Letter

Inflationary gravitational wave background as a tail effect

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The free propagator of a massless mode in an expanding universe can be written as a sum of two terms, a light cone and a tail part. The latter describes a subluminal (timelike) signal. We show that the inflationary gravitational wave background, influencing cosmic microwave background polarization and routinely used for constraining inflationary models through the so-called r ratio, originates exclusively from the tail part.

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I. INTRODUCTION

In four-dimensional flat spacetime, massless radiation propagates along light cones. This means that the corresponding retarded Green's function is proportional to the Dirac δ-function. On the other hand, in curved spacetime, massless fields develop a so-called tail. This means that the Green's function is nonzero inside the light cone.

The appearance of a tail was first observed in a mathematical analysis of second order partial differential equations [1,2]. Subsequently, it has been studied in a multitude of physical contexts [3–25], suggesting possible observable signatures. However, so far there is no direct observation of the tail.

A particularly relevant curved spacetime is the cosmological Friedmann-Lemaître-Robertson-Walker (FLRW) universe. A fundamental observation from its late period is the gravitational wave (GW) signal from mergers of black holes [26]. The main signal arrives on the light cone and is theoretically well understood [27]. The corresponding tail has not yet been observed, but its magnitude has been computed [24] (see also Refs. [19–23]). The tail signal arrives long after the main merger signal, but its magnitude is surprisingly large.

The FLRW universe also contains earlier periods. A particularly important one is that of inflation, where the seeds for structure formation, as well as the anisotropies that are observed in the cosmic microwave background (CMB) [28], are believed to have been generated. At linear order these originate from so-called scalar perturbations, but neither the scalar field driving inflation nor its perturbations are massless. However, inflation also generates tensor perturbations [29–32], which manifest themselves as gravitational waves. The gravitational wave background propagates until today, in a well-understood fashion [33], and leads to potentially observable consequences, through the polarization of the CMB photons [34]. In principle, primordial GWs could also be observed directly, e.g., via pulsar timing arrays [35–38], even if the sensitivity is not sufficient for the simplest inflationary models [39].

As the spacetime curvature is large during inflation, we may also anticipate a tail contribution from this epoch. We now proceed to describing the computation of the inflationary GW background, quantifying subsequently the tail's role in it.

II. GRAVITATIONAL WAVE BACKGROUND FROM DE SITTER VACUUM FLUCTUATIONS

The way that the inflationary GW background is computed is that we first consider the wave equation for tensor perturbations in de Sitter spacetime. The time dependence of the solution is easily found, but its normalization needs also to be fixed. This can be done by considering a fixed comoving momentum k. Thanks to the expansion of the universe, the corresponding physical wavelength is very small at early times. Therefore, it is within a causally connected domain, "inside the horizon."

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There its normalization can be fixed like in a Minkowskian vacuum, a well-established problem in quantum field theory.

Once the normalization and time evolution have been found, we can follow the momenta until late times. At some point, the modes "exit the horizon," i.e. $k \ll aH$ (or, in terms of physical momenta $p \equiv k/a$, $p \ll H$). Here H is the Hubble rate, which is constant in de Sitter spacetime, and a is the scale factor, which grows exponentially in physical time.

Once the modes exit the horizon, they "freeze out," i.e. their amplitude becomes constant. Up to overall normalization, the absolute value squared of the amplitude constitutes the primordial tensor power spectrum that we are interested in ($\mathcal{P}_{\mathrm{T}}$).

The modes do not stay forever outside of the horizon. As inflation ends, the Hubble constant starts to decrease, and the scale factor grows less rapidly. At some point, a given momentum mode reenters the horizon. Then it starts to oscillate again. As a result, it carries energy density, which is in principle observable. It also influences other modes propagating through the cosmological history, notably CMB photons. The determination of these postinflationary features amounts to the determination of a "transfer function" from the primordial to the current era [33]. We will not concern ourselves with the complicated postinflationary physics, only the primordial power spectrum.

In order to compute $\mathcal{P}_{\mathrm{T}}$, we find it illuminating to carry out the computation with the formalism of stochastic inflation [40]. As illustrated in Fig. 1, the full time evolution is then divided into two parts. While the early part is effectively the same as in the standard quantum-mechanical analysis, the latter part is simpler, as it can be viewed as a classical problem. This facilitates the identification of the light cone and tail contributions to the final result.

