Nonequilibrium thermodynamics perspectives for the monotonicity of the renormalization group flow

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We investigate the monotonicity of the renormalization group (RG) flow from the perspectives of nonequilibrium thermodynamics. Applying the Martin-Siggia-Rose formalism to the Wilsonian RG transformation, we incorporate the RG flow equations manifestly in an effective action, where all coupling functions are dynamically promoted. As a result, we obtain an emergent holographic dual effective field theory, where an extra dimension appears from the Wilsonian RG transformation. We observe that Becchi-Rouet-Stora-Tyutin (BRST)-type transformations play an important role in the bulk effective action, which give rise to novel Ward identities for correlation functions between the renormalized coupling fields. As generalized fluctuation-dissipation theorems in the semiclassical nonequilibrium dynamics can be understood from the Ward identities of such BRST symmetries, we find essentially the same principle for the RG flow in the holographic dual effective field theory. Furthermore, we discuss how these "nonequilibrium work identities" can be related to the monotonicity of the RG flow, for example, the c-theorem. In particular, we introduce an entropy functional for the dynamical coupling field and show that the production rate of the total entropy functional is always positive, indicating the irreversibility of the RG flow.

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I. INTRODUCTION

The monotonicity of the renormalization group (RG) flow serves as one of the fundamental constraints for the dynamics of elementary degrees of freedom in quantum field theories [1–6]. This RG monotonicity is formulated as c-theorem, where c_{UV} at a UV fixed point should be larger than c_{IR} at an IR one. Here, c is the central charge, representing the number of degrees of freedom of a conformal field theory describing the corresponding fixed point. The c-theorem states that the entanglement entropy has to decrease along the RG flow whatever perturbations are applied to the original fixed point [7–14].

In this study, we revisit the monotonicity of the RG flow from the perspectives of nonequilibrium thermodynamics [15–20]. In this perspective, UV and IR fixed points can be regarded as equilibrium states while the RG flow may be regarded as a nonequilibrium path connecting these states. Viewing the RG flow as a dynamical process is a useful perspective, and explored, e.g., in Ref. [21]. In nonequilibrium statistical mechanics, it has been shown that the arrow of time in nonequilibrium dynamics can be formulated as generalized fluctuation-dissipation theorems such as Jarzynski's equality [22,23] and more microscopically, the Crooks relation [24–26]. More directly, the so-called entropy production has been shown to be responsible for such nonequilibrium work identities [27]. Here, we find essentially the same principle for the RG flow and discuss how these "nonequilibrium work identities" can be related to the monotonicity of the RG flow, for example, the c-theorem.

We point out that generalized fluctuation-dissipation theorems or nonequilibrium work identities can be derived from the symmetry principle in the Schwinger-Keldysh path integral formulation [28–33]. Here, the number of elementary degrees of freedom is doubled to cause some redundancies in the path integral description. As a result, certain topological symmetries involved with unitarity appear to be described by two types of Becchi-Rouet-Stora-Tyutin (BRST) symmetries. In addition to these topological symmetries, there are microscopic time-reversal symmetries referred to as Kubo-Martin-Schwinger (KMS) ones if the initial state is in thermal equilibrium. These KMS symmetries can be described by two additional fermion-type symmetries. It turns out that these four types of fermion symmetries form an extended $\mathcal{N}=2$

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equivariant cohomology algebra [34]. This $\mathcal{N}=2$ supersymmetry gives strong constraints to the nonequilibrium thermodynamics.

To avoid any possible confusion and uncertainties, we limit ourselves to the semiclassical nonequilibrium dynamics, for example, Langevin-type dynamics. Then, the Schwinger-Keldysh formulation is reduced into the Martin-Siggia-Rose (MSR) formalism [35–37] for the description of a nonequilibrium protocol with two boundary conditions (equilibrium states) [38]. In this case, the four kinds of BRST symmetries have been more well established to give Ward identities for correlation functions [15–19]. These Ward identities can be translated as nonequilibrium work identities, i.e., Jarzynski's and Crooks' identities.

In the present study, we apply this strategy to the Wilsonian RG transformation. Applying the MSR formalism to the Wilsonian RG transformation, we incorporate the RG flow equations manifestly in an effective action, where all coupling functions are dynamically promoted to be coupling fields. As a result, we obtain an emergent holographic dual effective field theory, where an extra dimension appears from the Wilsonian RG transformation [39–48]. Here, the holography is realized by the appearance of the extra dimension, identified with an RG scale, and the duality indicates that the effective field theory is written in terms of collective order parameter fields instead of the original fields as the Landau-Ginzburg free energy functional. We furthermore turn on irrelevant perturbations at the UV scale, which plays the role of noise in the language of stochastic dynamics. We observe that there exist four kinds of BRST-type emergent symmetries in the bulk effective action, which give rise to novel Ward identities for correlation functions between the renormalized coupling fields. As a result, we find a generalized fluctuation-dissipation theorem for the RG flow, where the standard form of the theorem is modified by the RG transformation. Based on this thermodynamics perspective, we discuss the monotonicity of the RG flow, introducing an effective entropy functional in terms of the coupling field. It turns out that the rate of the total entropy functional is always positive, indicating the irreversibility of the RG flow. This indicates how the generalized fluctuation-dissipation theorem can be related to the monotonicity of the RG flow, for example, the c-theorem.

II. A REVIEW ON STOCHASTIC THERMODYNAMICS IN THE LANGEVIN SYSTEM

Since the main objective of the present study is to reformulate the monotonicity or irreversibility of the RG flow from the stochastic thermodynamics perspective, it would be helpful to review some mathematical constructions for the stochastic thermodynamics [15–20,27,35–37]. As a prototypical example, we consider the overdamped dynamics of a particle in one dimension subject to a force, described by the Langevin equation,

$$\partial_t x(t) = \mu F(x(t), \lambda(t)) + \xi(t). \tag{1}$$

Here, $F(x(t), \lambda(t)) = -\partial_x V(x(t), \lambda(t)) + f(x(t), \lambda(t))$ is the force, where $V(x(t), \lambda(t))$ is a conservative potential and $f(x(t), \lambda(t))$ is an external force. These force sources may be time-dependent through an external control parameter $\lambda(t)$ varied according to some prescribed experimental protocol from $\lambda(0) = \lambda_0$ to $\lambda(t_f) = \lambda_f$. μ is the mobility of the particle. $\xi(t)$ serves as stochastic increments modeled as Gaussian white noise,

$$\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'), \tag{2}$$

where D is the diffusion constant, given by the Einstein relation $D = \beta^{-1}\mu$ at temperature $T = \beta^{-1}$ in equilibrium.

To investigate the symmetries of the Langevin equation, it is more convenient to consider a generating functional for physical observables, analogous to the partition function. Here, the following identity is essential,

$$1 = \int_{x_i}^{x_f} Dx(t)\delta(\partial_t x(t) - \mu F(x(t), \lambda(t)) - \xi(t))\det(\partial_t - \mu\partial_x F(x(t), \lambda(t))). \tag{3}$$

This δ-function identity is nothing but the Faddeev-Popov procedure for the path integral quantization of gauge fields [49]. As a result, one may propose a generating functional for the overdamped Langevin dynamics subject to a force as follows [15–20,35–37]

$$\begin{aligned}\mathcal{W} = \mathcal{N}\int_{x_i}^{x_f} Dx(t)Dp(t)Dc(t)D\bar{c}(t)\int D\xi(t)\exp\left(-\frac{1}{4D}\int_{t_i}^{t_f} dt\xi^2(t)\right) \\ \times\exp\left[-\int_{t_i}^{t_f} dt\{ip(t)(\partial_t x(t) - \mu F(x(t), \lambda(t)) - \xi(t)) + \bar{c}(t)(\partial_t - \mu\partial_x F(x(t), \lambda(t)))c(t)\}\right].\end{aligned} \tag{4}$$

Here, $p(t)$ is a Lagrange multiplier, identified with a canonical momentum to the position $x(t)$, and $c(t)$ is a fermion variable to take the Jacobian factor with its canonical conjugate partner $\bar{c}(t)$. $\mathcal{N}$ is a normalization constant to reproduce

Eq. (2). Performing the average with respect to random noise fluctuations, we obtain

$$\mathcal{W} = \mathcal{N}\int_{x_i}^{x_f} Dx(t)Dp(t)Dc(t)D\bar{c}(t)\exp\Big[-\int_{t_i}^{t_f} dt\{ip(t)(\partial_t x(t) - \mu F(x(t),\lambda(t))) + Dp^2(t) + \bar{c}(t)(\partial_t - \mu\partial_x F(x(t),\lambda(t)))c(t)\}\Big]. \quad (5)$$

Based on this path integral formulation, Refs. [15–20] investigated BRST symmetries and discussed Ward identities. In this study, we apply this framework to the RG flow and reveal symmetries of the RG flow.

One important ingredient involved with the monotonicity of the RG flow is entropy production in the Langevin system. Introducing the following probability distribution,

$$\begin{aligned} p(x,t) &= \langle\delta(x - x(t))\rangle \\ &= \mathcal{N}\int D\xi(t')\exp\Big(-\frac{1}{4D}\int_{t_i}^{t} dt'\xi^2(t')\Big)\delta(x - x(t)), \end{aligned} \quad (6)$$

where the average of random noise fluctuations is taken. Then, one obtains the Fokker-Planck equation for the probability distribution function to find the particle at x and at time t,

$$\partial_t p(x,t) = -\partial_x j(x,t) = -\partial_x[(\mu F(x,\lambda) - D\partial_x)p(x,t)]. \quad (7)$$

$j(x,t) = (\mu F(x,\lambda) - D\partial_x)p(x,t)$ is the conserved current. This partial differential equation must be augmented by a normalized initial distribution, $p(x,0) = p_0(x)$. In Appendix A, we show our intuitive derivation for this Fokker-Planck equation. It is straightforward to see the formal path integral expression for the probability distribution function as follows

$$p(x,t) = \frac{\mathcal{N}}{\mathcal{W}}\int D\xi(t')\exp\Big(-\frac{1}{4D}\int_{t_i}^{t} dt'\xi^2(t')\Big)\int_{x_i}^{x} Dx(t')Dp(t')D\bar{c}(t')Dc(t') \times\exp\Big[-\int_{t_i}^{t} dt'\{ip(t')(\partial_{t'}x(t') - \mu F(x(t'),\lambda(t')) - \xi(t')) + \bar{c}(t')(\partial_{t'} - \mu\partial_x F(x(t'),\lambda(t')))c(t')\}\Big], \quad (8)$$

where the normalization constant or the generating functional is given by Eq. (5). One can verify

$$\int_{x_i}^{x_f} dx p(x,t) = 1. \quad (9)$$

To discuss the entropy production in the forced overdamped Langevin dynamics, Ref. [27] proposed a trajectory-dependent entropy for the particle or system as

$$s_{\text{sys}}(x,t) = -\ln p(x,t). \quad (10)$$

This definition is consistent with the common definition of a nonequilibrium Gibbs entropy, given by

$$S_{\text{sys}}(t) = \langle s_{\text{sys}}(x,t)\rangle = -\int_{x_i}^{x_f} dx p(x,t)\ln p(x,t). \quad (11)$$

This microscopic entropy gives rise to the macroscopic thermodynamic entropy for an equilibrium Boltzmann distribution at fixed λ,

$$s_{\text{sys}}(x,t) = \beta[V(x,\lambda) - F(\lambda)], \quad (12)$$

where the equilibrium free energy $F(\lambda)$ is $F(\lambda) = -\beta^{-1}\ln\int_{x_i}^{x_f} dx e^{-\beta V(x,\lambda)}$ with the conserved potential $V(x,\lambda)$ introduced before. Then, it is natural to consider the rate of heat dissipation in the environment as

$$\partial_t q(x,t) = F(x,\lambda)\partial_t x(t) = \beta^{-1}\partial_t s_{\text{env}}(x,t). \quad (13)$$

Accordingly, one may identify the exchanged heat with an increase in the environment entropy $s_{\text{env}}(x,t)$ at temperature $\beta^{-1} = D/\mu$.

