frac{dx A^{aa'}(x)}{2\pi i \bar{\omega}_n - x} \frac{(1-u)(1-u^{-1})}{(i\bar{\omega}_n - i\omega_m)(i\bar{\omega}_n - i\omega_{m'})} \Sigma^{a'b'}(i\omega_{m'}) \right] \psi_b(i\omega_{m'}).$$

After integrating out the shifted Matsubara frequency, the action becomes

$$\begin{aligned} \mathcal{S} = \mathcal{S}_a + \frac{1}{\beta^2} \sum_{m,m'} \bar{\psi}_b(i\omega_m) \left[-[G^{-1}(i\omega_m)]^{BB'} \delta_{m,m'} \right. \\ \left. + \int \frac{dx K_u(x) \Sigma^{ba}(i\omega_m) A^{aa'}(x) \Sigma^{a'b}(i\omega_{m'})}{(i\omega_m - x)(i\omega_{m'} - x)} \right] \psi_b(i\omega_{m'}), \end{aligned} \quad (\text{A22})$$

where $-[G^{-1}(i\omega_m)]^{BB'}$ is

$$-[G^{-1}(i\omega_m)]^{bb'} + \int \frac{dx \Sigma^{ba}(i\omega_m) A^{aa'}(x) \Sigma^{a'b'}(i\omega_m)}{i\omega_n - x}.$$

Then the whole second part in action \mathcal{S} gives $(\mathcal{G}_u^{BB'})^{-1}$, which is

$$[G_m^{-1}]^{BB'} \delta_{mm'} - \int \frac{dx K_u(x) \Sigma_m^{ba} A^{aa'}(x) \Sigma_{m'}^{a'b'}}{2\pi \beta (i\omega_m - x)(i\omega_{m'} - x)}.$$

The determinant $\det^{-\zeta}[(-\mathcal{G}_u^{BB'})^{-1}]$ now is

$$\det^{-\zeta} \left[[G_m^{-1}]^{BB'} \delta_{mm'} - \int \frac{dx K_u(x) \Sigma_m^{ba} A^{aa'}(x) \Sigma_{m'}^{a'b'}}{2\pi \beta (i\omega_m - x)(i\omega_{m'} - x)} \right].$$

Now we are facing a determinant,

$$\det^{-\zeta} [D_{mm'} \delta_{mm'} + \sum_x w_m(x) v_m^T(x)].$$

This is nothing but

$$\begin{array}{c} N_x \times N_a \\ N_m \end{array} \left\{ \begin{array}{c} \overbrace{\left(\begin{array}{c|c} \delta(x, y) \delta_{aa'} & v_m^T(x, a) \\ \hline -w_m(x, a) & D_{mm'} \delta_{mm'} \end{array} \right)}^{N_x \times N_a} \\ \overbrace{\left(\begin{array}{c|c} \delta(x, y) \delta_{aa'} & v_m^T(x, a) \\ \hline -w_m(x, a) & D_{mm'} \delta_{mm'} \end{array} \right)}^{N_m} \end{array} \right.$$

where N_x , N_a and N_m refer to the number of real frequencies, number of A modes, and number of Matsubara frequencies, respectively. Then the determinant is

$$\det^{-\zeta} \left[\delta(x, y) \delta_{aa'} + \sum_m v_m^T(x, a) D_{mm} w_m(y, a') \right].$$

Using Eq. (A18), and after the shuffling the determinant becomes

$$\det^{-\zeta} \left[\delta(x, y) - \frac{1}{\beta} \sum_m K_u(x) A^a(x) \frac{\Delta x}{2\pi} \frac{\Sigma_m^{ab} G_m^{BB'} \Sigma_m^{ba}}{(i\omega_m - x)(i\omega_m - y)} \right].$$

This motivates defining [z is the complex frequency]

$$\mathbb{R}(z) \equiv \Sigma^{ab}(z) G^{BB'}(z) \Sigma^{ba}(z), \quad (\text{A23})$$

in terms of which

$$r(x) \equiv \frac{1}{\beta} \sum_n \frac{\mathbb{R}(i\omega_n)}{i\omega_n - x} \quad (\text{A24})$$

$$= \int \frac{d\omega}{2\pi} \mathbb{A}_R(\omega) \frac{n_\zeta(x) - n_\zeta(\omega)}{x - \omega}. \quad (\text{A25})$$

