Letter

Localizing wrapped M5-branes and gravitational blocks

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We consider $d = 2$, $\mathcal{N} = (0, 2)$ SCFTs that can arise from M5-branes wrapping four-dimensional, complex, toric manifolds and orbifolds. We use equivariant localization to compute the off-shell central charge of the dual supergravity solutions, obtaining a result that can be written as a sum of gravitational blocks and precisely agrees with a field theory computation using anomaly polynomials and c-extremization.

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Introduction. Supersymmetric wrapped branes continue to provide a fertile arena for exploring the AdS/CFT correspondence. They give rise to rich classes of novel SCFTs in various spacetime dimensions, and they also provide a concrete framework for obtaining a microstate interpretation of the Bekenstein–Hawking entropy for asymptotically AdS black holes. In addition, the supersymmetric AdS solutions of supergravity associated with wrapped branes give rise to novel geometric structures, which are of interest in their own right.

In a recent paper [1], a new calculus was introduced for supersymmetric solutions of supergravity that have an R-symmetry. For several general classes of such solutions, it was shown there exists a set of equivariantly closed differential forms, which can be constructed from Killing spinor bilinears. Furthermore, various BPS observables can then be computed using localization via the Berline–Vergne–Atiyah–Bott (BVAB) fixed point formula [2,3], without solving the supergravity equations of motion. Here we want to further develop these tools for a general class of AdS_3 solutions of $D = 11$ supergravity that arise from M5-branes wrapping four-dimensional manifolds and orbifolds. The preserved supersymmetry is such that the AdS_3 solutions are dual to $d = 2$, $\mathcal{N} = (0, 2)$ SCFTs and, in particular, they have an R-symmetry.

More precisely, we focus on the class of supersymmetric $\mathrm{AdS}_3 \times M_8$ solutions of $D = 11$ supergravity considered in [4] and further analyzed in [5]. We construct a set of equivariantly closed forms and show that they can be used to compute the central charge of the dual SCFT, as well as the conformal dimensions of operators in the SCFT that are dual to supersymmetric wrapped probe M2-branes. To illustrate the formalism, we focus on examples where M_8 is an S^4 fibration over a toric B_4, which are associated with M5-branes wrapping B_4. Focusing on toric B_4 is of interest since we can both compare with some known and conjectured field theory results, as well as obtain results that provide new field theory predictions. As we shall see, the localization results are remarkably simple for these examples because the fixed points of the R-symmetry, which is linear combination of the $U(1)^2$ action on S^4 and the $U(1)^2$ action on B_4, are a set of isolated points on M_8. We can use the BVAB formula to implement flux quantization as well as obtain an *off-shell* expression for the central charge. After extremizing over the undetermined R-symmetry data, we then obtain an on-shell expression for the central charge. As explained in more detail in [6], it is important to emphasize that this will give the correct central charge, without solving the supergravity equations, just assuming the supergravity solution actually exists, or equivalently, that the low energy limit of M5-branes wrapped on the specific toric B_4 does indeed flow to a SCFT in the IR, in the large N limit. The formalism therefore provides a geometric, off-shell version of c-extremization [7,8].

The off-shell expression for the central charge that we derive takes the form of a sum of gravitational blocks [9]. The terminology "gravitational block" is perhaps somewhat confusing, as in many works, it is not referring to a computation in gravity, but rather is a conjecture that should arise for some unspecified gravity computation. More precisely various off-shell expressions for BPS quantities have been proposed, which either can be derived in field theory, for example, using anomaly polynomials, or alternatively have been noted to give the correct on-shell result for some specific, explicitly known supergravity solutions. For the former, invoking AdS/CFT, there is then an expectation that there will be a corresponding off-shell computation within gravity that leads to the same off-shell result for the central charge. However, it is not at all clear, in general, how one should go off-shell on the gravity side. That being said, in the setting of GK geometry, for Sasaki–Einstein fibrations over spindles, this was recently achieved in [10]. The results of this paper, as well as [1,6], indicate that the equivariant calculus of [1] provides a universal way of deriving gravitational blocks within a gravitational context. Moreover, the new results make it clear that the origin of gravitational blocks is when a trial R-symmetry has isolated fixed points on the space that the brane is wrapping.

In [6] we provide some further details of the equivariant calculus for the general class of AdS_3 solutions of $D=11$ supergravity discussed here. In addition, we will also analyze other examples of wrapped M5-branes, where the R-symmetry fixed point set no longer consists of isolated points, and, in particular, gravitational blocks are not relevant.

