Letter

Hairy black holes: Nonexistence of short hairs and a bound on the light ring size

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Several hairy black hole solutions are known to violate the original version of the celebrated no-hair conjecture. This prompted the development of a new theorem that establishes a universal lower bound on the extension of hairs outside any four-dimensional black hole solutions of general relativity. Our work presents a novel generalization of this "no-short hair" theorem, which notably does not use gravitational field equations and is valid for arbitrary spacetime dimensions ($D \geq 4$). Consequently, irrespective of the underlying theory of gravity, the "hairosphere" must extend to the innermost light ring of the black hole spacetime. Various possible observational implications of this intriguing theorem are discussed, and other useful consequences are explored.

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I. INTRODUCTION

Einstein's theory of general relativity (GR) is the cornerstone of our current understanding of classical gravitational physics. One of the most striking predictions of GR is the existence of black holes (BHs), which exhibit remarkable simplicity in stark contrast to other relativistic configurations. For example, in four dimensions, the stationary and asymptotically flat spacetime outside a vacuum BH solution of GR is represented by a Kerr metric characterized by only two parameters, namely the mass M and spin a. As a consequence, all higher multipoles of Kerr BHs are uniquely determined by M and a [1–4].

Even in the presence of matter, this important result finds its extension in the form of a well-motivated belief, known as the no-hair hypothesis [5,6] (see also [7]): Irrespective of the nature of the matter content, the end product of a gravitational collapse within the framework of GR can be completely specified by conserved charges such as mass, angular momentum, and electric charge measured at asymptotic infinity without any additional parameters (referred to as "hairs"). The motivation behind this conjecture stemmed from previous uniqueness theorems concerning BH solutions within the framework of GR [8,9]. Additionally, a heuristic physical interpretation was suggested in Ref. [10] that any matter fields existing outside a newly formed BH would either be emitted away to spatial infinity over time or absorbed by the BH itself unless those fields were associated with conserved charges at asymptotic infinity. The initial support for the no-hair conjecture came from the work of Bekenstein [11,12], which states that a stationary BH cannot be endowed with any exterior scalar, vector, or spin-2 meson fields. These results led to a belief that the no-hair conjecture is true irrespective of the nature of the matter content.

The first counterexample to this conjecture was provided by the discovery of the "colored" BH solution in Einstein-Yang-Mills theory, which contains an additional integer parameter that is not associated with any conserved charges like mass and spin [13]. It was soon established that this original version of the no-hair conjecture is invalid, and there are several hairy BH solutions, which include BHs with skyrmion [14], dilaton hairs [15], and axion [16].

Moreover, beyond the framework of GR, the presence of any putative modifications might lead to the violation of the no-hair property, resulting in BHs with extra hairs. Over the years, such hairy modifications of Kerr BHs have been extensively studied, and their observational signatures have been searched for [17–24]. However, these distinctive non-Kerr features arising from the presence of extra hairs can be suitably captured in observations probing only the far-away regions of spacetime, if the hairs extend sufficiently outside the horizon. Thus, the essential question of observational relevance is as follows: Can BHs have short hair confined only in the near-horizon region? An affirmative answer to this question would essentially imply that BHs with short hairs mimic Kerr-like signatures when probed in the far-field regions, though the near-horizon structure could be very different from that of a Kerr BH. Given this crucial observational relevance, we want to investigate this question in detail.

Given the violation of the no-hair conjecture, it is only natural to search for the crucial physical attribute which led to the existence of these solutions. In this regard, an

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important observation made in Ref. [10] is that the nonlinear character of the matter fields plays a fundamental role in the construction of the hairy black-hole solutions. Using the Einstein field equations, the weak energy condition (WEC), and the nonpositive trace condition on matter, it was shown that BHs cannot have short hairs and the region (referred to as "hairosphere") having the nonlinear behavior must extend at least up to three-halves of the horizon radius [10], which curiously corresponds to the location of the light ring (LR) in $D = 4$ dimensions [25].

This "no-short hair" theorem in GR provides a lower bound to the extent of the hairosphere outside the BH horizon. Thus, to detect the presence of hair around BHs, it is sufficient to probe till the near-LR region alone, which is greatly exemplified by the BH shadow observations. In other words, since the hairy configuration is extended at least up to the LR, it is possible to probe the presence of the hair in the image of the BHs [24,26,27].