Combining these two contributions, Ref. [27] found the trajectory-dependent total entropy production rate as follows

$$\begin{aligned} \partial_t s_{\text{tot}}(x,t) &= \partial_t s_{\text{env}}(x,t) + \partial_t s_{\text{sys}}(x,t) \\ &= \frac{\partial_x j(x,t)}{p(x,t)} + \frac{j(x,t)}{Dp(x,t)}\partial_t x(t). \end{aligned} \quad (14)$$

Taking the ensemble average, Ref. [27] showed that the averaged total entropy production rate is always positive, given by

$$\partial_t S_{\rm tot}(t) = \langle \partial_t s_{\rm tot}(x,t) \rangle = \int_{x_i}^{x_f} dx \frac{j^2(x,t)}{Dp(x,t)} \geq 0, \qquad (15)$$

where the equality holds in equilibrium only. The ensemble-averaged entropy production rate of the environment is given by

$$\partial_t S_{\rm env}(x,t) = \langle \partial_t s_{\rm env}(x,t) \rangle = \beta \int_{x_i}^{x_f} dx F(x,t) j(x,t), \quad (16)$$

where the force $F(x,t)$ and the conserved current $j(x,t)$ have been introduced above. In this study, we discuss the entropy production of the RG flow, following this line of thought.

III. TO MANIFEST THE RENORMALIZATION GROUP FLOW IN THE LEVEL OF AN EFFECTIVE ACTION

A. Wilsonian renormalization group transformation

We consider a partition function as follows

$$Z(\Lambda_{uv}) = \int D\psi_\sigma(x;\Lambda_{uv}) \exp\left\{ -\int d^D x \mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\lambda_a(\Lambda_{uv})\}; \Lambda_{uv}] \right\}. \qquad (17)$$

Here, Λ_{uv} is a UV cutoff, where the corresponding effective Lagrangian $\mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\lambda_a(\Lambda_{uv})\}; \Lambda_{uv}]$ is defined. $\psi_\sigma(x;\Lambda_{uv})$ is a dynamical matter field at a given spacetime x, where σ denotes its flavor index $\sigma = 1, \ldots, N$. $\{\lambda_a(\Lambda_{uv})\}$ represents a set of coupling functions such as velocity, interaction coefficients, etc., denoted by the subscript a.

Performing the Wilsonian RG transformation, we obtain the following expression for the partition function

$$Z(z_f) = \int D\psi_\sigma(x;z_f) \exp\left\{ -\int d^D x \left(\mathcal{L}[\psi_\sigma(x;z_f); \{\lambda_a(x,z_f)\}; z_f] + N \int_{\Lambda_{uv}}^{z_f} dz \mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z] \right) \right\}, \qquad (18)$$

where the UV cutoff Λ_{uv} is lowered to be z_f. In other words, all the dynamical fields $\psi_\sigma(x;z_f)$ and all the coupling functions $\lambda_a(x,z_f)$ are defined at a lower cutoff z_f, where the dynamical fields between z_f and Λ_{uv} are integrated over to introduce an effective potential $N \int_{\Lambda_{uv}}^{z_f} dz \mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z]$ into the partition function [39–41].

Considering that the partition function is invariant under the RG transformation, regardless of the cutoff scale, we observe that the effective potential is

$$\mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z] = -\frac{1}{N} \ln \int_{\Lambda(z)} D\psi_\sigma(x;z) \exp\left\{ -\int d^D x \mathcal{L}[\psi_\sigma(x;z); \{\lambda_a(x,z)\}; z] \right\}, \qquad (19)$$

at a given scale z. Accordingly, all the coupling functions are renormalized to be

$$\frac{\partial \lambda_a(x,z)}{\partial z} = \beta_a[\{\lambda_a(x,z)\}; z]. \qquad (20)$$

Here, the RG β-function for a coupling function $\lambda_a(x,z)$ is given by the first-order derivative of the effective potential with respect to $\lambda_a(x,z)$ as follows

$$\beta_a[\{\lambda_a(x,z)\}; z] = -\frac{\partial \mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z]}{\partial \lambda_a(x,z)}. \qquad (21)$$

Implementing this calculation explicitly, we see that the RG β-function is given by a renormalized vertex function,

$$N\beta_a[\{\lambda_a(x,z)\}; z] = \frac{1}{Z(z)} \int_{\Lambda(z)} D\psi_\sigma(x;z) \left(\frac{\partial}{\partial \lambda_a(x,z)} \int d^D x \mathcal{L}[\psi_\sigma(x;z); \{\lambda_a(x,z)\}; z] \right) \times \exp\left\{ -\int d^D x \mathcal{L}[\psi_\sigma(x;z); \{\lambda_a(x,z)\}; z] \right\}, \qquad (22)$$

where $Z(z)$ is an effective partition function at a given scale z,

$$Z(z) = \int_{\Lambda(z)} D\psi_\sigma(x;z) \exp\left\{ -\int d^D x \mathcal{L}[\psi_\sigma(x;z); \{\lambda_a(x,z)\}; z] \right\}. \tag{23}$$

B. To manifest the renormalization group flow in the level of an effective action

To manifest the RG flow at the level of an effective action, we consider the following identity

$$1 = \int D\lambda_a(x,z) \delta(\partial_z \lambda_a(x,z) - \beta_a[\{\lambda_a(x,z)\}; z]) \det\left(\partial_z \delta_{ab} - \frac{\partial \beta_a[\{\lambda_a(x,z)\}; z]}{\partial \lambda_b(x,z)} \right). \tag{24}$$

Here, $\det(\partial_z \delta_{ab} - \frac{\partial \beta_a[\{\lambda_a(x,z)\};z]}{\partial \lambda_b(x,z)})$ may be regarded as a Jacobian factor for the functional integral. Introducing this δ-function identity into the partition function, we obtain

$$Z(z_f) = \int D\psi_\sigma(x,z_f) D\lambda_a(x,z) \delta(\partial_z \lambda_a(x,z) - \beta_a[\{\lambda_a(x,z)\}; z]) \det\left(\partial_z \delta_{ab} - \frac{\partial \beta_a[\{\lambda_a(x,z)\}; z]}{\partial \lambda_b(x,z)} \right)$$
$$\times \exp\left\{ -\int d^D x \left(\mathcal{L}[\psi_\sigma(x,z_f); \{\lambda_a(x,z_f)\}; z_f] + N \int_{\Lambda_{uv}}^{z_f} dz \mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z] \right) \right\}. \tag{25}$$

Now, the coupling function is promoted to be a dynamical coupling field, which appears as the path integral formulation with the δ-function constraint. This is essentially the same as the Faddeev-Popov procedure for the path integral quantization of gauge fields [49], also applied to the semiclassical nonequilibrium physics, for example, the path integral formulation of Langevin dynamics, and referred to as the MSR formalism [35–37] discussed before. Here, the RG flow corresponds to the Langevin equation.

It is straightforward to exponentiate the δ-function constraint as follows

$$Z(z_f) = \int D\psi_\sigma(x,z_f) D\lambda_a(x,z) D\pi_a(x,z) D\bar{c}_a(x,z) Dc_a(x,z) \exp\Bigg[-\int d^D x \mathcal{L}[\psi_\sigma(x,z_f); \{\lambda_a(x,z_f)\}; z_f]$$
$$- N \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \Bigg\{ \pi_a(x,z)(\partial_z \lambda_a(x,z) - \beta_a[\{\lambda_a(x,z)\}; z]) + \bar{c}_a(x,z) \left(\partial_z \delta_{ab} - \frac{\partial \beta_a[\{\lambda_a(x,z)\}; z]}{\partial \lambda_b(x,z)} \right) c_b(x,z)$$
$$+ \mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z] \Bigg\} \Bigg]. \tag{26}$$

$\pi_a(x,z)$ is a Lagrange multiplier field to impose the RG flow constraint, which corresponds to the canonical momentum of the coupling field $\lambda_a(x,z)$. In the Schwinger-Keldysh formulation, it is identified with a quantum field denoted by the subscript a or qu in the standard notation. $c_a(x,z)$ ($\bar{c}_a(x,z)$) is an auxiliary fermion field to take care of the Jacobian factor, referred to as the Faddeev-Popov ghost. z is an RG scale, which serves as a cutoff scale for the Wilsonian RG transformation. Interestingly, this RG scale plays the role of an extra dimension, which reminds us of the holographic duality conjecture [50–56], where $\mathcal{S}_{\rm eff}[\{\pi_a(x,z), \lambda_a(x,)\}, \{\bar{c}_a(x,z), c_a(x,z)\}; z_f, \Lambda_{uv}] = N \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \{\pi_a(x,z)(\partial_z \lambda_a(x,z) - \beta_a[\{\lambda_a(x,z)\}; z]) + \bar{c}_a(x,z)(\partial_z \delta_{ab} - \frac{\partial \beta_a[\{\lambda_a(x,z)\};z]}{\partial \lambda_b(x,z)}) c_b(x,z) + \mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z]\}$ corresponds to an effective bulk action, supported by an effective boundary action of $\int d^D x \mathcal{L}[\psi_\sigma(x,z_f); \{\lambda_a(x,z_f)\}; z_f]$. Here, the duality means that the bulk effective action is written in terms of the coupling fields $\{\lambda_a(x,z)\}$, regarded to as collective dual fields to the corresponding matter composites. In other words, $\lambda_a(x,z)$ is dual to $\frac{\partial \mathcal{L}[\psi_\sigma(x;z); \{\lambda_a(x,z)\}; z]}{\partial \lambda_a(x,z)}$.

