

Black hole singularity resolution in $D = 2$ via duality-invariant α' correctionsTomas Codina^{1,*}, Olaf Hohm^{1,†} and Barton Zwiebach^{2,‡}¹*Institute for Physics, Humboldt University Berlin, Zum Großen Windkanal 6, D-12489 Berlin, Germany*²*Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA* (Received 19 September 2023; accepted 2 November 2023; published 4 December 2023)

Starting with the two-derivative limit of $D = 2$ string theory, we explore the space of T -duality invariant α' corrections, a space that contains a point representing the fully α' -corrected classical string theory. Using a parametrization introduced by Gasperini and Veneziano we obtain black hole solutions in this theory space. We prove that the dual of a solution with a regular horizon must have a curvature singularity. We find regions in the theory space where the black hole is deformed while preserving the horizon and the singularity, and regions where no black hole appears to exist. Furthermore, we find subregions in this theory space, probably not containing string theory, in which the black hole geometry exhibits a horizon leading to an interior that, having no singularity in the metric, curvature or dilaton, is a regular cosmology.

DOI: [10.1103/PhysRevD.108.126006](https://doi.org/10.1103/PhysRevD.108.126006)**I. INTRODUCTION**

Black holes are perhaps the most mysterious objects predicted by general relativity. They are characterized by an event horizon that forms, for instance, when the size of a collapsing star falls below its Schwarzschild radius. It follows from the singularity theorems of Penrose that in general relativity the gravitational collapse inevitably leads to a singularity at which the spacetime curvature becomes infinite [1]. Since singularities cannot exist in nature we learn that general relativity breaks down close to the black hole singularity. In contrast, general relativity should be applicable at the event horizon, since the curvature there might be relatively small. In addition, we have excellent reasons to believe that black holes with an event horizon exist in nature. This then leads to the problem of resolving the black hole singularity: finding a new theory that replaces general relativity in the appropriate regime so that there are solutions with an event horizon but no spacetime singularity in the interior [2–6].

In this paper we ask if the black hole (BH) singularity can be resolved in string theory by means of the higher-derivative α' corrections that are already a feature of classical string theory. We consider noncritical string theory

in two dimensions (2D). Apart from its simplicity this case is particularly promising in that there is an exact worldsheet conformal field theory whose target space interpretation is that of a 2D BH [7–9], although the 2D case is of course somewhat limited in view of the horizon reducing to a single point in space. Thus there should be an exact BH solution to all orders in α' . We were motivated by the recent result of Gasperini and Veneziano [10] who showed that, remarkably, the big bang singularity may be resolved in the α' -complete theory. Since the BH interior may be visualized as a cosmology, one could envision that a resolution of the BH singularity is feasible. Recent approaches trying to remove the BH singularity using the setup developed in [11,12] for cosmological backgrounds were studied in [13–16]. An old result in this direction is the $D = 2$ black-hole geometry of [17], which was suggested to be α' exact. The maximally extended spacetime geometry and regularity of such ansatz was studied in [18–20] and an action describing this background and its (singular) dual was built in [21].

This paper is based on our recent results in [22], in which we classify the possible higher-derivative corrections that are compatible with an $O(1, 1)$ duality symmetry that string theory is known to possess for such backgrounds to all orders in α' [23–27]. In this we generalized the earlier work in [11,12] on cosmology. Specifically, we consider the class of 2D string backgrounds with metrics $ds^2 = -m^2(x)dt^2 + n^2(x)dx^2$ that admit a timelike isometry since the metric components are independent of time t . The outside region of the 2D BH is of this form. For any local solution of this form, in string theory there is also the T -dual solution obtained by sending $m \rightarrow \frac{1}{m}$, while n and the dilaton Φ , defined in terms of the standard dilaton via

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$\sqrt{-g}e^{-2\phi} = ne^{-\Phi}$, are invariant. By exploiting all possible perturbative field redefinitions that are consistent with duality we showed, under certain assumptions on the falloff behavior of the coefficients, that the most general duality invariant action is of the form

$$I = \int dx n e^{-\Phi} [1 + (D\Phi)^2 + F(M)], \quad (1.1)$$

where $D := \frac{1}{n} \partial_x$ is a covariant derivative, $M := D \log m$, and

$$F(M) \equiv \sum_{i=0}^{\infty} \epsilon_i M^{2i+2} = -M^2 + \epsilon_1 M^4 + \dots, \quad \epsilon_0 \equiv -1. \quad (1.2)$$

The ϵ_i are the free coefficients that are not fixed by duality and that must be determined by other methods from string theory (see for instance [28–30]). We currently do not have the techniques to determine them and hence the function $F(M)$ to all orders. Nevertheless, here we explore the question whether there are points in the “theory space” of all choices of $F(M)$ for which there are BH solutions that carry an event horizon but are regular everywhere.

In order to deal efficiently with the equations following from the α' -complete action (1.1) we will use a powerful representation recently introduced by Gasperini and Veneziano [10], to study the resolution of the big bang singularity (for earlier work in this direction see [31,32]). In the present black-hole context, instead of seeking a solution given by explicit functions $m(x)$ and $\Phi(x)$ (say fixing the gauge $n = 1$) one seeks a solution of the form $m(f)$, $\Phi(f)$ and $x(f)$, with f a new parameter. Remarkably, such a parametrized form can be obtained by inverting the function $f(M) := F'(M)$ to obtain $M(f)$, which parametrizes the metric variable M in terms of f . For the two-derivative theory $f(M) = -2M$ and $M(f) = -\frac{1}{2}f$. While perturbatively α' corrections build up the function $f(M)$ as a power series, the inverse function $M(f)$ affords a more direct solution of the equations of motion. We are thus led to think of the “theory space” of duality-invariant derivative corrections as one described by the possible choices of the function $M(f)$. This leads to subtle questions on how to encode a complete solution by such a Gasperini-Veneziano parametrization, which in general requires the use of several contours in “ f space.” Each contour represents a patch in coordinate space, and together they represent the whole space. In fact, the equations of motion imply the existence of branch points and branch cuts in the f plane, and the underlying structure of the solution is determined by these features. The relation $x(f)$ is given in terms of an integral $x = \int^f \dots$. We divide a complete solution into *exterior* and *interior* regions, a division that has a clear meaning in the context of black hole solutions. We will use

