Fluid-gravity correspondence and causal first-order relativistic viscous hydrodynamics

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The fluid-gravity correspondence is a duality between anti–de Sitter Einstein gravity and a relativistic fluid living at the conformal boundary. We show that one can accommodate the causal first-order viscous hydrodynamics recently developed by Bemfica, Disconzi, Noronha, and Kovtun in this framework, by requiring a set of natural conditions for the geometric data at the horizon. The latter hosts an induced Carrollian fluid, whose equations of motion are shown to be tightly tied to the ones describing the fluid at the boundary. Functional expressions for the transport coefficients are found—with those associated to viscosity and heat flux uniquely determined—satisfying a set of known causality requirements for the underlying equations of motion.

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I. INTRODUCTION

In a series of works by Bemfica, Disconzi, Noronha, and Kovtun (BDNK), a formulation for viscous, relativistic hydrodynamics has been introduced where dissipative corrections are accounted for via first-order derivatives of the energy density and flow velocity [1–3], see also [4–6], and where causality of the resulting equations of motion is achieved when choosing transport coefficients within particular bounds. Such a formulation, at first sight, is in tension with standard results where at least second-order corrections are required to account for viscous relativistic hydrodynamics [7–11]. A key observation is that such results require a strictly non-negative entropy change, while the first-order formulation does so up to higher order in gradients. Arguably, this is not necessarily a significant shortcoming as such higher order terms should be subleading in the effective field theory regime where such theory can be written. Further, another key aspect of BDNK is formulating such theory in a more general frame than the typical Landau or Eckart frames where causality is violated in the first order formulation.

The Landau frame [12] was introduced requiring that the heat current vanishes, such that the fluid velocity is an eigenvector of the energy-momentum tensor. On the other hand, in the Eckart frame [13] the fluid velocity is aligned with the particle number flux, such that the equations are similar to those of nonrelativistic hydrodynamics. As pointed out by BDNK, the frame discussed above should not be chosen driven by aesthetics, but instead requiring that the resulting hydrodynamic equations lead to well-posed problems, thus the equations of motions should be hyperbolic and causal.

In a parallel development, the celebrated fluid-gravity correspondence [14–19] has linked the behavior of perturbed black holes (with asymptotically anti-de Sitter boundary conditions) to viscous relativistic hydrodynamics in one lower dimension. This remarkable correspondence, was fully developed to second order in gradients, but specialized in the Landau frame by judicious choices made when solving Einstein equations for a perturbed black brane. Under restricted assumptions on the bulk duals, the Landau frame was abandoned in [20–22], where the heat current was considered in the fluid-gravity correspondence.

In the current work, with the aim of shedding light on the connection between the fluid-gravity correspondence and the BDNK first order, viscous relativistic hydrodynamics, we first show that the fluid-gravity correspondence is well-suited to accommodate the full first order hydrodynamic frame spectrum. This freedom was already present in [14], but the correspondence was fully developed in the Landau frame. Given this freedom, are there reasonable choices–in particular, from a gravity perspective–leading to BDNK? To answer this question, we study the properties of the bulk projected on the horizon, which is a null hypersurface.

It is known since the original "membrane paradigm" [23,24] that the Einstein equations projected to the horizon are conservation equations of a fluid, which has been recently understood to be a Carrollian fluid [25–31]. We show that the Carrollian equations of motion, at first order,

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are equal to those of perfect fluid conservation for a conformal fluid.[1] We also observe that requiring the null vector generating the horizon to be aligned with the fluid velocity at first order in the derivative expansion selects exactly the BDNK form of the heat current. Similarly, the energy density at first order is the one used by BDNK if it is proportional to the horizon expansion. Under these assumptions we derive the values induced by Einstein gravity of most transport coefficients and a condition on a remaining one for a conformal equation of state. We find that the transport coefficients than can be fully fixed this way are within the causality and stability bounds. These observations may open the door to unraveling a deeper connection between the horizon Carrollian fluid and relativistic conformal fluids.

The rest of the manuscript is organized as follows. In Sec. II we review the construction of the fluid-gravity correspondence of [14], and extrapolate the boundary energy-momentum tensor using the holographic prescription [32,33]. In Sec. III we discuss the horizon geometry using the framework of [34], and study it at zeroth and first order derivative expansion. We then make our geometrical choices, that we show in Sec. IV lead to BDNK for the boundary fluid. We conclude with final remarks in Sec. V.