Here, we have used the spectral representation of the $R(z)$ function, defined as $\mathbb{A}_R(\omega) = i[\mathbb{R}(\omega + i\eta) - \mathbb{R}(\omega - i\eta)]$. We also define

$$\mathbb{J}(x, y) \equiv \frac{r(x) - r(y)}{x - y}. \quad (\text{A26})$$

Note that the $x \rightarrow \omega$ limit in Eq. (A25) and $x \rightarrow y$ limit in Eq. (A26) needs to be treated using L'Hôpital's rule. In terms of the \mathbb{J} matrix, we find

$$\det^{-\zeta} [\mathbb{1} + \tilde{\zeta} K_u(\omega) \mathbb{A}^a(\omega) \mathbb{J}^A(\omega, \omega')],$$

Finally, we get the u -sector partition function (8),

$$Z[u] = Z_a[u] Z_B \det^{-\zeta} [\mathbb{1} + \tilde{\zeta} K_u(\omega) \mathbb{A}^a(\omega) \mathbb{J}^A(\omega, \omega')]. \quad (\text{A27})$$

6. Noninteracting limit of our formalism

In this part, we will show that our approach can be connected to results in the noninteracting limit. In the noninteracting case, the self-energy in the action (4) is just a frequency independent constant V . Thus,

$$\begin{aligned} A_R^{aa'}(\omega) &= \sum_{b,b'} \text{Im}[V^{ab} G^{BB'}(\omega + i\eta) V^{b'a'}] \\ &= \sum_{b,b'} V^{ab} A^{BB'} V^{b'a'}. \end{aligned}$$

In the noninteracting limit, the spectral function is

$$A^{aa'}(\omega) = 2\pi \sum_\epsilon \phi_\epsilon(a) \phi_\epsilon^*(a') \delta(\omega - \epsilon),$$

so that the last determinant in (8) reduces to

$$\det^{-\zeta} [\phi_\epsilon^\dagger(a') (\delta(x, y) - \tilde{\zeta} K_u(x) \delta(x - \epsilon) J^{a'a}(x, y)) \phi_\epsilon(a)].$$

The $\phi_\epsilon(a)$ and $\phi_\epsilon^\dagger(a)$ in the determinant plays the role of unitary transformation from a modes to ϵ modes, which together with the $\delta(x - \epsilon)$ reduces the dimension of the determinant,

$$\det^{-\zeta} [\delta(\epsilon, \epsilon') - \tilde{\zeta} K_u(\epsilon) J^{\epsilon\epsilon'}(\epsilon, \epsilon')], \quad (\text{A28})$$

where $J(\epsilon, \epsilon')$ can be written as

$$J(\epsilon, \epsilon') = \frac{r(\epsilon) - r(\epsilon')}{\epsilon - \epsilon'}, \quad (\text{A29})$$

in terms of $r(\epsilon)$

$$r(\epsilon) = - \int \frac{d\omega}{2\pi} A_R(\omega) \frac{n_{\tilde{\zeta}}(\epsilon) - n_{\tilde{\zeta}}(\omega)}{\epsilon - \omega}. \quad (\text{A30})$$

The diagonal terms $J(\epsilon, \epsilon')$ need to be treated in a limiting procedure. In this context, $J(\epsilon, \epsilon')$ becomes

$$J(\epsilon, \epsilon') = - \int \frac{d\omega}{2\pi} \frac{A_R(\omega)}{\epsilon - \epsilon'} \left[\frac{n_{\tilde{\zeta}}(\epsilon) - n_{\tilde{\zeta}}(\omega)}{\epsilon - \omega} - (\epsilon \rightarrow \epsilon') \right].$$

With the help of $V G_B V = G_a^{-1} (G_A - G_a) G_a^{-1}$ which is shown in (A21), we can further show

$$\begin{aligned} V^{ab} A^{BB'} V^{b'a'} &= i V^{ab} (G_R^{BB'} - G_{Ad}^{BB'}) V^{b'a'} \\ &= (\omega - \epsilon) A^{AA'}(\omega) (\omega - \epsilon'). \end{aligned}$$