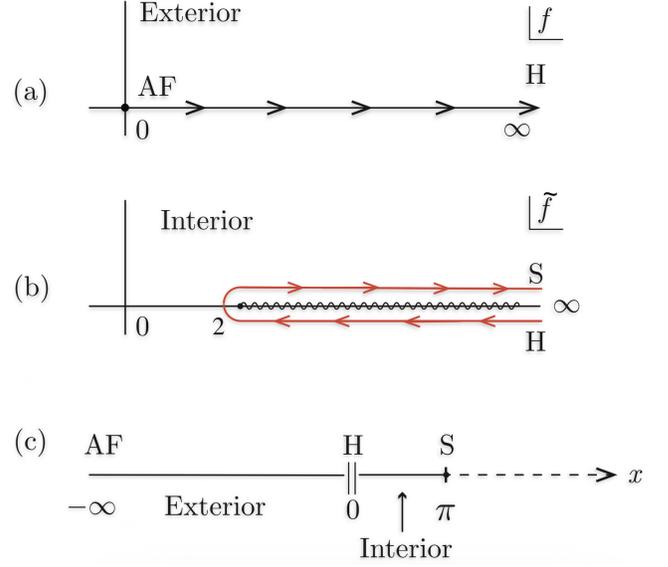


FIG. 1. The two-derivative black hole. (a) The exterior is produced by a contour $f \in (0, \infty)$ joining the AF region to the horizon (H). (b) The interior requires a contour going under and then over the cut, with branch point $\tilde{f} = 2$ and the horizon and singularity at $\tilde{f} = \infty$, under and over the cut, respectively. (c) The x -space representation of the black hole. The exterior is in $(-\infty, 0)$ and the interior in $(0, \pi)$.

f to parametrize the exterior and \tilde{f} for the interior, to make clear they represent different patches.

For the two-derivative BH the *exterior* geometry is obtained with an f plane contour going from zero to infinity on the real line [Fig. 1(a)]. Here $f = 0$ is the asymptotically flat (AF) region and $f = \infty$ is the horizon. For the interior, we find that the \tilde{f} plane has a branch point at $\tilde{f} = 2$ and a cut running from this point to infinity. The contour that gives the interior solution begins under the cut at infinity goes down to the branch point and then returns to infinity at the top of the cut [Fig. 1(b)]. The horizon and the singularity are at $\tilde{f} = \infty$, below and above the cut, respectively. The x -space picture has the exterior going from $x = -\infty$ (AF) to $x = 0$ (the horizon). The interior, drawn on the same picture goes from the horizon at $x = 0$ to the singularity at $x = \pi$ [Fig. 1(c)]. It must be emphasized that, in general, different x -space coordinate patches are used to cover the BH solution. For instance, in the coordinate systems we mainly use for the standard BH, the x coordinate ranges from $-\infty$ to 0 and then from 0 to π , which are two different patches, despite being displayed in the same graph.

It is worth noting that since the roles of space and time are interchanged when passing the event horizon, the isometry becomes spacelike, and hence the interior of a black hole is actually a cosmology [18]. For this cosmology the time coordinate is x , with time $x = 0$ corresponding to the event horizon. Since the curvature is finite at the event

horizon, the cosmology so obtained does not have a big bang singularity. In contrast, after a finite time the scale factor becomes infinite, with infinite curvature, leading to a “big rip.”

In this paper we show that T duality implies that the dual of a solution with a regular horizon must have a curvature singularity. This result, valid for any solution in the full theory space, is a formalization of the early observation of Giveon [9] noting this fact in the context of the two-derivative version of the string theory black hole. It has some implications in our work; in particular, two options seem available. First, solutions may be self-dual in the sense that the full BH (extended) geometry is invariant under T duality. In such a case, the BH must have a curvature singularity somewhere if it has a horizon. This is the case for the two-derivative black hole solution. Second, a complete solution with a horizon *and* no curvature or dilaton singularity would not be self-dual, but it would still be physically equivalent (in the sense of string theory) to its T -dual, which necessarily would exhibit a curvature singularity. It should be stressed that throughout this paper we employ the standard general relativity notion of singularities, which is probed by point particles. In string theory, however, spacetime is probed by strings, which requires further investigation.

We show that for a large class of functions $M(f)$ there are black hole solutions that, just like that of the two-derivative theory, are not regular and necessarily carry a singularity. More precisely, parametrizing the space of functions as $M(f) = -\frac{f}{2}(1 + h(f^2))$, we find a number of constraints that the function h has to obey in order to obtain solutions with a horizon. These conditions thus pose rather stringent constraints on the α' corrections for string theory to permit BH solutions with a horizon. Conversely, if one could show that the functions h arising in string theory do not obey these constraints one could prove that string theory does not permit BH solutions.¹

The singularity in this class of theories can be understood using the above horizon/singularity duality. For these black holes, the interior is represented by a contour in \tilde{f} space that goes both under and above a branch cut extending to infinity and ending at a branch point at some finite $\tilde{f} = f_0$ [Fig. 2(a)]. The interior solution is self-dual here, with the above and below the branch contours mapped into each other. As a result, the singularity is present (at infinity of the top contour) because the horizon is present (at infinity on the bottom contour).