II. SETUP

Following the presentation in [14], we consider a boosted black brane in $4+1$ dimensions with asymptotically anti–de Sitter boundary conditions. In the stationary case—zeroth order in the gradient expansion—the spacetime metric is given by

$$\mathrm{d}s^2 = -2u_a \mathrm{d}x^a \mathrm{d}r + r^2(P_{ab} - f(br)u_a u_b)\mathrm{d}x^a \mathrm{d}x^b; \quad (1)$$

with $f(r) = 1 - r^{-4}$, $u^a u^b \eta_{ab} = -1$ and $P_{ab} = \eta_{ab} + u_a u_b$ the projector orthogonal to u^a, $u^a P_{ab} = 0$. The vector u^a is constant and defines the boost, while the function $f(br)$ describes a black brane with radius $r_H = b^{-1}$. Perturbed solutions, in terms of a gradient expansion (also known as derivative expansion) can be obtained by considering (b, u^a) as slowly varying functions of x^a,[2] inserting into Einstein equations and solving for nontrivial functions of r (see [14]).

As opposed to the treatment in that work, we refrain from adopting the specific choice of vanishing "zero-modes" (as also considered in [6]). In other terms, we allow for small coordinate changes in the x^a sector, which we capture in terms of a scalar function $\chi(x^a)$ and a vector $q^a(x^b)$ transverse to u^a, $u^a q_a = 0$. The resulting solution, at first order in the gradient expansion is[3]

$$\begin{aligned}\mathrm{d}s^2 = {} & -2u_a \mathrm{d}x^a \mathrm{d}r + r^2(P_{ab} - f(br)u_a u_b)\mathrm{d}x^a \mathrm{d}x^b \\ & - \frac{2}{b^4 r^2} u_{(a} q_{b)} \mathrm{d}x^a \mathrm{d}x^b + \frac{\chi}{b^4 r^2} u_a u_b \mathrm{d}x^a \mathrm{d}x^b \\ & + 2r^2 b F(br)\sigma_{ab} \mathrm{d}x^a \mathrm{d}x^b + \frac{2}{3} r\theta u_a u_b \mathrm{d}x^a \mathrm{d}x^b \\ & - 2r u_{(a} a_{b)} \mathrm{d}x^a \mathrm{d}x^b, \end{aligned} \quad (2)$$

where, asymptotically,[4] $F(br) = \frac{1}{br} - \frac{1}{4b^4 r^4}$, and we introduced the shear, expansion, and acceleration of u^a,

$$\sigma^{ab} = P^{c(a}\partial_c u^{b)} - \frac{1}{3} P^{ab}\partial_c u^c \quad (3)$$

$$a_a = u^b \partial_b u_a, \qquad \theta = \partial_a u^a, \quad (4)$$

satisfying $u^a \sigma_{ab} = 0$, $\sigma_a{}^a = 0$ $(\eta_{ab}P^{ab} = 3)$, $u^a a_a = 0$.

Armed with this solution, we can now obtain the stress tensor induced at the timelike asymptotic boundary. To do so, we follow the holographic prescription discussed in [32,33], of which we now recap the salient ingredients. Given a bulk metric $g_{\mu\nu}$, we introduce a hypersurface at fixed r and its projector $h_{\mu\nu}$. Using the normalized normal form $N = N_\mu \mathrm{d}x^\mu = \frac{\mathrm{d}r}{\sqrt{g^{rr}}}$, the projector reads

$$h_{\mu\nu} = g_{\mu\nu} - N_\mu N_\nu \qquad h_\mu{}^\nu N_\nu = 0. \quad (5)$$

The extrinsic curvature (second fundamental form) is defined as

$$K_{ab} = h_a{}^\mu h_b{}^\nu \frac{1}{2} \pounds_N g_{\mu\nu} \quad (6)$$

$$= h_a{}^\mu h_b{}^\nu \frac{1}{2}(\nabla_\mu N_\nu + \nabla_\nu N_\mu), \quad (7)$$

where ∇ is the bulk Levi-Civita connection. The induced inverse metric is

$$\bar{g}^{ab} = h_\mu{}^a h_\nu{}^b g^{\mu\nu}, \quad (8)$$

which can be used to define the trace on the hypersurface

$$K = \bar{g}^{ab} K_{ab}. \quad (9)$$

The traceless part of the extrinsic curvature defines the boundary stress tensor

$$T^a{}_b = -2\lim_{r\to\infty} r^4 \left(K^a{}_b - \frac{K}{4}\delta^a_b\right), \quad (10)$$

[1] In particular, in agreement with the boundary fluid equations of motion for the perfect fluid.