Before going further, we point out an essential approximation in this effective field theory. First of all, nonlocal terms are neglected in the resulting effective action that manifests the RG flows of the coupling fields. We recall that the Wilsonian RG transformation generates nonlocal terms inevitably. Here, the RG β-function Eq. (21) is given by a Green's function of the corresponding matter field at a given energy scale z. Since the Green's function is bilocal, i.e., depending on x and x', nonlocality is unavoidable. Such emergent nonlocal interactions can however be "localized" at the cost of introducing higher-spin fields to decompose them in a local fashion based on the corresponding group structure [57–64]. In other words, integrating over such higher-spin fields can give rise to an effective gravity theory including only up to the spin two fields, but in the presence of effective nonlocal interactions between gravitons. In most cases, we will work with a proper local truncation of these RG-generated nonlocal terms, keeping the original form of the effective Lagrangian as in the conventional RG transformation [39–48]. Here, based on the gradient expansion in the limit of $\Delta x = x - x' \to 0$, we have only local terms in the resulting effective action. This issue was well summarized in Ref. [41].

C. To promote coupling functions to dynamical fields: Irrelevant deformations

Although the above reformulation for the Wilsonian RG transformation is rather analogous to the holographic dual effective field theory, there exists one important difference: The coupling field $\lambda_a(x,z)$ is not fully dynamical, whose dynamics is semiclassical, given by the RG flow equation. To promote the coupling field to be fully dynamical, we consider the following UV deformation,

$$Z(\Lambda_{uv}) = \int D\psi_\sigma(x;\Lambda_{uv}) D\lambda_a(x;\Lambda_{uv}) \exp\left\{ -\int d^D x \left(\mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\lambda_a(x;\Lambda_{uv})\}; \Lambda_{uv}] + \frac{1}{2\Gamma_a} [\lambda_a(x;\Lambda_{uv}) - \bar{\lambda}_a(\Lambda_{uv})]^2 \right) \right\}. \tag{27}$$

Here, we introduced random fluctuations of the coupling fields at UV, where Γ_a denotes the variance around the mean value $\bar{\lambda}_a(\Lambda_{uv})$. Taking the $\Gamma_a \to 0$ limit, we recover the previous formulation, where $\lambda_a(\Lambda_{uv})$ is replaced with $\bar{\lambda}_a(\Lambda_{uv})$. This UV deformation can be thought of as averaging over theories with different coupling constants [39] or turning on some irrelevant perturbations akin to the $T\bar{T}$ deformation [65].

To understand the physical meaning of this UV deformation more precisely, we perform the Gaussian integral with respect to $\lambda_a(x;\Lambda_{uv})$. Then, we obtain

$$Z(\Lambda_{uv}) = \int D\psi_\sigma(x;\Lambda_{uv}) \exp\left[-\int d^D x \left\{ \mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\bar{\lambda}_a(\Lambda_{uv})\}; \Lambda_{uv}] + \frac{\Gamma_a}{2} \left(\frac{\partial \mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\lambda_a(x;\Lambda_{uv})\}; \Lambda_{uv}]}{\partial \lambda_a(x;\Lambda_{uv})} \right)^2 \right\} \right]. \tag{28}$$

Suppose the Gross-Neveu model for spontaneous chiral symmetry breaking as $\mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\lambda_a(x;\Lambda_{uv})\}; \Lambda_{uv}] = \bar{\psi}_\sigma(x;\Lambda_{uv}) i\gamma^\mu \partial_\mu \psi_\sigma(x;\Lambda_{uv}) + \frac{\lambda_\chi(x;\Lambda_{uv})}{2} \bar{\psi}_\sigma(x;\Lambda_{uv}) \psi_\sigma(x;\Lambda_{uv}) \bar{\psi}_{\sigma'}(x;\Lambda_{uv}) \psi_{\sigma'}(x;\Lambda_{uv})$. Then, the last term is $(\frac{\partial \mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\lambda_a(x;\Lambda_{uv})\}; \Lambda_{uv}]}{\partial \lambda_a(x;\Lambda_{uv})})^2 \sim [\bar{\psi}_\sigma(x;\Lambda_{uv}) \psi_\sigma(x;\Lambda_{uv})]^4$ [48], generally irrelevant at the Gaussian fixed point and expected not to change the RG flow as long as weak Γ_χ is concerned. Considering these random fluctuations of the coupling functions at UV and performing the Wilsonian RG transformation, we obtain [39–41,48]

$$\begin{aligned} Z(z_f) = & \int D\psi_\sigma(x,z_f) D\lambda_a(x,z) D\pi_a(x,z) D\bar{c}_a(x,z) Dc_a(x,z) \\ & \times \exp\left[-\int d^D x \left(\mathcal{L}[\psi_\sigma(x,z_f); \{\lambda_a(x,z_f)\}; z_f] + \frac{1}{2\Gamma_a} [\lambda_a(x,\Lambda_{uv}) - \bar{\lambda}_a(\Lambda_{uv})]^2 \right) \right. \\ & - N \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left\{ \pi_a(x,z) (\partial_z \lambda_a(x,z) - \beta_a[\{\lambda_a(x,z)\}; z]) - \frac{\Gamma_a}{2} \pi_a^2(x,z) \right. \\ & \left. \left. + \bar{c}_a(x,z) \left(\partial_z \delta_{ab} - \frac{\partial \beta_a[\{\lambda_a(x,z)\}; z]}{\partial \lambda_b(x,z)} \right) c_b(x,z) + \mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z] \right\} \right], \end{aligned} \tag{29}$$

where $-\frac{\Gamma_a}{2} \pi_a^2(x,z)$ appeared to give the dynamics to $\lambda_a(x,z)$ in the extra dimension. It is trivial to check out that the $\Gamma_a \to 0$ limit reproduces Eq. (26).

In this study, we claim that the holographic dual effective field theory [Eq. (29)] enjoys essentially the same structure as the MSR formalism of the Langevin dynamics except for the following two aspects: One is the presence of the effective potential $\mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z]$, which arises from quantum fluctuations in the Wilsonian RG transformation, and the other is the existence of the boundary action, which defines both UV and IR boundary conditions for the bulk effective action. In spite of these two different aspects, we derive a generalized fluctuation-dissipation theorem for the RG flow and show the monotonicity of the RG flow to hold as the Langevin dynamics, where the existence of the effective potential gives rise to some corrections in the formulas.

IV. EMERGENT BRST "SYMMETRIES" IN THE RG FLOW

A. Four types of BRST transformations

Following the MSR formalism of the Langevin dynamics, we investigate emergent BRST symmetries of the RG-flow manifested effective field theory,

$$Z(z_f) = \int D\psi_\sigma(x,z_f) D\lambda(x,z) D\pi(x,z) D\bar{c}(x,z) Dc(x,z) \exp\bigg[-\int d^D x\bigg(\mathcal{L}[\psi_\sigma(x,z_f);\lambda(x,z_f);z_f] + \frac{N}{2\Gamma}[\lambda(x,\Lambda_{uv}) - \bar{\lambda}(\Lambda_{uv})]^2\bigg) - N\int_{\Lambda_{uv}}^{z_f} dz \int d^D x \bigg\{\pi(x,z)(\partial_z\lambda(x,z) - \beta[\lambda(x,z);z]) - \frac{\Gamma}{2}\pi^2(x,z) + \bar{c}(x,z)\bigg(\partial_z - \frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\bigg)c(x,z) + \mathcal{V}_{rg}[\lambda(x,z);z]\bigg\}\bigg]. \tag{30}$$

Here, we considered the case of one coupling field for simplicity.

One may consider four types of BRST transformations in this holographic dual effective field theory as the case of the Langevin dynamics [15–20]. We recall that in the Schwinger-Keldysh formulation, the first two BRST symmetries with their charges Q and $\bar{Q}$ are topological in origin, related with the unitarity. These two BRST symmetries do not commute with the KMS ones [33]. Considering both the BRST and KMS symmetries, we have to introduce additional two fermion-type symmetries with their charges D and $\bar{D}$. Although D and $\bar{D}$ correspond to the superderivatives in the superspace formulation as discussed in Appendix B, we also call these additional fermionic symmetries BRST-type symmetries. The first two BRST transformations lead the bulk kinetic energy $\pi(x,z)(\partial_z\lambda(x,z) - \beta[\lambda(x,z);z]) - \frac{\Gamma}{2}\pi^2(x,z) + \bar{c}(x,z)(\partial_z - \frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)})c(x,z)$ to be invariant while the last two do not. The effective potential $\mathcal{V}_{rg}[\lambda(x,z);z]$ does transform under all these BRST transformations. As a result, there do not exist any BRST-type emergent symmetries in this RG flow, precisely speaking. However, such BRST noninvariant terms are expressed in a "universal" way. As a result, we can derive generalized Ward identities from these four types of BRST transformations and find some constraints for correlation functions of the coupling field.

The first BRST transformation is given by

$$\delta_Q\lambda(x,z) = \epsilon[Q,\lambda(x,z)] = \epsilon c(x,z), \tag{31}$$

$$\delta_Q\pi(x,z) = \epsilon[Q,\pi(x,z)] = 0, \tag{32}$$

$$\delta_Q\bar{c}(x,z) = \epsilon[Q,\bar{c}(x,z)] = -\epsilon\pi(x,z), \tag{33}$$

$$\delta_Q c(x,z) = \epsilon[Q,c(x,z)] = 0, \tag{34}$$

where the first BRST charge Q is

$$Q = c(x,z)\frac{\delta}{\delta\lambda(x,z)} - \pi(x,z)\frac{\delta}{\delta\bar{c}(x,z)}. \tag{35}$$

Here, the infinitesimal parameter ϵ is fermionic. Then, the bulk effective Lagrangian is transformed into

$$\delta_Q\bigg\{\pi(x,z)(\partial_z\lambda(x,z) - \beta[\lambda(x,z);z]) - \frac{\Gamma}{2}\pi^2(x,z) + \bar{c}(x,z)\bigg(\partial_z - \frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\bigg)c(x,z) + \mathcal{V}_{rg}[\lambda(x,z);z]\bigg\} = \delta_Q\mathcal{V}_{rg}[\lambda(x,z);z] = -\epsilon c(x,z)\beta[\lambda(x,z);z], \tag{36}$$

which is not invariant due to the effective potential $\mathcal{V}_{rg}[\lambda(x,z);z]$.