Our strategy to construct a BH solution with a horizon but without a singularity consists in modifying the nature of

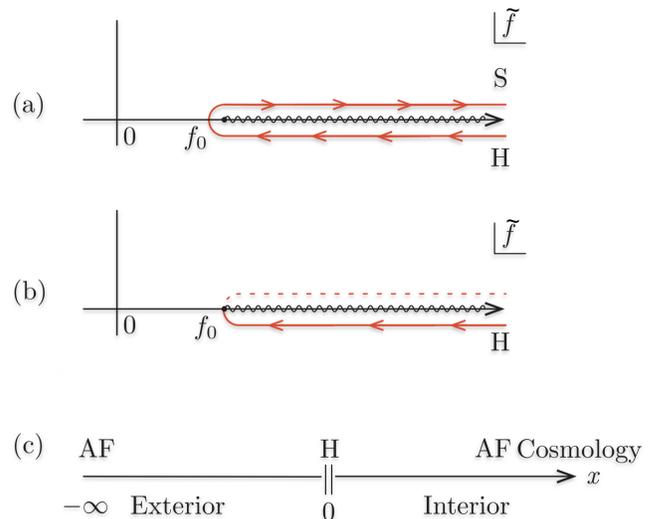


FIG. 2. (a) The interior of a deformed black hole with a full contour containing a horizon H and a singularity S . (b) The interior of the regularized black hole, without a singularity. The bottom contour containing the horizon ends at f_0 and the top contour (dotted) is not needed for completeness. (c) The x space representation of the regularized black hole.

the branch point f_0 in such a way that an \tilde{f} contour going *only* under the cut suffices to describe a complete geometry [Fig. 2(b)]. We identify a smaller class of functions $M(f)$ that can be engineered so that the branch point f_0 has the desired properties and the singularity is avoided. The solution is complete, but not self-dual. This black hole singularity resolution, which is nonperturbative in α' and closely analogous to the mechanism introduced in [11] to obtain de Sitter vacua in string cosmology, has the following general features. The interior is again a cosmology with no big bang singularity at $x = 0$. In contrast to the standard black hole, however, it describes an expanding universe that approaches flat Minkowski space in the infinite future [Fig. 2(c)]. Moreover, the dilaton goes to a fixed value in that infinite future. Such an asymptotic background, however, defines a $c = 2$ matter conformal field theory, which is not a critical string theory background. It thus appears that these regular black holes are solutions of duality-invariant theories, but apparently not of string theory.

In exploring the geometry of the regular black hole we emphasize that, in a subtle but important way, the f parametrization *extends* the definition of the α' corrections beyond what the series expansion $f(M)$, which might be an asymptotic series, can tell us. Indeed, we are postulating an $M(f)$ that is a well defined function, and for the case of the regular black hole, this function is *not* invertible [$M(f)$ is also not invertible for the cases considered in [10]]. Thus, $f(M)$ would have a finite radius of convergence. The physics of the resolved singularity involves values of f that are not achieved within its convergence region. This allows

¹The work of Ying [14] appears to imply that a cubic $M(f)$ would yield a regular BH. Our analysis of cubic deformations in this paper, however, finds no such regular solution. In fact, [15] indicates that this regular solution requires changing the value of the string theory constant term in the action.

the late time cosmology to have a fixed dilaton instead of a rolling one. In our reexamination of the derivation of the equations of motion in the f parametrization (Sec. III A) we discuss in detail how the issue of noninvertibility affects the analysis.

This paper is organized as follows. In Sec. II we review the general framework for the 2D black hole and the structure of higher-derivative duality-invariant corrections, considering the equations of motion relevant for the exterior and interior geometries. Section III introduces the new Gasperini-Veneziano parametrization by f and shows how the equations of motion are solved in this framework. We use the example of the two-derivative BH to understand the basic structure of branch points and cuts in f space. The next step is displaying a large class of solutions that can be viewed as a small general deformation of the two-derivative black hole function $M(f) = -f/2$. We do this in Sec. IV, where we also examine cubic deformations of the function $M(f)$, which happen not to yield black holes. In Sec. V we give the proof that the T -dual of a geometry with a regular horizon must have a curvature singularity. This result is used to explain why the back-and-forth contour in \tilde{f} space used for the interior solution necessarily leads to an interior with a singularity. Finally, in Sec. VI we find functions $M(f)$ for which the exterior solution is largely preserved but the interior solution has no singularity. This is done by adjusting $M(f)$ in such a way that x as a function of \tilde{f} diverges at the branch point. We discuss our results and some possible extensions in Sec. VII.

II. BLACK HOLE IN 2D STRING THEORY

Here we begin by reviewing our notation for the two-dimensional geometry, giving the general formulas for curvature and the action of T duality on the field variables (for more details see [22]). We reconsider the two-derivative black hole and discuss the equations of motion on the exterior and in the interior, the relevant equations being related by a number of sign changes. Finally, we consider general α' corrections and, again, formulate the exterior and interior versions of the equations of motion.

A. Metric, curvature and duality

We consider the target space theory of bosonic strings in two dimensions of Lorentzian signature, with the field content restricted to the universal massless sector. Since the antisymmetric b field is trivial in two dimensions, we thus include the metric $g_{\mu\nu}$ and the dilaton ϕ . Denoting coordinates by $x^\mu = (t, x)$, we further impose that the fields, and hence the gauge parameters, do not depend on time t . This truncated theory possesses a global $O(1, 1; \mathbb{R})$ duality symmetry together with one-dimensional diffeomorphism invariance: reparametrizations of x . We can assume a diagonal ansatz for the metric

$$g_{\mu\nu} = \begin{pmatrix} -m^2(x) & 0 \\ 0 & n^2(x) \end{pmatrix}, \quad ds^2 = -m^2(x)dt^2 + n^2(x)dx^2. \quad (2.1)$$

Here, $m(x)$ and the lapse function $n(x)$ are real. Under coordinate transformations of x , the field $m(x)$ is a scalar while $n(x)$ is a scalar density. The metric $m(x)$ enters the theory through

$$M \equiv \frac{Dm}{m}, \quad (2.2)$$

in terms of the covariant derivative

$$D \equiv \frac{1}{n} \frac{d}{dx}. \quad (2.3)$$

The scalar dilaton $\phi(x)$ is related to the duality invariant dilaton $\Phi(x)$ via

$$\phi(x) = \frac{1}{2}(\Phi(x) + \log |m(x)|), \quad (2.4)$$

which makes the measure of the effective theory to be of the standard form $\sqrt{-\tilde{g}}e^{-2\phi} = ne^{-\Phi}$. The scalar curvature R of the two-dimensional metric is given by

$$R = -2(M^2 + DM) = -2\frac{D^2m}{m}. \quad (2.5)$$

The curvature R encodes the full Riemann tensor and hence allows one to detect geometric singularities.