[2] We use bulk coordinates $x^\mu = (r, x^a)$, such that x^a are coordinates of fixed-r hypersurfaces, and in particular of the boundary.

[3] Using the convention $A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba})$.

[4] $F(r)$ is given by a transcendental expression, see Eq. (4.20) in [14] for details.

where the r pre-factor comes from the holographic dictionary and ensures its finiteness approaching the boundary.

Applying this procedure to our line element (2), the final result is

$$T_{ab} = 3\frac{1+\chi}{b^4}u_a u_b + \frac{1+\chi}{b^4}P_{ab} - \frac{2}{b^3}\sigma_{ab} - \frac{8}{b^4}u_{(a}q_{b)}. \quad (11)$$

This is the stress tensor of a conformal viscous fluid with fluid velocity u^a, which accounts for heat-flux through q^a, and a correction to the perfect fluid energy density through χ. Since a generic stress tensor is decomposed as

$$T_{ab} = \mathcal{E}u_a u_b + \mathcal{P}P_{ab} - 2\eta\sigma_{ab} + u_a Q_b + Q_a u_b, \quad (12)$$

one has,

$$\mathcal{E} = 3\frac{1+\chi}{b^4}, \qquad \mathcal{P} = \frac{1+\chi}{b^4}$$
$$\eta = \frac{1}{b^3}, \qquad Q_a = -\frac{4}{b^4}q_a. \quad (13)$$

We note that $\mathcal{E} = 3\mathcal{P}$ is the conformal equation of state implied by the asymptotic conformal symmetry, streaming from the vanishing of the stress tensor trace. Notice that at equilibrium the temperature is given by $T = \frac{1}{b}$.

From here, one could straightforwardly identify conditions for (χ, q^a) to recover BDNK. This would amount to using a particular formulation of viscous-relativistic hydrodynamics to fix conditions on the gravitational sector. However, our goal is to go in the opposite way, namely to consider arguably natural choices on the gravitational sector—specifically at the horizon—and explore what they correspond to in the hydrodynamic side.

III. CHOICES

To consistently deal with a degenerate metric at the null hypersurface describing the horizon, we adopt the null Rigging formalism described in [34]. The horizon is generically located at $r = r_H(x)$, and thus the one-form normal to the horizon is

$$\underline{n} = \tilde{\alpha}\mathrm{d}(r - r_H(x)), \quad (14)$$

and we adopt $\tilde{\alpha} = 1$ in the following. Next, we introduce the vector $k = \partial_r$ with the defining properties

$$\underline{n}(k) = 1, \qquad \underline{k}(k) = 0. \quad (15)$$

This vector is called the *null Rigging vector*. We can then define the Rigging projector as

$$\Pi_\mu{}^\nu = \delta_\mu^\nu - n_\mu k^\nu, \quad (16)$$

such that

$$\Pi_\mu{}^\nu n_\nu = 0 \qquad k^\nu \Pi_\nu{}^\mu = 0. \quad (17)$$

The Rigging projector projects to the null hypersurface, since indeed the form $\underline{n}$ and the vector k are normal to it.

The bulk metric duals n and $\underline{k} = -u_a \mathrm{d}x^a$ satisfy

$$\Pi_\mu{}^\nu k_\nu = k_\mu \qquad \ell^\mu = n^\nu \Pi_\nu{}^\mu. \quad (18)$$

Furthermore, the projected metric is given by

$$q_{\mu\nu} = \Pi_\mu{}^\rho \Pi_\nu{}^\sigma g_{\rho\sigma} = g_{\mu\nu} - n_\mu k_\nu - k_\mu n_\nu. \quad (19)$$

The components intrinsic to the hypersurface, (k_a, ℓ^a, q_{ab}), form the ruled Carrollian structure discussed in [31] (with the same conventions). In particular, ℓ^a is the Carrollian vector field, k_a is the Ehresmann connection, and q_{ab} is the degenerate Carrollian metric satisfying

$$\ell^a q_{ab} = 0 \quad (20)$$

at the horizon.

