Letter

Explaining nonlinearities in black hole ringdowns from symmetries

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It has been recently pointed out that nonlinear effects are necessary to model the ringdown stage of the gravitational waveform produced by the merger of two black holes giving rise to a remnant Kerr black hole. We show that this nonlinear behavior is explained, both on the qualitative and quantitative level, by near-horizon symmetries of the Kerr black hole within the Kerr/CFT correspondence.

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I. INTRODUCTION

Quasinormal modes (QNMs) provide a unique tool to investigate the properties of black holes (BHs) whose understanding is one of the major goals of gravitational wave astronomy [1]. In the merger of two BHs, during the final stage called ringdown, they dominate the BH response to any kind of disturbance, and their frequencies are uniquely determined by the BH mass, spin, and charge.

Gravitational waves (GWs) during ringdown are well described by a superposition of exponentially damped QNMs, labeled by two angular harmonic numbers (ℓ, m) and an overtone number n. Their amplitude is denoted $A_{(\ell,m,n)}$, while their oscillation frequency and decay timescale are given by the real and imaginary parts of $\omega_{(\ell,m,n)}$. The GW strain far from the BH source can be decomposed as

$$h(u,\theta,\phi) = \sum_{\ell\geq 2}\sum_{|m|\leq\ell} h_{(\ell,m)}(u)\,{}_{-2}Y_{(\ell,m)}(\theta,\phi),$$
$$h_{(\ell,m)}(u) = \sum_{n\geq 0} A_{(\ell,m,n)} e^{-i\omega_{(\ell,m,n)}(u-u_{\rm pk})}, \tag{1}$$

where $u = (t - r)$ is the retarded or Bondi time, $u_{\rm pk}$ is the time at which the strain achieves its maximum value, and ${}_{-2}Y_{(\ell,m)}$ are the spin-weight $s = -2$ spherical harmonics (the $s = +2$ mode with outgoing boundary conditions is subleading at infinity [2]). The GW strain produced is generically modeled using first-order BH perturbation theory. However, nonlinearities are an intrinsic property of general relativity and indeed it has been recently pointed out that second-order effects are relevant to describe ringdowns from BH merger simulations [3,4] (see also Refs. [5–7]).

In particular, the second-order mode amplitude $A^{(2,2,0)\times(2,2,0)}_{(4,4)}$ obtained from the square of the first-order fundamental mode $(\ell, m) = (2, 2)$ is comparable to or even larger than that of the fundamental linear mode (4,4). For the numerical simulations of quasicircular mergers giving rise to a Kerr BH with spin 0.7 (in units of the BH mass and we set from now on $G_N = 1$) Ref. [3] found

$$\frac{|A^{(2,2,0)\times(2,2,0)}_{(4,4)}|}{|A_{(2,2,0)}|^2} = 0.1637 \pm 0.0018, \tag{2}$$

where we have neglected the milder dependence on the BH mass ratio of the two BH mergers. This result is consistent with what is found in Ref. [4] which quotes values in the interval (0.15–0.2).

Restricting ourselves to the fundamental modes and noting that the second-order QNM (4,4) is sourced at second order from the square of the (2,2) mode with frequency $2\omega_{(2,2,0)}$, Eq. (2) can be written in the suggestive form

$$\frac{\langle h_{(2,2)}h_{(2,2)}h_{(4,4)}\rangle}{\langle h^2_{(2,2)}\rangle^2} \simeq 0.1637 \pm 0.0018. \tag{3}$$

Similarly, Ref. [3] found

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$$\frac{|A^{(2,2,0)\times(3,3,0)}_{(5,5)}|}{|A_{(2,2,0)}||A_{(3,3,0)}|} = 0.4735 \pm 0.0062, \quad (4)$$

again for a Kerr BH remnant of spin ∼0.7. In order to correctly model the BH ringdown one needs therefore to include nonlinear effects.

