Letter

Critical superflows and thermodynamic instabilities in superfluids

Blaise Goutéraux[*] and Filippo Sottovia[†]
CPHT, CNRS, École polytechnique, Institut Polytechnique de Paris, 91120 Palaiseau, France

Eric Mefford[‡]
Department of Physics and Astronomy, University of Victoria, Victoria, British Columbia V8W 3P6, Canada

(Received 20 January 2023; revised 30 April 2023; accepted 11 October 2023; published 27 October 2023)

In this work, we study the linear stability of superfluid phases of matter irrespective of the nature of microscopic degrees of freedom and the strength of interactions between them. Famously, assuming invariance under Galilean boosts and a phonon-roton single-particle dispersion relation, Landau predicted superfluid helium 4 would become unstable for large enough superfluid velocities. Here, we demonstrate that such instabilities generically follow from a change of sign of one of the eigenvalues of the matrix of second derivatives of the free energy. Our only assumption is the existence of static thermodynamic equilibrium, irrespective of any invariance under boosts or microscopic statistics. Turning on dissipation, we show that a linear dynamical instability also develops, leading to exponential growth in time of perturbations around equilibrium. Specializing to Galilean superfluids and assuming the existence of bosonic quasiparticles, our criterion reproduces Landau's critical velocity for Bose-Einstein condensates. Our criterion also reduces to the well-known maximal supercurrent in weakly coupled superconductors described either by Landau-Ginzburg or Bardeen-Cooper-Schrieffer theory. Further, it correctly reproduces the onset of the instability in relativistic, strongly coupled superfluids without quasiparticles at zero as well as finite temperature, which we construct using gauge/gravity duality. As a less trivial application of our criterion, we show that in dirty superfluids the instability manifests itself first in the thermal diffusion mode instead of the superfluid sound mode. Our work provides a simple, comprehensive, and unified description of the large superflow instability of superfluids and superconductors at any temperature independent of the microscopic details of the system and the strength of interactions.

DOI: 10.1103/PhysRevD.108.L081903

Introduction. In its simplest incarnation, a superfluid phase of matter is formed when a global U(1) symmetry is spontaneously broken. Superfluids (and associated superconducting phases when the symmetry is local) are found across energy scales in many systems, such as ^{3,4}He [1], quark matter, neutron stars [2], and ultracold atomic gases [3,4], as well as metals at low temperatures [5].

Famously, Landau showed that superfluids are unstable for large enough values of the superflow, $\mathbf{v_s} \equiv \nabla\varphi$, expressed in terms of the Goldstone field φ [1]. Upon using a Galilean transformation to boost the system to the superfluid rest frame, the energy of elementary quasiparticle excitations $\epsilon_q \mapsto \epsilon_q + \mathbf{q}\cdot\mathbf{v_s}$, with $q = |\mathbf{q}|$. For large enough $\mathbf{v_s}$ oriented antiparallel to the wave vector $\mathbf{q}$, the quasiparticle energy in the superfluid rest frame becomes negative, leading to the creation of particles and so to loss of superfluidity. The critical velocity is found by solving the equation $\partial_q(\epsilon_q/q) = 0$, given the quasiparticle energy ϵ_q,

$$v_L \equiv \min_q (\epsilon_q/q). \tag{1}$$

This argument successfully predicts that, at sufficiently large superfluid velocities, excitation of the roton mode will destroy superfluidity in helium 4.[1] In the context of weakly interacting Bose-Einstein condensates, the superfluidity of the system is typically analyzed using the nondissipative Gross-Pitaevskii equation, derived from the microscopic Hamiltonian using a mean-field, Hartree-Fock approximation [3,4] (see [9,10] for experimental reports of the critical velocity in these systems).

[*]blaise.gouteraux@polytechnique.edu
[†]filippo.sottovia@polytechnique.edu
[‡]ericmefford@uvic.ca

[1]Experiments designed explicitly to avoid vortex creation [6,7] match the critical velocity predicted from the roton spectrum. See [8] for an analogous result in helium 3-B.

While very intuitive, (1) rests on the assumption that the low-energy excitations of the system are bosonic quasiparticles and does not apply to fermionic superfluids [11] or to systems without quasiparticles or without invariance under Galilean boosts. Relativistic superfluid phases [12–18] are expected in the quark matter found in neutron stars or compact stars. In the absence of dissipation, they are described by a universal effective field theory [19,20], which matches standard superfluid hydrodynamics [21]. They have also been extensively investigated in the context of gauge/gravity duality applications to strongly correlated condensed matter systems, with systems without long-lived quasiparticles such as high T_c superconductors in mind [22–24]. "Dirty" superfluids, where translations are explicitly broken, are also of interest. The main purpose of this work is to formulate a criterion for the instability of superfluids at large superflow which does not rely on the microscopic details of the system or invariance under boosts and which will therefore be applicable to the wide variety of superfluids found in nature.