In these 2D backgrounds with a timelike isometry, the duality group acts on the fields as

$$\begin{aligned} m(x) &\rightarrow \hat{m}(x) = \frac{1}{m(x)}, & n(x) &\rightarrow \hat{n}(x) = n(x), \\ \Phi(x) &\rightarrow \hat{\Phi}(x) = \Phi(x). \end{aligned} \quad (2.6)$$

We will use hatted fields for the fields after duality. Note that the covariant derivative D is unchanged by duality. As a consequence of the transformation of m one also has

$$M(x) \rightarrow \hat{M}(x) = -M(x), \quad (2.7)$$

The dual scalar curvature, namely, the curvature of the dual metric, is then given by

$$R \rightarrow \hat{R} = -2(\hat{M}^2 + D\hat{M}) = -2(M^2 - DM), \quad (2.8)$$

so we have

$$\hat{R} = -2(M^2 - DM) = -R - 4M^2 = R + 4DM. \quad (2.9)$$

A useful gauge choice. We can pick a gauge for one-dimensional diffeomorphism invariance by fixing the lapse function $n(x)$. A familiar gauge choice takes

$$n(x) = 1. \quad (2.10)$$

In this gauge the field $M(x)$, if known, is easily integrated to find $m(x)$:

$$M(x) = \frac{\partial_x m}{m} = \partial_x \ln m \Rightarrow m(x) = \exp\left(\int_c^x M(x') dx'\right), \quad (2.11)$$

and consequently

$$\frac{m(x_2)}{m(x_1)} = \exp\left(\int_{x_1}^{x_2} M(x') dx'\right). \quad (2.12)$$

The scalar curvature (2.5) now reads

$$R = -2(M^2 + \partial_x M) = -2\frac{\partial_x^2 m}{m}. \quad (2.13)$$

It is worth emphasizing that all relations above hold for arbitrary 2D metrics of the form (2.1), independent of any equations of motion, and are hence applicable to generic higher-derivative theories.

B. The two-derivative, two-dimensional black hole

Before considering α' corrections, in the following we recall the 2D string equations with up to two derivatives and review the BH solution, both for the exterior region and the interior region. We will set up the solutions, both in the $n = 1$ gauge in such a way that the exterior solution is in the coordinate interval $(-\infty, 0)$ and the interior solution uses the coordinate interval $(0, \pi)$. While in both cases we use the label x to denote the coordinate, these two solutions do not form together a single solution over the full real line; the two x s are really different.

The target space action for the above string backgrounds reads to leading order

$$I = \int dx n e^{-\Phi} [1 + (D\Phi)^2 - M^2], \quad (2.14)$$

where we use a unit-free notation that absorbs factors of α' , denoted by a bar in [22] that we drop in this paper in order not to clutter the equations. Sometimes we refer to the zero-derivative term as the cosmological term, although it is not a cosmological constant in the standard sense. The equations of motion for m , Φ and n are equivalent to

$$\frac{d}{dx}(e^{-\Phi} M) = 0, \quad (2.15a)$$

$$\left(\frac{d\Phi}{dx}\right)^2 - M^2 - 1 = 0, \quad (2.15b)$$

$$\frac{d^2\Phi}{dx^2} - M^2 = 0, \quad (2.15c)$$

where we picked $n = 1$ gauge after performing the variation.

Exterior BH solution. The above equations admit a unique solution describing the outside region of a 2D BH [7–9]. We use the region $x \in (-\infty, 0)$, and the metric and duality-invariant dilaton take the form

$$m(x) = -\tanh\frac{x}{2}, \quad \Phi(x) = -\log|\sinh x| + \Phi_0, \quad x < 0, \quad (2.16)$$

with Φ_0 an integration constant, and the sign of $m(x)$ is fixed to get the familiar asymptotic behavior. The scalar curvature (2.13) is given by

$$R(x) = \frac{1}{\cosh^2\frac{x}{2}}. \quad (2.17)$$

Here $x = 0$ is the position of the horizon, $x \rightarrow -\infty$ is the AF region with m approaching one.² Indeed, at $x = 0$ the metric vanishes and the curvature is finite

$$m(0) = 0, \quad R(0) = 1, \quad (2.18)$$

while in the AF region we have

$$\lim_{x \rightarrow -\infty} m(x) = 1, \quad \lim_{x \rightarrow -\infty} R(x) = 0. \quad (2.19)$$

On the other hand, the duality-invariant dilaton $\Phi(x)$ becomes infinite at both extremes

$$\lim_{x \rightarrow 0^-} \Phi(x) = \infty, \quad \lim_{x \rightarrow -\infty} \Phi(x) = -\infty. \quad (2.20)$$

This does not necessarily indicate a pathology since Φ is not the scalar dilaton ϕ that determines the string coupling as $g_s = e^\phi$. Using (2.4), we get

$$\phi = -\frac{1}{2} \log \cosh^2\frac{x}{2} + \phi_0 \Rightarrow \phi(0) = \phi_0 = \frac{1}{2}(\Phi_0 - \log 2), \quad (2.21)$$

and so the dilaton and hence the string coupling are finite at the horizon. We also infer $\lim_{x \rightarrow \infty} \phi(x) = -\infty$, so the string coupling g_s goes to zero in the AF region.