Interestingly, with the advent of unprecedented technological progress, the question stated above becomes very relevant for observational purposes as we are gradually probing the near-field regime of BHs with ever-increasing precision. The detection of gravitational waves by the LIGO-Virgo collaborations [28] and the imaging of BH shadows by the Event Horizon Telescope (EHT) [29] have opened up a new era in BH physics. These observations may allow us to test the no(-short) hair conjecture and find the observational signature of hairy BHs.

Motivated by both the theoretical and observational importance of the aforesaid results in GR [10,25], we would like to ask the following question: Do the field equations or dimensionality of the spacetime play any fundamental role in determining the length of the hair being short (confined solely to near-horizon regime) or long (extends "sufficiently" beyond the horizon)? In particular, we want to know whether a similar result to no-short hair theorem can be obtained *independent* of the gravitational field equations in any dimensions ($D \geq 4$). If true, this will provide a novel generalization of the theorem to any theory of gravity which admits hairy BH solutions.

Intriguingly, the answer to the above question is negative. We show that for any static, spherically symmetric, and asymptotically flat hairy BH in D-dimensions, regardless of the theory of gravity under consideration, the hairosphere must extend at least up to the innermost LR. Such an extension is important on two fronts. First, it helps us understand the features of BH solutions of modified gravity in a unified theory-agnostic way. Second, such a generalization shows that the no-short hair result cannot be conceived as a null test of GR.

Other possible consequences of our analysis, such as existence of short hairs on horizonless compact objects is outlined generalizing the work of Ref. [30] in a theory-agnostic way, and various generalizations of the works of Refs. [31,32] about size of LRs in several higher-dimensional/modified theories of gravity are explored.

II. HAIRY BLACK HOLES

A general static, spherically symmetric BH spacetime in D dimensions is described by the metric

$$ds^2 = -f(r)dt^2 + \frac{1}{k(r)}dr^2 + h(r)d\Omega^2_{D-2}, \qquad (1)$$

where r is a Schwarzschild-like radial coordinate such that the outermost horizon is located at $r = r_H$ with $k(r_H) = 0$, and $h(r)$ is a strictly increasing function in the region $r \geq r_H$. Then, staticity implies that the norm of the timelike Killing vector must vanish at $r = r_H$, which implies $f(r_H) = 0$ [33]. This event horizon is considered to be nonextremal ($f'(r_H) \neq 0$), and regular so that all physical quantities such as curvature scalars are nondivergent there. We also assume that the spacetime is asymptotically flat, which requires that $f(r) \to 1 - C/r^{D-3} + \mathcal{O}(r^{-(D-2)})$, $k(r) \to 1 - C/r^{D-3} + \mathcal{O}(r^{-(D-2)})$, $h(r) \sim r^2$ as $r \to \infty$. Though C is related to the ADM mass of the spacetime, we will not assume any particular sign of C. Moreover, since $r = r_H$ denotes the outermost nonextremal horizon, asymptotic flatness implies that $f'(r_H) > 0$, and $k'(r_H) > 0$.

The metric described by Eq. (1) is a solution of a theory sourced by an arbitrary energy-momentum tensor $T_{\mu\nu}$. Due to the spherical symmetry, T^{μ}_{ν} should remain invariant under any rotation in the $(D-2)$-dimensional compact space with coordinates $\{\theta_i\}, i = 1, 2, (D-2)$. Therefore, $T^t_{\theta_i} = T^r_{\theta_i} = 0$, as they transform like vectors under such rotations. Also, spherical symmetry further implies $T^{\theta_1}_{\theta_1} = T^{\theta_2}_{\theta_2} = \cdots = T^{\theta_{D-2}}_{\theta_{D-2}}$, and all of the off-diagonal transverse components vanish. Thus, we have only four nonzero independent components of the T^{μ}_{ν}, namely $\{T^t_t \coloneqq -\rho; T^t_r; T^r_r \coloneqq p; T^{\theta_1}_{\theta_1} \coloneqq p_T\}$, where ρ, p, and p_T are identified as the energy density, radial, and tangential pressure, respectively. All of these components are functions of r only and should be nondivergent at the event horizon, which is assumed to be regular so that physical invariants such as $T^{\mu}_{\nu}T^{\nu}_{\mu}$ are finite there. Now, the radial component of the energy-momentum conservation equation $\nabla_{\mu}T^{\mu}_{\nu} = 0$, leads to