The goal of this paper is to explain the nonlinearities of the Kerr BH remnant from symmetry arguments. Our starting point is the realization that the QNMs are produced in the proximity of the BH horizon [8]. In the region very close to the horizon of an extreme Kerr BH one can set consistent boundary conditions such that the asymptotic symmetry generators identify one copy of the Virasoro algebra [9]. This implies that the near-horizon quantum states can be identified with those of (a chiral half of) a two-dimensional conformal field theory (CFT) with finite temperature $T = 1/2\pi$. This goes under the name of the Kerr/CFT correspondence [10]. Although the CFT details are not exactly known, several nontrivial checks have been studied in the past (see, for instance, Refs. [11–13]), consolidating the idea that there is a relation between universal properties of BHs and CFTs, and trying also to extend the results to nonextremal cases [14–17].

If the correspondence is correct, then the correlators of bulk fields can be computed through the correlators of the corresponding boundary operators. In turn, the latter are dictated by the CFT. In the case of the spin-2 strain, the associated boundary operator is the two-dimensional energy momentum tensor, whose correlator amplitudes are fixed by the central charge. This simple reasoning will allow us to calculate the three-point correlators to estimate the level of nonlinearities in the ringdowns from symmetry arguments.

Before launching ourselves in the technicalities, we briefly summarize the basics of the Kerr/CFT correspondence. The expert reader on the subject can skip this part.

II. KERR/CFT

The Kerr BH with mass M and angular momentum J is described in Boyer-Lindquist coordinates by the metric

$$\mathrm{d}s^2 = -\frac{\Delta}{\rho^2}(\mathrm{d}\hat{t} - a\sin^2\theta)^2 + \frac{\rho^2}{\Delta}\mathrm{d}\hat{r}^2 + \frac{\sin^2\theta}{\rho^2}[(\hat{r}^2 + a^2)\mathrm{d}\hat{\phi} - a\mathrm{d}\hat{t}]^2 + \rho^2\mathrm{d}\theta^2, \quad (5)$$

where

$$\Delta = \hat{r}^2 - 2M\hat{r} + a^2, \quad \rho^2 = \hat{r}^2 + a^2\cos^2\theta, \quad a = \frac{J}{M}. \quad (6)$$

The two horizons are defined as the solutions to $\Delta(r) = 0$:

$$\hat{r}_\pm = M \pm \sqrt{M^2 - a^2}, \quad (7)$$

and the Hawking temperature can be written as

$$T_H = \frac{\hat{r}_+ - \hat{r}_-}{8\pi M\hat{r}_+}. \quad (8)$$

We consider the extremal case $a = M$ and the change of coordinates [18]

$$t = \frac{\lambda\hat{t}}{2M}, \quad r = \frac{\hat{r} - M}{\lambda M}, \quad \phi = \hat{\phi} - \frac{\hat{t}}{2M}, \quad (9)$$

such that, in the limit $\lambda \to 0$ and keeping fixed the coordinates (t, r, ϕ, θ), one can zoom into a small region around the BH event horizon $\hat{r} = M$. The resulting metric is the near horizon extreme Kerr (NHEK),

$$\mathrm{d}s^2 = 2J\Gamma(\theta)\left[-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2 + \Lambda^2(\theta)(\mathrm{d}\phi + r\mathrm{d}t)^2\right],$$
$$\Gamma(\theta) = \frac{1 + \cos^2\theta}{2}, \quad \Lambda(\theta) = \frac{2\sin\theta}{1 + \cos^2\theta}, \quad (10)$$

with $\phi \sim \phi + 2\pi$ and $0 \leq \theta \leq \pi$. For a generic value of θ one deals with the geometry of a warped AdS_3 with a $\mathrm{SL}(2,\mathbb{R}) \otimes U(1)$ symmetry group. At the specific value θ_0 such that $\Lambda(\theta_0) = 1$, the metric is that of AdS_3 [19,20].