The second BRST transformation is given by

$$\delta_{\bar{Q}}\lambda(x,z) = \bar{\epsilon}[\bar{Q},\lambda(x,z)] = \bar{\epsilon}\,\bar{c}(x,z), \tag{37}$$

$$\delta_{\bar{Q}}\pi(x,z) = \bar{\epsilon}[\bar{Q},\pi(x,z)] = \bar{\epsilon}\frac{2}{\Gamma}\partial_z\bar{c}(x,z), \tag{38}$$

$$\delta_{\bar{Q}}\bar{c}(x,z) = \bar{\epsilon}[\bar{Q},\bar{c}(x,z)] = 0, \tag{39}$$

$$\delta_{\bar{Q}}c(x,z) = \bar{\epsilon}[\bar{Q},c(x,z)] = \bar{\epsilon}\bigg(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\bigg), \tag{40}$$

where the second BRST charge $\bar{Q}$ is

$$\bar{Q} = \bar{c}(x,z)\frac{\delta}{\delta\lambda(x,z)} + \frac{2}{\Gamma}[\partial_z\bar{c}(x,z)]\frac{\delta}{\delta\pi(x,z)} + \bigg(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\bigg)\frac{\delta}{\delta c(x,z)}. \tag{41}$$

As a result, the bulk effective Lagrangian is transformed into

$$\delta_{\bar{Q}}\left\{\pi(x,z)(\partial_z\lambda(x,z)-\beta[\lambda(x,z);z])-\frac{\Gamma}{2}\pi^2(x,z)+\bar{c}(x,z)\left(\partial_z-\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\right)c(x,z)+\mathcal{V}_{rg}[\lambda(x,z);z]\right\}$$
$$=\bar{\epsilon}\frac{d}{dz}\left\{\frac{2}{\Gamma}\bar{c}(x,z)(\partial_z\lambda(x,z)-\beta[\lambda(x,z);z])-\bar{c}(x,z)\pi(x,z)\right\}-\bar{\epsilon}\,\bar{c}(x,z)\beta[\lambda(x,z);z], \tag{42}$$

where $\delta_{\bar{Q}}\mathcal{V}_{rg}[\lambda(x,z);z]=-\bar{\epsilon}\,\bar{c}(x,z)\beta[\lambda(x,z);z]$. The total derivative term does not affect the corresponding Ward identity to be discussed below.

The third BRST transformation is given by

$$\delta_D\lambda(x,z)=\varepsilon[D,\lambda(x,z)]=\varepsilon\bar{c}(x,z), \tag{43}$$

$$\delta_D\pi(x,z)=\varepsilon[D,\pi(x,z)]=0, \tag{44}$$

$$\delta_D\bar{c}(x,z)=\varepsilon[D,\bar{c}(x,z)]=0, \tag{45}$$

$$\delta_D c(x,z)=\varepsilon[D,c(x,z)]=\varepsilon\pi(x,z), \tag{46}$$

where the third BRST charge D is

$$D=\bar{c}(x,z)\frac{\delta}{\delta\lambda(x,z)}+\pi(x,z)\frac{\delta}{\delta c(x,z)}. \tag{47}$$

Accordingly, the bulk Lagrangian transforms as

$$\delta_D\left\{\pi(x,z)(\partial_z\lambda(x,z)-\beta[\lambda(x,z);z])-\frac{\Gamma}{2}\pi^2(x,z)+\bar{c}(x,z)\left(\partial_z-\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\right)c(x,z)+\mathcal{V}_{rg}[\lambda(x,z);z]\right\}$$
$$=2\varepsilon\pi(x,z)\partial_z\bar{c}(x,z)-\varepsilon\partial_z(\bar{c}(x,z)\pi(x,z))-\varepsilon\bar{c}(x,z)\beta[\lambda(x,z);z], \tag{48}$$

where $\delta_D\mathcal{V}_{rg}[\lambda(x,z);z]=-\varepsilon\bar{c}(x,z)\beta[\lambda(x,z);z]$. We point out an additional noninvariant term $2\varepsilon\pi(x,z)\partial_z\bar{c}(x,z)$.

The last BRST transformation is given by

$$\delta_{\bar{D}}\lambda(x,z)=\bar{\varepsilon}[\bar{D},\lambda(x,z)]=\bar{\varepsilon}c(x,z), \tag{49}$$

$$\delta_{\bar{D}}\pi(x,z)=\bar{\varepsilon}[\bar{D},\pi(x,z)]=\bar{\varepsilon}\frac{2}{\Gamma}\partial_z c(x,z), \tag{50}$$

$$\delta_{\bar{D}}\bar{c}(x,z)=\bar{\varepsilon}[\bar{D},\bar{c}(x,z)]=-\bar{\varepsilon}\left(\pi(x,z)-\frac{2}{\Gamma}\partial_z\lambda(x,z)\right), \tag{51}$$

$$\delta_{\bar{D}}c(x,z)=\bar{\varepsilon}[\bar{D},c(x,z)]=0, \tag{52}$$

where the last BRST charge $\bar{D}$ is

$$\bar{D}=c(x,z)\frac{\delta}{\delta\lambda(x,z)}+\frac{2}{\Gamma}[\partial_z c(x,z)]\frac{\delta}{\delta\pi(x,z)}-\left(\pi(x,z)-\frac{2}{\Gamma}\partial_z\lambda(x,z)\right)\frac{\delta}{\delta\bar{c}(x,z)}. \tag{53}$$

The bulk effective Lagrangian transforms as

$$\delta_{\bar{D}}\left\{\pi(x,z)(\partial_z\lambda(x,z)-\beta[\lambda(x,z);z])-\frac{\Gamma}{2}\pi^2(x,z)+\bar{c}(x,z)\left(\partial_z-\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\right)c(x,z)+\mathcal{V}_{rg}[\lambda(x,z);z]\right\}$$
$$=-2\bar{\varepsilon}\pi(x,z)[\partial_z c(x,z)]+\bar{\varepsilon}\frac{4}{\Gamma}[\partial_z c(x,z)][\partial_z\lambda(x,z)]-\bar{\varepsilon}\frac{2}{\Gamma}\partial_z(c(x,z)\beta[\lambda_a(x,z);z])-\bar{\varepsilon}c(x,z)\beta[\lambda(x,z);z], \tag{54}$$

where $\delta_{\bar{D}}\mathcal{V}_{rg}[\lambda(x,z);z] = -\bar{\varepsilon}c(x,z)\beta[\lambda(x,z);z]$. We point out an additional noninvariant term $-2\bar{\varepsilon}\pi(x,z)[\partial_z c(x,z)] + \bar{\varepsilon}\frac{4}{\Gamma}[\partial_z c(x,z)][\partial_z\lambda(x,z)]$.

Before discussing the Ward identities for correlation functions of the coupling field, we check out anticommutators between BRST charges. The anticommutator for the first two BRST charges is given by

$$[Q,\bar{Q}]_+ = Q\bar{Q} + \bar{Q}Q = -\frac{2}{\Gamma}\left(\pi(x,z)\partial_z\frac{\delta}{\delta\pi(x,z)} + \partial_z\lambda(x,z)\frac{\delta}{\delta\lambda(x,z)} + c(x,z)\partial_z\frac{\delta}{\delta c(x,z)} + [\partial_z\bar{c}(x,z)]\frac{\delta}{\delta\bar{c}(x,z)}\right), \tag{55}$$

and that of the last two BRST ones is given by

$$[D,\bar{D}]_+ = D\bar{D} + \bar{D}D = \frac{2}{\Gamma}\left(\pi(x,z)\partial_z\frac{\delta}{\delta\pi(x,z)} + \partial_z\lambda(x,z)\frac{\delta}{\delta\lambda(x,z)} + [\partial_z c(x,z)]\frac{\delta}{\delta c(x,z)} + \bar{c}(x,z)\partial_z\frac{\delta}{\delta\bar{c}(x,z)}\right). \tag{56}$$

In Appendix B, we revisit these two anticommutation relations between BRST charges in the superspace formulation.

B. Generalized fluctuation-dissipation theorems for the RG flows of correlation functions of the coupling functions

To derive the Ward identities from these BRST transformations, we consider an action for sources as follows [18,49]

$$\begin{aligned}
\mathcal{S}_{Source} &= N\int_{\Lambda_{uv}}^{z_f} dz\int d^D x(\bar{T}(x,z)\lambda(x,z) + \pi(x,z)T(x,z) + \bar{G}(x,z)c(x,z) + \bar{c}(x,z)G(x,z)) \\
&= N\int_{\Lambda_{uv}}^{z_f} dz\int d^D x\left(\bar{T}(x,z)\frac{\partial}{\partial\bar{T}(x,z)} + T(x,z)\frac{\partial}{\partial T(x,z)} + \bar{G}(x,z)\frac{\partial}{\partial\bar{G}(x,z)} - \frac{\partial}{\partial G(x,z)}G(x,z)\right) \\
&= N\int_{\Lambda_{uv}}^{z_f} dz\int d^D x\left(\frac{\partial}{\partial\lambda(x,z)}\lambda(x,z) + \frac{\partial}{\partial\pi(x,z)}\pi(x,z) - \frac{\partial}{\partial c(x,z)}c(x,z) + \bar{c}(x,z)\frac{\partial}{\partial\bar{c}(x,z)}\right).
\end{aligned} \tag{57}$$

Here, $\bar{T}(x,z)$ ($T(x,z)$) is the bosonic source field for $\lambda(x,z)$ ($\pi(x,z)$), and $\bar{G}(x,z)$ ($G(x,z)$) is the fermionic source field for $c(x,z)$ ($\bar{c}(x,z)$). Accordingly, the four BRST charges are represented as follows

$$Q = c(x,z)\frac{\delta}{\delta\lambda(x,z)} - \pi(x,z)\frac{\delta}{\delta\bar{c}(x,z)} = \bar{T}(x,z)\frac{\partial}{\partial\bar{G}(x,z)} - G(x,z)\frac{\partial}{\partial T(x,z)}, \tag{58}$$

$$\begin{aligned}
\bar{Q} &= \bar{c}(x,z)\frac{\delta}{\delta\lambda(x,z)} + \frac{2}{\Gamma}[\partial_z\bar{c}(x,z)]\frac{\delta}{\delta\pi(x,z)} + \left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right)\frac{\delta}{\delta c(x,z)} \\
&= -\bar{T}(x,z)\frac{\partial}{\partial G(x,z)} - \frac{2}{\Gamma}T(x,z)\left(\partial_z\frac{\partial}{\partial G(x,z)}\right) - \bar{G}(x,z)\left(\frac{\partial}{\partial T(x,z)} - \frac{2}{\Gamma}\partial_z\frac{\partial}{\partial\bar{T}(x,z)}\right),
\end{aligned} \tag{59}$$

and

$$D = \bar{c}(x,z)\frac{\delta}{\delta\lambda(x,z)} + \pi(x,z)\frac{\delta}{\delta c(x,z)} = -\bar{T}(x,z)\frac{\partial}{\partial G(x,z)} - \bar{G}(x,z)\frac{\partial}{\partial T(x,z)}, \tag{60}$$

$$\begin{aligned}
\bar{D} &= c(x,z)\frac{\delta}{\delta\lambda(x,z)} + \frac{2}{\Gamma}[\partial_z c(x,z)]\frac{\delta}{\delta\pi(x,z)} - \left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right)\frac{\delta}{\delta\bar{c}(x,z)} \\
&= \bar{T}(x,z)\frac{\partial}{\partial\bar{G}(x,z)} + \frac{2}{\Gamma}T(x,z)\left(\partial_z\frac{\partial}{\partial\bar{G}(x,z)}\right) - G(x,z)\left(\frac{\partial}{\partial T(x,z)} - \frac{2}{\Gamma}\partial_z\frac{\partial}{\partial\bar{T}(x,z)}\right),
\end{aligned} \tag{61}$$

respectively.