$$\hat{P}'(r) = \frac{h^{D/2-1}}{2f}(p+\rho)\Delta + \frac{h^{D/2-1}}{2}h'T, \qquad (2)$$

where $'$ represents radial derivative, $\hat{P} = h^{D/2}p$, $\Delta = (fh' - hf')$, and T stands for the trace of the energy-momentum tensor. This equation generalizes the similar result of Ref. [25] in the context of four-dimensional GR to an arbitrary dimension ($D \geq 4$) in a theory-independent way. That is, we have not used any gravitational field equations to derive Eq. (2).

We will assume that the matter content present in this BH spacetime satisfies the following conditions [10,25]:

(i) The WEC implying the energy density to be positive semidefinite, $\rho \geq 0$. It also bounds the radial pressure via the inequality $p + \rho \geq 0$.

(ii) The trace of the energy-momentum tensor is nonpositive, $T \leq 0$ implying $p + (D-2)p_T \leq \rho$. This assumption plays a crucial role in the existence of hair.

(iii) The energy density ρ and the radial pressure p approaches zero faster than r^{-D} as $r \to \infty$. This condition naturally rules out the existence of any extra conserved charges [10,25]. Therefore, using the relation $\hat{P} = h^{D/2} p$, we have the boundary condition $\hat{P}(r) \sim r^D p(r) \to 0$ as $r \to \infty$. This also implies that we are working with a hairy BH solution. As was the case in Ref. [10], the hair under consideration is not "secondary" (see the terminology of Ref. [34]).

Let us now study the behavior of $\hat{P}(r)$ in the vicinity of the BH horizon. We define a local coordinate system near the horizon as $dx = k^{-1/2} dr$, known as proper radial distance. The equivalence principle ensures that the proper radial distance is a well-defined coordinate. In terms of this new coordinate, Eq. (2) becomes

$$\frac{d\hat{P}}{dx} = \frac{h^{D/2-1}}{2f}(p+\rho)\left(f\frac{dh}{dx} - h\frac{df}{dx}\right) + \frac{h^{D/2-1}}{2} T\left(\frac{dh}{dx}\right). \tag{3}$$

Then, the regularity of the horizon implies that the left-hand side of Eq. (3) is also finite in the limit $r \to r_H$. Assuming a nonextremal BH with $f'(r_H) > 0$, WEC and the finiteness of the right-hand side of Eq. (3) at the horizon imply that

$$p(r_H) = -\rho(r_H) \leq 0. \tag{4}$$

Note that our assumption of nonextremal BH is very crucial for the validity of the above equation. It is because for an extremal BH, the term $(1/f)(df/dx)$ (which diverges at a nonextremal horizon) in Eq. (3) is an indeterminate $0/0$ form that might lead to a finite limit and then, it is not required to set $\rho + p = 0$ at $r = r_H$. The above equation along with the fact that $k(r) > 0$ outside the event horizon, gives the following results:

$$\hat{P}(r) \leq 0, \quad \text{and} \quad \hat{P}'(r) < 0, \tag{5}$$

in the vicinity ($r \to r_H$) of the BH horizon. Using the conditions derived above, we now proceed to prove the key theorem of our paper.

Theorem.—If the matter content satisfies all three conditions stated above and there exists a nonempty interval $r_H \leq r \leq r_p$ where the function $\Delta(r) = (fh' - hf') \leq 0$, we must have $\hat{P}'(r_H \leq r \leq r_p) \leq 0$.

The proof of this statement can be derived using Eqs. (2) and (5). The WEC and the nonpositivity of $\Delta(r)$ in the region $r_H \leq r \leq r_p$ imply the first term in the right-hand side of Eq. (2) is nonpositive. The same is true for the second term as well due to the trace condition $T \leq 0$ and $h'(r) > 0$. Then, it immediately follows that $\hat{P}'(r) \leq 0$ in the region $r_H \leq r \leq r_p$.