At extremality, rotational perturbations along the angular azimuthal direction correspond to excitations of the left sector of the dual CFT. The right modes are not excited (unless one goes above extremality). On the other hand, the left modes cannot be associated with the Hawking temperature T_H, which vanishes for extremal Kerr BHs. Indeed, the Hartle-Hawking vacuum for quantum fields in the region outside the Schwarzschild BH horizon (which gives rise to a density matrix $e^{-\omega/T_H}$) cannot be used for the Kerr spacetime. This is because in spacetimes lacking a globally defined timelike Killing vector, such as the Kerr geometry, the Hartle-Hawking vacuum does not exist. One can, however, use the Frolov-Thorne vacuum [21], which is appropriately defined in the vicinity of a spinning BH horizon. Quantum fields can be expanded in asymptotic energy and angular momentum eigenstates of the operators $\partial_{\hat{t}}$ and $\partial_{\hat{\phi}}$ as

$$\Phi_s(\hat{t}, \hat{r}, \theta, \hat{\phi}) = \sum_{\omega,\ell,m} \Phi_{\omega\ell m} e^{-i\omega\hat{t} + im\hat{\phi}} R_{\ell m}(\hat{r}, \theta). \quad (11)$$

In the near horizon coordinates (t, r, θ, ϕ), the asymptotic energy-angular momentum eigenstates are expressed as

$$e^{-i\omega\hat{t} + im\hat{\phi}} = e^{-in_R t + in_L\phi}, \quad (12)$$

where

$$n_L = m, \qquad n_R = \frac{1}{\lambda}(2\omega M - m). \tag{13}$$

Then the Frolov-Thorne vacuum gives rise to a density matrix with a Boltzmann weighting factor,

$$e^{-\frac{n_L}{T_L}-\frac{n_R}{T_R}}, \tag{14}$$

where the left- and right-temperatures are [10]

$$T_L = \frac{r_+ - M}{2\pi(r_+ - a)}, \qquad T_R = \frac{r_+ - M}{2\pi r_+ \lambda}. \tag{15}$$

In the extremal case (keeping λ small, but finite), we end up with the left-moving sector in the Frolov-Thorne temperature,

$$T_L = \frac{1}{2\pi}, \tag{16}$$

and a Boltzmann suppression factor $e^{-2\pi n_L}$, while the right sector has $T_R = 0$. At the same time, the geometry (10) has a boundary at $r \to \infty$ (where boundary fields are a function of t and ϕ) and an asymptotic symmetry group (with specific boundary conditions for the perturbations of the metric in AdS), which extends the symmetry to half of a Virasoro algebra [10]. This result led to the Kerr/CFT conjecture, where the NHEK BH can be described by a dual chiral CFT. According to the correspondence, for every bulk field Φ there is a corresponding boundary local operator $\mathcal{O}$ coupled to the boundary field Φ_b such that

$$Z_{\mathrm{AdS,eff}}[\Phi] = e^{iS_{\mathrm{eff}}[\Phi]} = \langle \mathrm{T} e^{\int_{\partial \mathrm{AdS}} \Phi_b \mathcal{O}} \rangle_{\mathrm{CFT}}, \tag{17}$$

where T is the time-ordering operator. In particular, for the helicity-2 gravitational strain the associated boundary local operator is the stress energy-momentum tensor. The correlators of the QNMs can be therefore inferred from the correlators of the boundary energy-momentum tensor, as we will show in the following.

III. NONLINEARITIES OF THE QNMs FROM THE KERR/CFT

The energy-momentum tensor exists in any local CFT. In two-dimensions, its components are denoted as T and $\bar{T}$, and they have weights $(h, \bar{h}) = (2, 0)$ and (0,2), respectively. The central charge $c = 12J$ completely fixes their correlators [10]. By defining the energy-momentum tensor as $T_{\mu\nu}$ as the response of the Hamiltonian $\mathcal{H}$ to the transformation $x^\mu \to x^\mu + \xi^\mu$, as