The stochastic formalism is employed in different variants in the literature. In the following, we implement it in a way that is not an approximation but just a mathematical reorganization of the usual quantum-mechanical computation (cf., e.g., Refs. [41–43] and references therein). Apart from a clear physical picture, the advantage of this approach is that it introduces an arbitrary parameter, denoted by ϵ, whose cancellation offers a nice cross-check, analogously to the role played by the gauge parameter in Yang-Mills theories.

Let h be a canonically normalized massless field. Denoting by $\tau \in (-\infty, 0)$ the conformal time, de Sitter spacetime with a constant Hubble parameter H has the scale factor $a = -1/(H\tau)$. The solution of the wave equation $h'' - \frac{2}{\tau}h' - \nabla^2 h = 0$ reads

$$h = \int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{(2\pi)^3}} \left[w_{\mathbf{k}} \underbrace{h_k(\tau)}_{\frac{iH}{\sqrt{2k^3}}(1+ik\tau)e^{-ik\tau}} e^{i\mathbf{k}\cdot\mathbf{x}} + \mathrm{H.c.} \right], \qquad (1)$$

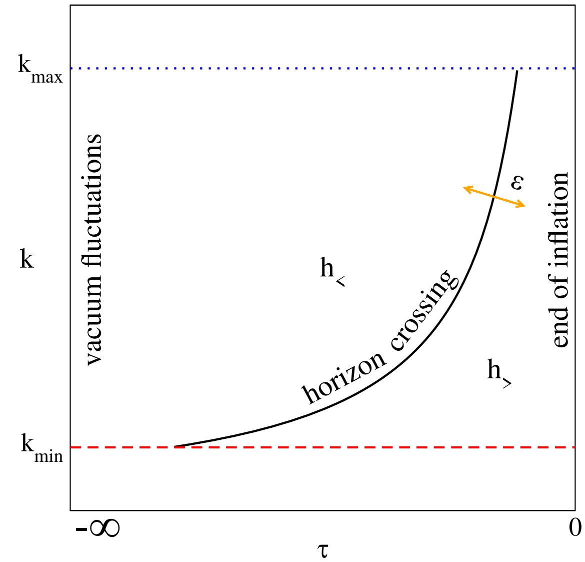

FIG. 1. An illustration of the computation of the gravitational wave background with the stochastic formalism, in the plane of conformal time (τ) and comoving momentum (k). The quantum mechanical solution at early times is denoted by $h_<$. A chosen "horizon crossing" hypersurface (whose position $k = -\epsilon/\tau$ depends on a parameter ϵ that cancels from final results) is used for matching $h_<$ onto a long-wavelength field $h_>$. The latter can be treated as a classical perturbation.

where $\mathbf{k}$ and $\mathbf{x}$ are a comoving momentum and coordinate, respectively, $w_{\mathbf{k}}$ is an annihilation operator of the distant-past (Bunch-Davies) vacuum, and the canonical commutation relation takes the form $[w_{\mathbf{k}}, w_{\mathbf{l}}^\dagger] \equiv \delta^{(3)}(\mathbf{k} - \mathbf{l})$.

The mode h is now divided into short-distance ($h_<$) and long-distance parts ($h_>$), $h = h_> + h_<$, by defining

$$h_< \equiv \int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{(2\pi)^3}} \underbrace{W_k(\tau)}_{\theta(k+\frac{\epsilon}{\tau})} \left[w_{\mathbf{k}} h_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + \mathrm{H.c.} \right], \qquad (2)$$

where the window function W_k selects large momenta. The parameter ϵ is arbitrary and must drop out from physical results [44]. Inserting Eq. (2) into the equation of motion yields

$$h''_> - \frac{2}{\tau} h'_> - \nabla^2 h_> = \varrho_{\mathrm{Q}}, \qquad (3)$$

$$\varrho_{\mathrm{Q}} \equiv -\left(\partial_\tau^2 - \frac{2}{\tau}\partial_\tau - \nabla^2 \right) h_<, \qquad (4)$$

where the "quantum noise" has the form

$$\varrho_{\mathrm{Q}}(\tau,\mathbf{x}) = -\int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{(2\pi)^3}}\left[w_{\mathbf{k}} \underbrace{f_k(\tau)}_{(W_k''-\frac{2}{\tau}W_k')h_k+2W_k'h_k'} e^{i\mathbf{k}\cdot\mathbf{x}} + \mathrm{H.c.}\right]. \quad (5)$$

We note that, with W_k from Eq. (2), the function f_k in Eq. (5) contains the Dirac-δ or its derivative. Therefore, the differential operator acting on $h_<$ amounts to a holographic projection, localizing the information from early times onto the horizon crossing hypersurface (cf. Fig. 1).