Only assuming the existence of thermodynamic equilibrium but no particular invariance under boosts, our main result is to show that a local thermodynamic instability arises when the second derivative of the free energy with respect to the superfluid velocity becomes negative, $\partial^2 f/\partial v_s^2 < 0$, and is accompanied by a dynamical instability, signaled by the crossing of one of the gapless poles of the retarded Green's functions to the upper half complex frequency plane. We formally turn on dissipation by including dissipative gradient corrections in a hydrodynamic approximation. Next, we revisit the case of Galilean superfluids with bosonic quasiparticle excitations and show our criterion reproduces (1). For superconductors, it coincides with the well-known result that an instability develops when the supercurrent is maximal [11]. It also matches previous results in ideal [17,25,26] and dissipative [27,28] relativistic superfluids, the latter of which were constructed using gauge/gravity duality.[2] Finally, we consider a dirty relativistic superfluid with a slowly relaxing normal fluid velocity: there, the instability first appears in the thermal diffusion mode rather than in superfluid fourth sound, contrary to intuition from (1). We give some technical details in the Supplemental Material [34], while the details of our holographic analyses will appear in a companion paper [54].

Some of these results appeared under different forms in previous literature. The connection between the Landau criterion and thermodynamic stability was pointed out for a Galilean superfluid at nonzero temperature in [55,56]. The Landau instability is usually referred to as an energetic instability (the speed of sound vanishes), distinct from a dynamical instability leading to exponential growth in time of perturbations. The link between the Landau instability and a dynamical instability was discussed in [57,58], setting however the temperature to zero and only including a subset of dissipative terms. Here we substantially expand on these previous analyses and place them in a unified perspective, emphasizing that this energetic instability is always accompanied by a dynamical instability and the connection to positivity of entropy production.

Throughout this work, we adopt units where $\hbar = k_B = c = e = 1$.

Superfluid hydrodynamics. The hydrodynamics of superfluids is well known [12–17,59,60]. It is governed by the conservation equations following from invariance under time translations, space translations, and U(1) global transformations,

$$\partial_t \epsilon + \partial_i j_\epsilon^i = 0, \quad \partial_t g^i + \partial_j \tau^{ji} = 0, \quad \partial_t n + \partial_i j^i = 0, \tag{2}$$

together with the Josephson relation $\partial_t \varphi + v_n^i \partial_i \varphi = -\mu$ which follows from gauge invariance. ϵ, n, and g^i are the energy, charge, and momentum densities, j_ϵ and j are the energy and charge currents, and τ is the spatial stress tensor.

The thermodynamics of the system in the grand-canonical ensemble follows from the static partition function expressed as a local functional of the temperature T, the chemical potential μ, and the norm of the superfluid velocity $v_s = |\mathbf{v_s}|$. In order to facilitate the navigation between our different examples, we do not impose any particular boost symmetry at this point (see [35,36,61–63] for previous work on hydrodynamics of fluids without boosts) and work in the laboratory rest frame. The first law of thermodynamics in the laboratory rest frame is

$$d\epsilon = T ds + \mu dn + \mathbf{v_n} \cdot d\mathbf{g} + \mathbf{h} \cdot d\mathbf{v_s}, \tag{3}$$

where s is the entropy density, while $\mathbf{v_n}$ is the normal fluid velocity and $\mathbf{h} \equiv n_s(\mathbf{v_s} - \mathbf{v_n})$ is the conjugate quantity to the superfluid velocity. n_s is the superfluid charge density, which quantifies the fraction of the density that participates in dissipationless superflow.

The local second law is expressed as the divergence of an entropy current $T\partial_t s + T\partial_i(s v_n^i + \tilde{j}_q^i/T) \equiv \Delta \geq 0$, which in combination with (3) gives

$$\Delta = -\tilde{j}_q^i \partial_i T/T - \tilde{j}^i \partial_i \mu - \tilde{\tau}^{ji} \partial_i v_{nj} - \tilde{X} \partial_i h^i, \tag{4}$$

together with the constitutive relations for the expectation values of the currents in the equilibrium thermal, finite density state

[2] Relations between thermodynamic and dynamical instabilities have been discussed previously in the context of gauge/gravity duality [29–33].

$$j^i = n v_n^i + h^i + \tilde{j}^i, \qquad \tau^{ji} = p\delta^{ij} + v_n^i g^j + h^i v_s^j + \tilde{\tau}^{ji},$$
$$j_\epsilon^i = (\epsilon + p) v_n^i - \partial_t \varphi h^i + \tilde{j}_q^i + \mu \tilde{j}^i + v_{nj} \tilde{\tau}^{ji}. \quad (5)$$

The pressure obeys the relation $p = -\epsilon + sT + n\mu + v_n^i g_i$ so that the first law can also be written $dp = sdT + nd\mu + g_i dv_n^i - h_i dv_s^i$. Symmetry under rotations implies that the stress tensor τ^{ij} is symmetric, and so $g^i = \rho v_n^i + h^i$, where ρ is an undetermined function of all thermodynamic parameters.[3] All tilded quantities are dissipative corrections to the ideal order constitutive relations.[4] The Josephson relation is also corrected

$$\partial_t \varphi + v_n^i \cdot \partial_i \varphi = -\mu - \tilde{X}. \quad (6)$$

At ideal order, $\Delta = 0$, but in general, positivity of entropy production requires $\Delta \geq 0$, which provides powerful constraints on the constitutive relations.