So that $J(\epsilon, \epsilon')$ becomes

$$\begin{aligned} J(\epsilon, \epsilon') &= \int \frac{d\omega}{2\pi} A^{AA'}(\omega) n_{\tilde{\zeta}}(\omega) + \delta(\epsilon, \epsilon') \frac{n_{\tilde{\zeta}}(\epsilon) \epsilon' - n_{\tilde{\zeta}}(\epsilon') \epsilon}{\epsilon - \epsilon'} \\ &\quad - \int \frac{d\omega}{2\pi} \omega A^{AA'}(\omega) \frac{n_{\tilde{\zeta}}(\epsilon) - n_{\tilde{\zeta}}(\epsilon')}{\epsilon - \epsilon'} \\ &= -\tilde{\zeta} \frac{1}{\beta} \sum_n e^{-i\omega_n 0^-} G_A(i\omega_n) + n'_{\tilde{\zeta}}(\epsilon) \epsilon - n_{\tilde{\zeta}}(\epsilon) - n'_{\tilde{\zeta}}(\epsilon') \epsilon \\ &= -\tilde{\zeta} G_A(\tau=0^-) - n_{\tilde{\zeta}}(\epsilon) \\ &= \langle a^\dagger a \rangle - n_{\tilde{\zeta}}(\epsilon), \end{aligned}$$

where we use $\int \frac{d\omega}{2\pi} \omega A^{AA'}(\omega) = \epsilon \delta(\epsilon, \epsilon')$. And we can further define $\mathbb{C} = \langle a^\dagger a \rangle$ for both Bosons and Fermions, the $\text{Tr}[\rho_A^M]$ becomes

$$\begin{aligned} \text{Tr}[\rho_A^M] &= \prod_u \frac{Z_a(u)}{Z_a} \det^{-\zeta} [\delta(\epsilon, \epsilon') - \tilde{\zeta} K_u(\epsilon) (\mathbb{C} - n_{\tilde{\zeta}}(\epsilon))] \\ &= \det^{-\zeta} [\mathbb{C}^M + \mathbb{D}^M], \end{aligned}$$

where $\mathbb{D} = \mathbb{1} + \tilde{\zeta} \mathbb{C} = \langle a a^\dagger \rangle$. The term $\prod_u [Z_a(u)/Z_a]$ can be just absorbed into the determinant to get the final expression because the dimension of the determinant in noninteracting case is finite, but one does not have this luxury again in the interacting case. Finally, the von Neumann entropy and Rényi entropy are

$$S_A^{\text{vN}} = \tilde{\zeta} \text{Tr}[\mathbb{D} \ln \mathbb{D} + \tilde{\zeta} \mathbb{C} \ln \mathbb{C}]$$

and

$$S_A^{\text{RM}} = \frac{-\zeta}{1-M} \ln \det[\mathbb{D}^M + \tilde{\zeta} \mathbb{C}^M],$$

which are exactly Casini's results using the reduced density matrix method.

7. The perturbative limit of our formalism

The EE $S_A = S_a + \Delta S_A$ has a thermal part S_a and a quantum correction. The latter can be expressed as

$$\Delta S_A = \int dc \Delta \rho(c) g^{\text{RM}}(c) \quad (\text{A31})$$

is expressed in terms of

$$g^{\text{RM}}(c) \equiv \frac{-\zeta}{1-M} \log[(1 + \tilde{\zeta} c)^M - \tilde{\zeta} c^M], \quad (\text{A32})$$

and the entanglement density of states,

$$\Delta \rho(c) = -\frac{1}{\pi} \partial_c \text{Im}[\text{Tr} \log\{(c+1 - \mathbb{C}_0)^{-1} (c+1 - \mathbb{C})\}].$$