The long-wavelength mode $h_>$ can subsequently be determined from Eq. (3) with a retarded Green's function. The Green's function satisfies

$$\left(\partial_\tau^2 - \frac{2}{\tau}\partial_\tau - \nabla_{\mathbf{x}}^2\right) G_{|\mathbf{x}-\mathbf{z}|}(\tau,\tau_i) = \delta(\tau-\tau_i)\delta^{(3)}(\mathbf{x}-\mathbf{z}), \quad (6)$$

with the boundary conditions

$$G_{|\mathbf{x}-\mathbf{z}|}(\tau,\tau_i) \overset{\tau\le\tau_i}{\equiv} 0, \quad (7)$$

$$\lim_{\tau\to\tau_i^+} \partial_\tau G_{|\mathbf{x}-\mathbf{z}|}(\tau,\tau_i) = \delta^{(3)}(\mathbf{x}-\mathbf{z}). \quad (8)$$

Here τ and τ_i are the time arguments of observation and source, respectively.

Given the Green's function, we can determine the late-time solution as

$$h_>(\tau,\mathbf{x}) = \int_{\mathbf{z}}\int_{-\infty}^{\tau} \mathrm{d}\tau_1 G_{|\mathbf{x}-\mathbf{z}|}(\tau,\tau_1)\varrho_{\mathrm{Q}}(\tau_1,\mathbf{z}), \quad (9)$$

where $\int_{\mathbf{z}}$ is a spatial integral over source locations. This can be viewed as an antiholographic mapping, from the hypersurface unto late times. Our next task is then to specify the properties of the Green's function.

III. LIGHT CONE AND TAIL PARTS OF THE GREEN'S FUNCTION

The Green's function from Eqs. (6)–(8) is easily solved by representing it in momentum space,

$$G_x(\tau,\tau_i) = \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} G_k(\tau,\tau_i), \quad (10)$$

where $\int_{\mathbf{k}} \equiv \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}$. The momentum-space solution is split up into two parts, called the light cone (lc) and the tail parts (tail), defined as

$$G_k(\tau,\tau_i) = \theta(\tau-\tau_i)\left[g_{k,\mathrm{lc}}(\tau,\tau_i) + g_{k,\mathrm{tail}}(\tau,\tau_i)\right], \quad (11)$$

$$g_{k,\mathrm{lc}}(\tau,\tau_i) \equiv \frac{\tau\sin[k(\tau-\tau_i)]}{k\tau_i}, \quad (12)$$

$$g_{k,\mathrm{tail}}(\tau,\tau_i) \equiv \frac{\sin[k(\tau-\tau_i)] - k(\tau-\tau_i)\cos[k(\tau-\tau_i)]}{k^3\tau_i^2}. \quad (13)$$

The rationale behind this nomenclature becomes clear when we go back to configuration space. Then we obtain

$$g_{x,\mathrm{lc}}(\tau,\tau_i) \underset{x>0}{\overset{\tau>\tau_i}{=}} \frac{\tau\delta(\tau-\tau_i-x)}{4\pi x\tau_i}, \quad (14)$$

$$g_{x,\mathrm{tail}}(\tau,\tau_i) \underset{x>0}{\overset{\tau>\tau_i}{=}} \frac{\theta(\tau-\tau_i-x)}{4\pi\tau_i^2}. \quad (15)$$

The light cone part describes a signal arriving at the speed of light, the tail part arrives later. It can be verified that the sum of the two parts, though not the parts separately, satisfies the s-wave equation

$$\left(\partial_\tau^2 - \frac{2}{\tau}\partial_\tau - \partial_x^2 - \frac{2}{x}\partial_x\right)(g_{x,\mathrm{lc}} + g_{x,\mathrm{tail}}) = 0. \quad (16)$$

The s-wave solution is the relevant one, because the source in Eq. (6) is a monopole.

