Letter

Algorithm for symbol integrations for loop integrals

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We derive an algorithm for computing the total differentials of multiloop integrals expressed as onefold integrals of multiple polylogarithms, which can involve square roots of polynomials up to degree 4 and may evaluate to (elliptic) multiple polylogarithms [(e)MPLs]. This gives simple algebraic rules for computing the $(W-1,1)$ coproduct of the resulting weight-W functions up to period terms, and iterating it gives the symbol without actually performing any integration. In particular, our algorithm generalizes existing MPL integration rules and sidesteps the complicated rationalization procedure in the presence of square roots. We apply our algorithm to conformal double-D-gon integrals in D dimensions with generic kinematics and possibly massive circumferential propagators. We directly compute, for the first time, the total differential and symbol (up to period terms) of the $D=3$ double triangle and the $D=4$ double box, which in the special case with massless propagators represent the first appearance of eMPL functions in (two-loop) scattering amplitudes of Aharony-Bergman-Jafferis-Maldacena theory and $\mathcal{N}=4$ super-Yang-Mills theory, respectively.

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I. INTRODUCTION

The key for precise predictions in perturbative quantum field theory (QFT) lies in the analytic computation of Feynman integrals, which often reveals rich and unexpected structures of QFT itself. At least for simple kinematics, a systematic method to compute (dimensionally regularized) Feynman integrals is via differential equations [1–5]. For complicated kinematics, other than direct integration [6–15], it is often possible to bootstrap a Feynman integral [16–20] once we have control over its analytic structure.

The simplest class of functions that Feynman integrals evaluate to are multiple polylogarithms (MPLs) [21–26]. Their analytic structure is well understood due to powerful mathematical tools such as the "symbol" and the more general "coproduct" [27–31], which manifest singularity structures and trivialize function identities. More complicated Feynman integrals evaluate to more complicated functions (see Ref. [32] and references therein), the simplest case involving elliptic multiple polylogarithms (eMPLs) [33–59], for which one can define the symbol as well [48]. For MPLs and eMPLs alike, the symbol maps a complicated function to a tensor of simpler objects, the "symbol letters." It is defined recursively by total differentials [or, equivalently, $(W-1,1)$ coproducts],

$$\mathrm{d}\mathcal{I}=\sum_\alpha \mathcal{I}_\alpha \mathrm{d}w_\alpha \Rightarrow \mathcal{S}(\mathcal{I})=\sum_\alpha \mathcal{S}(\mathcal{I}_\alpha)\otimes w_\alpha, \quad (1)$$

where $\mathcal{I}$ and $\mathcal{I}_\alpha$ have "transcendental weight" W and $(W-1)$. The eMPL letters w_α are onefold integrals of rational functions on genus-one curves, while MPL letters are their genus-zero degenerations, namely, logarithms.

In this Letter, we propose an algorithm for the direct computation of the symbol of MPLs and eMPLs expressed as onefold integrals $\mathcal{I}=\int F(t)\mathrm{d}t$ of MPLs $F(t)$, which applies to a large class of Feynman integrals [60]. Algorithms exist [61] that compute $\mathrm{d}\mathcal{I}$ and iteratively $\mathcal{S}(\mathcal{I})$ in terms of $\mathcal{S}(F(t))$, as long as singularities of $F(t)$ involve linear factors of t only. However, it was previously unknown how to perform such "symbol integrations" when singularities of $F(t)$ involve square roots of polynomials of t. We take an important step in solving this

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long-standing problem by deriving algebraic rules for $\mathrm{d}\mathcal{I}$ and iteratively $\mathcal{S}(\mathcal{I})$, given $\mathrm{d}F(t)$. Our method sidesteps rationalization and gives the MPL symbol in the presence of square roots of quadratic polynomials. In the presence of square roots of cubics/quartics, it computes the eMPL symbol up to "period terms" where $w_\alpha = \tau$, the modular parameter of the elliptic curve. The restriction to nonperiod terms, which is also the goal of elliptic symbol bootstrap [59], is often convenient in the study of eMPL symbols, since the period terms can be formally reconstructed via the "symbol prime" [57].

We apply our new method to the symbol integration of an important class of conformal integrals, double-D-gon integrals in D dimensions [62,63] (see also Refs. [64–66]), which are weight D and can be expressed as onefold integrals of deformed $2(D-1)$-gons [29,67–70]. We compute their total differentials, or $(D-1,1)$ coproducts, even in the presence of massive circumferential propagators. For the most general $D=3$ double triangle and $D=4$ double box, up to period terms, we obtain all last entries w_α, as well as the (symbol of) accompanying weight-$(D-1)$ integrals which evaluate to MPLs. In the special case with massless propagators, they reduce to the first eMPL contributions to scattering amplitudes in Aharony-Bergman-Jafferis-Maldacena (ABJM) and $\mathcal{N}=4$ super-Yang-Mills theory, respectively. For higher D, these weight-$(D-1)$ integrals involve elliptic or even higher-genus curves, and we leave their explicit computation to future work.