Though it remains to be shown that such a radial interval exists where $\Delta(r) \leq 0$, which we shall consider in subsequent paragraphs, let us first discuss its consequence. The above theorem along with Eq. (5) suggests that $\hat{P}(r)$ is nonpositive at the horizon and then it decreases at least up to $r = r_p$. Now following Refs. [10,25], let us define the extent of the "hairosphere" $r_{\rm hair}$ to be the radius at which $|\hat{P}(r)|$ has a local maximum, then we must have $r_{\rm hair} \geq r_p$.

Thus, the hair on an asymptotically flat, static, and spherically symmetric BH solution of any theory of gravity cannot be shorter than r_p. Thus, it is essential to know whether r_p has any physical characteristic of the BH spacetime. Though it is not apparent, we shall now show that r_p corresponds to the location of the innermost LR.

To show this, we follow a similar analysis provided in Ref. [25] and study the timelike and null geodesics in the BH spacetime described by Eq. (1). The motion of such a particle with energy E and angular momentum L moving in the equatorial plane is described by $\dot{r}^2 = k(r)[\frac{E^2}{f(r)} - \frac{L^2}{h(r)} - \epsilon]$, where a dot denotes a derivative with respect to some affine parameter. Also, $\epsilon = 0$ represents null geodesics and $\epsilon = 1$ represents timelike geodesics. Circular geodesics are characterized by $\dot{r}^2 = 0 = (\dot{r}^2)'$. Solving these two equations, we get

$$\Delta(r)E^2 = \epsilon f^2(r)h'(r), \qquad \Delta(r)L^2 = \epsilon h^2(r)f'(r). \tag{6}$$

Since for timelike/null orbits, E^2 and L^2 should be nonnegative, we must have the quantity $\Delta(r) > 0$ for timelike circular geodesic, and $\Delta(r) = 0$ for null circular geodesics. Note that the zeros of the function $\Delta(r)$ denote the locations of the light rings of the spacetime. In order to analyze the behavior of $\Delta(r)$ in the region $[r_H, \infty)$, we consider two auxiliary functions: $L(r) = f(r)h'(r)$ and $R(r) = h(r)f'(r)$ such that $\Delta = L - R$. The behaviors of these functions are as follows:

(i) At the horizon $r = r_H$, $L(r_H) = 0$, and $R(r_H) > 0$. This follows from the conditions that $f(r_H) = 0$, $f'(r_H) > 0$, and $h'(r_H) > 0$.

(ii) For $r \to \infty$, $L(r) \sim r$, and $R(r) \sim r^{-(D-4)}$. Here, we have used the asymptotic flatness, which implies $f(r) \to 1$ with $f'(r) \sim \text{sgn}(C) r^{-(D-2)}$ and $h(r) \sim r^2$ with $h'(r) \sim r$ as $r \to \infty$.

The above conditions for $D \geq 4$ suggest that $\Delta(r)$ has an odd number of zeroes [irrespective of the sign of C, denoted by $\mathrm{sgn}(C)$], which correspond to the LRs outside the outermost horizon. These LRs divide the interval $[r_H, \infty)$ into an even number of regions. Since in the outermost region the quantity $\Delta(r) \geq 0$, we must have $\Delta(r) \leq 0$ in the innermost region $r_H \leq r \leq r_\gamma^1$, where r_γ^1 denotes the location of the innermost LR. This concludes our proof that there exists a radial interval $r_H \leq r \leq r_p$ with $\Delta(r) \leq 0$, provided that we identify r_p with r_γ^1. Therefore, the hairosphere of a BH must extend at least up to the innermost LR, $r_{\rm hair} \geq r_\gamma^1$. In other words, we have shown that BHs cannot have short hairs confined only in the near-horizon region.

Some comments are as follows. Though our analysis runs parallel to the work of Ref. [25], the novelty of our work lies in its generality. Namely, we have not used any field equations to obtain the result and it is valid for any spacetime dimensions $D \geq 4$. Now, we shall discuss few interesting consequences of Eq. (2) in the following subsections.