Taking the first two BRST transformations to the partition function, we find

$$\int_{\Lambda_{uv}}^{z_f} dz\int d^D x\left(\bar{T}(x,z)\frac{\partial}{\partial\bar{G}(x,z)} - G(x,z)\frac{\partial}{\partial T(x,z)}\right)Z(z_f) = \frac{1}{2}\int_{\Lambda_{uv}}^{z_f} dz\int d^D x\beta\left(\frac{\partial}{\partial\bar{T}(x,z)};z\right)\frac{\partial}{\partial\bar{G}(x,z)}Z(z_f), \tag{62}$$

$$\int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left\{ \bar{T}(x,z) \frac{\partial}{\partial G(x,z)} + \frac{2}{\Gamma} T(x,z) \left(\partial_z \frac{\partial}{\partial G(x,z)} \right) + \bar{G}(x,z) \left(\frac{\partial}{\partial T(x,z)} - \frac{2}{\Gamma} \partial_z \frac{\partial}{\partial \bar{T}(x,z)} \right) \right\} Z(z_f)$$
$$= \frac{1}{2} \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left[-\partial_z \left\{ \frac{2}{\Gamma} \left(\partial_z \frac{\partial}{\partial \bar{T}(x,z)} - \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \right) \frac{\partial}{\partial G(x,z)} - \frac{\partial}{\partial T(x,z)} \frac{\partial}{\partial G(x,z)} \right\} + \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \frac{\partial}{\partial G(x,z)} \right] Z(z_f). \tag{63}$$

Considering the second two BRST transformations to the partition function, we obtain

$$\int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left(\bar{T}(x,z) \frac{\partial}{\partial G(x,z)} + \bar{G}(x,z) \frac{\partial}{\partial T(x,z)} \right) Z(z_f)$$
$$= \frac{1}{2} \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left[\partial_z \left(\frac{\partial}{\partial T(x,z)} \frac{\partial}{\partial G(x,z)} \right) - 2 \frac{\partial}{\partial T(x,z)} \partial_z \frac{\partial}{\partial G(x,z)} + \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \frac{\partial}{\partial G(x,z)} \right] Z(z_f), \tag{64}$$

$$\int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left\{ \bar{T}(x,z) \frac{\partial}{\partial \bar{G}(x,z)} + \frac{2}{\Gamma} T(x,z) \left(\partial_z \frac{\partial}{\partial \bar{G}(x,z)} \right) - G(x,z) \left(\frac{\partial}{\partial T(x,z)} - \frac{2}{\Gamma} \partial_z \frac{\partial}{\partial \bar{T}(x,z)} \right) \right\} Z(z_f)$$
$$= \frac{1}{2} \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left[\frac{2}{\Gamma} \partial_z \left\{ \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \frac{\partial}{\partial \bar{G}(x,z)} \right\} + 2 \frac{\partial}{\partial T(x,z)} \partial_z \frac{\partial}{\partial \bar{G}(x,z)} - \frac{4}{\Gamma} \left(\partial_z \frac{\partial}{\partial \bar{T}(x,z)} \right) \partial_z \frac{\partial}{\partial \bar{G}(x,z)} + \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \frac{\partial}{\partial \bar{G}(x,z)} \right] Z(z_f). \tag{65}$$

Equations (62), (63), (64), and (65) are one of the main results of this study. Based on these four types of equations, one can derive various Ward identities for correlation functions of the coupling field. Here, we demonstrate some of them. Applying $\frac{\partial}{\partial G(x',z')} \frac{\partial}{\partial \bar{T}(x'',z'')}$ to Eq. (62) and $\frac{\partial}{\partial \bar{G}(x',z')} \frac{\partial}{\partial \bar{T}(x'',z'')}$ to Eq. (63), respectively, we obtain

$$\left(\frac{\partial}{\partial G(x',z')} \frac{\partial}{\partial \bar{G}(x'',z'')} - \frac{\partial}{\partial \bar{T}(x'',z'')} \frac{\partial}{\partial T(x',z')} \right) Z(z_f) = \frac{1}{2} \frac{\partial}{\partial G(x',z')} \frac{\partial}{\partial \bar{T}(x'',z'')} \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \frac{\partial}{\partial \bar{G}(x,z)} Z(z_f), \tag{66}$$

$$\left(\frac{\partial}{\partial \bar{G}(x',z')} \frac{\partial}{\partial G(x'',z'')} + \frac{\partial}{\partial \bar{T}(x'',z'')} \frac{\partial}{\partial T(x',z')} - \frac{2}{\Gamma} \frac{\partial}{\partial \bar{T}(x'',z'')} \partial_{z'} \frac{\partial}{\partial \bar{T}(x',z')} \right) Z(z_f)$$
$$= \frac{1}{2} \frac{\partial}{\partial \bar{G}(x',z')} \frac{\partial}{\partial \bar{T}(x'',z'')} \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \frac{\partial}{\partial G(x,z)} Z(z_f), \tag{67}$$

where all other source fields were set to zero. These two equations lead to

$$\langle \bar{c}(x',z') c(x,z) \rangle + \langle \lambda(x,z) \pi(x',z') \rangle = -\frac{1}{2} \langle \bar{c}(x',z') \lambda(x,z) \int_{\Lambda_{uv}}^{z_f} dw \int d^D y \beta[\lambda(y,w); w] c(y,w) \rangle, \tag{68}$$

$$\langle c(x',z') \bar{c}(x,z) \rangle - \langle \lambda(x,z) \pi(x',z') \rangle + \frac{2}{\Gamma} \langle \lambda(x,z) \partial_{z'} \lambda(x',z') \rangle = \frac{1}{2} \langle c(x',z') \lambda(x,z) \int_{\Lambda_{uv}}^{z_f} dw \int d^D y \beta[\lambda(y,w); w] \bar{c}(y,w) \rangle, \tag{69}$$

respectively.

Considering the ghost Green's function $(\partial_z - \frac{\partial \beta[\lambda(x,z);z]}{\partial \lambda(x,z)}) \langle c(x,z) \bar{c}(x',z') \rangle = -\delta^{(D)}(x-x') \delta(z-z')$, we obtain

$$\langle \lambda(x',z') \pi(x,z) \rangle - \langle \lambda(x,z) \pi(x',z') \rangle = -\frac{2}{\Gamma} \langle \lambda(x,z) \partial_{z'} \lambda(x',z') \rangle - \langle \lambda(x',z') \left(\partial_z - \frac{\partial \beta[\lambda(x,z);z]}{\partial \lambda(x,z)} \right)^{-1} \beta[\lambda(x,z);z] \rangle \tag{70}$$

from Eqs. (68) and (69). If we consider a fixed point defined by $\beta[\lambda(x,z);z] = 0$, we obtain

$$\langle\lambda(x',z')\pi(x,z)\rangle-\langle\lambda(x,z)\pi(x',z')\rangle=-\frac{2}{\Gamma}\langle\lambda(x,z)\partial_{z'}\lambda(x',z')\rangle.$$

This is essentially the same as the fluctuation-dissipation theorem of the Langevin dynamics in equilibrium. Away from the fixed point, there is an RG flow given by the RG β-function, which plays the role of the nonequilibrium work in the dynamics, reflected in the last term of Eq. (70).

C. RG flow of an on-shell effective action: Hamilton-Jacobi equation

To discuss the RG flow of an IR effective action, we take the large N limit and obtain equations of motion with boundary conditions. We recall the holographic dual field theory,

$$Z(z_f)=\int D\psi_\sigma(x,z_f)D\lambda(x,z)D\pi(x,z)D\bar{c}(x,z)Dc(x,z)\exp\Bigg[-\int d^Dx\bigg(\mathcal{L}[\psi_\sigma(x,z_f);\lambda(x,z_f);z_f]+\frac{N}{2\Gamma}[\lambda(x,\Lambda_{uv})-\bar{\lambda}(\Lambda_{uv})]^2\bigg)$$
$$-N\int_{\Lambda_{uv}}^{z_f}dz\int d^Dx\bigg\{\pi(x,z)(\partial_z\lambda(x,z)-\beta[\lambda(x,z);z])-\frac{\Gamma}{2}\pi^2(x,z)+\bar{c}(x,z)\bigg(\partial_z-\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\bigg)c(x,z)+\mathcal{V}_{rg}[\lambda(x,z);z]\bigg\}\Bigg].$$

Taking the large N limit and performing variations with respect to $\pi(x,z)$ and $\lambda(x,z)$, we obtain the Hamiltonian equation of motion as follows

$$\pi(x,z)=\frac{1}{\Gamma}(\partial_z\lambda(x,z)-\beta[\lambda(x,z);z]),\tag{71}$$

$$\partial_z\pi(x,z)=-\pi(x,z)\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}-\bigg(\partial_z-\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\bigg)^{-1}\frac{\partial^2\beta[\lambda(x,z);z]}{\partial\lambda^2(x,z)}-\beta[\lambda(x,z);z].\tag{72}$$

We recall

$$\beta[\lambda(x,z);z]=-\frac{\partial\mathcal{V}_{rg}[\lambda(x,z);z]}{\partial\lambda(x,z)}.$$

Just for completeness, we write down the Lagrangian formulation for the partition function,

$$Z(z_f)=\int D\psi_\sigma(x,z_f)D\lambda(x,z)D\bar{c}(x,z)Dc(x,z)\exp\Bigg[-\int d^Dx\bigg(\mathcal{L}[\psi_\sigma(x,z_f);\lambda(x,z_f);z_f]+\frac{N}{2\Gamma}[\lambda(x,\Lambda_{uv})-\bar{\lambda}(\Lambda_{uv})]^2\bigg)$$
$$-N\int_{\Lambda_{uv}}^{z_f}dz\int d^Dx\bigg\{\frac{1}{2\Gamma}(\partial_z\lambda(x,z)-\beta[\lambda(x,z);z])^2+\bar{c}(x,z)\bigg(\partial_z-\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\bigg)c(x,z)+\mathcal{V}_{rg}[\lambda(x,z);z]\bigg\}\Bigg],\tag{73}$$

and obtain the corresponding Lagrange equation of motion,

$$\frac{1}{\Gamma}\partial_z^2\lambda(x,z)=-\beta[\lambda(x,z);z]+\frac{1}{\Gamma}\beta[\lambda(x,z);z]\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}-\bigg(\partial_z-\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\bigg)^{-1}\frac{\partial^2\beta[\lambda(x,z);z]}{\partial\lambda^2(x,z)}.\tag{74}$$