II. DERIVING RULES FOR (ELLIPTIC) SYMBOL INTEGRATIONS

A. 2-forms and MPL symbol integrations without rationalization

Before studying elliptic integrals, let us first derive the symbol integration for MPL functions. We use $\delta \coloneqq \delta u \partial_u$ for the differential with respect to variables $\{u\}$ parametrizing the kinematic space $\mathcal{K}$, to distinguish it from the differential $\mathrm{d} \coloneqq \mathrm{d}t\partial_t$ with respect to the integration variable t. It is helpful to consider the big space $\mathcal{M}$ parametrized by $\{u,t\}$ with total differential operator $D=\delta+\mathrm{d}$. Differential forms $\Omega^p(\mathcal{M}) = \bigoplus_{r=0}^{p}\Omega^{r,p-r}(\mathcal{M})$ on $\mathcal{M}$ are graded into a bicomplex by δ and d,

$$\Omega^{r+1,p-r}(\mathcal{M}) \xleftarrow{\mathrm{d}} \Omega^{r,p-r}(\mathcal{M}) \xrightarrow{\delta} \Omega^{r,p-r+1}(\mathcal{M}),$$

with $\Omega^{r,p-r}(\mathcal{M}) = \varnothing$ for $r\geq 2$ since there is only one t variable. Importantly, each kinematic point $\{u\}$ locates a Riemann t-sphere in $\mathcal{M}$, and a (1,1)-form can be viewed as an $\Omega^1(\mathcal{K})$-valued 1-form on the sphere. The line integral operator $\int_{a(u)}^{b(u)}: \Omega^1(S^2)\to\mathbb{C}$ can be extended to a linear map $\Omega^1(S^2)\otimes\Omega^1(\mathcal{K})\to\Omega^1(\mathcal{K})$, which defines an integration of (1, 1)-forms.

To warm up, consider the total differential $\delta\mathcal{T}(u)$ of

$$\mathcal{T}(u) = \int_{a(u)}^{b(u)} F(t;u)\mathrm{d}\log(t+c(u)), \tag{2}$$

where $DF(t)=H(t)D\log(t+d)$ is known [71]. Integrating by parts, $\delta\mathcal{T}$ has boundary contributions that are trivial to compute, as well as integral terms,

$$\delta\mathcal{T} = F(b)\delta\log(b+c) - F(a)\delta\log(a+c) + \int_a^b H(t)\omega^{(1,1)}. \tag{3}$$

Here, $\omega^{(1,1)}$ is the (1, 1) component of the 2-form $\omega \coloneqq D\log(t+c)\wedge D\log(t+d)$. To obtain the symbol integration rule, we need only separate the t dependence of $\omega^{(1,1)}$. This is done purely algebraically by matching residues, because $\omega^{(1,1)}\in\Omega^1(S^2)\otimes\Omega^1(\mathcal{K})$ is a meromorphic 1-form on the t-sphere, which is determined by residues. Matching the residues at $t=-c$ and $t=-d$,

$$\omega^{(1,1)} = \mathrm{d}\log\frac{t+c}{t+d}\wedge\delta\log(c-d). \tag{4}$$

This way, we obtain the integral term contribution to $\delta\mathcal{T}$,

$$\int_a^b H(t)\omega^{(1,1)} = \left(\int_a^b H(t)\mathrm{d}\log\frac{t+c}{t+d}\right)\delta\log(c-d). \tag{5}$$

By definition, the above rule computes the $(W-1,1)$ coproduct of the weight-W function $\mathcal{T}$, and iterating it yields the well-known symbol integration rule for linear symbol entries [61,72].

Now we move to MPL symbol integrations involving square roots of quadratic polynomials, which usually requires rationalization and gets complicated when there are multiple square roots [73]. We show that no explicit rationalization is needed from the 2-form perspective, and the method can be readily extended to elliptic cases. Our prototype is the integral

$$\mathcal{T} = \int_a^b F(t)\mathrm{d}\log r(t), \quad DF(t) = H(t)D\log r_\Delta(t), \tag{6}$$

where $r(t) = \frac{A(t)+\sqrt{R(t)}}{A(t)-\sqrt{R(t)}}$ and $r_\Delta(t) = \frac{B(t)+\sqrt{R(t)}\sqrt{\Delta(t)}}{B(t)-\sqrt{R(t)}\sqrt{\Delta(t)}}$. Here, $A(t)$, $B(t)$, and even $R(t)$ can be arbitrary polynomials of t, but, crucially, $\Delta(t)$ is quadratic. Again, boundary terms are trivial to compute, and the key is to separate the t dependence of $\omega^{(1,1)}$, where $\omega \coloneqq D\log r(t)\wedge D\log r_\Delta(t)$. Note that it is parity-even under $\sqrt{R(t)}\to -\sqrt{R(t)}$. Hence, it is single valued near $R(t)=0$ despite the apparent dependence on $\sqrt{R(t)}$, and the only branch points appear at $\Delta(t)=0$. Therefore, $\tilde{\omega}^{(1,1)} \coloneqq \sqrt{\Delta(t)}\omega^{(1,1)}$

is single-valued and meromorphic on the t-sphere. By matching residues of $\tilde{\omega}^{(1,1)}$, we obtain

$$\omega^{(1,1)} = \sum_{t_0 \in \{\text{poles of } \tilde{\omega}\}} \frac{\sqrt{\Delta(t_0)}\mathrm{d}t}{(t-t_0)\sqrt{\Delta(t)}} \wedge \operatorname*{Res}_{t-t_0=0} \omega. \quad (7)$$