$\mathrm{AdS}_3\times M_8$ *solutions.* We consider supersymmetric solutions of $D=11$ supergravity of the form

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{e}^{2\lambda}[\mathrm{d}s^2(\mathrm{AdS}_3)+\mathrm{d}s^2(M_8)], \\ G &= \mathrm{e}^{3\lambda}F + \mathrm{vol}(\mathrm{AdS}_3)\wedge \mathrm{e}^{3\lambda} f, \end{aligned} \tag{1}$$

where λ, F and f are a function, a four-form and a one-form on M_8, respectively. In addition, $\mathrm{d}s^2(\mathrm{AdS}_3)$ is the metric on a unit radius AdS_3 and $\mathrm{vol}(\mathrm{AdS}_3)$ is the corresponding volume form. The Bianchi identity implies $\mathrm{d}(\mathrm{e}^{3\lambda}f)=0$, and it is convenient to introduce a function a_0, locally defined in general, via $\mathrm{e}^{3\lambda}f=\mathrm{d}a_0$.

We assume that the preserved supersymmetry is such that the dual $d=2$ SCFTs have $\mathcal{N}=(0,2)$ supersymmetry. We will focus on the class of solutions classified in [4]. Following the conventions of [5], there is then a complex spinor ϵ on M_8, with $\bar\epsilon\epsilon=1=\bar\epsilon^c\epsilon$ as well as $\bar\epsilon^c\gamma_9\epsilon=0$, where $\gamma_9\equiv\gamma_1\ldots\gamma_8$. There is an R-symmetry Killing vector ξ, with a dual one-form $\xi^\flat$ which can be constructed as a bilinear:

$$\xi^\flat = -\frac{\mathrm{i}}{2}\bar\epsilon\gamma_9\gamma_{(1)}\epsilon. \tag{2}$$

We have introduced the notation $\gamma_{(r)}=\frac{1}{r!}\gamma_{\mu_1\cdots\mu_r}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_r}$ and have normalized ξ so that $\mathcal{L}_\xi\epsilon=\frac{\mathrm{i}}{2}\epsilon$. We also define a scalar, two two-forms and a four-form bilinear

$$\begin{aligned} \sin\alpha &= \bar\epsilon\gamma_9\epsilon, \qquad J=-\mathrm{i}\bar\epsilon\gamma_{(2)}\epsilon, \\ \omega &= -\mathrm{i}\bar\epsilon\gamma_9\gamma_{(2)}\epsilon, \qquad \Psi=\bar\epsilon\gamma_{(4)}\epsilon, \end{aligned} \tag{3}$$

and introduce the locally defined function y, given by $y=\frac{1}{2}(\mathrm{e}^{3\lambda}\sin\alpha-a_0)$.

These ingredients can be used to define the following polyforms on M_8:

$$\begin{aligned} \Phi &= \mathrm{e}^{9\lambda}\mathrm{vol}_8+\frac{1}{4}\mathrm{e}^{9\lambda}*J-\frac{1}{8}y\mathrm{e}^{6\lambda}\Psi-\frac{1}{16}y^2\mathrm{e}^{3\lambda}F \\ &\quad +\frac{1}{32}y^2\mathrm{e}^{3\lambda}\omega+\frac{1}{192}y^3, \\ \Phi^F &= \mathrm{e}^{3\lambda}F-\frac{1}{2}\mathrm{e}^{3\lambda}\omega-\frac{1}{4}y, \\ \Phi^{*F} &= \mathrm{e}^{6\lambda}*F-a_0\mathrm{e}^{3\lambda}F-\frac{1}{2}(\mathrm{e}^{6\lambda}J-a_0\mathrm{e}^{3\lambda}\omega)-\frac{1}{4}y^2, \end{aligned} \tag{4}$$

where $*$ denotes the Hodge dual, and vol_8 is the volume form on M_8. A key result [11] is that the differential and algebraic conditions satisfied by the above bilinears, along with the Bianchi identity and equation of motion for the four-form, imply these polyforms are equivariantly closed: $\mathrm{d}_\xi\Phi=\mathrm{d}_\xi\Phi^F=\mathrm{d}_\xi\Phi^{*F}=0$, where $\mathrm{d}_\xi\equiv\mathrm{d}-\xi\lrcorner$. Thus, we can compute their integrals on closed cycles using the BVAB formula. In particular, the integral of Φ^F on a four-cycle Γ_4 represents the flux of the four-form of 11-dimensional supergravity, which (in the large N limit) should be quantized as