Interior BH solution. The solution (2.16) describes only the exterior region of the BH. Therefore, it has no information on the BH singularity, which lies in the interior. This region, together with others, can be obtained by analytic extension of the exterior solution. Moreover, for the 2D BH, it is possible to obtain the maximally extended solution using a single patch, by the so-called

²The solution can also be used for $x \in (0, \infty)$, but our later work is simplified by this choice of range of x .

Kruskal-Szekeres coordinates. This coordinate system covers not only the exterior and interior region of the BH, but also a white hole region and a disconnected asymptotically flat space. Since we are mainly interested in the BH singularity, for now it is sufficient to describe the interior region only.

The most straightforward approach to get the interior solution is to look for a transformation that preserves the measure and changes the metric signature or, more precisely, that maps a metric with timelike isometry to a metric with spacelike isometry as follows:

$$ne^{-\Phi} \rightarrow \tilde{n}e^{-\tilde{\Phi}}, \quad (2.22)$$

$$\begin{aligned} ds^2 &= -m^2(x)dt^2 + n^2(x)dx^2 \rightarrow ds^2 \\ &= \tilde{m}^2(x)d\tilde{t}^2 - \tilde{n}^2(x)d\tilde{x}^2. \end{aligned} \quad (2.23)$$

This can be achieved by setting

$$m = i\tilde{m}, \quad n = -i\tilde{n}, \quad \Phi = \tilde{\Phi} - \log i, \quad (2.24)$$

which preserves the measure and the product mn . Therefore, from (2.2) and (2.3) we have

$$D = i\tilde{D}, \quad M = i\tilde{M}, \quad (2.25)$$

with $\tilde{D} = \frac{1}{\tilde{n}} \frac{d}{d\tilde{x}}$ and $\tilde{M} = \frac{1}{\tilde{m}\tilde{n}} \partial_{\tilde{x}} \tilde{m}$. Applying these substitutions to the curvature (2.5) we find $R \rightarrow \tilde{R}$ with

$$\tilde{R} = 2(\tilde{M}^2 + \tilde{D}\tilde{M}) = 2\frac{\tilde{D}^2\tilde{m}}{\tilde{m}}. \quad (2.26)$$

Fixing the gauge $\tilde{n}(x) = 1$,

$$\tilde{R} = 2(\tilde{M}^2 + \partial_{\tilde{x}}\tilde{M}) = 2\frac{\partial_{\tilde{x}}^2\tilde{m}}{\tilde{m}}. \quad (2.27)$$

The scalar dilaton in (2.4) is mapped to

$$\tilde{\phi}(x) = \frac{1}{2}(\tilde{\Phi}(x) + \log|\tilde{m}|). \quad (2.28)$$

From the original action (2.14), we obtain

$$\tilde{I} = \int dx \tilde{n}e^{-\tilde{\Phi}} [1 - (\tilde{D}\tilde{\Phi})^2 + \tilde{M}^2]. \quad (2.29)$$

Varying this action and fixing the gauge $\tilde{n} = 1$, the equations of motion read

$$\frac{d}{dx}(e^{-\tilde{\Phi}}\tilde{M}) = 0, \quad (2.30a)$$

$$\left(\frac{d\tilde{\Phi}}{dx}\right)^2 - \tilde{M}^2 + 1 = 0, \quad (2.30b)$$

$$\frac{d^2\tilde{\Phi}}{dx^2} - \tilde{M}^2 = 0. \quad (2.30c)$$

The solution in this case is given by trigonometric functions

$$\tilde{m}(x) = \tan \frac{x}{2},$$

$$\tilde{\Phi}(x) = -\log \sin x + \Phi_0 \Rightarrow \tilde{\phi}(x) = -\frac{1}{2} \log \cos^2 \frac{x}{2} + \phi_0, \quad (2.31)$$

where we pick the same integration constant Φ_0 as for the exterior solution in (2.16), and hence we got the same ϕ_0 as in (2.21). The curvature is obtained by inserting this solution into the right-hand side of (2.27):

$$\tilde{R}(x) = \frac{1}{\cos^2 \frac{x}{2}}. \quad (2.32)$$

This solution is now valid for the finite range $x \in (0, \pi)$. For $x = 0$, the metric, scalar dilaton and curvature take the values

$$\tilde{m}(0) = 0, \quad \tilde{\phi}(0) = \phi_0, \quad \tilde{R}(0) = 1, \quad (2.33)$$

which is consistent with a horizon interpretation. More importantly, the curvature and scalar dilaton on both sides match. At the other end $x = \pi$, all fields and curvature diverge

$$\tilde{m}(\pi) = \infty, \quad \tilde{\phi}(\pi) = \infty, \quad \tilde{R}(\pi) = \infty, \quad (2.34)$$

signaling that this is the position of the BH singularity.

Dual BH solutions. As shown in [9], under T duality the exterior region of the 2D BH is mapped to a region beyond the singularity, while the interior region is mapped into itself. Here we revisit briefly these phenomena.

Starting from the exterior solution (2.16) and performing the duality transformation (2.6), one obtains the new solution, also valid for $x \in (-\infty, 0)$:

$$\hat{m}(x) = -\coth \frac{x}{2}, \quad \hat{\Phi}(x) = -\log |\sinh x| + \Phi_0,$$

$$\hat{\phi}(x) = -\frac{1}{2} \log \sinh^2 \frac{x}{2} + \phi_0, \quad (2.35)$$

where the dual metric is the inverse of the original one and the duality-invariant dilaton remains the same. Inserting the dual metric into the dual curvature (2.27) we get

$$\hat{R}(x) = -\frac{1}{\sinh^2 \frac{x}{2}}. \quad (2.36)$$

In this dual solution, the former horizon $x = 0^-$ is mapped to a curvature singularity

$$\lim_{x \rightarrow 0^-} \hat{m}(x) = \infty, \quad \lim_{x \rightarrow 0^-} \hat{\phi}(x) = \infty, \quad \lim_{x \rightarrow 0^-} \hat{R}(x) = -\infty, \quad (2.37)$$

while the asymptotic region is again a flat spacetime which lies beyond the BH singularity.