We now linearize the equations of motion around an equilibrium state characterized by background values of all thermodynamic quantities and associated sources, $(n, s, \mathbf{g}, \mathbf{v_s}) = (\bar{n}, \bar{s}, \bar{\mathbf{g}}, \bar{\mathbf{v}}_\mathbf{s}) + e^{-i\omega t + i\mathbf{q}\cdot x}(\delta n, \delta s, \delta \mathbf{g}, \delta \mathbf{v_s})$ and $(\mu, T, \mathbf{v_n}, \mathbf{h}) = (\bar{\mu}, \bar{T}, \bar{\mathbf{v}}_\mathbf{n}, \bar{\mathbf{h}}) + e^{-i\omega t + i\mathbf{q}\cdot x}(\delta\mu, \delta T, \delta \mathbf{v_n}, \delta \mathbf{h})$, which are related by a matrix of static susceptibilities, $\chi_{AB} = \delta O_A / \delta s_B = \delta^2 W / \delta s_B \delta s_A$, defined as the variation of the expectation value O_A of operator A with respect to the source s_B of operator B holding other sources fixed, or equivalently, the second variation of the static thermal free energy $W \equiv -T \log Z$ (where Z is the static partition function). For the purposes of linearizing and solving the equations of motion, it is convenient to treat $\mathbf{v_s}$ as a thermal ensemble expectation value and $\mathbf{h}$ as a source. However, from the perspective of the first law (and also for practical applications), it is more convenient to vary the superfluid velocity $\mathbf{v_s}$. We provide the correspondence between the thermodynamic derivatives in these two choices of ensemble in [34], denoting with a tilde the static susceptibilities in the fixed $\mathbf{h}$ ensemble.

Upon linearizing and transforming to Fourier space, the equations of motion take the form

$$\partial_t \delta O_A + \Big(i q v_n \tilde{\chi}_{AB} + \tilde{M}_{AB}(q) \Big) \delta s_B = 0. \quad (7)$$

Due to the nonzero background normal velocity v_n, the fluctuations are dragged at a velocity v_n. Technically, this is an immediate consequence of the terms proportional to v_n in the ideal constitutive relations. If the theory has boost invariance, we can boost to a frame where $v_n = 0$. In the absence of boosts, we can simply consider changing to coordinates $x \to x - v_n t$ and redefining the frequency of perturbations to $\omega \to \tilde{\omega} = \omega - v_n q$. $\tilde{M}$ is a matrix which is expanded order by order in gradients $\tilde{M}(q) = iq\tilde{M}_1 + q^2 \tilde{M}_2 + O(q^3)$, where ideal terms are contained in $\tilde{M}_1$ and dissipative terms at first order in gradients in $\tilde{M}_2$. The $\tilde{M}_i$ are real and do not depend on q.[5]

The spectrum of collective modes is obtained by solving the equation $\det(-i\tilde{\omega} + \tilde{M} \cdot \tilde{\chi}^{-1}) = 0$. Upon increasing the superfluid velocity, an instability can only occur if one of the eigenvalues of the matrix $-i\tilde{\omega} + \tilde{M} \cdot \tilde{\chi}^{-1}$ becomes zero and then changes sign, i.e., $\det(\tilde{M} \cdot \tilde{\chi}^{-1}) = \det \tilde{M} / \det \tilde{\chi} = 0$. $\det \tilde{M}$ cannot vanish. First, we observe that $\det \tilde{M}_1 > 0$. Then, we notice that $\tilde{M}_2$ is related to the quadratic form appearing in the divergence of the entropy current by $\Delta = q^2 \delta s_A (\tilde{M}_2)_{AB} \delta s_B$ (see [34]). Imposing positivity of the quadratic form $\Delta \geq 0$ then implies $\det \tilde{M}_2 > 0$. Thus, for an instability to occur, $\det \tilde{\chi}$ must diverge and change sign. We find this happens when $\chi_{hh} \equiv 1/\tilde{\chi}_{v_s v_s} = n_s - w\chi_{n_s h} = 0$. In [34], we show this in the limit when $\mathbf{v_n}$, $\mathbf{v_s}$, and the wave vector $\mathbf{q}$ are collinear,[6] since the critical velocity is minimized in this limit.

We expect four collective modes (one for each independent fluctuation in the longitudinal sector) with a dispersion relation $\tilde{\omega}_i = c_i q - i\Gamma_i q^2 + O(q^3)$. We can insert this expression in the determinant and solve it order by order in q. $\chi_{hh} = 0$ implies that one of the modes has a vanishing velocity and attenuation, after which it crosses into the upper half plane. The expressions for the modes are rather involved, so we do not reproduce them here, but they are straightforward to obtain given the $\tilde{M}$ and $\tilde{\chi}$ matrices, see [34]. This leads to a perturbation growing exponentially with time and so to a dynamical instability.[7]

The vanishing of χ_{hh} defines the critical superfluid velocity at which the instability develops,[8]

[3]This can also be derived by noticing that $v_s^i = v_n^i + h^i / n_s$ and imposing that the matrix of static susceptibility is Onsager symmetric [34].

[4]We are using a thermodynamic frame [36], which automatically incorporates hydrostatic gradient corrections originating from a gradient expansion of the static partition function into a redefinition of the densities of conserved charges appearing in the equations of motion.

[5]We always use the equations of motion at one order lower to get rid of time derivatives in dissipative terms, which greatly simplifies writing expressions for the modes.

[6]Lifting this assumption presents no conceptual obstacle but explicit expressions become very unwieldy.