$$N_{\Gamma_4}\equiv\frac{1}{(2\pi\ell_p)^3}\int_{\Gamma_4}\Phi^F\in\mathbb{Z}, \tag{5}$$

where ℓ_p is the Planck length. By computing the effective three-dimensional Newton constant, one can show that the integral of Φ is proportional to the trial central charge

$$c=\frac{3}{2^5\pi^7\ell_p^9}\int_{M_8}\Phi. \tag{6}$$

M5-branes wrapped on B_4. Within the above setup, we are interested in solutions that describe holographic duals to M5-branes wrapping a holomorphic four-cycle B_4 inside a Calabi–Yau four-fold. A local model for the Calabi–Yau is given by the sum of two line bundles $\mathcal{L}_1\oplus\mathcal{L}_2\to B_4$ subject to the condition

$$c_1(\mathcal{L}_1)+c_1(\mathcal{L}_2)+c_1(TB_4)=0, \tag{7}$$

which also guarantees the supersymmetry of a wrapped M5-brane. For the associated supergravity solutions (in the near horizon limit), we take M_8 to be an S^4 bundle over B_4,

$$S^4\hookrightarrow M_8\to B_4. \tag{8}$$

Here we write $S^4\subset\mathbb{C}_1\oplus\mathbb{C}_2\oplus\mathbb{R}$, where the $\mathbb{C}_i$ factors are twisted by the respective line bundles $\mathcal{L}_i$. We will assume that the solutions have $U(1)^2 \subset SO(5)$ isometry of the S^4, as well as the isometries of B_4. In the following, we will first consider the B_4 base to be a complex toric surface and compute the corresponding trial central charge using equivariant localization. Later we will consider cases when the toric B_4 has orbifold singularities, and we will then also slightly generalize the Calabi-Yau condition (7). Other classes of B_4 are considered in [6].

For the toric B_4 examples considered here, the R-symmetry will only have isolated fixed points, and as a consequence, the BVAB formula takes a particularly simple form. On M_8, and even-dimensional invariant submanifolds $M_{2k} \subset M_8$, the integral of a general equivariantly closed polyform Φ is given by a sum of contributions from the fixed points x_ℓ

$$\int_{M_{2k}} \Phi = \sum_\ell \frac{1}{d_\ell} \frac{(2\pi)^k}{\epsilon_1^\ell \cdots \epsilon_k^\ell} \Phi \bigg|_{x_\ell}. \tag{9}$$

Here M can have orbifold singularities, where the normal space to the point x_ℓ is $\mathbb{R}^{2k}/\Gamma_\ell$, and d_ℓ is the order of the finite group Γ_ℓ. On this normal space ξ generates a linear isometric action with weights ϵ_i^ℓ.

In the sequel, for simplicity, we impose one other condition on the class of solutions that we are considering, namely, that a certain flux integral threading the S^4 vanishes [12]:

$$\int_{S^4} (\mathrm{e}^{6\lambda} * F - a_0 \mathrm{e}^{3\lambda} F) = 0. \tag{10}$$

Smooth toric base. The first family of solutions we focus on is when the base B_4 is a toric complex surface, with B_4 having $U(1)^2$ isometry. The R-symmetry Killing vector ξ on M_8 generically mixes the $U(1)^2 \subset SO(5)$ isometry of the S^4 with the $U(1)^2$ of B_4, and so we can write

$$\xi = \sum_{i=1}^2 b_i \partial_{\varphi_i} + \sum_{A=1}^2 \varepsilon_A \partial_{\psi_A}, \tag{11}$$

where b_i, ε_A are constants. Here ∂_{φ_i} rotate the two copies of $\mathbb{C}_i$ in $S^4 \subset \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \mathbb{R}$, with weight 1, and ∂_{ψ_A} are a lift of the generators of the torus isometry of B_4 to M_8.

For generic b_i, ε_A, the fixed points of the action of ξ on M_8, where $\|\xi\| = 0$, are isolated, as noted above. Concretely, the $U(1)^2$ action on the S^4 has two fixed points, at the north and south pole. If we take the $U(1)^2$ isometry on B_4 to have d isolated fixed points, we then have a total of $2d$ fixed points on M_8. These are labeled by (N/S, a), where N/S refers to the north or south pole of S^4, and $a = 1, \ldots, d$ labels the isolated fixed points on B_4.