The dual solution to the interior solution (2.31) is given by

$$\begin{aligned}\hat{m}(x) &= \cot \frac{x}{2}, & \hat{\Phi}(x) &= -\log \sin x + \Phi_0, \\ \hat{\phi}(x) &= -\frac{1}{2} \log \sin^2 \frac{x}{2} + \phi_0,\end{aligned}\quad (2.38)$$

together with the curvature

$$\hat{R}(x) = \frac{1}{\sin^2 \frac{x}{2}}. \quad (2.39)$$

This geometry still corresponds to the interior region, with the horizon and the singularity exchanged. This can be made clear by noting that³ duality plus the “time reversal” transformation $x \rightarrow \pi - x$ leaves the interior solution invariant: $\hat{m}(\pi - x) = \hat{m}(x)$.

C. General α' corrections

In [22] we considered higher-derivative modifications to (2.14) and pointed out that, due to the presence of the cosmological term, contributions with more derivatives are generally not subleading relative to terms with less derivatives. A naive perturbative expansion is therefore not meaningful unless one assumes that the numerical coefficients of the higher-derivative terms obey certain falloff conditions so that terms with more derivatives are subleading after all. Under this assumption we classified in [22] the higher-derivative terms modulo field redefinitions that preserve the falloff behavior. This leads to the minimal all-order action

$$I = \int dx n e^{-\Phi} [1 + (D\Phi)^2 + F(M)], \quad (2.40)$$

where

$$F(M) \equiv \sum_{i=0}^{\infty} \epsilon_i M^{2i+2} = -M^2 + \epsilon_1 M^4 + \dots, \quad \epsilon_0 \equiv -1. \quad (2.41)$$

Note that, as one consequence of the classification in [22], there are no higher-derivative corrections for the dilaton. The falloff conditions we assumed amount to the following constraints for the coefficients ϵ_i

$$e \equiv \epsilon_1 \ll 1, \quad \epsilon_i \sim (e)^i, \quad i \geq 1, \quad \epsilon_p \epsilon_k \sim \epsilon_{p+k}, \quad (2.42)$$

³We thank M. Gasperini and G. Veneziano for pointing this out.

where \sim indicates equality up to numerical constants of order 1.

Taking the variation of (2.40) with respect to m , Φ and n , combining the dilaton and lapse equations and fixing $n = 1$, we arrive at the all-order extension of (2.15)

$$\frac{d}{dx} (e^{-\Phi} f(M)) = 0, \quad (2.43a)$$

$$\left(\frac{d\Phi}{dx} \right)^2 + \check{g}(M) - 1 = 0, \quad (2.43b)$$

$$\frac{d^2\Phi}{dx^2} + \frac{1}{2} M f(M) = 0. \quad (2.43c)$$

Here we introduced⁴

$$\begin{aligned}f(M) &\equiv F'(M) = \sum_{i=0}^{\infty} (2i+2) \epsilon_i M^{2i+1} = -2M + 4\epsilon M^3 \\ &\quad + \mathcal{O}(\epsilon^2), \\ \check{g}(M) &\equiv \sum_{i=0}^{\infty} (2i+1) \epsilon_i M^{2i+2} = -M^2 + 3\epsilon M^4 + \mathcal{O}(\epsilon^2),\end{aligned}\quad (2.44)$$

which satisfy the following relation

$$\check{g}'(M) = M f'(M), \quad (2.45)$$

where $'$ denotes the derivative with respect to M .

For the two-derivative theory it was straightforward to solve (2.15) to obtain (2.16), the exterior region of the BH. While it is much harder to find solutions of (2.43) for generic $f(M)$ and $g(M)$, assuming that we have such a solution describing the exterior of a BH, we can obtain the solution describing the interior by performing the “signature change” discussed above. To this end we apply (2.24)–(2.40) to get

$$\tilde{I} = \int dx \tilde{n} e^{-\tilde{\Phi}} [1 - (\tilde{D}\tilde{\Phi})^2 + \tilde{F}(\tilde{M})], \quad (2.46)$$

where we have defined

$$\tilde{F}(\tilde{M}) \equiv F(i\tilde{M}) = \sum_{i=0}^{\infty} (-1)^{i+1} \epsilon_i \tilde{M}^{2i+2} = \tilde{M}^2 + \epsilon \tilde{M}^4 + \dots \quad (2.47)$$

The equations of motion in the $\tilde{n} = 1$ gauge are now given by

$$\frac{d}{dx} (e^{-\tilde{\Phi}} \tilde{f}(\tilde{M})) = 0, \quad (2.48a)$$

⁴In our previous work [22], we used $g(M)$ for what is now written as $\check{g}(M)$. This change of notation is useful to deal with the issue of noninvertibility of $M(f)$.

$$\left(\frac{d\tilde{\Phi}}{dx}\right)^2 - \tilde{g}(\tilde{M}) + 1 = 0, \quad (2.48b)$$

$$\frac{d^2\tilde{\Phi}}{dx^2} + \frac{1}{2}\tilde{M}\tilde{f}(\tilde{M}) = 0. \quad (2.48c)$$

The new function $\tilde{f}(\tilde{M})$ is defined analogously to (2.44):

$$\tilde{f}(\tilde{M}) \equiv \tilde{F}'(\tilde{M}), \quad (2.49)$$

while $\tilde{g}(\tilde{M})$ can be defined via the identity

$$\tilde{g}'(\tilde{M}) = \tilde{M}\tilde{f}'(\tilde{M}). \quad (2.50)$$

One can check that these functions are related to the ones in (2.44) via

$$\tilde{f}(\tilde{M}) = if(i\tilde{M}), \quad \tilde{g}(\tilde{M}) = \check{g}(i\tilde{M}). \quad (2.51)$$

Equations (2.43) and (2.48) govern the exterior and interior of higher-derivative BH solutions, respectively.