[7]Here all velocities are generally nonzero, but it is not necessarily always the case. The argument goes through since the exponential growth in time is caused by the imaginary part changing sign.

[8]After our work appeared, [64] pointed out that, in the absence of boosts, an extra contribution to (3) proportional to $d(\mathbf{v_n} \cdot \mathbf{v_s})$ is needed. Now v_s and v_n no longer appears in h with the same coefficient, which changes the specific expression for χ_{hh} but not the result that the instability is driven by the change of its overall sign.

$$\partial_{v_s}((v_s - v_n)n_s)|_{v_s=v_s^c} = 0. \quad (8)$$

This is our main result. We now proceed to demonstrate that it exactly matches the Landau criterion in Galilean bosonic superfluids, the condition of maximal supercurrent in superconductors [11], and correctly predicts the onset of the instability in holographic superfluids.

Galilean limit and superfluids with quasiparticle excitations. To connect to the Landau criterion, we consider Galilean superfluids in the superfluid rest frame. In the Galilean limit, we impose that $g^i = j^i$ (setting the electron charge and the particle mass $e = m = 1$). Going to the superfluid rest frame involves boosting from the laboratory frame to a frame moving with velocity $\overline{\mathbf{v}_\mathbf{s}}$, parametrized by coordinates $t' = t$, $\mathbf{x}' = \mathbf{x} - \bar{\mathbf{v}}_s t$ with $\partial'_t = \partial_t - \bar{v}_s^i \partial'_i, \partial'_i = \partial_i$. From here on we drop the upper bar on $\mathbf{v}_\mathbf{s}$.

The entropy-producing, dissipative corrections to the ideal order constitutive relations for Galilean superfluids were first written down in [12,13] with an additional dissipative coefficient being identified in [15,16,65]. The gradient corrections are invariant under Galilean boosts and so the conclusions drawn in the laboratory frame continue to hold in the rest frame up to a shift in the velocity, $\omega' = \tilde{\omega} - v_s q$. In particular, as we show in [34], as $\chi_{hh} \to 0$ there is a mode in the rest frame with dispersion

$$\omega_\star = (v_n - v_s)q + \chi_{hh}\left[\hat{v}q - i\hat{\Gamma}q^2 + O(q^3)\right], \quad (9)$$

where $\hat{v}$ and $\hat{\Gamma}$ are nonzero constants. The linear dependence of the attenuation on χ_{hh} confirms that this mode becomes dynamically unstable as χ_{hh} changes sign. There is a subtlety in the superfluid rest frame, however. Using the rest frame identities $\mu_0 = \mu + \mathbf{v}_\mathbf{s} \cdot \mathbf{v}_\mathbf{n} - \mathbf{v}_\mathbf{s}{}^2/2$, $\mathbf{g}_0 = n_n \mathbf{w}$, and $\mathbf{w} \equiv \mathbf{v}_\mathbf{n} - \mathbf{v}_\mathbf{s}$ the relative velocity, the thermodynamics is ignorant of h and $\mathbf{v}_\mathbf{s}$ [59], e.g.,

$$dp = sdT + nd\mu_0 + \mathbf{g}_0 \cdot d\mathbf{w}. \quad (10)$$

To reconcile this, we note that under a Galilean transformation $p \mapsto p$, but $\tilde{p} \mapsto \tilde{p} + \mathbf{h} \cdot \mathbf{v}_s = p$, or similarly, we identify $d\tilde{p} = dp$ for states with $\bar{\mathbf{v}}_s = 0$. This identifies $\tilde{p}$ as the natural thermodynamic ensemble to compare states with the same $\bar{\mathbf{v}}_s$, as we should in the superfluid rest frame. The susceptibilities $\tilde{\chi}$ are constructed from variations of $\tilde{p}$ and, in [34], we show that every entry in $\tilde{\chi}$ diverges at the critical velocity. Hence, it suffices to consider a common microscopic condition for such a divergence.

In the superfluid rest frame, we write the pressure

$$\tilde{p} = -T \int \frac{d^d q}{(2\pi)^d} \ln(1 - e^{-(\epsilon_q - \mathbf{q}\cdot\mathbf{w})/T}). \quad (11)$$

$\epsilon_q(\mu_0, T)$ is the dispersion relation of the quasiparticle excitations [1,17], which we expect to be a smooth function of its parameters, and we have boosted to a frame where the normal fluid composed of the thermal excitations moves relative to the superfluid at velocity $\mathbf{w}$.

It is clear from the form of the pressure that as

$$\mathbf{w} \to \mathbf{w_c} = \min_q \frac{\epsilon_q(\mu_0, T)}{q} \quad (12)$$

the susceptibilities will diverge, e.g.,

$$\lim_{w\to w_c} \tilde{\chi}_{ij} \propto \int \frac{d^d q}{(2\pi)^d} \frac{\mathcal{F}_{ij}(q, \mu_0, T)}{(\epsilon_q - \mathbf{q}\cdot\mathbf{w})^2}. \quad (13)$$

Here, $\mathcal{F}_{ij}$ depends on the particular susceptibility and smoothness of $\epsilon_q(T, \mu_0)$ implies that $\mathcal{F}$ is also smooth near $\mathbf{w}_c$. Hence, (12) exactly reproduces the Landau criterion connecting instabilities of the macroscopic superfluid to microscopic excitations. In [55,56], the critical velocity for which $\chi_{hh} = 0$ was related to the Landau critical velocity defined by (1) in a different thermodynamic ensemble, but only found equality at zero temperature, assuming a phonon-roton quasiparticle dispersion relation. Here we see that we do not need to assume any specific dispersion relation.