The other relevant quantities for the horizon physics are the surface gravity, expansion, Hájiček connection, and acceleration. They are defined in the bulk by,

(1) Surface gravity[5]:

$$\ell^\mu \nabla_\mu \ell^\nu = \kappa \ell^\nu, \qquad k_\nu \ell^\mu \nabla_\mu \ell^\nu = \kappa; \quad (21)$$

(2) Expansion:

$$\Theta = q_\nu{}^\mu \nabla_\mu n^\nu; \quad (22)$$

(3) Hájiček connection:

$$\pi_\mu = q_\mu{}^\nu k_\rho \nabla_\nu n^\rho; \quad (23)$$

(4) Acceleration:

$$\varphi_\mu = n^\nu \nabla_{[\mu} k_{\nu]}. \quad (24)$$

We now proceed to compute these quantities, for clarity we do so first in the stationary solution (zeroth order), and then in the first order perturbed case.

[5]This quantity should be called inaffinity, but for nonexpanding horizons these two concepts coincide. Here, by construction at zeroth order and as a consequence of the equations of motion at first order, the horizon expansion vanishes so we are in this framework.

A. Zeroth order

At this order, the location of the horizon, and associated normal form and vector are

$$r_H = \frac{1}{b}, \qquad \underline{n} = \mathrm{d}r, \qquad k = \partial_r. \tag{25}$$

The bulk metric duals are

$$n = r^2 f(br)\partial_r + u^a\partial_a, \qquad \underline{k} = -u_a \mathrm{d}x^a \tag{26}$$

and thus the Carrollian vector is exactly given by the boundary fluid congruence (which is constant at zeroth order)

$$\ell^\mu = n^\nu \Pi_\nu{}^\mu = u^a \delta_a^\mu. \tag{27}$$

This implies

$$\ell^\mu k_\mu = -u^a u_a = 1. \tag{28}$$

The degenerate metric on the null surface is

$$q_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & r^2(P_{ab} - f(br)u_a u_b) \end{pmatrix} \xrightarrow{r=r_H} \begin{pmatrix} 0 & 0 \\ 0 & \frac{P_{ab}}{b^2} \end{pmatrix} \tag{29}$$

which indeed satisfies at the horizon

$$\ell^\mu q_{\mu\nu} = u^a q_{ab} \delta_\nu^b \xrightarrow{r=r_H} u^a \frac{P_{ab}}{b^2} = 0. \tag{30}$$

With the above quantities, it is straightforward to obtain:

$$\kappa = \frac{2}{b}, \qquad \Theta = 0, \qquad \pi_\mu = 0, \qquad \varphi_\mu = 0. \tag{31}$$

Exactly like the relativistic conformal fluid at the boundary, the Carrollian fluid at the horizon is a perfect fluid at zeroth order. Delving into the properties of the Carrollian fluid on the horizon and its connection to the boundary fluid would bring us too afield from the subject of this manuscript. We leave this exploration to a future work.

B. First order

We now perturb the stationary solution using the first order gradient expansion. Details on how to establish the location of the perturbed horizon are in [15] (in particular subsection II. 3), so we just report the result here. At first order, the horizon and associated normal form are

$$r_H = \frac{1}{b} + \frac{\theta}{6} + \frac{\chi}{4b} - \frac{u^a\partial_a b}{2b}, \qquad \underline{n} = \mathrm{d}r + \frac{\mathrm{d}b}{b^2}, \tag{32}$$

where θ, χ, and $\mathrm{d}b$ are first order quantities.

Following the steps described above, we gather

$$\underline{k} = -u_a \mathrm{d}x^a, \tag{33}$$

$$\ell^a = u^a - b a^a - q^a + P^{ab}\partial_b b, \quad \ell^r = -u^a \frac{\partial_a b}{b^2}; \tag{34}$$

where the indices of the various quantities (a^a, q^a, and P_{ab}) are raised using η_{ab}, and we note that ℓ^r is nonvanishing due to the fact that the horizon position is now a function of x^a.