For later purposes we need to find out how the inverse functions f^{-1} and \tilde{f}^{-1} are related: if f^{-1} is known, what is \tilde{f}^{-1} ? We claim that acting on a variable y we have

$$\tilde{f}^{-1}(y) = -if^{-1}(-iy). \quad (2.52)$$

This can be checked using the first relation in (2.51) as follows:

$$\begin{aligned} \tilde{f}^{-1}(\tilde{f}(\tilde{M})) &= -if^{-1}(-i\tilde{f}(\tilde{M})) = -if^{-1}(-i \cdot if(i\tilde{M})) \\ &= -if^{-1}(f(i\tilde{M})) = -i \cdot i\tilde{M} = \tilde{M}. \end{aligned} \quad (2.53)$$

Noting that we have $f(M)$ and $\tilde{f}(\tilde{M})$ functions, the inverse f^{-1} is the function $M(f)$ and the inverse \tilde{f}^{-1} is the function $\tilde{M}(\tilde{f})$. With this notation, (2.52) reads

$$\tilde{M}(\tilde{f}) = -iM(-i\tilde{f}). \quad (2.54)$$

III. THE GASPERINI-VENEZIANO PARAMETRIZATION

In order to study solutions of (2.43) and (2.48) we will use a useful parametrization inspired by recent work by Gasperini and Veneziano in the context of pre-big bang string cosmology [10]. Instead of describing the fields as functions of the coordinate x , they are described as functions of a new parameter, which arises as follows. One considers the function $f(M)$ in (2.44), a typically infinite series in the metric variable M , and inverts it to find $M(f)$. In $M(f)$ the metric variable is parametrized by f , the Gasperini-Veneziano parameter. One can then find $x(f)$ by performing an integral analytically, or numerically if needed. We will distinguish the parameter for each

region, using f to describe the exterior and \tilde{f} for the interior.

There is a shift in perspective here. While one originally would view the theory space as described by the set of possible functions $f(M)$, we now consider the theory space as described by the set of functions $M(f)$. The invertibility issues are actually central to the development and we will argue that the f parametrization leads to a natural *extension* of the original equations of motion, beyond their original domain of validity. Clarifying this will be our first task in this section.

In this section we show how to solve the exterior and interior equations of motion using this parametrization. We then do this explicitly for the two-derivative BH. Here the nature of the contours required in f space becomes evident.

A. Multivalued functions and equations of motion

As mentioned above and reviewed in Sec. II C, the conventional definition of α' corrections begins by stating that the Lagrangian contains a function $F(M)$ given as an infinite power series expansion in (even) powers of M , each successive term containing an additional power of α' . It is quite possible, perhaps even likely, that this series only has a finite radius of convergence, making the definition of the action incomplete. From $F(M)$ one defines $f(M) \equiv F'(M)$ and one usually works with $f(M)$. The viewpoint adopted here is that α' corrections are described by the *function* $M(f)$ which we assume to be well defined (single valued), and that $f(M)$ is just its inverse. Thus we have

$$M(f) \text{ is a well-defined function (single valued)}. \quad (3.1)$$

It follows that the inverse $f(M)$ is usually multivalued. Indeed, assume $M(f)$ grows monotonically from zero at $f = 0$ to some maximum M_* at some value $f_* > 0$, and then falls down. Then the inverse $f(M)$ is not single valued. In fact if one expands $M(f)$ as a series in f and perturbatively inverts it, $f(M)$ will only converge up to $M = M_*$, where it has a branch point. In general, let M_* denote the maximum value of M for which the series $f(M)$ converges. In summary, we have two important additional facts:

The inverse $f(M)$ of $M(f)$ is not single valued.
The series definition of $f(M)$ converges for $M \in [0, M_*]$ and $f(0) = 0$.

The convergent series is a precise definition of $f(M)$ in this range.

The equations of motion also feature a quantity $\check{g}(M)$ defined by two relations

$$\frac{d\check{g}}{dM} = M \frac{df}{dM} \quad \text{and} \quad \check{g}(0) = 0. \quad (3.2)$$

Both of these are implemented by the integral definition

$$\check{g}(M) \equiv \int_{u=0}^{u=M} u \frac{df(u)}{du} du, \quad M \in [0, M_*]. \quad (3.3)$$

The condition on the range of validity of the definition is needed because \check{g} is a multivalued function. Indeed, df/du appearing inside the integral, is multivalued, a property it inherits from $f(u)$. But now, in this range we can change the variable of integration from u to f . In doing so u must be thought as $u(f)$, where this is the inverse function of $f(u)$. The integral then becomes

$$\check{g}(M) \equiv \int_{f(u=0)}^{f(u=M)} u(f) df = \int_0^{f(M)} u(f') df', \quad (3.4)$$

so that, recalling that we call M the inverse function of f ,

$$\check{g}(M) \equiv \int_0^{f(M)} M(f') df', \quad M \in [0, M_*]. \quad (3.5)$$

The above definition suggests the following construction of a $g(f)$ well defined for all f :

$$g(f) \equiv \int_0^f M(f') df'. \quad (3.6)$$

It follows immediately from the last two equations that

$$\check{g}(M) = g(f(M)), \quad M \in [0, M_*]. \quad (3.7)$$

This states that in the domain of definition, $\hat{g}(M)$ is given by the well-defined $g(f)$.