To obtain boundary conditions, we consider an effective boundary action as

$$\mathcal{S}_{\rm eff}(z_f)=N\int d^Dx\bigg(\mathcal{V}_{rg}[\lambda(x,z_f);z_f]+\pi(x,z_f)\lambda(x,z_f)+\bar{c}(x,z_f)c(x,z_f)$$
$$+\frac{N}{2\Gamma}[\lambda(x,\Lambda_{uv})-\bar{\lambda}(\Lambda_{uv})]^2-\pi(x,\Lambda_{uv})\lambda(x,\Lambda_{uv})-\bar{c}(x,\Lambda_{uv})c(x,\Lambda_{uv})\bigg).\tag{75}$$

Here, the boundary effective potential $\mathcal{V}_{rg}[\lambda(x,z_f);z_f]$ comes from

$$\mathcal{V}_{rg}[\lambda(x,z_f);z_f]=-\frac{1}{N}\ln\int_{\Lambda(z_f)}D\psi_\sigma(x;z_f)\exp\bigg\{-\int d^Dx\mathcal{L}[\psi_\sigma(x,z_f);\lambda(x,z_f);z_f]\bigg\},\tag{76}$$

and $\pi(x,z_f)\lambda(x,z_f)+\bar{c}(x,z_f)c(x,z_f)-\pi(x,\Lambda_{uv})\lambda(x,\Lambda_{uv})-\bar{c}(x,\Lambda_{uv})c(x,\Lambda_{uv})$ results from the bulk action $\pi(x,z)\partial_z\lambda(x,z)+\bar{c}(x,z)\partial_z c(x,z)$ by integration by parts. Taking variations of this effective boundary action with respect to $\lambda(x,z_f)$, $c(x,z_f)$ and $\lambda(x,\Lambda_{uv})$, $c(x,\Lambda_{uv})$, we find the IR boundary conditions

$$\pi(x,z_f)=\beta[\lambda(x,z_f);z_f], \qquad \bar{c}(x,z_f)=0, \tag{77}$$

and UV ones

$$\pi(x,\Lambda_{uv})=\frac{N}{\Gamma}[\lambda(x,\Lambda_{uv})-\bar{\lambda}(\Lambda_{uv})], \quad \bar{c}(x,\Lambda_{uv})=0. \tag{78}$$

Combined with the canonical momentum Eq. (71), we obtain both UV and IR boundary conditions for $\lambda(x,z)$, which support the second order Lagrange equation of motion for $\lambda(x,z)$.

We emphasize again that the partition function is invariant under the RG transformation, formulated as

$$-\frac{d}{dz_f}\ln Z(z_f)=0. \tag{79}$$

This equation gives rise to

$$-\left(\frac{\Gamma}{2}+1\right)\left(\frac{\partial\mathcal{V}_{rg}[\lambda(x,z_f);z_f]}{\partial\lambda(x,z_f)}\right)^2+\mathcal{V}_{rg}[\lambda(x,z_f);z_f]+\partial_{z_f}\mathcal{V}_{rg}[\lambda(x,z_f);z_f]=0, \tag{80}$$

where the IR boundary condition $\pi(x,z_f)=\beta[\lambda(x,z_f);z_f]$ has been used. This is nothing but the Hamilton-Jacobi equation to determine the IR renormalized effective potential $\mathcal{V}_{rg}[\lambda(x,z_f);z_f]$ [40,41]. One may regard this Hamilton-Jacobi equation as a signature to guarantee self-consistency of the present framework [40,41], where the IR effective potential is given by Eq. (76).

D. "Entropy production" in the RG flow

To investigate the monotonicity or "irreversibility" of the RG flow, we discuss "entropy production" in the RG flow, following the procedure for the overdamped Langevin system in Ref. [27]. We recall the effective partition function to manifest the RG flow as an effective bulk action with an extra dimension,

$$\begin{aligned}Z(z_f)=&\int_{\xi_{uv}}^{\xi_{ir}}D\xi(x,z)\exp\left\{-N\int_{\Lambda_{uv}}^{z_f}dz\int d^Dx\frac{1}{2\Gamma}\xi^2(x,z)\right\}\int_{\lambda_{uv}}^{\lambda_{ir}}D\lambda(x,z)D\pi(x,z)D\bar{c}(x,z)Dc(x,z)\\&\times\exp\left[-N\int_{\Lambda_{uv}}^{z_f}dz\int d^Dx\left\{\pi(x,z)(\partial_z\lambda(x,z)-\beta[\lambda(x,z);z]-\xi(x,z))+\bar{c}(x,z)\left(\partial_z-\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\right)c(x,z)\right.\right.\\&\left.\left.+\mathcal{V}_{rg}[\lambda(x,z);z]\right\}\right].\end{aligned} \tag{81}$$

Here, random noise fluctuations were explicitly introduced by the Hubbard-Stratonovich transformation for the bulk canonical momentum. The IR boundary conditions were assumed, where both λ_{ir} and ξ_{ir} can be determined by the IR boundary conditions of $\lambda(x,z_f)$ and $\pi(x,z_f)$. Accordingly, the "Hamiltonian equation of motion" is given by

$$\partial_z\lambda(x,z)=\beta[\lambda(x,z);z]+\xi(x,z), \tag{82}$$

$$\partial_z\pi(x,z)=-\beta[\lambda(x,z);z]-\pi(x,z)\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}-\left(\partial_z-\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\right)^{-1}\frac{\partial^2\beta[\lambda(x,z);z]}{\partial\lambda^2(x,z)}. \tag{83}$$

The first equation corresponds to the overdamped Langevin equation, where the coupling field $\lambda(x,z)$ and the RG scale z may be identified with the position of a particle and time.

The "probability distribution" function for the coupling field is defined as follows

$$\begin{aligned}\rho(\lambda,z)&=\langle\delta(\lambda-\lambda(x,z))\rangle\\&=\mathcal{N}\int D\xi(x,z')\exp\left\{-N\int_{\Lambda_{uv}}^{z}dz'\int d^Dx\left(\frac{1}{2\Gamma}\xi^2(x,z')+\mathcal{V}_{rg}[\lambda(x,z');z']\right)\right\}\delta(\lambda-\lambda(x,z)),\end{aligned} \tag{84}$$

where $\mathcal{N}$ is a normalization constant to be specified below. We emphasize that there appears a correction in the RG flow, given by $\mathcal{V}_{rg}[\lambda(x,z');z']$. Then, the path integral expression of this probability distribution function is given by

$$\rho(\lambda, z) = \frac{1}{Z(z_f)} \int_{\lambda_{uv}}^{\lambda} D\lambda(x, z') D\pi(x, z') D\bar{c}(x, z') Dc(x, z') \times \int D\xi(x, z') \exp\left\{-N \int_{\Lambda_{uv}}^{z} dz' \int d^D x \left(\frac{1}{2\Gamma}\xi^2(x, z') + \mathcal{V}_{rg}[\lambda(x, z'); z']\right)\right\} \times \exp\left[-N \int_{\Lambda_{uv}}^{z} dz' \int d^D x \left\{\pi(x, z')(\partial_{z'}\lambda(x, z') - \beta[\lambda(x, z'); z'] - \xi(x, z')) + \bar{c}(x, z')\left(\partial_{z'} - \frac{\partial\beta[\lambda(x, z'); z']}{\partial\lambda(x, z')}\right)c(x, z')\right\}\right], \tag{85}$$

where the normalization constant is given by the partition function introduced above. One can check out

$$\mathrm{tr}\rho(\lambda, z) = \int_{\lambda_{uv}}^{\lambda_{ir}} d\lambda \rho(\lambda, z) = 1. \tag{86}$$

Following the standard procedure to derive the Fokker-Planck equation from the Langevin equation, we obtain

$$(\partial_z - \mathcal{V}_{rg}(\lambda, z))\rho(\lambda, z) = -\partial_\lambda\left\{\left(\beta(\lambda, z) - \frac{\Gamma}{2}\partial_\lambda\right)\rho(\lambda, z)\right\}, \tag{87}$$

where the RG effective potential $\mathcal{V}_{rg}(\lambda, z)$ serves as the "time" component of a background gauge field. The conserved current is given by

$$j(\lambda, z) = \left(\beta(\lambda, z) - \frac{\Gamma}{2}\partial_\lambda\right)\rho(\lambda, z), \tag{88}$$

which shares essentially the same structure as that of the overdamped Langevin dynamics, discussed before. In Appendix A, we show our intuitive derivation for this Fokker-Planck equation.

Following Ref. [27], it is natural to introduce the entropy of a system, given by

$$s_{\mathrm{sys}}(\lambda, z) = -\ln\rho(\lambda, z). \tag{89}$$

Then, the ensemble average of the system or bulk entropy is

$$S_{\mathrm{sys}}(z) = \langle s_{\mathrm{sys}}(\lambda, z)\rangle = -\int_{\lambda_{uv}}^{\lambda_{ir}} d\lambda \rho(\lambda, z)\ln\rho(\lambda, z), \tag{90}$$

as expected.

The time evolution of the bulk entropy is given by

$$\partial_z s_{\mathrm{sys}}(z) = -\frac{\partial_z\rho(\lambda, z)}{\rho(\lambda, z)} - \frac{\partial_\lambda\rho(\lambda, z)}{\rho(\lambda, z)}\partial_z\lambda(x, z). \tag{91}$$

Resorting to the Fokker-Planck equation and considering the definition of the conserved current, we rewrite the above expression as follows

$$\partial_z s_{\mathrm{sys}}(z) = \frac{\partial_\lambda j(\lambda, z)}{\rho(\lambda, z)} - \mathcal{V}_{rg}(\lambda, z) + \frac{2}{\Gamma}\frac{j(\lambda, z)}{\rho(\lambda, z)}\partial_z\lambda(x, z) - \frac{2}{\Gamma}\beta(\lambda, z)\partial_z\lambda(x, z). \tag{92}$$

Here, we introduce the time evolution of the "environment" entropy in a similar way as Ref. [27],

$$\partial_z s_{\mathrm{env}}(\lambda, z) = \partial_z q(\lambda, z) = \frac{2}{\Gamma}\beta(\lambda, z)\partial_z\lambda(x, z) + \mathcal{V}_{rg}(\lambda, z). \tag{93}$$

$\partial_z q(\lambda, z)$ is the rate of heat dissipation in the medium, where we identify the exchanged heat with an increase in entropy of the medium.