We immediately obtain the integration rule in the same way as the linear-entry case,

$$\begin{aligned}\delta\mathcal{T} = &\sum_{t_0 \in \{\text{poles of } \tilde{\omega}\}} \left(\int_a^b H(t) \frac{\sqrt{\Delta(t_0)}\mathrm{d}t}{(t-t_0)\sqrt{\Delta(t)}} \right) \operatorname*{Res}_{t-t_0=0} \omega \\ &+ F(b)\delta \log r(b) - F(a)\delta \log r(a), \end{aligned} \quad (8)$$

where the integration kernel can be nicely written as a d log form since $\Delta(t)$ is quadratic, facilitating further iterations. A similar reasoning shows that the kernel becomes $\mathrm{d}\log(t-t_0)$ when there is no “net” square root $\sqrt{\Delta(t)}$ in the 2-form ω.

Since the square root $\sqrt{\Delta(t)}$ is carried along in our rules of symbol integration, no explicit rationalization (or any related subtleties [72]) is involved. Moreover, our method generalizes existing ones and applies whenever the net square root $\sqrt{\Delta(t)}$ of ω has quadratic $\Delta(t)$. The organization of results is nicely suited for analyzing symbol structures of Feynman integrals. The “parity” of every square root is manifest, and a basis of independent last entries is obtained after only one iteration.

We have applied (8) to various two-loop MPL integrals with square roots of quadratic polynomials, previously computed only through canonical differential equations, such as the five-mass double box [Fig. 1(a), I_{17} in [20]] and the four-point double box with equal circumferential masses [Fig. 1(b), g_{10} in [74]]. Starting from the deformed hexagon representation (see the Appendix), we reproduce their symbols with very little work.

B. Elliptic symbol integrations

Next we consider the simplest elliptic integrals, where the prototype involves an elliptic curve $\mathcal{E} = \{(t,y) | y^2 = P(t)\} \subseteq \mathbb{CP}^2$ and $P(t)$ is an irreducible cubic or quartic polynomial,

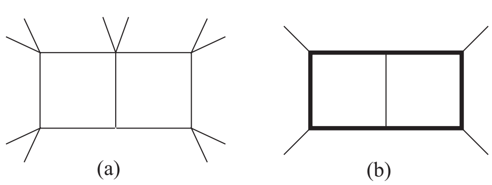


FIG. 1. Examples of MPL double-boxes, which depend on (a) 5 and (b) 2 kinematic variables.

$$\mathcal{I} = \int_a^b F(t) \frac{\mathrm{d}t}{y(t)}, \qquad DF(t) = H(t) D \log r'_\Delta(t), \quad (9)$$

where $r'_\Delta(t) \coloneqq \frac{B(t)+y(t)\sqrt{\Delta(t)}}{B(t)-y(t)\sqrt{\Delta(t)}}$, where $B(t)$ is an arbitrary polynomial and $\Delta(t)$ is quadratic. The key difference from MPL cases is that the integration kernel is no longer a d log form, but we can still write it as a total differential $\frac{\mathrm{d}t}{y(t)} = \mathrm{d}W(t)$ with $W(t) \coloneqq \int_*^t \frac{\mathrm{d}t'}{y(t')}$ for any reference point $*$. As in the MPL case, we obtain

$$\delta\mathcal{I} = F(b)\delta W(b) - F(a)\delta W(a) + \int_a^b H(t)\omega^{(1,1)}. \quad (10)$$

There is an ambiguity in the definition of $W(t)$ even after fixing an initial point $*$. Because the natural domain of the integrand $\mathrm{d}t'/y(t')$ is topologically a torus, we can freely add any multiple of two independent cycles $\gamma_{1,2}$ to the contour, leading to definitions of $W(t)$ that differ by multiples of $\omega_{1,2} = \oint_{\gamma_{1,2}} \mathrm{d}t/y(t)$. Practically, after performing a birational change of variables $(t,y) \mapsto (T,Y)$ to put $\mathcal{E}$ into Weierstrass form $Y^2 = 4T^3 - g_2 T - g_3$, we choose $W(t) = \wp^{-1}(T; g_2, g_3)$ for some branch of $\wp^{-1}$. We also renormalize $\mathcal{I}$ and $W(t)$ with ω_1 to obtain a pure function: $\mathcal{T} \coloneqq \frac{1}{\omega_1}\mathcal{I}$, $w(t) \coloneqq \frac{1}{\omega_1}W(t)$, and $\delta\mathcal{T}$ is of the same form (10) as $\delta\mathcal{I}$, except $W(t) \to w(t)$.