$$\delta\mathcal{H} = -\int \mathrm{d}^2 x \partial_\mu \xi_\nu T^{\mu\nu}, \tag{18}$$

the two- and three-point correlators on the complex plane turn out to be [22]

$$\langle T(z_1)T(z_2)\rangle = \frac{1}{(2\pi)^2}\frac{c/2}{z_{21}^4}, \tag{19}$$

$$\langle T(z_1)T(z_2)T(z_3)\rangle = \frac{1}{(2\pi)^3}\frac{c}{z_{21}^2 z_{32}^2 z_{13}^2}, \tag{20}$$

where $z_{ij} = (z_i - z_j)$ and $T = T_{zz}$. To take into account that the CFT is at finite temperature T_L for the left movers, we map the complex plane to the cylinder with complex coordinate $w = x^1 + i\tau$ by [9]

$$z = e^{2\pi T_L w}, \qquad \bar{z} = e^{2\pi T_L \bar{w}}, \tag{21}$$

and identify τ with $\tau + 1/T_L$. Then, using the transformation property of the energy-momentum tensor under conformal transformations $z \to w(z)$,

$$T(z) \to T(w) = (\partial_z w)^2 \left(T(z) - \frac{c}{12}\{w; z\} \right), \tag{22}$$

where $\{w; z\}$ is the Schwarzian derivative, we find that the two- and three-point functions at finite temperature T_L, after Wick rotating $\tau \to ix^0$, are

$$\langle T(x_1^-)T(x_2^-)\rangle = \frac{c/2}{(2\pi)^2}\left(\frac{\pi T_L}{\sinh(\pi T_L x_{21}^-)}\right)^4, \tag{23}$$

$$\begin{aligned}\langle T(x_1^-)T(x_2^-)T(x_3^-)\rangle = \frac{c}{(2\pi)^3}\left(\frac{\pi T_L}{\sinh(\pi T_L x_{12}^-)}\right)^2 \\ \times \left(\frac{\pi T_L}{\sinh(\pi T_L x_{23}^-)}\right)^2 \\ \times \left(\frac{\pi T_L}{\sinh(\pi T_L x_{31}^-)}\right)^2,\end{aligned} \tag{24}$$

where $x^- = (x^1 - x^0)$. The bulk isometries ∂_ϕ and ∂_t are identified, up to a scale, with the left and right translations in the CFT, implying $x^- = \phi$ and $x^+ = t$. Going to momentum space and from Eqs. (12) and (13), one identifies therefore the frequency of the left movers with the azimuthal number m [11].

We can write the above connected correlators in momentum space as

$$\langle T_{m_1} T_{m_2}\rangle = \frac{1}{2}\int \prod_{i=1}^{2} \mathrm{d}x_i^- e^{im_i x_i^-} \langle T(x_1^-)T(x_2^-)\rangle,$$

and

$$\langle T_{m_1} T_{m_2} T_{m_3}\rangle = \int \prod_{i=1}^{3} \frac{\mathrm{d}x_i^-}{2^{3/2}} e^{im_i x_i^-} \langle T(x_1^-)T(x_2^-)T(x_3^-)\rangle,$$

where the T_m's are the boundary duals to the modes of the gravitational strain with angular momentum m. We obtain [11,23,24]

$$\langle T_{m_1} T_{m_2} \rangle' = \frac{c}{24} \frac{(2\pi T_L)^3}{(2\pi)^2} e^{m_1/(2T_L)} \left| \Gamma\left(2 + i \frac{m_1}{2\pi T_L}\right) \right|^2,$$

and

$$\langle T_{m_1} T_{m_2} T_{m_3} \rangle' = -\frac{c}{2\sqrt{2}} \frac{(2\pi T_L)^4}{(2\pi)^3} e^{-(m_1+m_2)/(2T_L)} \times G^{3,3}_{3,3}\left(\begin{matrix} -i\frac{m_1}{2\pi T_L}, & 0, & i\frac{m_2}{2\pi T_L} \\ 1 - i\frac{m_1}{2\pi T_L}, & 1, & 1 + i\frac{m_2}{2\pi T_L} \end{matrix} \middle| e^{i\pi} \right), \tag{25}$$