Our criterion also holds for weakly coupled superconductors. Indeed, within Bardeen-Cooper-Schrieffer theory, the superfluid current in $d = 3$ peaks at a critical value $v_{s,\max}$ slightly above $v_L = \Delta/q_F$ at low temperatures, which can be seen by an analysis of the free energy assuming the existence of long-lived Cooper pairs [11]. However, depending on the geometry of the Fermi surface, the number of spatial dimensions, or increasing temperature, $v_{s,\max}$ can differ more significantly from v_L. The correct instability condition is in fact $dJ_s/dv_s = 0$ rather than (1), as verified experimentally in ^{3}He [37]. In our conventions, restricting to static superfluids and neglecting dissipative effects, we can define $J_s = h$, so that $v_{s,\max}$ follows from our (8).

Relativistic superfluids. We review relativistic superfluids in [34]. In the zero temperature effective field theory [20], (8) reproduces the known result that the Landau instability onsets when the superfluid velocity exceeds the speed of sound [25,26]. We illustrate the validity of (8) at finite temperatures using a gauge/gravity duality model of a $2+1$-dimensional superfluid [22–24,38–40] with finite background superflow [15,27,66,67], which we outline in [34] and the details of which will appear in a companion paper [54]. The hydrodynamic modes [15,16] correspond to the quasinormal modes of the dual black hole [41–44].

The phase diagram we obtain is depicted in Fig. 1. We restrict ourselves to superfluid velocities antiparallel to the

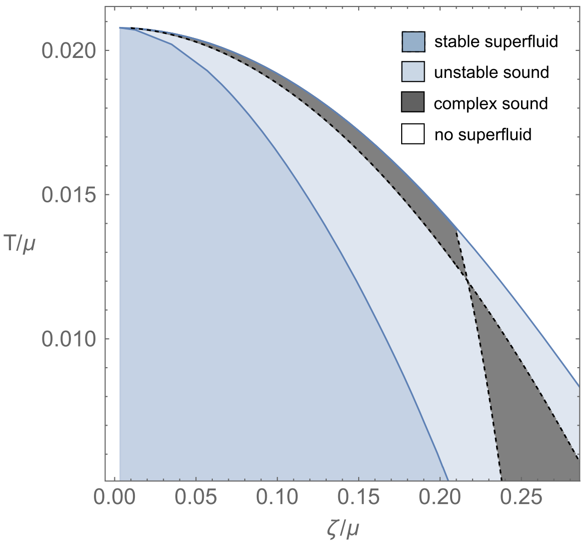

FIG. 1. Phase diagram for a holographic relativistic superfluid, with $\zeta = \sqrt{\zeta_\mu \zeta^\mu}$, $\zeta^\mu = P^{\mu\nu}\partial_\mu\varphi$. The boundaries are obtained from computing the static susceptibilities and matching with the behavior of the quasinormal modes. Starting from the stable phase, the first instability appears as an unstable sound mode at the critical velocity given by (8).

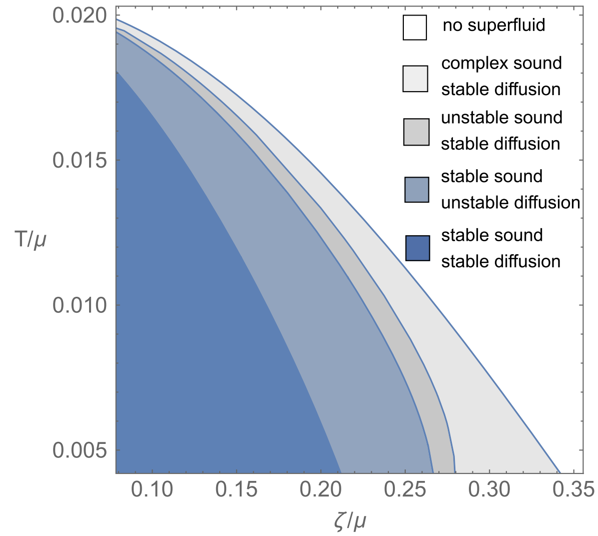

FIG. 2. Phase diagram for a holographic relativistic superfluid with weakly broken translations such that $\Gamma_n = \frac{1}{100} \times \frac{s}{4\pi(\mu n_n + sT)} \ll 1/T$, and $\zeta = \sqrt{\zeta_\mu \zeta^\mu}$, $\zeta^\mu = P^{\mu\nu}\partial_\mu\varphi$. The boundaries are obtained from computing the static susceptibilities and matching with the quasinormal modes. Starting from the stable phase, the first instability appears in the diffusion mode at the critical velocity given by (8).

wave vector, for which the instability is expected to arise for the lowest critical value of the superfluid velocity. At low temperatures and superfluid velocity, all hydrodynamic modes are stable. For fixed temperature, as the superfluid velocity increases, one of the sound modes crosses to the upper half plane, signaling the onset of an instability. The critical value for the superfluid velocity is precisely given by condition (8).

There are also regions where two of the sound modes acquire complex velocities. This has been interpreted previously as a "two-stream" instability [57,58,68].