Summing over these two contributions, we obtain

$$\partial_z s_{\mathrm{tot}}(\lambda, z) = \partial_z s_{\mathrm{env}}(\lambda, z) + \partial_z s_{\mathrm{sys}}(\lambda, z) = \frac{\partial_\lambda j(\lambda, z)}{\rho(\lambda, z)} + \frac{2}{\Gamma}\frac{j(\lambda, z)}{\rho(\lambda, z)}\partial_z\lambda(x, z), \tag{94}$$

fully consistent with that of the overdamped Langevin dynamics [27], although there exists a clear modification in the Fokker-Planck equation, Eq. (87). As a result, we find the irreversibility of the RG flow, given by the total entropy function,

$$\partial_z S_{\mathrm{tot}}(z) = \langle\partial_z s_{\mathrm{tot}}(\lambda, z)\rangle = \int_{\lambda_{uv}}^{\lambda_{ir}} d\lambda \frac{2}{\Gamma}\frac{j^2(\lambda, z)}{\rho(\lambda, z)} \geq 0, \tag{95}$$

where the ensemble average has been taken, and the following current conservation has been used,

$$\left\langle\frac{\partial_\lambda j(\lambda, z)}{\rho(\lambda, z)}\right\rangle = \int_{\lambda_{uv}}^{\lambda_{ir}} d\lambda \partial_\lambda j(\lambda, z) = 0. \tag{96}$$

More explicitly, we have

$$\langle\partial_z s_{\mathrm{tot}}(\lambda, z)\rangle = \int_{\lambda_{uv}}^{\lambda_{ir}} d\lambda \rho(\lambda, z)\left\{\frac{2}{\Gamma}[\beta(\lambda, z)]^2 + \frac{\Gamma}{2}(\partial_\lambda\ln\rho(\lambda, z))^2 - 2\beta(\lambda, z)\partial_\lambda\ln\rho(\lambda, z)\right\} \geq 0. \tag{97}$$

V. SUMMARY AND DISCUSSION

In this study, we applied the MSR formulation to the RG flow and obtained the holographic dual effective field theory to manifest the RG flow at the level of an effective bulk action. Here, we observed that four types of BRST transformations can give some constraints to the RG flow of the coupling field although the RG-generated effective potential breaks such BRST symmetries. Resorting to the BRST transformations, we derived Ward identities for correlation functions of the coupling field. In particular, we found that the fluctuation-dissipation theorem is modified by the RG β-function, analogous to the nonequilibrium work relation. This becomes more transparent in the superspace formulation.

It is natural to apply the present framework to the holographic renormalization [54–56]. To consider the holographic renormalization, we introduce the Arnowitt-Deser-Misner (ADM) formalism for general relativity [66], where the coordinate system is given by

$$ds^2 = (\mathcal{N}^2(x,z) + \mathcal{N}_\mu(x,z)\mathcal{N}^\mu(x,z))dz^2 + 2\mathcal{N}_\mu(x,z)dx^\mu dz + g_{\mu\nu}(x,z)dx^\mu dx^\nu. \quad (98)$$

Here, $\mathcal{N}(x,z)$ is the lapse function and $\mathcal{N}_\mu(x,z)$ is the shift vector. We consider the Gaussian normal coordinate system, given by gauge fixing of $\mathcal{N}(x,z) = 1$ and $\mathcal{N}_\mu(x,z) = 0$. Then, the holographic bulk effective action is

$$F = -\frac{1}{\beta}\ln\int Dg_{\mu\nu}(x,z)D\pi^{\mu\nu}(x,z)D\bar{c}^{\mu\nu}(x,z)Dc^{\rho\gamma}(x,z)\exp\Bigg[-N_c^2\int_0^{z_f}dz\int d^Dx\Bigg\{\pi^{\mu\nu}(x,z)(\partial_z g_{\mu\nu}(x,z) - \beta_{\mu\nu}[g_{\mu\nu}(x,z);z]) - \frac{\kappa}{2}\frac{1}{\sqrt{g(x,z)}}\pi^{\mu\nu}(x,z)\mathcal{G}_{\mu\nu\rho\gamma}(x,z)\pi^{\rho\gamma}(x,z) + \bar{c}^{\mu\nu}(x,z)\left(\partial_z\mathcal{G}_{\mu\nu\rho\gamma}(x,z) - \frac{\partial}{\partial g^{\rho\gamma}(x,z)}\beta_{\mu\nu}[g_{\mu\nu}(x,z);z]\right)c^{\rho\gamma}(x,z) + \frac{1}{2\kappa}\sqrt{g(x,z)}(R(x,z) - 2\Lambda)\Bigg\}\Bigg]. \quad (99)$$

Here, N_c is the number of color degrees of freedom in dual quantum field theories. $\pi^{\mu\nu}(x,z)$ is the canonical momentum of the metric tensor $g_{\mu\nu}(x,z)$, and $\mathcal{G}_{\mu\nu\rho\gamma}(x,z)$ is de Witt supermetric [67],

$$\mathcal{G}_{\mu\nu\rho\gamma}(x,z) \equiv g_{\mu\rho}(x,z)g_{\nu\gamma}(x,z) - \frac{1}{D-1}g_{\mu\nu}(x,z)g_{\rho\gamma}(x,z). \quad (100)$$

The RG β-function is introduced for the appearance of the RG flow as follows

$$\beta_{\mu\nu}[g_{\mu\nu}(x,z);z] = -\frac{\partial}{\partial g^{\mu\nu}(x,z)}\left\{\frac{1}{2\kappa}\sqrt{g(x,z)}(R(x,z) - 2\Lambda)\right\} = -\frac{1}{2\kappa\sqrt{g(x,z)}}\left(R_{\mu\nu}(x,z) - \frac{1}{2}R(x,z)g_{\mu\nu}(x,z) + \Lambda g_{\mu\nu}(x,z)\right), \quad (101)$$

which modifies the holographic dual effective field theory [40,41].

Following the strategy of the present study, we propose the following Ward identity

$$\langle g_{\mu\nu}(x',z')\pi^{\mu\nu}(x,z)\rangle - \langle g_{\mu\nu}(x,z)\pi^{\mu\nu}(x',z')\rangle = -\frac{2}{\kappa}\langle g_{\mu\nu}(x,z)\mathcal{G}^{\mu\nu\rho\gamma}(x',z')\partial_{z'}g_{\rho\gamma}(x',z')\rangle - \langle g_{\mu\nu}(x',z')(\partial_z\mathcal{G}_{\mu\nu\rho\gamma}(x,z) - \frac{\partial}{\partial g^{\rho\gamma}(x,z)}\beta_{\mu\nu}[g_{\mu\nu}(x,z);z])^{-1}\beta_{\rho\gamma}[g_{\mu\nu}(x,z);z]\rangle, \quad (102)$$

which is analogous to the fluctuation-dissipation theorem away from equilibrium. Here, the Einstein tensor $G_{\mu\nu}(x,z) \sim \beta_{\mu\nu}[g_{\mu\nu}(x,z);z]$ [40,41] would result in entropy production in the holographic RG flow.

All these thermodynamics perspectives motivate us to find an effective entropy functional in terms of the coupling field, expected to show its monotonicity during the RG evolution along the extra dimension. Following Ref. [27], we discussed the entropy production during the RG flow. First, we introduced a probability distribution function for the coupling field, where the effective Fokker-Planck equation has been modified by the RG effective potential. Based on this probability distribution function, we proposed a microscopic definition for the entropy of the bulk system, and considered the RG evolution of the system entropy, resorting to the modified Fokker-Planck equation. This leads us to introduce the rate of heat dissipation in the medium, identified with the rate of environmental entropy. Combining these two contributions, we could find that the total entropy production rate is always positive after the ensemble averaging. This positive total entropy production rate confirms the monotonicity or irreversibility of the RG flow.

We would like to point out that our nonequilibrium thermodynamics perspectives for the monotonicity of the RG flow may have an interesting geometrical interpretation. It has been demonstrated that the holographic RG flow is given by the Ricci flow equation, where the extradimensional coordinate plays the role of time in the evolution of the geometry from UV to IR [54–56,68,69]. We recall that the general RG flow equation can be made manifest in the level of an effective action with the introduction of an emergent extradimensional space, regarded to be emergent dual holography [39–44]. It has been also shown that the Ricci flow [70–72] is a gradient flow [73], where the evolution of the induced metric in the ADM hypersurface is given by a gradient of a functional. Indeed, G. Perelman constructed the so-called "entropy" functional and showed that the Ricci flow belongs to the gradient flow with positive definite metric, extremizing his entropy functional [73–75]. He was able to show the monotonicity of the Ricci flow based on his entropy functional. We speculate that our entropy functional constructed from the probability distribution function serves as a microscopic description for the macroscopic thermodynamics entropy functional, analogous to Perelman's entropy functional [76–78]. We hope to clarify this aspect in our future study.

Unfortunately, we could not reveal a clear connection from the monotonicity of the RG flow based on our holographic dual effective field theory to the holographic c—theorem of Refs. [13,14]. In the holographic c—theorem, the so-called holographic c-function has been constructed from geometry and shown to have monotonicity. It would be interesting to understand how these three frameworks, (1) our microscopic construction of the entropy functional, (2) the macroscopic description of the Perelman's entropy functional, and (3) the geometric construction of the holographic c-function, are related.

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APPENDIX A: DERIVATION OF THE FOKKER-PLANCK EQUATION

We recall the generating functional for the overdamped Langevin dynamics,

$$\mathcal{W}=\mathcal{N}\int_{x_i}^{x_f}Dx(t)Dp(t)Dc(t)D\bar{c}(t)\exp\left[-\int_{t_i}^{t_f}dt\{ip(t)(\partial_t x(t)-\mu F[x(t)])+Dp^2(t)+\bar{c}(t)(\partial_t-\mu\partial_x F[x(t)])c(t)\}\right]. \quad \text{(A1)}$$

This generating functional indicates that the "Hamiltonian" would be given by

$$\mathcal{H}=-ip(t)\mu F[x(t)]+Dp^2(t)\rightarrow-\frac{\partial}{\partial x}\mu F(x)+D\frac{\partial^2}{\partial x^2}. \quad \text{(A2)}$$

Hinted from

$$\partial_t\Psi(x,t)=\left(-\frac{\partial}{\partial x}\mu F(x)+D\frac{\partial^2}{\partial x^2}\right)\Psi(x,t), \quad \text{(A3)}$$

we obtain

$$\partial_t p(x,t)=-\partial_x j(x,t)=-\partial_x[(\mu F(x)-D\partial_x)p(x,t)], \quad \text{(A4)}$$

where $\Psi(x,t)$ is identified with $p(x,t)$.