We wish to algebraically separate the t dependence of the (1,1) component of the 2-form $\omega \coloneqq Dw(t) \wedge D \log r'_\Delta(t)$. Since ω is parity-even under $y(t) \to -y(t)$, the net square root is $\sqrt{\Delta(t)}$, and defining $\tilde{\omega}^{(1,1)} \coloneqq \sqrt{\Delta(t)}\omega^{(1,1)}$ eliminates the branch points at $\Delta(t) = 0$. However, unlike MPL cases where $\tilde{\omega}^{(1,1)}$ is rational, the presence of $w(t) = \wp^{-1}(T)/\omega_1$ introduces extra branch cuts. Discontinuities of $\delta w(t)$ across the branch cuts are proportional to $\delta\tau$, where $\tau \coloneqq \omega_2/\omega_1$. Ultimately, the reason is that genus-one curves have nontrivial moduli that depend on the kinematics.

It is not clear how to proceed directly, so we follow the proposal in [59] and get around this problem by restricting to the subspace $\underline{\mathcal{K}}$ of $\mathcal{K}$ defined by $\delta\tau = 0$. In other words, we focus on the elliptic symbol/coproduct up to period terms containing τ. Notationally, we use $\underline{\delta}$ to indicate the differential operator on $\underline{\mathcal{K}}$, and $\underline{D} = \mathrm{d} + \underline{\delta}$. The restricted (1, 1) component

$$\tilde{\omega}^{(1,1)}|_{\underline{\mathcal{K}}} = \sqrt{\Delta(t)}[\underline{D}w(t) \wedge \underline{D} \log r'_\Delta(t)]^{(1,1)} \quad (11)$$

is indeed an $\Omega^1(\underline{\mathcal{K}})$-valued meromorphic 1-form on the t-sphere. Incidentally, the restriction frees us from explicitly specifying the branch of $\wp^{-1}$ when defining $w(t)$.

We can now determine $\omega^{(1,1)}|_{\underline{\mathcal{K}}}$ by matching residues of $\tilde{\omega}^{(1,1)}|_{\underline{\mathcal{K}}}$, and the residue computation is surprisingly easy: since $\mathrm{d}w(t)$ is holomorphic, the first term does not contribute at all, and all contributions come from singularities of

$\log r'_\Delta(t)$. Denoting such singularities as $t_\pm$, which satisfy $B(t_\pm) \mp y(t_\pm)\sqrt{\Delta(t_\pm)} = 0$, we have

$$\omega^{(1,1)}|_{\underline{\mathcal{K}}} = \sum_{t_\pm} \pm \frac{\sqrt{\Delta(t_\pm)}\mathrm{d}t}{(t-t_\pm)\sqrt{\Delta(t)}} \wedge \underline{\delta} w(t_\pm), \quad (12)$$

which gives the final result,

$$\underline{\delta}\mathcal{T} = \sum_{t_\pm} \pm \left(\int_a^b H(t) \frac{\sqrt{\Delta(t_\pm)}\mathrm{d}t}{(t-t_\pm)\sqrt{\Delta(t)}} \right) \underline{\delta} w(t_\pm) + F(b)\underline{\delta}w(b) - F(a)\underline{\delta}w(a). \quad (13)$$

III. APPLICATION TO DOUBLE-*D*-GON INTEGRALS IN *D* DIMENSIONS

A. Double-triangle integrals in $D=3$

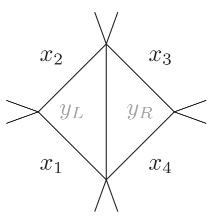
Consider the $D=3$ double triangle, which can be represented as the integral of a deformed box (A7),

$$\mathcal{I}_3 = \int_0^\infty \frac{\mathrm{d}s}{\sqrt{-\mathcal{Q}(s^2)}} \langle\langle Q(s^2) \rangle\rangle, \quad (14)$$

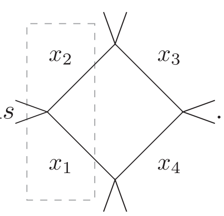
where we have performed a change of variable $t = s^2$ to get rid of the $\sqrt{t}$ in the denominator. The notation $\langle\langle Q(s^2)\rangle\rangle$ denotes the pure function (A8) defined by the quadric $Q(s^2)$, and $\mathcal{Q}(s^2) := \det Q(s^2)$.