IV. COMPUTATION OF THE PRIMORDIAL TENSOR POWER SPECTRUM

In cosmology, the fundamental objects are equal-time 2-point correlation functions, such as those of the temperature of the CMB photons. What we are interested in here is the 2-point correlator of the tensor perturbations. For the $h_>$ field, from Eq. (9), evaluating the expectation value in the distant-past vacuum, the equal-time correlator can be expressed as

$$\begin{aligned}&\langle h_>(\tau,\mathbf{x})h_>(\tau,\mathbf{y})\rangle \\ &\quad = \int_{\mathbf{z},\mathbf{w}}\int_{-\infty}^{\tau}\mathrm{d}\tau_1\int_{-\infty}^{\tau}\mathrm{d}\tau_2 G_{|\mathbf{x}-\mathbf{z}|}(\tau,\tau_1)G_{|\mathbf{y}-\mathbf{v}|}(\tau,\tau_2) \\ &\qquad \times \langle 0|\varrho_{\mathrm{Q}}(\tau_1,\mathbf{z})\varrho_{\mathrm{Q}}(\tau_2,\mathbf{v})|0\rangle. \end{aligned} \quad (17)$$

The quantum mechanics of the problem is now hidden in the autocorrelator of ϱ_{Q}. Inserting Eq. (5), it becomes

$$\langle 0|\varrho_{\mathrm{Q}}(\tau_1,\mathbf{z})\varrho_{\mathrm{Q}}(\tau_2,\mathbf{v})|0\rangle = \int_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{z}-\mathbf{v})} f_k(\tau_1) f_k^*(\tau_2). \quad (18)$$

The correlator of Eq. (17) is most easily evaluated in comoving momentum space (the configuration space computation is described in the Supplemental Material [45]). After a Fourier transform, we obtain

$$\begin{aligned}&\langle h_>(\tau,\mathbf{k})h_>(\tau,\mathbf{q})\rangle \\ &\quad = \tilde{\delta}(\mathbf{k}+\mathbf{q})\left|\int_{-\infty}^{\tau}\mathrm{d}\tau_i G_k(\tau,\tau_i) f_k(\tau_i)\right|^2, \end{aligned} \quad (19)$$

where $\int_{\mathbf{k}} \tilde{\delta}(\mathbf{k}) \equiv 1$.

Making use of f_k from Eq. (5) and carrying out a partial integration, the integral in Eq. (19) can be expressed as

$$\begin{aligned}&\int_{-\infty}^{\tau} \mathrm{d}\tau_i G_k(\tau,\tau_i) f_k(\tau_i)\\&\quad = \int_{-\infty}^{\tau} \mathrm{d}\tau_i W_k'(\tau_i)\Big\{-\partial_{\tau_i} G_k(\tau,\tau_i) h_k(\tau_i)\\&\qquad + G_k(\tau,\tau_i)\Big[h_k'(\tau_i) - \frac{2}{\tau_i} h_k(\tau_i)\Big]\Big\}. \end{aligned} \tag{20}$$

With $W_k' = -\frac{\epsilon}{\tau^2}\delta(k + \frac{\epsilon}{\tau})$, the light cone part yields

$$\begin{aligned}&\int_{-\infty}^{\tau} \mathrm{d}\tau_i g_{k,\mathrm{lc}}(\tau,\tau_i) f_k(\tau_i)\\&= \frac{iHk\tau}{\sqrt{2k^3}}\frac{e^{i\epsilon}}{\epsilon}\Big[\cos(k\tau+\epsilon)(1-i\epsilon) + \sin(k\tau+\epsilon)\Big(\frac{1}{\epsilon} - i - \epsilon\Big)\Big],\end{aligned} \tag{21}$$

whereas for the tail part we obtain

$$\begin{aligned}&\int_{-\infty}^{\tau} \mathrm{d}\tau_i g_{k,\mathrm{tail}}(\tau,\tau_i) f_k(\tau_i)\\&\quad = \frac{iHk\tau}{\sqrt{2k^3}}\frac{e^{i\epsilon}}{\epsilon}\Big[\cos(k\tau+\epsilon)\Big(-1 - \frac{\epsilon}{k\tau}\Big)\\&\qquad + \sin(k\tau+\epsilon)\Big(-\frac{1}{\epsilon} + i + \frac{i\epsilon}{k\tau}\Big)\Big]. \end{aligned} \tag{22}$$