Following this strategy, we derive the Fokker–Planck equation for the holographic dual effective field theory. The partition function is given by

$$\mathcal{Z}(z_f) = \int D\lambda(x,z)D\pi(x,z)D\bar{c}(x,z)Dc(x,z)\exp\left[-N\int_{\Lambda_{uv}}^{z_f}dz\int d^Dx\left\{i\pi(x,z)(\partial_z\lambda(x,z)-\beta[\lambda(x,z);z]) + \frac{\Gamma}{2}\pi^2(x,z) + \bar{c}(x,z)\left(\partial_z - \frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\right)c(x,z) + \mathcal{V}_{rg}[\lambda(x,z);z]\right\}\right]. \tag{A5}$$

This expression gives the "Hamiltonian" as

$$\mathcal{H} = \mathcal{V}_{rg}[\lambda(x,z);z] - i\pi(x,z)\beta[\lambda(x,z);z] + \frac{\Gamma}{2}\pi^2(x,z) \to \mathcal{V}_{rg}(\lambda,z) - \frac{\partial}{\partial\lambda}\beta(\lambda,z) + \frac{\Gamma}{2}\frac{\partial^2}{\partial\lambda^2}. \tag{A6}$$

As a result, it is natural to propose the Fokker–Planck equation of the RG flow as

$$(\partial_z - \mathcal{V}_{rg}(\lambda,z))\rho(\lambda,z) = -\partial_\lambda\left\{\left(\beta(\lambda,z) - \frac{\Gamma}{2}\partial_\lambda\right)\rho(\lambda,z)\right\}. \tag{A7}$$

We emphasize that this Fokker-Planck equation of the RG flow is semiclassical, i.e., justified in the large N-limit.

APPENDIX B: SUPERSPACE FORMULATION

In this appendix, we reformulate the holographic dual effective field theory in superspace, following Refs. [15–20]. The superspace formulation shows transparently that both the boundary action and the RG-generated effective potential are two sources to break the previously introduced BRST symmetries, more precisely, $\mathcal{N}=2$ supersymmetry. This point clarifies that the RG-generated effective potential is responsible for the entropy production during the RG flow.

First, we introduce a superfield,

$$\Phi(x,z,\theta,\bar{\theta}) = \lambda(x,z) + \theta\bar{c}(x,z) + c(x,z)\bar{\theta} + \theta\bar{\theta}\pi(x,z), \tag{B1}$$

and its source field,

$$J(x,z,\theta,\bar{\theta}) = T(x,z) + \theta\bar{G}(x,z) + G(x,z)\bar{\theta} + \theta\bar{\theta}\bar{T}(x,z), \tag{B2}$$

respectively. Here, θ and $\bar{\theta}$ are Grassmann coordinates in superspace, where $\int_{\Lambda_{uv}}^{z_f}dz\int d^Dx$ is replaced with $\int_{\Lambda_{uv}}^{z_f}dz\int d^Dx d\bar{\theta}d\theta$.

In this superspace formulation, the first two BRST charges Q and $\bar{Q}$ are represented by

$$\begin{aligned} Q\Phi(x,z,\theta,\bar{\theta}) &= \left(c(x,z)\frac{\delta}{\delta\lambda(x,z)} - \pi(x,z)\frac{\delta}{\delta\bar{c}(x,z)}\right)(\lambda(x,z) + \theta\bar{c}(x,z) + c(x,z)\bar{\theta} + \theta\bar{\theta}\pi(x,z)) \\ &= c(x,z) + \theta\pi(x,z) = -\partial_{\bar{\theta}}(\lambda(x,z) + \theta\bar{c}(x,z) + c(x,z)\bar{\theta} + \theta\bar{\theta}\pi(x,z)), \end{aligned} \tag{B3}$$

$$\begin{aligned} \bar{Q}\Phi(x,z,\theta,\bar{\theta}) &= \left\{\bar{c}(x,z)\frac{\delta}{\delta\lambda(x,z)} + \frac{2}{\Gamma}[\partial_z\bar{c}(x,z)]\frac{\delta}{\delta\pi(x,z)} + \left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right)\frac{\delta}{\delta c(x,z)}\right\}(\lambda(x,z) + \theta\bar{c}(x,z) \\ &\quad + c(x,z)\bar{\theta} + \theta\bar{\theta}\pi(x,z)) = \bar{c}(x,z) + \bar{\theta}\left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right) + \frac{2}{\Gamma}\theta\bar{\theta}\partial_z\bar{c}(x,z) \\ &= \left(\partial_\theta - \frac{2}{\Gamma}\bar{\theta}\partial_z\right)(\lambda(x,z) + \theta\bar{c}(x,z) + c(x,z)\bar{\theta} + \theta\bar{\theta}\pi(x,z)), \end{aligned} \tag{B4}$$

respectively. The second two BRST charges D and $\bar{D}$ are given by

$$\begin{aligned} D\Phi(x,z,\theta,\bar{\theta}) &= \left(\bar{c}(x,z)\frac{\delta}{\delta\lambda(x,z)} + \pi(x,z)\frac{\delta}{\delta c(x,z)}\right)(\lambda(x,z) + \theta\bar{c}(x,z) + c(x,z)\bar{\theta} + \theta\bar{\theta}\pi(x,z)) \\ &= \bar{c}(x,z) + \bar{\theta}\pi(x,z) = \partial_\theta(\lambda(x,z) + \theta\bar{c}(x,z) + c(x,z)\bar{\theta} + \theta\bar{\theta}\pi(x,z)), \end{aligned} \tag{B5}$$

$$\bar{D}\Phi(x,z,\theta,\bar{\theta}) = \left\{c(x,z)\frac{\delta}{\delta\lambda(x,z)} + \frac{2}{\Gamma}[\partial_z c(x,z)]\frac{\delta}{\delta\pi(x,z)} - \left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right)\frac{\delta}{\delta\bar{c}(x,z)}\right\}(\lambda(x,z) + \theta\bar{c}(x,z) + c(x,z)\bar{\theta} + \theta\bar{\theta}\pi(x,z)) = c(x,z) + \theta\left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right) + \theta\bar{\theta}\frac{2}{\Gamma}[\partial_z c(x,z)] = -\left(\partial_{\bar{\theta}} + \frac{2}{\Gamma}\theta\partial_z\right)(\lambda(x,z) + \theta\bar{c}(x,z) + c(x,z)\bar{\theta} + \theta\bar{\theta}\pi(x,z)), \tag{B6}$$

respectively.

Anticommutation relations of these BRST charges are given by

$$[Q,\bar{Q}]_+ = Q\bar{Q} + \bar{Q}Q = -\partial_{\bar{\theta}_a}\left(\partial_{\theta_a} - \frac{2}{\Gamma_a}\bar{\theta}_a\partial_z\right) - \left(\partial_{\theta_a} - \frac{2}{\Gamma_a}\bar{\theta}_a\partial_z\right)\partial_{\bar{\theta}_a} = \frac{2}{\Gamma_a}\partial_z, \tag{B7}$$

$$[D,\bar{D}]_+ = D\bar{D} + \bar{D}D = -\partial_{\theta_a}\left(\partial_{\bar{\theta}_a} + \frac{2}{\Gamma_a}\theta_a\partial_z\right) - \left(\partial_{\bar{\theta}_a} + \frac{2}{\Gamma_a}\theta_a\partial_z\right)\partial_{\theta_a} = -\frac{2}{\Gamma_a}\partial_z, \tag{B8}$$

both of which lead to the translation operator along the extra dimension.

Based on these constructions, we rewrite the partition function as follows

$$Z(z_f) = \int D\psi_\sigma(x,z_f)D\Phi(x,z,\theta,\bar{\theta})\exp\Big[-\int d^Dx\Big\{\mathcal{L}[\psi_\sigma(x,z_f);\lambda(x,z_f);z_f\Big] + \frac{N}{2\Gamma}[\lambda(x,\Lambda_{uv}) - \bar{\lambda}(\Lambda_{uv})]^2\Big\} - N\int_{\Lambda_{uv}}^{z_f}dz\int d^Dxd\bar{\theta}d\theta\Big\{\frac{\Gamma}{2}D\Phi(x,z,\theta,\bar{\theta})\bar{D}\Phi(x,z,\theta,\bar{\theta}) + \mathcal{V}_{rg}[\Phi(x,z,\theta,\bar{\theta});z] + J(x,z,\theta,\bar{\theta})\Phi(x,z,\theta,\bar{\theta})\Big\} - N\int_{\Lambda_{uv}}^{z_f}dz\int d^Dx\mathcal{V}_{rg}[\lambda(x,z);z]\Big]. \tag{B9}$$

The kinetic energy is checked out as

$$\int d^Dxd\bar{\theta}d\theta\frac{\Gamma}{2}D\Phi(x,z,\theta,\bar{\theta})\bar{D}\Phi(x,z,\theta,\bar{\theta}) = \int d^Dxd\bar{\theta}d\theta\frac{\Gamma}{2}\left\{\bar{c}(x,z) + \bar{\theta}\pi(x,z)\right\}\left\{c(x,z) + \theta\left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right) + \theta\bar{\theta}\frac{2}{\Gamma}[\partial_z c(x,z)]\right\} = \int d^Dx\left\{\pi(x,z)\partial_z\lambda(x,z) - \frac{\Gamma}{2}\pi^2(x,z) + \bar{c}(x,z)\partial_z c(x,z)\right\}. \tag{B10}$$

The RG β function is generated by the effective potential

$$\mathcal{V}_{rg}[\Phi(x,z,\theta,\bar{\theta});z] = -\frac{1}{N}\ln\int_{\Lambda(z)}D\psi_\sigma(x,z)\exp\left\{-\int d^Dxd\bar{\theta}d\theta\mathcal{L}[\psi_\sigma(x,z);\Phi(x,z,\theta,\bar{\theta});z]\right\} \tag{B11}$$

in the superspace. The source-field coupling action is

$$\int d^Dxd\bar{\theta}d\theta J(x,z,\theta,\bar{\theta})\Phi(x,z,\theta,\bar{\theta}) = \int d^Dxd\bar{\theta}d\theta(T(x,z) + \theta\bar{G}(x,z) + G(x,z)\bar{\theta} + \theta\bar{\theta}\bar{T}(x,z))(\lambda(x,z) + \theta\bar{c}(x,z) + c(x,z)\bar{\theta} + \theta\bar{\theta}\pi(x,z)) = \int d^Dx(\bar{T}(x,z)\lambda(x,z) + T(x,z)\pi(x,z) + \bar{G}(x,z)c(x,z) + \bar{c}(x,z)G(x,z)). \tag{B12}$$

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