We first consider the special case with massless propagators, which depends on conformal cross ratios $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ with $x_{ij}^2 := (x_i - x_j)^2$,

$$\mathcal{I}_3 =$$



$$= \int_0^\infty \mathrm{d}s$$



$. \quad (15)$

The dashed box indicates deformation (A6), and after some rescalings using projectivity, the quadric reads

$$Q(s^2) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & v & 1 \\ 1 & v & 0 & (1+s^2)u \\ 1 & 1 & (1+s^2)u & 0 \end{pmatrix}. \quad (16)$$

As usual, introduce $z, \bar{z} = \frac{1}{2}(1 + (1+s^2)u - v \pm \sqrt{\Delta(s)})$ and $\Delta(s) = (1-(1+s^2)u - v)^2 - 4(1+s^2)uv$ such that $(1+s^2)u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$. Then, $\langle\langle Q(s^2)\rangle\rangle$ is precisely the (deformed) four-mass box function [75],

$$\langle\langle Q(s^2)\rangle\rangle = \log\frac{z\bar{z}}{v}\log\frac{1-z}{1-\bar{z}} - 2\mathrm{Li}_2\left(\frac{-z}{1-z}\right) + 2\mathrm{Li}_2\left(\frac{-\bar{z}}{1-\bar{z}}\right). \quad (17)$$

The integral $\mathcal{I}_3$ is elliptic, involving the curve $y^2 = -\mathcal{Q}(s^2)$. Define $\mathcal{T}_3 := \mathcal{I}_3/\omega_1$. It can be shown [76] that $w(0), w(\infty) = 0 \bmod \langle 1,\tau\rangle/2$, where $\langle 1,\tau\rangle/2 := \frac{1}{2}\mathbb{Z} + \frac{1}{2}\mathbb{Z}\tau$ is the lattice generated by $1,\tau$ together with half lattice points. Therefore, $\underline{\delta}\mathcal{T}_3$ has no contribution from the boundary terms. Note that the last entries of $\mathcal{S}(\langle\langle Q(s^2)\rangle\rangle)$ and the kernel $\frac{\mathrm{d}s}{\sqrt{-\mathcal{Q}(s^2)}} = i\frac{\mathrm{d}s}{\sqrt{\Delta(s)}}$ are both odd under $\sqrt{\Delta(s)} \to -\sqrt{\Delta(s)}$, so there are no net square roots. Applying the rules from the previous section,

$$\underline{\delta}\mathcal{T}_3 = \left(\int_0^\infty \log v \mathrm{d}\log(s-i)\right)\underline{\delta}w(i) + (i \to -i). \quad (18)$$

However, the elliptic curve $y^2 = -\mathcal{Q}(s^2)$ is even under $s \to -s$, which implies $w(i) + w(-i) = 0 \mod \langle 1,\tau\rangle$. Hence, there is only one independent last entry of $\underline{\delta}\mathcal{T}_3$,

$$\underline{\delta}\mathcal{T}_3 = \left(\int_0^\infty \log v \mathrm{d}\log\frac{s-i}{s+i}\right)\underline{\delta}w(i) = (i\pi \log v)\underline{\delta}w(i). \quad (19)$$

In the last step, we have chosen to perform the integral on the function level, instead of using our symbol integration rules. Of course, the symbol integration rules still apply in this case, yielding a vanishing result because the symbol of $i\pi \log v$ as a weight-two function is zero. The fact that $\underline{\delta}\mathcal{T}_3$ turns out to be proportional to π is not unfamiliar for (MPL) integrals in three dimensions [77–80].

The computation of the double triangle with massive circumferential propagators [Fig. 2(a)] is entirely similar. Here, we merely record the result,

$$\underline{\delta}\mathcal{T}_3^{\text{massive}} = i\pi \log\frac{(1+u_{14})(1+u_{23})}{(1+u_{13})(1+u_{24})}\underline{\delta}w(i) - i\pi\sum_{i=1}^4 \log\frac{X_i + i\sqrt{R_i}}{X_i - i\sqrt{R_i}}\underline{\delta}w\left(\frac{\sqrt{R_i}}{\sqrt{U_i^2 - 1}}\right), \quad (20)$$

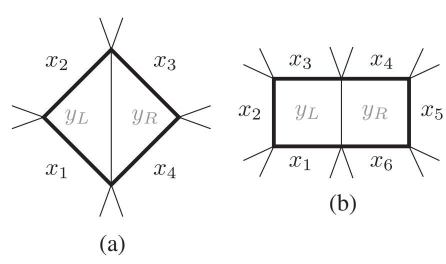


FIG. 2. (a) Double triangle and (b) double box with massive circumferential propagators, which depend on (a) 6 and (b) 15 cross ratios.

where $u_{ij} = \frac{x_{ij}^2+m_i^2+m_j^2}{2m_i m_j}$, and

$$U_i = \begin{cases} u_{34}, & i \in \{1,2\} \\ u_{12}, & i \in \{3,4\} \end{cases}, \qquad X_i = 1 + \sum_{\substack{j,k\neq i \\ j<k}} u_{jk}, \tag{21}$$

$$R_i = \mathcal{Q}_i^i(0) = 1 - \sum_{\substack{j,k\neq i \\ j<k}} u_{jk}^2 + 2\prod_{\substack{j,k\neq i \\ j<k}} u_{jk}. \tag{22}$$

Here, $\mathcal{Q}_i^i(0)$ is the minor of $Q(0)$ with the ith row and column deleted. As a consistency check, $\underline{\delta}\mathcal{T}_3^{\text{massive}}$ has branch points at $u_{ij} = -1$ or $x_{ij}^2 = -(m_i + m_j)^2$, exactly as predicted by Cutkosky's rules.