We can now use the BVAB formula to compute the flux of Φ^F through the S^4 cycle over any of the d fixed points on the base. Since these cycles are all in the same homology class [13], using (5), we have

$$\begin{aligned} N_{S^4} &= \frac{1}{(2\pi \ell_p)^3} \int_{S^4} \Phi^F \\ &= -\frac{1}{(2\pi \ell_p)^3} \frac{1}{4} \frac{(2\pi)^2}{b_1^a b_2^a} (y_N^a - y_S^a), \end{aligned} \tag{12}$$

where N_{S^4} is the number of wrapped M5-branes. Here $y_{N/S}^a$ denotes the value of the function y at the fixed point (N/S, a), and b_i^a are the weights of the action of ξ on the normal space $\mathbb{R}^4 = \mathbb{C} \oplus \mathbb{C}$ to the fixed point in S^4. We can similarly compute the flux of Φ^{*F} through the same cycles which, recall from (10), we assumed to vanish. Utilizing the BVAB formula then immediately gives $(y_N^a)^2 = (y_S^a)^2$. Thus, requiring that $N_{S^4} \neq 0$, we conclude that

$$y_N^a = -y_S^a = -4\pi \ell_p^3 b_1^a b_2^a N_{S^4}. \tag{13}$$

We can similarly evaluate the central charge, with contributions from the $2d$ fixed points given by

$$\begin{aligned} c &= \frac{3}{2^5 \pi^7 \ell_p^9} \frac{1}{192} \sum_{a=1}^d \frac{(2\pi)^4}{\epsilon_1^a \epsilon_2^a b_1^a b_2^a} [(y_N^a)^3 - (y_S^a)^3] \\ &= \sum_{a=1}^d \frac{1}{\epsilon_1^a \epsilon_2^a} (b_1^a b_2^a)^2 (-N_{S^4})^3. \end{aligned} \tag{14}$$

Here the normal space to the fixed points in M_8 is $\mathbb{R}^8 = \mathbb{C}^{\oplus 4}$, with b_i^a, ϵ_A^a being the associated weights of the action of ξ on those four copies of $\mathbb{C}$.

It is remarkable how simply the key expression (14), as a sum of blocks, emerges from our formalism. In particular, we see that each block is related to the off-shell central charge for the $d = 6$, $\mathcal{N} = (0, 2)$ SCFT in the large N limit [14]. To obtain our final off-shell result, it remains to compute b_i^a, ϵ_A^a in terms of the R-symmetry vector (11), together with global topological data for M_8. In fact we will be able to do this straightforwardly, utilizing various standard results in the toric geometry literature.

Weights from toric geometry. We begin by recalling some key facts about complex toric four-manifolds B_4. By definition these are complex manifolds equipped with a holomorphic $(\mathbb{C}^*)^2$ action, which has a dense open orbit. There always exists a compatible Kähler metric, where $U(1)^2 \subset (\mathbb{C}^*)^2$ is an isometry, but we emphasize that no metric data enters the fixed point formulas we use.

Such a B_4 has a distinguished set of $a = 1, \ldots, d$ toric divisors $D_a \subset B_4$. By definition these are complex two-dimensional submanifolds, invariant under the $U(1)^2$ action, where the normal space to a given D_a is rotated by the $U(1) \subset U(1)^2$ subgroup specified by a vector with components $v_A^a \in \mathbb{Z}$, $A = 1, 2$. The set of v_A^a is referred to as the toric data for B_4. The toric divisors may be ordered cyclically, with $x_a = D_a \cap D_{a+1}$ precisely giving the set of d points that are fixed under the $U(1)^2$ action, the index a understood to be defined modulo d.