A. Size of static shells

Consider a static shell of finite thickness located entirely between the outer horizon r_H and the innermost LR r_γ^1, i.e., $r_H < r_1 < r_2 < r_\gamma^1$, where r_1 and r_2 are the inner and the outer radius of the shell. Now, if the matter field obeys WEC and $T \leq 0$ as discussed for the BH case, Eq. (2) implies that $\hat{P}'(r_1 \leq r \leq r_2) \leq 0$. Consequently, this implies that the pressure at the outer surface is lower than (or the same as) the pressure at the inner surface. However, this contradicts the requirement that the pressure must be zero at both surfaces, considering there is no matter present outside the shell. Therefore, without using any field equations, we conclude that a static shell of finite thickness can not exist entirely between the outer horizon and the innermost LR.

B. Bound on light rings in Einstein-Gauss-Bonnet gravity

One can utilize our theorem in the context of any particular theories of gravity (thereby, using the corresponding field equations) to get an upper bound on the size of the innermost LR by performing a similar calculation presented in Refs. [31,32]. For the purpose of illustration, let us consider the Einstein-Gauss-Bonnet (EGB) theory, which is studied extensively in literature. It is the unique quadratic curvature modification of GR so that the field equations remain second order in time and the theory is free from perturbative ghosts in any dimensions $D \geq 5$ [35]. Owing to these nice features, EGB theory provides us with a well-motivated model to study the properties of its BH solutions.

The Lagrangian of this theory is given by $\mathcal{L} = R + \hat{\alpha}(R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})$, which leads to the field equation:

$$G^{(1)}_{\alpha\beta} + \hat{\alpha} G^{(2)}_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{7}$$

The explicit forms of the $G^{(1)}_{\alpha\beta}$ and $G^{(2)}_{\alpha\beta}$ can be found in Ref. [36]. As a solution of Eq. (7), we consider a spherically symmetric, static, and asymptotically flat BH metric in the form of

$$ds^2 = -e^{-2\delta(r)}\mu(r)dt^2 + \frac{1}{\mu(r)}dr^2 + r^2 d\Omega^2_{D-2}. \tag{8}$$

We assume that there exists a regular nonextremal event horizon at $r = r_H$, so that $\mu(r_H) = 0$, $\mu'(r_H) > 0$, and $\delta(r)$ and its radial derivative is finite there [31]. Due to the asymptotic flatness, we must have $\mu(r) \to 1$ and $\delta(r) \to 0$ at infinity. Now, using the above field equations of EGB theory, we get

$$\begin{aligned} \delta' &= -\frac{8\pi r^3(p+\rho)}{(D-2)\mu[r^2 + 4\alpha(1-\mu)]}, \\ \mu' &= \frac{2r^3(D-3)}{r^2 + 4\alpha(1-\mu)}\left[\frac{1-\mu}{2r^2} + \frac{\alpha(D-5)(1-\mu)^2}{(D-3)r^4} \right. \\ &\quad \left. - \frac{8\pi\rho}{(D-2)(D-3)}\right], \end{aligned} \tag{9}$$

where $T^t_t = -\rho$, $T^r_r = p$, and $\alpha = (D-3)(D-4)\hat{\alpha}/2$. One can easily find the location of the innermost LR r_γ^1, which is given by the smallest positive root of the following equation:

$$2e^{-2\delta_\gamma}\mu_\gamma - r_\gamma^1 (e^{-2\delta}\mu)'_{r_\gamma^1} = 0. \tag{10}$$

Here, μ_γ and δ_γ are shorthand for $\mu(r_\gamma^1)$ and $\delta(r_\gamma^1)$, respectively. Then, using Eqs. (9) and (10), we get an identity,

$$\begin{aligned} N(r_\gamma^1) &:= (D-1)\mu_\gamma - (D-3) + \frac{8\alpha\mu_\gamma(1-\mu_\gamma)}{(r_\gamma^1)^2} \\ &\quad - \frac{2\alpha(D-5)(1-\mu_\gamma)^2}{(r_\gamma^1)^2} = \frac{16\pi r_\gamma^2 p(r_\gamma)}{D-2}. \end{aligned} \tag{11}$$