where the primes indicate we have removed the Dirac delta function $(2\pi)\delta(\sum_i m_i)$, and $G^{3,3}_{3,3}$ is a Meijer-G function. The two- and three-point correlators of the graviton are then inferred from the expressions [25]

$$\langle h_m h_{-m} \rangle' = -\frac{1}{2\mathrm{Re}\langle T_m T_{-m} \rangle'},$$
$$\langle h_{m_1} h_{m_2} h_{m_3} \rangle' = \frac{2\mathrm{Re}\langle T_{m_1} T_{m_2} T_{m_3} \rangle'}{\prod_{i=1}^3 (-2\mathrm{Re}\langle T_{m_i} T_{-m_i} \rangle')}, \tag{26}$$

from where it follows that (setting finally $T_L = 1/2\pi$)

$$\frac{\langle h_{m_1} h_{m_2} h_{m_3} \rangle'}{\langle h_{m_1} h_{-m_1} \rangle' \langle h_{m_2} h_{-m_2} \rangle'} = -\frac{\mathrm{Re}\langle T_{m_1} T_{m_2} T_{m_3} \rangle'}{\mathrm{Re}\langle T_{m_3} T_{-m_3} \rangle'} = \frac{6\sqrt{2}}{2\pi} \frac{G^{3,3}_{3,3}\left(\begin{matrix} -im_1, & 0, & im_2 \\ 1 - im_1, & 1, & 1 + im_2 \end{matrix} \middle| e^{i\pi} \right)}{|\Gamma(2 + im_3)|^2}, \tag{27}$$

which critically does not depend upon the central charge c. The last passage is to integrate over the remaining part of the spin-weighted spherical harmonics in the polar angle θ.[1] We obtain the general expression

$$\frac{\langle h_{(\ell_1,m_1)} h_{(\ell_2,m_2)} h_{(\ell_1+\ell_2,m_1+m_2)} \rangle}{\langle h^2_{(\ell_1,m_1)} \rangle \langle h^2_{(\ell_2,m_2)} \rangle} = \frac{6\sqrt{2}}{2\pi} {}_{-2}C^{m_1,m_2,m_1+m_2}_{\ell_1,\ell_2,\ell_1+\ell_2} \frac{G^{3,3}_{3,3}\left(\begin{matrix} -im_1, & 0, & im_2 \\ 1 - im_1, & 1, & 1 + im_2 \end{matrix} \middle| e^{i\pi} \right)}{|\Gamma(2 - i(m_1+m_2))|^2}, \tag{28}$$

where

$$ {}_{-2}C^{m_1,m_2,m_3}_{\ell_1,\ell_2,\ell_3} = 2\pi \int_0^\pi \mathrm{d}\theta \sin\theta \, {}_{-2}Y_{(\ell_1,m_1)} \, {}_{-2}Y_{(\ell_2,m_2)} \, {}_{-2}\bar{Y}_{(\ell_3,m_3)} = \frac{\Gamma(-2 + \sum_{i=1}^3 \frac{|m_i|}{2}) \Gamma(4 + \sum_{i=1}^3 \frac{|m_i|}{2})}{2\sqrt{\pi}\Gamma(2 + \sum_{i=1}^3 |m_i|)} \times \left(\prod_{i=1}^3 \frac{(2|m_i| + 1)!}{(|m_i| + 2)!(|m_i| - 2)!} \right)^{1/2}, \tag{29}$$

with ${}_s\bar{Y}_{(\ell,m)} = (-1)^{s-m} {}_{-s}Y_{(\ell,-m)}$, valid for $\ell_1 = m_1$, $\ell_2 = m_2$, and $\ell_3 = m_3 = (m_1 + m_2)$. Taking $m_1 = m_2 = 2$, we get[2]