Dirty superfluids. Next, we study the instability when translations are explicitly broken, which is relevant for dirty superfluids. The momentum of the normal fluid relaxes at a rate $\Gamma_n \ll T$ which enters in the equations of motion as[9]

$$\partial_t g^i + \partial_j \tau^{ji} = -\Gamma_n v_n^i. \tag{14}$$

For simplicity, we assume that the normal fluid has a vanishing background velocity, $\overline{\mathbf{v_n}} = 0$. Then, the spectrum of collective excitations contains two sound modes (usually called fourth sound), one gapped mode $\omega = -i\Gamma_n + O(q)$, and a thermal diffusion mode

$$\omega = -i\frac{s^2\chi_{hh}}{(\chi_{sh}^2 + \chi_{ss}\chi_{hh})\Gamma_n} q^2, \tag{15}$$

which lies in the upper half plane when $\chi_{hh} < 0$.

We illustrate this using a gauge/gravity duality model of superfluids with broken translations based on [45,70] (see [39,46–48,71,72] for previous investigations of such models). Going through the same exercise as in the translation-invariant case, we produce the phase diagram in Fig. 2. Our results confirm that the leading instability is given by the condition (8), upon which the thermal diffusion mode crosses to the upper half plane. As we further increase the superfluid velocity, this mode crosses back into the lower half plane, and instead one of the sound modes becomes unstable. This happens when $\tilde{\chi}_{ss} = \chi_{ss} + \chi_{sh}^2/\chi_{hh}$ vanishes. This is allowed since in this region $\chi_{hh} < 0$. Increasing further the superfluid velocity, both sound modes acquire complex velocities.

Discussion and outlook. In this work, we have demonstrated that superfluids are linearly dynamically unstable whenever the superfluid velocity becomes too large, and that this instability is of thermodynamic origin: one of the eigenvalues of the matrix of static susceptibility changes sign. Further, we have shown that this coincides with the Landau criterion for the critical velocity when we assume invariance under Galilean boosts and the existence of quasiparticles.

As advertised, our instability criterion does not rely on the nature of the microscopic details of the system or the

[9]A more careful analysis along the lines of [69] is warranted, but the extra dissipative transport coefficients there are subleading compared to the effect of Γ_n.

strength of the coupling between them, although of course these are implicitly contained in the specific dependence of the static partition function on thermodynamic parameters. It is equally valid whether the end point of the instability is the excitation of rotons or the nucleation of vortices. It would be interesting to connect it to super-Landau critical velocities reported in experiments with moving macroscopic defects [73] or other mechanisms [74,75].

A corollary of our analysis (under the simplifying assumption of collinearity) is that local thermodynamic stability together with positivity of entropy production is sufficient to guarantee linear dynamical stability: all hydrodynamic modes lie in the lower half complex frequency plane. We will expand on this in [76].

Acknowledgments. We are grateful to Andreas Schmitt and Benjamin Withers for discussions. The work of B. G. and F. S. was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant Agreement No. 758759). The work of E. M. was supported in part by NSERC and in part by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant Agreement No. 758759). We all gratefully acknowledge Nordita's hospitality during the Nordita program "Recent developments in strongly correlated quantum matter" where part of this work was carried out.

[1] L. Landau, E. Lifshitz, and L. Pitaevskii, *Course of Theoretical Physics: Statistical Physics, Part 2: by E.M. Lifshitz and L.P. Pitaevskii* (Pergamon, New York, 1980), Vol. 9.

[2] D. Page and S. Reddy, Dense matter in compact stars: Theoretical developments and observational constraints, Annu. Rev. Nucl. Part. Sci. **56**, 327 (2006).

[3] A. J. Leggett, Bose-Einstein condensation in the alkali gases: Some fundamental concepts, Rev. Mod. Phys. **73**, 307 (2001).

[4] C. J. Pethick and H. Smith, *Bose–Einstein Condensation in Dilute Gases*, 2nd ed. (Cambridge University Press, Cambridge, England, 2008), 10.1017/CBO9780511802850.

[5] A. J. Leggett, *Quantum Liquids: Bose Condensation and Cooper Pairing in Condensed-Matter Systems* (Oxford University Press, New York, 2006).

[6] L. Meyer and F. Reif, Ion motion in superfluid liquid helium under pressure, Phys. Rev. **123**, 727 (1961).

[7] P. McClintock, Ions in superfluid helium, Physica (Amsterdam) **127B+C**, 300 (1984).

[8] A. I. Ahonen, J. Kokko, O. V. Lounasmaa, M. A. Paalanen, R. C. Richardson, W. Schoepe, and Y. Takano, Mobility of negative ions in superfluid ^{3}He, Phys. Rev. Lett. **37**, 511 (1976).

[9] C. Raman, M. Köhl, R. Onofrio, D. S. Durfee, C. E. Kuklewicz, Z. Hadzibabic, and W. Ketterle, Evidence for a critical velocity in a Bose-Einstein condensed gas, Phys. Rev. Lett. **83**, 2502 (1999).

[10] R. Onofrio, C. Raman, J. M. Vogels, J. R. Abo-Shaeer, A. P. Chikkatur, and W. Ketterle, Observation of superfluid flow in a Bose-Einstein condensed gas, Phys. Rev. Lett. **85**, 2228 (2000).