If ∂_{ψ_A} denote vector fields generating the $U(1)^2$ isometry, then we may write a Killing vector on B_4 as $\sum_{A=1}^2 \varepsilon_A \partial_{\psi_A}$. The fixed point x_a has normal space $\mathbb{R}_a^4 = \mathbb{C}_1^a \oplus \mathbb{C}_2^a$, and the weights ϵ_A^a of this Killing vector on $\mathbb{C}_A^a$ are given by the standard toric geometry formulas

$$\epsilon_1^a = -\det(v^{a+1}, \varepsilon), \qquad \epsilon_2^a = \det(v^a, \varepsilon). \tag{15}$$

The internal space M_8 is in turn the total space of an S^4 bundle over B_4. By definition the vector fields ∂_{φ_i} in (11) rotate the two copies of $\mathbb{C}_i$ in $S^4 \subset \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \mathbb{R}$, with weight 1, but to define ξ, we must also choose a lifting of the ∂_{ψ_A} to M_8. This may be achieved by choosing a lifting to each line bundle $\mathcal{L}_i \to B_4$, making these equivariant line bundles. On the other hand, a basis of such equivariant line bundles L_a is naturally provided by the toric divisors D_a. The corresponding equivariant first Chern class $c_1^\xi(L_a)$, when restricted to the fixed point $x_b \in B_4$, is given by the formula

$$c_1^\xi(L_a)|_{x_b} = \delta_{a,b}\epsilon_1^a + \delta_{a,b+1}\epsilon_2^b, \tag{16}$$

where the weights ϵ_A^a are given by (15). We may thus write $\mathcal{L}_i = -\sum_{a=1}^d \mathfrak{p}_i^a L_a$, where $\mathfrak{p}_i^a \in \mathbb{Z}$ specify both the topology of $\mathcal{L}_1 \oplus \mathcal{L}_2 \to B_4$, and also a choice of lifting of the $U(1)^2$ isometry of B_4 to the total space. From (16) the weights of ξ on the two complex line fibres are then

$$b_i^a = b_i - \sum_{b=1}^d \mathfrak{p}_i^b c_1^\xi(L_b)|_{x_a} = b_i - \mathfrak{p}_i^a \epsilon_1^a - \mathfrak{p}_i^{a+1}\epsilon_2^a. \tag{17}$$

Having determined explicit formulas (15) and (17) for the weights of ξ at the fixed points, finally we must impose that ξ is an R-symmetry: That is, there is a Killing spinor ϵ satisfying $\mathcal{L}_\xi \epsilon = \frac{\mathrm{i}}{2}\epsilon$. This is where the Calabi–Yau condition (7) enters as a further set of constraints on our parameters.

For a toric complex manifold B_4, we have the standard toric geometry formula $c_1(TB_4) = \sum_{a=1}^d c_1(L_a)$. Since also by definition $\mathcal{L}_i = -\sum_{a=1}^d \mathfrak{p}_i^a L_a$, imposing the equivariant version of Eq. (7) gives

$$\sum_{a=1}^d \left(\sum_{i=1}^2 \mathfrak{p}_i^a - 1 \right) c_1^\xi(L_a) = 0. \tag{18}$$

The $c_1^\xi(L_a)$ are precisely a set of generators for the equivariant cohomology of B_4, with no relations, so the coefficients in (18) must all be zero: $\sum_{i=1}^2 \mathfrak{p}_i^a = 1$.

On the other hand, the resulting $SU(4)$-invariant chiral spinor on the Calabi–Yau fourfold satisfies a standard set of projection conditions $\gamma^{2j-1,2j}\epsilon = \mathrm{i}\epsilon$ in an orthonormal frame, for each $j = 1, 2, 3, 4$. The original local Calabi–Yau geometry is embedded inside M_8 as the normal bundle of the north pole section, in our conventions for the labeling of north/south poles. As shown in the Appendix, given the above projection conditions, the charge of the spinor at the point x_a^N in this north pole section is then

$$\mathcal{L}_\xi \epsilon = \frac{\mathrm{i}}{2}(b_1^a + b_2^a + \epsilon_1^a + \epsilon_2^a)\epsilon = \frac{\mathrm{i}}{2}(b_1 + b_2)\epsilon. \tag{19}$$

Here we have used (17), and then imposed the Calabi–Yau condition in the form (18). Thus, together we have the following constraints on our parameters:

$$b_1 + b_2 = 1, \qquad \sum_{i=1}^2 \mathfrak{p}_i^a = 1. \tag{20}$$

We will discuss a generalization of these constraints later.

Inserting the formulas (15) and (17) into (14) gives our final "gravitational block" formula for the trial central charge in supergravity, expressed in terms of the toric data v_A^a, $\mathfrak{p}_i^a$, and choice of R-symmetry vector field (11), subject to (20). Our resulting expression agrees precisely with the field theory formula in [15], obtained via equivariant localization of the M5-brane anomaly polynomial and provides a striking confirmation of AdS/CFT. To get the on-shell result, we need to extremize over the choice of R-symmetry; in [6] we show that on the gravity side this is indeed a necessary condition for imposing the supergravity equations of motion.