Then, our theorem implies $p(r_\gamma^1) \leq 0$, which leads to the following condition:

$$N(r_\gamma^1) \leq 0. \tag{12}$$

Now, to get an upper bound on the size of the innermost LR, we require an explicit form for $\mu(r)$. It is useful to define a mass function

$$m(r) = \frac{r_H}{2} + \Omega_{D-2}\int_{r_H}^{r} \rho(x)x^{D-2}dx, \tag{13}$$

where $\Omega_{D-2} = 2\pi^{(D-1)/2}/\Gamma[(D-1)/2]$ is the surface element of the unit $(D-2)$ sphere and we have chosen the boundary condition, $m(r_H) = r_H/2 > 0$. With these conditions, the μ' equation given by Eq. (9) admits the following solution:

$$\mu(r) = 1 + \frac{r^2}{4\alpha}\left[1 - \sqrt{1 + \frac{16\alpha M(r)}{r^{D-1}}}\right], \qquad (14)$$

where $M(r) = 8\pi m(r)/(D-2)\Omega_{D-2}$. Replacing Eq. (14) in Eq. (12), we get the following polynomial inequality:

$$(r_\gamma^1)^{2D-6} + 16\alpha M_\gamma (r_\gamma^1)^{D-5} - (D-1)^2 M_\gamma^2 \leq 0, \qquad (15)$$

where $M_\gamma = M(r_\gamma^1)$. The above equation has some important consequences. An immediate one would be the innermost LR of any D-dimensional BHs in GR ($\alpha = 0$) is bounded above by the radius (r_γ^{ST}) of the LR of Schwarzschild-Tangherlini (ST) BH [37] with mass $M_{\mathrm{ST}} \coloneqq M(r \to \infty)$. It follows directly from the $\alpha \to 0$ limit of the above inequality that $r^1_{\gamma,(\mathrm{GR})} \leq [(D-1)M_\gamma]^{1/(D-3)} = r_\gamma^{\mathrm{ST}}$, as $M_{\mathrm{ST}} \geq M_\gamma$. This generalizes the result of Ref. [31] for higher dimensional GR.

Now, for EGB gravity with $\alpha \neq 0$, we must choose α in such a way that there exists a horizon (nonextremal) to avoid naked singularity. In this context, it is possible to generalize the above result of GR and demonstrate that Boulware-Deser (BD) BH [38] has the largest LR. For this purpose, let us consider the following polynomials in $r \in [0, \infty)$ as

$$F_1(r) = r^{2D-6} + 16\alpha M_\gamma r^{D-5} - (D-1)^2 M_\gamma^2,$$
$$F_2(r) = r^{2D-6} + 16\alpha M_{\mathrm{BD}} r^{D-5} - (D-1)^2 M_{\mathrm{BD}}^2,$$

where $M_{\mathrm{BD}} \coloneqq M(r \to \infty) \geq M_\gamma$. For either signs of α, Descartes's rule of signs ensures that both F_1 and F_2 have single positive root r_m and r_γ^{BD}, respectively. Note that r_γ^{BD} denotes the location of the LR of the BD BH with mass M_{BD}. Whereas Eq. (15) suggests that the innermost LR of any BH solution of EGB gravity is bounded above by r_m, i.e., $r_\gamma^1 \leq r_m$.

Since $M_{\mathrm{BD}} \geq M_\gamma$, the function $F_2(r=0)$ has a value less than $F_1(r=0)$. Then, it is suggestive to evaluate $F_2(r)$ at the location of the root (r_m) of $F_1(r)$. Some simple algebraic manipulation gives us

$$F_2(r_m) = \left[\frac{r_m^{2D-6}}{M_\gamma} + (D-1)M_{\mathrm{BD}}\right](M_\gamma - M_{\mathrm{BD}}). \qquad (16)$$

Since $M_{\mathrm{BD}} \geq M_\gamma$, we have $F_2(r_m) \leq 0$. This, in turn, implies the following inequality must hold: $r_\gamma^{\mathrm{BD}} \geq r_m \geq r_\gamma^1$, which completes the proof that the size of the innermost LR of BH solutions of EGB gravity is bounded above by that of Boulware-Desser LR.