Summing together, many terms cancel, and the remaining ones can be factorized,

$$\begin{aligned}&\int_{-\infty}^{\tau} \mathrm{d}\tau_i G_k(\tau,\tau_i) f_k(\tau_i)\\&= \frac{iHk\tau}{\sqrt{2k^3}}(-e^{i\epsilon})\Big(i + \frac{1}{k\tau}\Big)\Big[\cos(k\tau+\epsilon) - i\sin(k\tau+\epsilon)\Big]\\&= -\frac{iH}{\sqrt{2k^3}} e^{-ik\tau}(1 + ik\tau). \end{aligned} \tag{23}$$

In the equations above, an implicit factor $\theta(1/\tau + k/\epsilon)$ has been suppressed for simplicity of notation.

An important property of Eq. (23) is that it is independent of the parameter ϵ and thus of the position of the horizon crossing hypersurface in Fig. 1. That is, the antiholographic mapping from the hypersurface onto late times erases all memory of the hypersurface itself.

We note that Eq. (23) is just the standard result for massless mode functions. The novelty of our computation is that the first term in the parentheses, dominant on superhorizon scales $k|\tau| \ll 1$, is seen to come exclusively from the tail, the second from the light cone.

For physical conclusions, we need the absolute value squared of Eq. (23), according to Eq. (19). The corresponding power spectrum is obtained by multiplying this with the measure $k^3/(2\pi^2)$, yielding

$$\mathcal{P}_h = \Big(\frac{H}{2\pi}\Big)^2 (1 + k^2\tau^2). \tag{24}$$

After adding the normalization factors for tensor perturbations and considering the limit $k|\tau| = k/(aH) \ll 1$, valid for momenta well outside of the horizon (these are the ones having an observable effect today), this reduces to the textbook tensor power spectrum,

$$\mathcal{P}_{\mathrm{T}} \overset{k|\tau|\ll 1}{\approx} \frac{16}{\pi}\Big(\frac{H}{m_{\mathrm{pl}}}\Big)^2, \tag{25}$$

where $m_{\mathrm{pl}} \equiv 1.22091 \times 10^{19}$ GeV is the Planck mass.

From the tensor power spectrum, once multiplied with the transfer function [33], we can obtain the current-day fractional gravitational energy density Ω_{GW}. Measuring this is an ongoing effort [35–38]. The ratio of the tensor power spectrum to the curvature one, $r \equiv \mathcal{P}_{\mathrm{T}}/\mathcal{P}_{\mathcal{R}}$, is already strongly constrained by Planck data on CMB, $r < 0.056$ [28].

We stress that the only terms left over in Eq. (25) are the ones enhanced by $1/(k\tau)$ in Eq. (22). In other words, Eq. (25) arises exclusively from the tail contribution. However, the light cone contribution is also conceptually important, for it guarantees the independence of the result of the parameter ϵ at finite values of $k\tau$ [46].

V. CONCLUSIONS

The purpose of this Letter has been to demonstrate that the primordial tensor power spectrum, cf. Eq. (25), which plays an important role in constraining inflationary models through CMB data [28], originates from the $1/(k\tau)$-enhanced terms in Eq. (22). In other words, the tails of the gravitational waves that cross the horizon are responsible for the physical phenomena observable today, implementing thereby a remarkable memory effect.

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[44] As alluded to above, in the literature the stochastic formalism is often implemented in an approximate fashion, in which case it is only correct for $\epsilon \ll 1$.

[45] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.108.L101503, for more details on the configuration space computation.

[46] We note that if we expand Eq. (23) in a small $k\tau$, then the leading term is the constant, and the next-to-leading term is of $\mathcal{O}(k^2\tau^2)$. If we take a time derivative, as is relevant for energy density, the leading term drops out. Thus, the tail part does not dominate the would-be energy density measured outside of the horizon (this is a purely theoretical construct). But it is the constant part that turns into the energy density observable today [33].