Other observables. As also shown in [1,6], other physical observables may similarly be computed using equivariant localization in supergravity.

For example, consider the four-cycles $\Gamma_i^a \subset M_8$ that are the total spaces of S_i^2 bundles over the toric divisors $D_a \subset B_4$, where $S_i^2 \subset \mathbb{C}_i \oplus \mathbb{R}$ is a linearly embedded two-sphere within the fibre S^4. There are four fixed points, namely the copies of x_a, x_{a-1} at the north and south poles of the fibre S_i^2. In fact x_a, x_{a-1} are also precisely the poles of $D_a \cong S_a^2$, with ϵ_2^a and $\epsilon_1^{a-1} = -\epsilon_2^a$ being the weights on the tangent space, respectively. Picking for instance $i = 1$, using localization, we compute the flux

$$\begin{aligned} N_1^a &= \frac{1}{(2\pi\ell_p)^3}\int_{\Gamma_1^a} \Phi^F \\ &= -\frac{1}{(2\pi\ell_p)^3}\frac{(2\pi)^2}{4}\left[\frac{y_N^a - y_S^a}{b_1^a\epsilon_2^a} + \frac{y_N^{a-1} - y_S^{a-1}}{b_1^{a-1}\epsilon_1^{a-1}}\right] \\ &= \frac{1}{\epsilon_2^a}[\mathfrak{p}_2^a(\epsilon_2^{a+1} + \epsilon_2^{a-1}) - (\mathfrak{p}_2^{a+1} + \mathfrak{p}_2^{a-1})\epsilon_2^a]N_{S^4} \\ &= -\mathfrak{q}_2^a N_{S^4}, \end{aligned} \tag{21}$$

and similarly $N_2^a = -\mathfrak{q}_1^a N_{S^4}$. To get the third line in (21), we substituted (13) and (17) and used (15). To get the final line, we evaluated the determinants in (15), subject to $\det(v^{a-1}, v^a) = \det(v^a, v^{a+1}) = 1$, which follow from B_4 being smooth. In (21) $\mathfrak{q}_i^a \in \mathbb{Z}$ is (minus), the integral of the first Chern class of $\mathcal{L}_i$ through the divisor D_a

$$\mathfrak{q}_i^a \equiv -\int_{D_a} c_1(\mathcal{L}_i) = \sum_{a,b} D_{ab}\mathfrak{p}_i^b, \tag{22}$$

where $D_{ab} = \int_{B_4} c_1(L_a)c_1(L_b)$ is the intersection matrix of the toric divisors

$$D_{ab} = \begin{cases} 1 & b = a \pm 1 \\ -\det(v^{a-1}, v^{a+1}) & b = a \\ 0 & \text{otherwise} \end{cases}. \tag{23}$$

A derivation of (21), using only algebraic topology, can be found in an Appendix to [6].

We can also compute the dimension of chiral primary operators dual to M2-branes wrapping submanifolds calibrated by ω. These are obtained by applying the BVAB formula to Φ^F restricted to Σ_2 [6]. Consider the (homologically trivial) S_i^2 just considered over a fixed point x_a; if it is calibrated by ω, then

$$\begin{aligned}\Delta(S_i^2) &= \frac{1}{(2\pi)^2\ell_p^3}\int_{S_i^2} \mathrm{e}^{3\lambda}\omega = \frac{1}{(2\pi)^2\ell_p^3}\frac{2\pi}{b_i^a}\frac{y_N^a - y_S^a}{2} \\ &= \frac{2b_1^a b_2^a}{b_i^a}(-N_{S^4}).\end{aligned} \tag{24}$$

One can also similarly consider the divisor D_a at, say, the north pole of the S^4, and if this is calibrated by ω, then (with an appropriate orientation choice)

$$\begin{aligned}\Delta(D_a) &= -\frac{1}{(2\pi)^2\ell_p^3}\int_{D_a} \mathrm{e}^{3\lambda}\omega \\ &= -\frac{1}{(2\pi)^2\ell_p^3}\frac{2\pi}{2}\left(\frac{y_N^a}{\epsilon_a^2} + \frac{y_N^{a-1}}{\epsilon_{a-1}^1}\right) \\ &= \frac{1}{\epsilon_a^2}(b_1^a b_2^a - b_1^{a-1}b_2^{a-1})N_{S^4} \\ &= (b_1\mathfrak{q}_2^a + b_2\mathfrak{q}_1^a + D_{abc}\mathfrak{p}_1^b\mathfrak{p}_2^c)(-N_{S^4}).\end{aligned} \tag{25}$$