With the above, through a direct, but naturally more involved calculation, one obtains:

$$\kappa = \frac{2}{b} - \frac{2\theta}{3} + \frac{\chi}{2b} + \frac{u^a\partial_a b}{b} \tag{35}$$

$$\Theta = \theta - 3\frac{u^a\partial_a b}{b} \tag{36}$$

$$\pi_a = 2\frac{q_a}{b} - \frac{P_a{}^b\partial_b b}{b} \tag{37}$$

$$\varphi_a = a_a. \tag{38}$$

With these, we are now ready to argue for some particular choices. First, one could demand that at first order, the component of the null vector ℓ^μ orthogonal to r should be aligned with u^a (just as in the zeroth order case). This allows one to still identify the Carrollian vector with the boundary fluid velocity, even at first order. Such a choice implies

$$q^a = -b a^a + P^{ab}\partial_b b. \tag{39}$$

This, as we shall discuss below, is precisely in line with the hydrodynamic choice in BDNK.

Before such discussion, we must address the choice of χ. First, note that r_H can be reexpressed as,

$$r_H = \frac{1}{b} + \frac{1}{6}\Theta + \frac{\chi}{4b}. \tag{40}$$

We shall now show that, to first order, $\Theta = 0$ as a consequence of the Einstein equations projected on the horizon, specifically the Raychaudhuri equation. Thus, the choice $\chi \propto \Theta$, conveniently keeps $r_H = 1/b$ on-shell. Note that, with this choice, κ receives nontrivial first-order corrections. We discuss the consequences of choosing it to remain unchanged in Appendix V A. Since κ depends on the generators' parameterization, we regard keeping r_H unchanged as the more natural choice.

To see that $\Theta = 0$ to first order, let us recall that Raychaudhuri's and Damour's equations in vacuum are,

$$(\pounds_\ell + \Theta)[\Theta] = \mu\Theta - \sigma_a{}^b\sigma_b{}^a, \tag{41}$$

$$q_a{}^b(\pounds_\ell + \Theta)[\pi_b] + \Theta\varphi_a = (\bar{D}_b + \varphi_b)(\mu q_a{}^b - \sigma_a{}^b) \tag{42}$$

where $\pounds_\ell$ is the Lie-derivative along ℓ, $\bar{D}_a$ the (Carrollian) covariant derivative associated to q_{ab}, $\mu = \Theta/2 + \kappa$ and we used the conventions of [31]. Since here we will be interested only in the first order expression, where most terms in these equations vanish, we refer to this reference for an explanation of all the quantities involved in general. Notice κ has an order 0 contribution, so, Eq. (41) implies that at first order $\Theta = 0$. A similar analysis of Eq. (42) implies ${q_a}^b \partial_b \kappa + \varphi_a \kappa = 0$, where ${q_a}^b$ is the projector orthogonal to ℓ^a at the horizon, which at zeroth order is simply ${P_a}^b$, and thus [using (38)] this equation is equal to $a_a = {P_a}^b \frac{\partial_b b}{b}$.

These observations have several consequences. First, since to the order we work, $\Theta = 0$, the choice stated above for χ indeed implies $r_H = 1/b$ to this order. Further, and importantly, they indicate that at first order Raychaudhuri's and Damour's equations are exactly equal to the conservation of the boundary perfect fluid stress tensor. Indeed, one can easily show that $\partial_a {T^a}_b = 0$, using (11) and the relationships (35), (36), and (38), gives exactly the Raychuadhuri's and Damour's equations, once projected on u^a and ${P_a}^b$, respectively. This is ultimately tied to the fact that these equations all come from the bulk Einstein equations and their particular hierarchical structure arising from the characteristic treatment along a timelike-null foliation of the spacetime.