C. Comments on horizonless compact objects

The scope of our discussion has been limited to BHs thus far. In this section, we will try to extend our analysis to the horizonless compact objects (i.e. objects with LRs outside, as we shall discuss later), described by the metric in Eq. (1). This time, in the absence of a horizon, we set the inner boundary conditions $R(0) = 0$ and $L(0) > 0$ at the center of the object $r = 0$ which is deduced from Ref. [39] with the assumption that $h(0) = 0$. However, the behaviors of the auxiliary functions $L(r)$ and $R(r)$ defined earlier remain unchanged at spatial infinity due to asymptotic flatness.

Following Ref. [39], regularity of the matter configurations requires various components of T^μ_ν such as ρ and p to be finite and well behaved at $r = 0$. Thus, we have $\hat{P}(r) = 0$ at the center of the compact object. Now, for the purpose of extending the no-short hair result for horizonless objects, we need to assume that the trace T of the energy-momentum tensor T^μ_ν is non-negative ($T \geq 0$) [30], which is exactly opposite to the BH scenario. While the other two assumptions on the matter, namely the WEC and asymptotic falloff of $\hat{P}(r)$, remain unchanged as in the case of BHs.

Just like the BH scenario, under these conditions the quantity $\Delta(r) \geq 0$ in the region $0 \leq r \leq r_\gamma^1$, where r_γ^1 is the location of the innermost LR. Then, we must have $\hat{P}'(r) \geq 0$ in the same region. This statement, along with Eq. (2) implies that $\hat{P}(r)$ starts with a zero value at the center of the compact object and then, it increases with r at least up to $r = r_\gamma^1$. If we define $r = r_{\mathrm{hair}}$ to be the radius at which $|\hat{P}(r)|$ has a local maximum, then we must have $r_{\mathrm{hair}} \geq r_\gamma^1$. Thus, for horizonless compact objects also, the hair is extended at least up to the innermost LR. This generalizes the results of Ref. [30] in a theory-agnostic fashion.

A few comments on this result are as follows. First, note that the condition $T > 0$ is not satisfied by "usual" matter content, since for those $\rho \gg (p, p_T)$. Therefore, for ordinary celestial objects, the no-short hair theorem is not applicable. However, for objects made of "exotic" matter, $T > 0$ condition may hold true and then, the no-short hair result can give us useful information about their structures. Another point to note is that, the asymptotic falloff condition of $\hat{P}(r)$ makes sure of the existence of LRs (an even number of them); otherwise in the absence of LRs, $\hat{P}(r)$ would increase monotonically with r.

An extension of the above result could be to consider hairy wormhole spacetimes by setting the inner boundary condition at the throat ($r = b > 0$) as long as $L(b) > R(b)$, and assuming other necessary conditions such as asymptotic flatness and WEC with $T > 0$ hold true.

III. CONCLUSION AND DISCUSSIONS

In this work, we have generalized the no-short hair theorem of GR discussed in Refs. [10,25] in a theory-agnostic way, which is valid for arbitrary spacetime dimensions ($D \geq 4$). Such a result has both theoretical and observational significance. On the theory side, it helps us understand some key features of BH solutions of modified gravity in a unified way. Additionally, this theorem has relevance for both EHT (shadow) and GW (ringdown quasinormal modes) observations that could probe the near-field region of a BH. Since the hairosphere cannot be confined solely in the near-horizon regime, exploring the near-LR region alone might give us important information about the signatures of BH hairs. Moreover, the presence of hairs near the LR may give rise to intriguing new phenomena, including gravitational lensing effects and potential modifications to the BH shadow.

Apart from these interesting results, we have discussed several useful consequences of our analysis. For example, we have shown that it is possible to set an upper bound on the size of the innermost LR of various theories of gravity including EGB theory in $D \geq 5$ dimensions. This bound may be translated to constrain both the shadow size [32] and the real part of the eikonal quasinormal modes of EGB BHs under perturbation [31]. Also, a few comments on the possible generalization of the no-short hair theorem for horizonless compact objects are outlined.

The implications of our findings are significant and warrant further consideration. For instance, it will be interesting to extend our work when various assumptions, such as spherical symmetry, asymptotic flatness, or WEC on matter are relaxed. Especially, it will be of great observational relevance if the short-hair behavior of rotating BHs could be investigated [40]. We leave such possible extensions for future study.

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