B. Double-box integrals in $D = 4$

For the $D = 4$ double box [Fig. 2(b)], the starting point is the deformed hexagon (see the Appendix),

$$\mathcal{I} = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{-\mathcal{Q}(t)}} \langle\langle Q(t)\rangle\rangle, \tag{23}$$

where $Q(t)$ is given by deforming the 6×6 Gram matrix G with one on the diagonal and $u_{ij} = \frac{x_{ij}^2+m_i^2+m_j^2}{2m_i m_j}$ off the diagonal. Read off the symbol (A8),

$$\mathcal{S}(\langle\langle Q(t)\rangle\rangle) = \sum_{1\leq i<j\leq 6} \mathcal{S}(\mathrm{Box}_{ij}(t)) \otimes \log R_{ij}(t), \tag{24}$$

where $\mathrm{Box}_{ij}(t) = \langle\langle Q_{ij}^{ij}(t)\rangle\rangle$ is obtained by deleting the ith and jth row and column of Q, and the last entries $R_{ij}(t) = \underline{ij}$ are given by (A9). Define the renormalized pure integral and last entries,

$$\mathcal{T} = \frac{1}{\omega_1}\mathcal{I}, \qquad w(t) = \frac{1}{\omega_1}\int^t \frac{\mathrm{d}t'}{\sqrt{-\mathcal{Q}(t')}} = \frac{1}{\omega_1}\wp^{-1}(T). \tag{25}$$

Again, it can be shown that $w(\infty) = 0 \bmod \langle 1, \tau\rangle$, so there is no boundary term at $t = \infty$. The boundary term at $t = 0$ is $-\mathcal{S}(\langle\langle Q(0)\rangle\rangle)\underline{\delta}w(0)$, representing the undeformed hexagon. For the integral terms, we need only consider singularities of $\log R_{ij}(t)$ located at $[\mathcal{Q}_j^i]^2 = -\mathcal{Q}_{ij}^{ij}\mathcal{Q} \Leftrightarrow \mathcal{Q}_i^i\mathcal{Q}_j^j = 0$, i.e., zeros of $\mathcal{Q}_i^i$; here, $\mathcal{Q}_J^I \coloneqq \det Q_J^I(t)$ is the minor of $Q(t)$ with the rows (columns) labeled by I (J) deleted.

Very nicely, the zeros of 5×5 minors $\mathcal{Q}_i^i$ are easy to obtain: for $i \in \{1,2,3\}$, the minor is quadratic in t and has two roots $\{r_i^{(1)}, r_i^{(2)}\}$; for $i \in \{4,5,6\}$, it is cubic with three roots $\{-1, r_i^{(1)}, r_i^{(2)}\}$ [81]. Therefore, we have 13 different singularities coming from all possible $R_{ij}(t)$, which implies that the integral terms contributing to $\underline{\delta}\mathcal{T}$ have 13 possible last entries $\underline{\delta}w(t_0)$, where $t_0 \in \{-1\} \cup \{r_i^{(1)}, r_i^{(2)}\}_{i=1}^6$. In total,

$$\underline{\delta}\mathcal{T} = -\mathcal{S}(\langle\langle Q(0)\rangle\rangle)\underline{\delta}w(0) + \sum_{t_0}\mathcal{S}(V_{t_0})\underline{\delta}w(t_0), \tag{26}$$

which is what we expect: the kinematical space $\mathcal{K}$ is 15 dimensional, and one of the degrees of freedom is captured by the unknown $\delta\tau$ term, leaving 14 functionally independent last entries.

We can immediately write down an integral representation of the (3,1) coproduct, as long as we keep track of the various signs,

$$V_{-1} = \sum_{\substack{i\in\{1,2,3\} \\ j\in\{4,5,6\}}} \int_0^\infty \mathrm{Box}_{ij}(t) \frac{\sqrt{\Delta_{ij}(-1)}\mathrm{d}t}{(t+1)\sqrt{\Delta_{ij}(t)}}, \tag{27}$$

$$V_{r_i^{(a)}} = \sum_{j\neq i} \pm \int_0^\infty \mathrm{Box}_{ij}(t) \frac{\sqrt{\Delta_{ij}(r_i^{(a)})}\mathrm{d}t}{(t - r_i^{(a)})\sqrt{\Delta_{ij}(t)}}, \tag{28}$$

where $\Delta_{ij}(t)$ is the box square root,

$$\Delta_{ij}(t) = \begin{cases} \mathcal{Q}_{ij}^{ij}(t), & \text{if } \mathcal{Q}_{ij}^{ij}(-1) \neq 0 \\ (t+1)^{-2}\mathcal{Q}_{ij}^{ij}(t), & \text{if } \mathcal{Q}_{ij}^{ij}(-1) = 0 \end{cases}. \tag{29}$$

Nothing stops us from iterating our rules to obtain $\mathcal{S}(V_{t_0})$ explicitly, though the calculation is a bit tedious. We content ourselves with computing the symbol in the special case where all propagators are massless: the 12 last entries $w(r_i^{(1,2)})$ satisfy linear relations and combine into six independent ones (modulo $\langle 1, \tau\rangle$). We have computed the six accompanying weight-three symbols and found perfect agreement with [59].