To summarize then, examining the resulting structure at the horizon, our choices are:

$$\ell^a = u^a \Leftrightarrow q^a = -b a^a + P^{ab} \partial_b b \tag{43}$$

$$r_H = \frac{1}{b} + (2 + 3\alpha) \frac{\Theta}{12} \Leftrightarrow \chi = \alpha b \Theta, \tag{44}$$

with α a proportionality function that remains to be specified. We reported these results off-shell of the conservation laws discussed above. If we now impose these conservation laws, we obtain $q^a = 0$ and $\chi = 0$. This is precisely the outcome of the intrinsic hydrodynamic BDNK analysis for a conformal relativistic fluid: the heat current and the first-order correction to the energy that implement causality are zero on-shell of the first order conservation law. In what follows, we discuss in detail the structure implied by the geometrical identifications/choices on the resulting hydrodynamical equations.

IV. CONSEQUENCES

We can now examine the consequences of these choices on the thermodynamic quantities obtained in (13). First, note that

$$\mathcal{E}^{(0)} = 3\mathcal{P}^{(0)} \equiv e = \frac{3}{b^4} \tag{45}$$

$$\mathcal{E}^{(1)} = 3\mathcal{P}^{(1)} = \frac{3\alpha}{b^3} \left(\partial_a u^a - \frac{3}{b} u^a \partial_a b \right) \tag{46}$$

$$Q^a = \frac{4}{b^3} \left(a^a - \frac{P^{ab} \partial_b b}{b} \right), \tag{47}$$

where we introduced $\{e, p = \frac{e}{3}\}$ to denote the zeroth order expressions for energy and pressure respectively.

The first order expressions can be reexpressed in terms of e and p as,

$$\mathcal{E}^{(1)} = \frac{3\alpha}{b^3} \left(\partial_a u^a + \frac{u^a \partial_a e}{(e+p)} \right) \tag{48}$$

$$Q^a = \frac{4}{b^3} \left(a^a + \frac{P^{ab} \partial_b e}{3(e+p)} \right) \tag{49}$$

(the expressions for the pressure is trivially set by the conformal condition). We can now compare with the expressions adopted by BDNK for the conformal case, as this is the one that corresponds to our case [3]. Namely, denoting with an overbar their choices,

$$\bar{\mathcal{E}}^{(1)} = \left(\chi_2 \partial_a u^a + \chi_1 \frac{u^a \partial_a e}{(e+p)} \right) \tag{50}$$

$$\bar{Q}^a = \lambda \left(a^a + \frac{P^{ab} \partial_b e}{3(e+p)} \right), \tag{51}$$

with λ, χ_i transport coefficients[6] that are chosen to ensure causality of the underlying equations, together with η defined in (12).

Remarkably, the functional form for the first order corrections are in excellent agreement with the proposed terms in [3]. Moreover, our choices motivated by considerations at the horizon also imply for the transport coefficients [for η we recall (13)],

$$\eta = \frac{1}{b^3}, \qquad \lambda = \frac{4}{b^3},$$
$$\chi_1 = \chi_2 = 3\chi_3 = 3\chi_4 = \frac{3\alpha}{b^3}, \tag{52}$$

where $\{\chi_3, \chi_4\}$ are linked to $\{\chi_1, \chi_2\}$ by conformality.

Not only do the transport coefficients have the temperature dependency of T^3 as expected from kinetic theory [3], but the shear viscosity and heat transport coefficients are uniquely determined.[7] In particular, they satisfy the criteria for causality $\lambda \geq \eta$ identified in [3]. Notice however our expressions make the transport coefficients χ_i all proportional to each other but do not completely fix them, nor provide bounds for them which need not be surprising.

[6]Which include $\{\chi_3, \chi_4\}$ analogously introduced for the first-order pressure $\mathcal{P}^{(1)}$.

[7]The value of the viscous transport coefficient is tied to the lowest-lying quasinormal modes of the perturbed black brane (see, e.g. [35]).

Namely, conditions on χ_i determined by the causality analysis of [3], effectively, come from the high frequency limit (through the standard analysis within PDE theory). This can be seen by examining the dispersion relations for the shear and sound modes and their dependency on $\{\eta, \lambda, \alpha\}$. Their roles appear at order k^2, k^4 and k^6 respectively. On the other hand, the fluid-gravity correspondence is obtained in the long wavelength regime of perturbed black holes in general relativity—which is a causal theory—thus it is natural to expect that in the regime where the duality can be established, conditions on relevant parameters on the hydrodynamic side can be obtained implying such property.