We can write the \mathbb{C} matrix as $\mathbb{C} = \mathbb{C}_0 + \mathbb{I}$, where

$$\mathbb{C}_0 = \tilde{\zeta} n_{\tilde{\zeta}}(\omega) \mathbb{1}, \quad \mathbb{I} = \sqrt{\mathbb{A}_a(\omega)} \mathbb{J}_A(\omega, \omega') \sqrt{\mathbb{A}_a(\omega')}. \quad (\text{A33})$$

In the perturbative the entanglement density of states is given by

$$\Delta \rho(c) = -\frac{1}{\pi} \partial_c \text{Im}[\text{Tr} \log\{\mathbb{1} - (c+1 - \mathbb{C}_0)^{-1} \mathbb{I}\}]. \quad (\text{A34})$$

This expression can be expanded perturbatively. The leading order term is

$$\Delta \rho(c) \approx \frac{1}{\pi} \partial_c \int \frac{d\omega}{2\pi} \text{Im} \left[\frac{\text{Tr} \mathbb{I}(\omega, \omega)}{c + i\eta - \tilde{\zeta} n(\omega)} \right], \quad (\text{A35})$$

$\text{Tr}[\mathbb{I}(\omega, \omega)]$ is real. Therefore.

$$\Delta \rho(c) = -\partial_c \int \frac{d\omega}{2\pi} \delta[c - \tilde{\zeta} n_{\tilde{\zeta}}(\omega)] \text{Tr} \mathbb{I}(\omega, \omega). \quad (\text{A36})$$

Note that in the limit $x \rightarrow y$ from Eq. (A26) we find $\mathbb{J}(\omega, \omega) = \partial_\omega \mathbb{r}(\omega)$. Instead the ω -integral, next we do the c -integral and find

$$\Delta S_A^{\text{RM}} = - \int \frac{d\omega}{2\pi} \tilde{g}^{\text{RM}}(\omega) \text{Tr}[\mathbb{A}^a(\omega) \partial_\omega \mathbb{r}(\omega)], \quad (\text{A37})$$

where $\tilde{g}^{\text{RM}}(\omega)$ is defined as

$$\tilde{g}^{\text{RM}}(c) \equiv \partial_c g^{\text{RM}}(c), \quad (\text{A38})$$

and using $c = \tilde{\zeta} n_{\tilde{\zeta}}(\omega) = 1/(e^{\beta\omega} - \tilde{\zeta})$, we can express it as

$$\tilde{g}^{\text{RM}}(\omega) = \zeta \frac{M}{M-1} \frac{n_{\tilde{\zeta}}(M\omega)}{n_{\tilde{\zeta}}[(M-1)\omega] n_{\tilde{\zeta}}(\omega)}. \quad (\text{A39})$$

Note that this expression has a well-behaved limit as $M \rightarrow 1^+$. The leading perturbative quantum correction to the Rényi entropy, Eq. (A37), can alternatively be extracted from Eq. (8). Perturbation theory in $J(\omega, \omega')$ gives

$$\begin{aligned} S_A^{\text{RM}} &= \frac{1}{1-M} \log \frac{Z_M}{Z_0^M} \\ &= \frac{|\zeta|}{M-1} \int \frac{d\omega}{2\pi} \text{Tr}[\mathbb{A}^a(\omega) \partial_{\omega} \mathbb{r}(\omega)] \sum_p K_p(\omega). \end{aligned} \quad (\text{A40})$$

$\sum_p K_{u_p}(\omega)$ can be computed using contour integration with $\partial_u \log(u^M - \tilde{\zeta})$, which gives

$$\sum_p K_{u_p}(\omega) = \tilde{\zeta} M \frac{n_{\tilde{\zeta}}(M\omega)}{n_{\tilde{\zeta}}(\omega) n_{\tilde{\zeta}}[(M-1)\omega]}, \quad (\text{A41})$$

which gives the same result.

8. Review of equilibrium path integral of the coupled-SYK model

For the sake of completeness, we list the saddle point equations and thermal dynamical properties of the coupled-SYK model. While it has been shown that such models can undergo a spontaneous symmetry breaking [54], we restrict our analysis to symmetry-preserving phases. We provide the self-consistency equations in the real-frequency so that they can be readily used with our real-frequency formalism.