The same expression holds for the divisor D_a at the south pole of S^4, up to an overall sign related to the choice of orientation. Here we introduced the intersection of three equivariant Chern classes (see, e.g., [16] for a review)

$$D_{abc} \equiv \int_{B_4} c_1^\xi(L_a)c_1^\xi(L_b)c_1^\xi(L_c) = \begin{cases} -\epsilon_2^{a-1} & a_i = a_j = a_k + 1 \equiv a \\ -\epsilon_1^a & a_i = a_j = a_k - 1 \equiv a \\ -(\epsilon_1^a)^2 - (\epsilon_2^{a-1})^2 & a_i = a_j = a_k \equiv a \\ 0 & \text{otherwise} \end{cases}. \tag{26}$$

Notice that the first two terms in the last line of (25), as well as the expression (24), are both independent of ε_A and indeed match the analogous formulas in the absence of $U(1)^2$ symmetry on B_4 discussed in [6]. In contrast, this is not true of D_{abc} in (25), and in fact $\Delta(D_a)$ is linear in ϵ_A^a.

Orbifolds and antitwists. The above discussion has some immediate generalizations, and this allows us to make a connection with the results of [16,17].

First, we may replace B_4 by a complex toric orbifold. Here the fixed points x_a of the $U(1)^2$ action are now orbifold points, with tangent space $\mathbb{C}^2/\mathbb{Z}_{d_a}$, where more generally $d_a = \det(v^a, v^{a+1})$. The latter is positive, with appropriately oriented cyclic ordering of the toric divisors, and is equal to 1 when B_4 is a smooth manifold.

Appropriate factors of $1/d_a$ then enter fixed point formulas, as also explained in [1,6]. In particular, the S_a^4 cycles over the fixed points x_a on B_4 do not belong to the same homology class, but instead the classes $[d_a S_a^4] \in H_4(M_8)$ are all equal. Correspondingly the fluxes through the cycles are not (12) but instead

$$N_{S_a^4} = -\frac{1}{8\pi\ell_p^3}\frac{1}{d_a}\frac{y_N^a - y_S^a}{b_1^a b_2^a}. \tag{27}$$

However, from the above remarks, the combinations $d_a N_{S_a^4}$ are necessarily all equal, and we label these by N_{S^4} so that (13) is not modified. The same orbifold order appears in the BVAB formula used to compute (14), which generalizes to

$$c = \sum_{a=1}^{d}\frac{1}{d_a}\frac{1}{\epsilon_1^a\epsilon_2^a}(b_a^1 b_a^2)^2(-N_{S^4})^3. \tag{28}$$

This result provides a gravitational derivation of the gravitational block conjecture for M5-branes wrapped on toric four-orbifolds in [17]. Similarly, the expressions (15) for the weights receive a factor of $1/d_a$, whereas (17) is formally unchanged. The expression (21) for the flux N_i^a is also formally unchanged, but if B_4 is not smooth, the intersection matrix D_{ab} reads

$$D_{ab} = \begin{cases} 1/d_{a-1} & b = a-1 \\ -\det(v^{a-1}, v^{a+1})/d_{a-1}d_a & b = a \\ 1/d_a & b = a+1 \\ 0 & \text{otherwise} \end{cases}, \quad (29)$$

with a similar expression for the generalization of (26) that may be found in [16].

Second, we may relax the Calabi-Yau condition (7), and in particular, (20). This is motivated by the so-called antitwist, discovered in [18] as a novel way to preserve supersymmetry for D3-branes wrapped on a two-dimensional orbifold known as a spindle, but which has since been generalized to many other setups. In particular the solutions [19], describing M5-branes wrapping orbifolds B_4 that are the total spaces of spindles fibred over spindles, have been further studied in [16,17,20,21], where it was proposed to relax the second condition in (20) to

$$\sum_{i=1}^{2} \mathfrak{p}_i^a = \sigma^a. \quad (30)$$

Here $\sigma^a = \pm 1$ is *a priori* chosen freely for each toric divisor. Again using (17), one can check that the following identity now holds, for each $a = 1, \ldots, d$:

$$b_1^a + b_2^a + \sigma^a \epsilon_1^a + \sigma^{a+1} \epsilon_2^a = 1. \quad (31)$$