For the unfixed parameter, we only demand $\alpha > 0$, as this choice ensures equilibrium can be reached, i.e. $\mathcal{E}^{(0)} + \mathcal{E}^{(1)} \to \mathcal{E}^{(0)}$ within a timescale given by α. Of course, one can choose a suitable value for α such that the full set of requirements for causality are satisfied (e.g. $\alpha = 4/3$, so that $\chi_{\{1,2\}} = 3\chi_{\{3,4\}} = \lambda$) but there is no geometric reason at this order we can demand to argue for a specific value.

V. FINAL WORDS

In this work we examined from a gravitation angle how the BDNK first order formulation of relativistic, viscous hydrodynamics is connected to the fluid-gravity correspondence. Such a formulation, which in practice is simpler to deal with than standard, second order viscous formulations [36,37], has received significant attention in recent years both at the theoretical level [2–5,38] and also in incipient numerical investigations (e.g. [39–41]). The results obtained also revealed new connections between relativistic and Carrollian hydrodynamics as well as with gravity.

Our analysis unearthed a natural way to motivate the BDNK formulation from a gravitational perspective. Further, the expected functional dependence of transport coefficients was obtained and, for the viscous and heat-flux coefficients, a unique expression was found. As well, our analysis revealed a connection between the effective Carrollian hydrodynamic description of null surfaces and the asymptotic relativistic fluid that is identified at the timelike infinity of perturbed black branes in AdS. Such connection implies that, at leading order, Raychaudhuri's and Damour's equations encode the conservation of a conformal perfect fluid. The analysis of higher orders and the exploration of Carrollian hydrodynamics from this perspective is an interesting task which we defer to future work.

In a similar vein, it would be interesting to explore the horizon physics deeper, as results there would also hold for asymptotically flat spacetimes. Importantly, in the latter case, there is also an interesting relation between the structure of interior null surfaces (like the horizon), and future null infinity. However, the relationship between the horizon membrane paradigm and the asymptotic (e.g. $\mathcal{I}^+$) null boundary Carrollian fluid is still largely unexplored. The latter fluid however enjoys Weyl symmetry, which makes it special. This could also help motivate a fluid interpretation of (particular) quasi-normal modes in asymptotically flat spacetimes. Another avenue for exploration is to consider a potential entropy current, both for the relativistic fluid at the boundary and the horizon Carrollian fluid. This current could help us connect with its microscopic origin and inform standing questions on Carrolllian hydrodynamics. Finally, a deeper understanding of potential connections between phenomena in nonlinear gravity and hydrodynamics can motivate new avenues to identify and study nonlinear gravitational behavior (e.g. [16,30,42–49]).

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APPENDIX: ALTERNATIVE CHOICE FOR χ

An alternative choice for χ, which fixes it completely, would be to demand that to first order $\kappa = 2/b$. This would imply

$$\chi = 2\left(\frac{2b}{3}\partial_a u^a - u^a\partial_a b\right) = \frac{2b}{3}\left(2\partial_a u^a + \frac{u^a\partial_a e}{(e+p)}\right), \tag{A1}$$

and as a consequence, $\chi_1 = 3\chi_3 = 2/b^3$, $\chi_2 = 3\chi_4 = 4/b^3$. These values however, (complemented by $\lambda = 4/b^3$, $\eta = 1/b^3$) are not within the causality bounds of [3]. Further, on-shell dynamical solutions have an associated energy density at first order which differs from that at zeroth order.

Going further, one could demand that the first order expression for μ be proportional to Θ, so first order perturbations ($\Theta \to \Theta + \delta\Theta$) of Raychauduri's equation would receive no corrections from the nonlinear terms except the shear contribution. In turn, this would require $\kappa = 2/b + \alpha\Theta$, thus

$$\chi = \frac{b}{3}\left((4+6\alpha)\partial_a u^a + (2+6\alpha)\frac{u^a \partial_a e}{(e+p)}\right). \qquad \text{(A2)}$$

Thus, $\chi_1 = 3\chi_3 = (2+6\alpha)/b^3$, $\chi_2 = 3\chi_4 = (4+6\alpha)/b^3$. For $\alpha \geq 1$ the causality conditions are satisfied, but again the associated energy-density at first order—on-shell—would differ from that at zeroth order.

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