The Hamiltonian of this coupled SYK model is

$$H = \frac{1}{4!} \sum_{\mu=A,B} \sum_{i,j,k,l} J_{ijkl}^{\mu} \chi_i^{\mu} \chi_j^{\mu} \chi_k^{\mu} \chi_l^{\mu} + \sum_{i,j=1}^{N_A} \sum_{k,l=1}^{N_B} V_{ijkl} \chi_i^A \chi_j^A \chi_k^B \chi_l^B,$$

and J_{ijkl}^{μ} , V_{ijkl} are Gaussian random variables with zero mean and variances,

$$\langle (J_{ijkl}^{\mu})^2 \rangle = \frac{3! J^2}{N_{\mu}^3}, \quad \langle V_{ijkl}^2 \rangle = \frac{V^2}{(N_A N_B)^{\frac{3}{2}}}.$$

In this case, the action is

$$\begin{aligned} S &= - \sum_{\mu} N_{\mu} \ln Pf(\partial_{\tau} + \Sigma_{\mu}) \\ &\quad - \int d\tau d\tau' \sum_{\mu} \frac{N_{\mu}}{2} (\Sigma_{\mu}(\tau, \tau') G_{\mu}(\tau', \tau) + \frac{J^2}{4} G_{\mu}(\tau, \tau')^4) \\ &\quad - \frac{V^2 \sqrt{N_A N_B}}{2} G_A(\tau, \tau')^2 G_B(\tau, \tau')^2. \end{aligned}$$

Self-consistent equations—In the large-N limit, we can get the saddle point solutions,

$$\begin{aligned} \Sigma_a(\tau, \tau') &= J^2 G_A^3(\tau, \tau') + 2V^2 \sqrt{p} G_B^2(\tau, \tau') G_A(\tau, \tau'), \\ \Sigma_b(\tau, \tau') &= J^2 G_B^3(\tau, \tau') + 2V^2 \sqrt{\frac{1}{p}} G_A^2(\tau, \tau') G_B(\tau, \tau'), \end{aligned}$$

where $p = N_B/N_A$.

These relations can be brought to real frequency by introducing $B_{\mu}(\tau, \tau') = G_{\mu}^2(\tau, \tau')$. Generally, we have the symmetries $G_{\mu}(\tau_1, \tau_2) = -G_{\mu}(\tau_2, \tau_1)$, which imply $G_{\mu}(i\omega_n) = -G_{\mu}(-i\omega_n)$ in Matsubara frequency. In the Matsubara frequency domain, they become

$$\begin{aligned} \Sigma_a(i\omega_n) &= \frac{J^2}{\beta^2} \sum_{n_1, n_2} G_A(i\omega_{n_1}) G_A(i\omega_{n_2}) G_A(i\omega_n - i\omega_{n_1} - i\omega_{n_2}) \\ &\quad + \frac{2V^2 \sqrt{p}}{\beta^2} \sum_{n_1, n_2} G_B(i\omega_{n_1}) G_B(i\omega_{n_2}) \\ &\quad \times G_A(i\omega_n - i\omega_{n_1} - i\omega_{n_2}), \end{aligned}$$

and

$$\begin{aligned} \Sigma_b(i\omega_n) &= \frac{J^2}{\beta^2} \sum_{n_1, n_2} G_B(i\omega_{n_1}) G_B(i\omega_{n_2}) G_B(i\omega_n - i\omega_{n_1} - i\omega_{n_2}) \\ &\quad + \frac{2V^2 \sqrt{\frac{1}{p}}}{\beta^2} \sum_{n_1, n_2} G_A(i\omega_{n_1}) G_A(i\omega_{n_2}) \\ &\quad \times G_B(i\omega_n - i\omega_{n_1} - i\omega_{n_2}). \end{aligned}$$

We define spectral bosonic and fermions spectral functions $A_{\mu}^B(\omega)$ and $A_{\mu}^G(\omega)$ as

$$\begin{aligned} A_{\mu}^G(\omega) &\equiv i[G_{\mu}(\omega + i\eta) - G_{\mu}(\omega - i\eta)], \quad \text{and} \\ A_{\mu}^B(\omega) &\equiv i[B_{\mu}(\omega + i\eta) - B_{\mu}(\omega - i\eta)]. \end{aligned} \quad (\text{A42})$$