C. Double-D-gon integrals in $D \geq 5$

The $D \leq 4$ and $D \geq 5$ cases are different. The embedding space vectors live in $(D+2)$ dimensions, which implies that all $(D+3) \times (D+3)$ minors of G vanish for $D \geq 5$ (no such minors exist for $D \leq 4$). Therefore, up to the $(D-5)$th derivatives vanish: $\mathcal{Q}(0) = \mathcal{Q}'(0) = \cdots = \mathcal{Q}^{(D-5)}(0) = 0$, which implies $\mathcal{Q}(t) = -t^{D-4}P(t)$, where $P(0) \neq 0$. Remarkably, for $D \geq 5$, the integration kernel of $\mathcal{I}_D$ remains elliptic,

$$\mathcal{I}_D = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{P(t)}} \langle\langle Q(t)\rangle\rangle, \qquad \deg P(t) = 3, \quad \forall\ D \geq 5. \tag{30}$$

Our method yields all the last entries of $\underline{\delta}\mathcal{T}_D$ together with the accompanying integrals for $\mathcal{T}_D = \mathcal{I}_D/\omega_1$.

The 2-form $\omega = \frac{Dt}{\sqrt{P(t)}} \wedge D\log\underline{\rho_2\rho_1}$ is proportional to $\sqrt{t^{D-4}\mathcal{Q}_{\rho_1\rho_2}^{\rho_1\rho_2}} = \sqrt{\Delta(t)} \times$ rational, and after taking complete squares out of the square root, the net square root

$\sqrt{\Delta(t)}$ is not necessarily quadratic. Hence, the kernel of the accompanying integral

$$\int_0^\infty \langle\langle Q^{\rho_1\rho_2}_{\rho_1\rho_2}(t)\rangle\rangle \frac{\sqrt{\Delta(t_0)}\mathrm{d}t}{(t-t_0)\sqrt{\Delta(t)}} \tag{31}$$

is not necessarily d log. Specifically, if $\Delta(t)$ is cubic or quartic, the accompanying integral itself is elliptic; and if $\deg\Delta(t)\geq 5$, which first appears at $D=7$, the accompanying integral involves higher-genus curves and their symbology has not been studied in the literature. Our method provides partial results about these integrals, but conceivably we would miss even more terms because higher-genus curves have more moduli.

IV. CONCLUSION AND OUTLOOK

We have proposed algebraic rules of (e)MPL symbol integration that efficiently computes the total differentials or $(W-1,1)$ coproducts of onefold integrals of MPLs up to period terms, which can be iterated to produce the symbol. By exploiting the 2-form, we are able to sidestep rationalization completely, thus greatly improving on the existing method. We have checked our algorithm by reproducing (within minutes on a laptop using a very rough code) the results of some (e)MPL Feynman integrals, previously obtained through indirect methods.

Our algorithm applies nicely to the family of conformal double-D-gon integrals in D dimensions, possibly with massive circumferential propagators. In particular, we have computed the (2,1) coproduct of the $D=3$ case on the function level and have obtained an integral representation of the $(D-1,1)$ coproduct for $D\geq 4$, up to period terms. Moreover, we have argued that, unlike $D=3$, 4 cases, the weight-$(D-1)$ integrals accompanying the last entries can involve elliptic and even higher-genus curves for large D. It would be extremely interesting to understand the symbol and the geometric interpretation of double polygons, much like the well-known (one-loop) polygons [29,67–70].

Our method brings (elliptic) symbol integrations within reach for numerous other integrals. For example, it can be applied to integrals beyond double triangles for higher-point two-loop amplitudes in ABJM theory [78], and the recently studied family of elliptic ladder integrals [82,83] can serve as an all-loop application of our method. Along this line, it would be highly desirable to systematize elliptic symbol integration to include different integration kernels [45] and period terms. Another important question is how to extend our symbol integration rules to the function level, first for MPLs, but eventually for eMPLs, now that we can avoid rationalization.

We expect that this computational method will reveal more structures of symbols and coproducts. The fact that symbol letters produced by our algorithm are closely related to singularities of the integrand may provide insight into the success of the recently proposed Schubert analysis [20,59,84,85] in predicting (e)MPL symbol letters and may further extend it to general spacetime dimensions. It would also be interesting to explore interpretations of the accompanying weight-$(W-1)$ integrals, along the lines of [74] or [69], which is related to the diagrammatic coaction [67,86–89].