[11] J. Bardeen, Critical fields and currents in superconductors, Rev. Mod. Phys. **34**, 667 (1962).

[12] A. J. Clark, On the hydrodynamics of superfluid helium, Ph.D. thesis, Massachusetts Institute of Technology, 1963.

[13] S. J. Putterman, *Superfluid Hydrodynamics* (North-Holland/American Elsevier, London, 1974), Vol. 3.

[14] I. M. Khalatnikov and I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity*, Frontiers in Physics (Benjamin, New York, NY, 1965) translated from Russian.

[15] J. Bhattacharya, S. Bhattacharyya, and S. Minwalla, Dissipative superfluid dynamics from gravity, J. High Energy Phys. 04 (2011) 125.

[16] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and A. Yarom, A theory of first order dissipative superfluid dynamics, J. High Energy Phys. 05 (2014) 147.

[17] A. Schmitt, *Introduction to Superfluidity: Field-Theoretical Approach and Applications* (Springer, Cham, Switzerland, 2015), Vol. 888, 10.1007/978-3-319-07947-9.

[18] L. V. Delacrétaz, D. M. Hofman, and G. Mathys, Superfluids as higher-form anomalies, SciPost Phys. **8**, 047 (2020).

[19] B. Carter and D. Langlois, The equation of state for cool relativistic two constituent superfluid dynamics, Phys. Rev. D **51**, 5855 (1995).

[20] D. T. Son, Low-energy quantum effective action for relativistic superfluids, arXiv:hep-ph/0204199.

[21] A. Nicolis, Low-energy effective field theory for finite-temperature relativistic superfluids, arXiv:1108.2513.

[22] S. S. Gubser, Breaking an Abelian gauge symmetry near a black hole horizon, Phys. Rev. D **78**, 065034 (2008).

[23] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, Building a holographic superconductor, Phys. Rev. Lett. **101**, 031601 (2008).

[24] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, Holographic superconductors, J. High Energy Phys. 12 (2008) 015.

[25] M. G. Alford, S. K. Mallavarapu, A. Schmitt, and S. Stetina, From a complex scalar field to the two-fluid picture of superfluidity, Phys. Rev. D **87**, 065001 (2013).