When analytically continued onto the real frequency axis, the self energies become

$$\begin{aligned} \Sigma_a(\omega + i\eta) &= J^2 \int \frac{d\omega'}{2\pi} [A_A^G(\omega') f(\omega') B_A(\omega + i\eta - \omega') + G_A(\omega + i\eta - \omega') n_B(-\omega') A_A^B(\omega)] \\ &\quad + 2V^2 \sqrt{p} \int \frac{d\omega'}{2\pi} [A_A^G(\omega') f(\omega') B_B(\omega + i\eta - \omega') + G_A(\omega + i\eta - \omega') n_B(-\omega') A_B^B(\omega)], \end{aligned}$$

$$\begin{aligned}\Sigma_b(\omega + i\eta) &= J^2 \int \frac{d\omega'}{2\pi} [A_B^G(\omega')f(\omega')B_B(\omega + i\eta - \omega') + G_B(\omega + i\eta - \omega')n_B(-\omega')A_B^B(\omega)] \\ &\quad + 2V^2 \sqrt{\frac{1}{p}} \int \frac{d\omega'}{2\pi} [A_B^G(\omega')f(\omega')B_A(\omega + i\eta - \omega') + G_B(\omega + i\eta - \omega')n_B(-\omega')A_A^B(\omega)],\end{aligned}$$

where

$$B_\mu(\omega + i\eta) = \int \frac{d\omega'}{2\pi} A_\mu(\omega')G_\mu(\omega + i\eta - \omega')[f(\omega') - f(-\omega')].$$

Free energy and thermal entropy—The free energy is equal to

$$\begin{aligned}\frac{\beta F}{N} &= -\frac{1}{1+p} \ln Pf(\partial_\tau \delta(\tau, \tau') + \Sigma_a(\tau, \tau')) - \frac{1}{2(1+p)} \int d\tau d\tau' \frac{3}{4} \Sigma_a(\tau, \tau') G_A(\tau', \tau) \\ &\quad - \frac{1}{1+1/p} \ln Pf(\partial_\tau \delta(\tau, \tau') + \Sigma_b(\tau, \tau')) - \frac{1}{2(1+1/p)} \int d\tau d\tau' \frac{3}{4} \Sigma_b(\tau, \tau') G_B(\tau', \tau).\end{aligned}\quad (\text{A43})$$

In Matsubara frequency domain and analytically continued onto the real frequency axis, the free energy density becomes

$$\frac{F}{N} = \int \frac{d\omega}{\pi} \left\{ \frac{1}{2} f(\omega) \text{Im} \sum_\mu \frac{1}{1 + \zeta_\mu^p} \left[\log(-\omega - i\eta + \Sigma_\mu) + \frac{3}{4} \Sigma_\mu(\omega + i\eta) G_\mu(\omega + i\eta) \right] \right\},$$

where $\zeta_\mu^p = p$ for $\mu = A$, $1/p$ for $\mu = B$. The thermal entropy is

$$\begin{aligned}\frac{S}{N} &= -\frac{1}{2} \int \frac{d\omega}{\pi} \left\{ \partial_T f(\omega) \text{Im} \sum_\mu \frac{1}{1 + \zeta_\mu^p} \left[\log(-\omega - i\eta + \Sigma_\mu) + \Sigma_\mu(\omega + i\eta) G_\mu(\omega + i\eta) \right] \right. \\ &\quad - \frac{J^2}{4} \sum_\mu B_\mu(-\omega - i\eta) [\partial_T n_B(\omega) B_\mu(\omega + i\eta) + (n_B(\omega) - n_B(-\omega)) \partial_T B_\mu(\omega + i\eta)] \\ &\quad \left. - \frac{V^2 \sqrt{p}}{1+p} B_A(-\omega - i\eta) [\partial_T n_B(\omega) B_B(\omega + i\eta) + (n_B(\omega) - n_B(-\omega)) \partial_T B_B(\omega + i\eta)] \right\}.\end{aligned}$$

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