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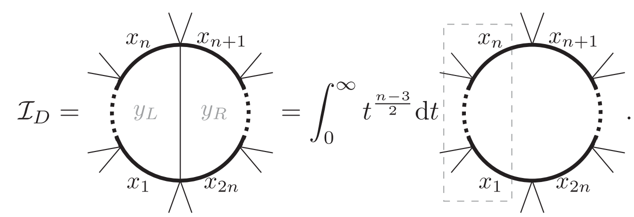
APPENDIX: THE DEFORMED POLYGON REPRESENTATION OF DOUBLE POLYGONS

In this appendix, we discuss the representation of double-D-gon integrals in D dimensions as an integral of a deformed $2n$-gon [82] with $D=n+1\geq 3$, where some of the $2n$ dual points may be identified. Schematically, we show that



(A1)

The dashed box indicates t deformation; see (A6).

Consider the most general double polygon, with generically massive circumferential propagators,

$$\mathcal{I}_D = \int \mathrm{d}^D y_L \mathrm{d}^D y_R D_{L,R} \prod_{i=1}^{n} D_{L,i} D_{R,n+i}, \tag{A2}$$

where $D^{-1}_{L,R}=(y_L-y_R)^2$ and $D^{-1}_{\ell,i}=(y_\ell-x_i)^2+m_k^2$ for $\ell=L,R$. Using the embedding formalism and performing a loop-by-loop Feynman parametrization,

$$\mathcal{I}_D = \int_0^\infty \frac{\langle \alpha \mathrm{d}^{2n-1}\alpha\rangle}{(R_1R_1)^{\frac{n-1}{2}}(R_2R_2)^{\frac{n+1}{2}}}, \tag{A3}$$

where

$$R_1=\alpha_1X_1+\cdots+\alpha_nX_n,\quad R_2=\alpha_1X_1+\cdots+\alpha_{2n}X_{2n}, \tag{A4}$$

and the embedding space vectors $X_i^M=(x_i^\mu;x_i^2+m_i^2,1)$ have inner products $(X_iX_j)=(x_i-x_j)^2+m_i^2+m_j^2$. Introducing a further Feynman parameter to combine the denominators,

$$\mathcal{I}_D = \int_0^\infty t^{\frac{n-3}{2}} \mathrm{d}t \int_0^\infty \frac{\langle \alpha \mathrm{d}^{2n-1}\alpha \rangle}{(\alpha \cdot Q(t) \cdot \alpha)^n}, \tag{A5}$$

where the quadric $\alpha \cdot Q(t) \cdot \alpha = t(R_1 R_1) + (R_2 R_2)$ represents a deformed $2n$-gon,

$$Q(t) = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad}^{n} & \overbrace{\qquad\qquad}^{n} \end{array} \\ \begin{pmatrix} (1+t) & 1 \\ 1 & 1 \end{pmatrix} \end{array} \odot G. \tag{A6}$$

Here, the symbol $\odot$ indicates elementwise multiplication, and the (i, j) entry of the $2n \times 2n$ Gram matrix G is $(X_i X_j)$. We will often omit the t dependence and denote $Q \equiv Q(t)$ and $G \equiv Q(0)$. Because of the projective nature of the quadric integral, we can freely rescale the ith row and the ith column by the same constant.

The result of the quadric integral is well known [68]. It evaluates to an MPL function $\langle\!\langle Q \rangle\!\rangle / \sqrt{-\mathcal{Q}}$ with nontrivial leading singularity, where $\mathcal{Q} \equiv \det Q$ and $\langle\!\langle Q \rangle\!\rangle$ is a pure function. In other words, we obtain the precise form of (A1),

$$\mathcal{I}_D = \int_0^\infty \frac{t^{\frac{n-3}{2}} \mathrm{d}t}{\sqrt{-\mathcal{Q}}} \langle\!\langle Q(t) \rangle\!\rangle. \tag{A7}$$

The symbol of $\langle\!\langle Q(t) \rangle\!\rangle$ can be read off from the quadric,

$$\mathcal{S}(\langle\!\langle Q \rangle\!\rangle) = \sum_\rho \log \underline{\rho_{2n}\rho_{2n-1}} \otimes \cdots \otimes \log \underline{\rho_2 \rho_1}, \tag{A8}$$

where ρ runs over all ordered partitions of $2n$ labels into n symmetric pairs, and the symbol entries

$$\underline{\rho_{2k}\rho_{2k-1}} = \frac{\mathcal{Q}^{\rho_{[2k-2]}\rho_{2k-1}}_{\rho_{[2k-2]}\rho_{2k}} + \sqrt{-\mathcal{Q}^{\rho_{[2k-2]}}_{\rho_{[2k-2]}} \mathcal{Q}^{\rho_{[2k]}}_{\rho_{[2k]}}}}{\mathcal{Q}^{\rho_{[2k-2]}\rho_{2k-1}}_{\rho_{[2k-2]}\rho_{2k}} - \sqrt{-\mathcal{Q}^{\rho_{[2k-2]}}_{\rho_{[2k-2]}} \mathcal{Q}^{\rho_{[2k]}}_{\rho_{[2k]}}}}. \tag{A9}$$

Here, $\rho_{[2k]}$ denotes the label set $\rho_1 \cdots \rho_{2k}$ and $\mathcal{Q}^I_J = \det Q^I_J$ is the minor of Q with the rows (columns) labeled by I (J) deleted.

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