[26] I. Kourkoulou, A. Nicolis, and K. Parmentier, Low-temperature thermal corrections to a superfluid's equation of state, arXiv:2212.12555.
[27] I. Amado, D. Areán, A. Jiménez-Alba, K. Landsteiner, L. Melgar, and I. Salazar Landea, Holographic superfluids and the Landau criterion, J. High Energy Phys. 02 (2014) 063.
[28] S. Lan, H. Liu, Y. Tian, and H. Zhang, Landau instability and soliton formations, arXiv:2010.06232.
[29] S. S. Gubser and I. Mitra, Instability of charged black holes in anti-de Sitter space, Clay Math. Proc. **1**, 221 (2002), https://inspirehep.net/literature/533658.
[30] S. S. Gubser and I. Mitra, The evolution of unstable black holes in anti-de Sitter space, J. High Energy Phys. 08 (2001) 018.
[31] J. J. Friess, S. S. Gubser, and I. Mitra, Counterexamples to the correlated stability conjecture, Phys. Rev. D **72**, 104019 (2005).
[32] A. Buchel, A holographic perspective on Gubser-Mitra conjecture, Nucl. Phys. **B731**, 109 (2005).
[33] A. Donos, J. P. Gauntlett, and V. Ziogas, Diffusion for holographic lattices, J. High Energy Phys. 03 (2018) 056.
[34] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.108.L081903 for critical superflows and thermodynamic instabilities in superfluids, which includes Refs. [11,15–17,20,21,26,35–53].
[35] I. Novak, J. Sonner, and B. Withers, Hydrodynamics without boosts, J. High Energy Phys. 07 (2020) 165.
[36] J. Armas and A. Jain, Effective field theory for hydrodynamics without boosts, SciPost Phys. **11**, 054 (2021).
[37] J. P. Eisenstein, G. W. Swift, and R. E. Packard, Observations of a critical current in ^{3}He-*b*, Phys. Rev. Lett. **43**, 1676 (1979).
[38] I. Amado, M. Kaminski, and K. Landsteiner, Hydrodynamics of holographic superconductors, J. High Energy Phys. 05 (2009) 021.
[39] D. Arean, M. Baggioli, S. Grieninger, and K. Landsteiner, A holographic superfluid symphony, J. High Energy Phys. 11 (2021) 206.
[40] A. Donos and P. Kailidis, Dissipative effects in finite density holographic superfluids, J. High Energy Phys. 11 (2022) 053.
[41] G. Policastro, D. T. Son, and A. O. Starinets, From AdS/CFT correspondence to hydrodynamics, J. High Energy Phys. 09 (2002) 043.
[42] D. T. Son and A. O. Starinets, Minkowski space correlators in AdS/CFT correspondence: Recipe and applications, J. High Energy Phys. 09 (2002) 042.
[43] G. Policastro, D. T. Son, and A. O. Starinets, From AdS/CFT correspondence to hydrodynamics. 2. Sound waves, J. High Energy Phys. 12 (2002) 054.
[44] C. P. Herzog, The hydrodynamics of M theory, J. High Energy Phys. 12 (2002) 026.
[45] T. Andrade and B. Withers, A simple holographic model of momentum relaxation, J. High Energy Phys. 05 (2014) 101.
[46] T. Andrade and S. A. Gentle, Relaxed superconductors, J. High Energy Phys. 06 (2015) 140.
[47] B. Goutéraux and E. Mefford, Normal charge densities in quantum critical superfluids, Phys. Rev. Lett. **124**, 161604 (2020).
[48] B. Goutéraux and E. Mefford, Non-vanishing zero-temperature normal density in holographic superfluids, J. High Energy Phys. 11 (2020) 091.
[49] V. Balasubramanian and P. Kraus, A stress tensor for anti-de Sitter gravity, Commun. Math. Phys. **208**, 413 (1999).
[50] K. Skenderis, Lecture notes on holographic renormalization, Classical Quantum Gravity **19**, 5849 (2002).
[51] J. Sonner and B. Withers, A gravity derivation of the Tisza-Landau model in AdS/CFT, Phys. Rev. D **82**, 026001 (2010).
[52] P. K. Kovtun and A. O. Starinets, Quasinormal modes and holography, Phys. Rev. D **72**, 086009 (2005).
[53] O. J. C. Dias, J. E. Santos, and B. Way, Numerical methods for finding stationary gravitational solutions, Classical Quantum Gravity **33**, 133001 (2016).
[54] B. Goutéraux, E. Mefford, and F. Sottovia, Holographic superfluids with finite superfluid velocities (to be published).
[55] A. F. Andreev and L. A. Melnikovsky, Thermodynamic inequalities in a superfluid, J. Exp. Theor. Phys. Lett. **78**, 574 (2003).
[56] A. F. Andreev and L. A. Melnikovsky, Thermodynamics of superfluidity, J. Low Temp. Phys. **135**, 411 (2004).
[57] A. Haber, A. Schmitt, and S. Stetina, Instabilities in relativistic two-component (super)fluids, Phys. Rev. D **93**, 025011 (2016).
[58] N. Andersson and A. Schmitt, Dissipation triggers dynamical two-stream instability, Particles **2**, 457 (2019).
[59] L. Landau and E. Lifshitz, *Fluid Mechanics*, 2nd ed., edited by L. Landau and E. Lifshitz (Pergamon, New York, 1987).
[60] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, England, 1995), 10.1017/CBO9780511813467.
[61] J. de Boer, J. Hartong, N. A. Obers, W. Sybesma, and S. Vandoren, Perfect fluids, SciPost Phys. **5**, 003 (2018).
[62] J. de Boer, J. Hartong, N. A. Obers, W. Sybesma, and S. Vandoren, Hydrodynamic modes of homogeneous and isotropic fluids, SciPost Phys. **5**, 014 (2018).
[63] J. de Boer, J. Hartong, E. Have, N. A. Obers, and W. Sybesma, Non-boost invariant fluid dynamics, SciPost Phys. **9**, 018 (2020).
[64] J. Armas and E. Have, Ideal fracton superfluids, arXiv:2304.09596.
[65] N. Banerjee, S. Dutta, and A. Jain, First order Galilean superfluid dynamics, Phys. Rev. D **96**, 065004 (2017).
[66] C. P. Herzog, An analytic holographic superconductor, Phys. Rev. D **81**, 126009 (2010).
[67] D. Arean, M. Bertolini, C. Krishnan, and T. Prochazka, Type IIB holographic superfluid flows, J. High Energy Phys. 03 (2011) 008.
[68] A. Schmitt, Superfluid two-stream instability in a microscopic model, Phys. Rev. D **89**, 065024 (2014).

[69] J. Armas, A. Jain, and R. Lier, Approximate symmetries, pseudo-Goldstones, and the second law of thermodynamics, Phys. Rev. D **108**, 086011 (2023).

[70] A. Donos and J. P. Gauntlett, Holographic Q-lattices, J. High Energy Phys. 04 (2014) 040.

[71] Y. Ling, P. Liu, C. Niu, J.-P. Wu, and Z.-Y. Xian, Holographic superconductor on Q-lattice, J. High Energy Phys. 02 (2015) 059.

[72] K.-Y. Kim, K. K. Kim, and M. Park, A simple holographic superconductor with momentum relaxation, J. High Energy Phys. 04 (2015) 152.

[73] D. I. Bradley, S. N. Fisher, A. M. Guénault, R. P. Haley, C. R. Lawson, G. R. Pickett, R. Schanen, M. Skyba, V. Tsepelin, and D. E. Zmeev, Breaking the superfluid speed limit in a fermionic condensate, Nat. Phys. **12**, 1017 (2016).

[74] A. Paris-Mandoki, J. Shearring, F. Mancarella, T. M. Fromhold, A. Trombettoni, and P. Krüger, Superfluid flow above the critical velocity, Sci. Rep. **7**, 9070 (2017).

[75] S. Autti, J. T. Mäkinen, J. Rysti, G. E. Volovik, V. V. Zavjalov, and V. B. Eltsov, Exceeding the Landau speed limit with topological Bogoliubov Fermi surfaces, Phys. Rev. Res. **2**, 033013 (2020).

[76] B. Goutéraux, E. Mefford, and F. Sottovia, Linear dynamical stability and positivity of entropy production (to be published).