From the discussion in the Appendix, we may interpret this as a necessary condition for the spinor ϵ to have R-charge $\frac{1}{2}$, but where the projection conditions on the spinor at different fixed points now depend on the choice of σ^a. In particular, the chirality of the spinor at x_a^N is determined by the sign of $\sigma^a \sigma^{a+1}$. This change of chirality at different fixed points is understood in detail for the spindle [22], and indeed the motivation for introducing σ^a in [17] was precisely that this describes the known supergravity solutions where B_4 is a spindle fibred over another spindle [19].

It would be interesting to understand better what global constraints there are on the choice of projection conditions in a general setup, but we leave this for future work.

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Appendix: Chirality, charge relation for killing spinors. We consider Killing spinors with definite charge under a Killing vector ξ, and show how this is related to the chirality of the spinor at a fixed point. Suppose the charge is q so that $\mathcal{L}_\xi \epsilon \equiv \xi^a \nabla_a \epsilon + \frac{1}{8}(\mathrm{d}\xi^\flat)_{ab}\gamma^{ab}\epsilon = \mathrm{i}q\epsilon$. Also suppose the Killing spinor satisfies a Killing spinor equation of the form $\nabla_a \epsilon = (\mathcal{M}\cdot\gamma)_a \epsilon + (\mathcal{N}\cdot\gamma)_a \epsilon^c$.

Near a fixed point, we introduce Cartesian coordinates $(x_1, y_1, \ldots, x_4, y_4)$ and polar coordinates (r_i, ϕ_i) for each plane parametrized by (x_i, y_i). The Killing vector is given by $\xi = \sum_{i=1}^4 \epsilon_i \partial_{\phi_i} = \sum_{i=1}^4 \epsilon_i(-y_i \partial_{x_i} + x_i \partial_{y_i})$. From the Killing spinor equation, we deduce that at a fixed point $\xi^a \nabla_a \epsilon = 0$, and hence, $\frac{1}{8}(\mathrm{d}\xi^\flat)_{ab}\gamma^{ab}\epsilon = \mathrm{i}q\epsilon$, or

$$\mathrm{i}q\epsilon = \frac{1}{2}\sum_{i=1}^{4} \epsilon_i \gamma^{2i-1,2i}\epsilon. \quad (A1)$$

Thus, ϵ is an eigenvector of each $\gamma^{2i-1,2i}$, which is square to -1 and thus has eigenvalues $\pm\mathrm{i}$. Therefore, a spinor with definite charge q is related to the weights ϵ_i via

$$q = \frac{1}{2}\sum_{i=1}^{4} s_i \epsilon_i, \quad (A2)$$

for some signs $s_i = \pm 1$. Define the chirality operator by $\gamma_9 \equiv \gamma_{12\cdots 8}$, with $\gamma_9^2 = 1$ and eigenvalues ± 1. Thus, when acting on a spinor with definite charge q, we also have

$$\gamma_9 \epsilon = \left(\prod_{i=1}^{4} s_i\right)\epsilon, \quad (A3)$$

so the chirality is equal to the product of the signs appearing in the charge.

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[12] This condition holds for the known solutions in [4,8] with M_8 an S^4 fibration over B_4. One can relax this condition and obtain off-shell expressions for the central charge that will depend on this flux. It would be interesting to know whether or not any such solutions exist.

[13] Take any curve connecting two fixed points x_a, $x_{a'}$ in B_4, together with the S^4 bundle over that curve. This is a five-dimensional submanifold of M_8 with boundaries S^4_a and $S^4_{a'}$ (the spheres over the fixed points), showing the latter are homologous.

[14] Recall that this is given by $a = \frac{256}{7}(b_1 b_2)^2 N^3$, subject to $b_1 + b_2 = 1$. This off-shell expression only assumes the existence of a $U(1)^2 \subset SO(5)$ R-symmetry and after extremizing a over the constrained b_i, one finds the on-shell weights $b_1 = b_2 = \frac{1}{2}$ and on-shell central charge $a = \frac{16}{7} N^3$, which is the correct result for the $d = 6$, $\mathcal{N} = (0,2)$ SCFT with its $SO(5)$ R-symmetry, in the large N limit. Further discussion in the context of equivariant localization can be found in Appendix B of [6].

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