$$\frac{\langle h_{(2,2)} h_{(2,2)} h_{(4,4)} \rangle}{\langle h^2_{(2,2)} \rangle^2} \simeq 0.62 \cdot \frac{5}{24} \sqrt{\frac{7}{\pi}} \simeq 0.19, \tag{30}$$

which is quite in good agreement with the numerical results of Refs. [3,4]. To get closer to the spin value of 0.7 considered in those simulations, we can partially approximate departure from extremality using the corresponding temperature T_L from Eq. (15) in our expressions. We obtain in Eq. (30) the numerical value of 0.17, which is even astonishingly closer to the numerical result in Ref. [3]. For the modes $m_1 = 2$ and $m_2 = 3$, we find (summing up the two terms from the permutation of the modes $m = 2$ and 3)

$$\frac{\langle h_{(2,2)} h_{(3,3)} h_{(5,5)} \rangle}{\langle h^2_{(2,2)} \rangle \langle h^2_{(3,3)} \rangle} \simeq 1.57 \cdot \frac{2}{3} \sqrt{\frac{7}{11\pi}} \simeq 0.47, \tag{31}$$

matching again the value found in Ref. [3] and giving confidence on the validity of Eq. (28) (we obtain 0.45 taking into account the corresponding temperature T_L from Eq. (15) for spins equal to 0.7). Notice that the CFT calculation does not select the fundamental mode, but it captures as well the overtones (which have roughly the same real part of the corresponding frequency) close to the horizon. The numerical agreement between our results and those in Refs. [3,4] indicates that the nonlinearities are indeed sourced mostly, and not surprisingly, by the (2,2) fundamental mode. It would be nice to check the expression (28) against numerical quasicircular simulations giving rise to fast spinning remnants from which the various multipole correlators may be extracted.

IV. CONCLUSIONS

In this paper we have offered an argument based on the Kerr/CFT correspondence to evaluate the nonlinearities of the Kerr BH ringdown. We have shown that the Kerr/CFT correspondence provides a simple way to both explain at

[1] One spin-weighted spheroidal harmonics are adopted to decouple QNM angular modes our final results change by at most $\mathcal{O}(10\%)$.

[2] The fits in Ref. [3] are obtained for positive azimuthal numbers. We thank E. Berti and M. Cheung for exchanges about this technical point. However, notice that our results are symmetric under the change of sign of the azimuthal numbers.

the qualitative level, the quadratic scaling of the second-order mode amplitude with the product of the amplitudes of the fundamental modes, and also to quantitatively predict the size of this effect in striking agreement with numerical results found in recent literature.

Our findings hold only in the extremal case; our next step will be to study departures from this condition. Numerical results are indeed found close to, but not exactly at, extremality, i.e., for remnant dimensionless spins $\simeq 0.7$ [3,4]. Some corrections might therefore intervene. For small deviations from extremality, a different set of boundary conditions lead to a second copy of the Virasoro algebra [14] with indications of a hidden dual CFT even far from extremality [26–28], where the new excitations correspond to right movers. However, the fact that our findings are so close to the numerical fits of Refs. [3,4] might indicate that the value of the remnant spin is not so relevant (as it also appears from numerical results) and that the right-mover sector is decoupled from the dynamics. We have, indeed, reasons to believe that the right sector is associated to the $s = +2$ degree of freedom whose contribution to the GW strain is subleading at infinity with respect to the $s = -2$ mode [29]. Large nonlinearities have been observed also in the case of head-on collisions, giving rise to nonspinning BHs [3]. This fact as well might be explained using similar symmetry arguments, but in the context of Schwarzschild BHs [27]. The CFT approach may also provide new consistency relations among the QNM amplitudes and useful predictions for the nonlinearities involved in the QNM dynamics. Finally, it would also be interesting to go beyond the three-point correlator to see if sizeable effects persist at higher orders.

We will investigate these issues in order to provide generic predictions in terms of the multipole numbers and spins of the remnants in future work [29].

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