Let us now consider the field equations, following the steps considered in [10], but in the light of the above observations. In the two-dimensional theory we have (2.43b) that reads

$$\left(\frac{d\Phi}{dx}\right)^2 = 1 - \check{g}(M), \quad M \in [0, M_*]. \quad (3.8)$$

Using the above definition of \check{g} we have

$$\left(\frac{d\Phi}{dx}\right)^2 = 1 - g(f(M)), \quad M \in [0, M_*]. \quad (3.9)$$

Here the dilaton derivative is effectively set equal to a function of $f(M)$. But then we may as well forget M , declaring that this equation sets the dilaton derivative equal to a function of f :

$$\left(\frac{d\Phi}{dx}\right)^2 = 1 - g(f) = 1 - \int_0^f M(\tilde{f}) d\tilde{f}, \quad \forall f. \quad (3.10)$$

This equation is now valid for all f , thus going beyond the limitations of the power series definition of the α'

corrections. The equation indeed yields a parametrization of the dilaton derivative in terms of f :

$$\frac{d\Phi}{dx}(f) = \pm \sqrt{P(f)}, \quad \text{with } P(f) \equiv 1 - \int_0^f M(f') df'. \quad (3.11)$$

The plus/minus signs correspond to using different branches of the square root. The equations of motion also imply that f ends up being a coordinate, which can be related to the x coordinate. Indeed, consider (2.43a) and distribute the x derivative

$$\frac{d\Phi}{dx} f(M) = \frac{df}{dx} \Rightarrow dx = \frac{df}{f \frac{d\Phi}{dx}(f)}. \quad (3.12)$$

Using (3.11), the second equality leads to an equation for the coordinate x as a function of f

$$dx = \pm \frac{df}{f \sqrt{P(f)}}. \quad (3.13)$$

Finally, consider again (2.43a), which states that the combination $e^{-\Phi} f(M)$ is a constant relative to the x coordinate. It is therefore a constant relative to f , and therefore,

$$\Phi(f) = \log |f| + \Phi_1, \quad (3.14)$$

with $\Phi_1 \equiv \Phi(f=1)$ an integration constant that absorbs the sign of f and encodes the mass of the black hole.

The above equations represent an extension of the original equations, and we will assume these provide the nonperturbative definition of the theory with α' corrections.

B. Parametrization in terms of f

In this section we consider the equations of motion in the f parametrization, both for the exterior and for the interior. Note that we have passed from using $\check{g}(M)$ to using $g(f)$, in the exterior, and from using $\check{g}(\tilde{M})$ to using $\check{g}(\tilde{f})$ in the interior. The analog of (3.10) for the interior follows from (2.43c) and becomes

$$\left(\frac{d\tilde{\Phi}}{dx}\right)^2 = -1 + \check{g}(\tilde{f}). \quad (3.15)$$

Exterior solution. Much of the work was already done above, so we just make some comments, consider a few extra equations and collect results. Consider again (3.11):

$$\frac{d\Phi}{dx}(f) = \pm \sqrt{P(f)}, \quad \text{with } P(f) \equiv 1 - \int_0^f M(f') df'. \quad (3.16)$$

The allowed range of f is determined by the condition $P(f) \geq 0$. We see that $P(f=0) = 1 > 0$, so by continuity there is at least an interval around $f=0$ where the solution exists. Moreover, if the integral appearing above is negative for all values of f (as will be the case for the standard BH), then the solution is valid for the whole real line $f \in \mathbb{R}$.

Using (3.13) in (2.12) we obtain an expression for the metric components

$$\frac{m(x_2)}{m(x_1)} = \frac{m(f_2)}{m(f_1)} = \exp\left(\pm \int_{f_1}^{f_2} \frac{df}{f} \frac{M(f)}{\sqrt{P(f)}}\right). \quad (3.17)$$

With $m(f)$, we can relate $\Phi(f)$ to the scalar dilaton $\phi(f)$ via (2.4), where now all quantities are parametrized by f . For the curvature, we have from (2.13),

$$R_{\pm}(f) = -2\left(M^2 + \frac{dM}{df} \frac{df}{dx}\right) = -2\left(M^2 \pm f \frac{dM}{df} \sqrt{P(f)}\right). \quad (3.18)$$

Collecting the results, for ease of reference, we have

$$\frac{d\Phi}{dx}(f) = \pm \sqrt{P(f)}, \quad \Phi(f) = \log|f| + \Phi_1, \quad (3.19a)$$

$$dx = \pm \frac{df}{f \sqrt{P(f)}}, \quad (3.19b)$$

$$\frac{m(x_2)}{m(x_1)} = \frac{m(f_2)}{m(f_1)} = \exp\left(\pm \int_{f_1}^{f_2} \frac{df}{f} \frac{M(f)}{\sqrt{P(f)}}\right), \quad (3.19c)$$

$$R_{\pm}(f) = -2\left(M^2 + \frac{dM}{df} \frac{df}{dx}\right) = -2\left(M^2 \pm f \frac{dM}{df} \sqrt{P(f)}\right). \quad (3.19d)$$

Here the argument $P(f)$ of the square roots is

$$P(f) \equiv 1 - \int_0^f M(f') df'. \quad (3.20)$$

Equation (3.19) encode the general solution to (2.43) in terms of f as a parameter. The solution is completely determined once an ansatz for $M(f)$ is given, and the physics depends on such choice. To make this point more emphatically, we can write the metric $ds^2 = -m^2(x)dt^2 + dx^2$ by passing from the x coordinate to the f coordinate using (3.19b), and writing $m(x)$ using (3.19c):

$$ds^2 = -\exp\left(\pm 2 \int^f \frac{du}{u} \frac{M(u)}{\sqrt{P(u)}}\right) (dt)^2 + \frac{(df)^2}{f^2 P(f)},$$

$$\Phi(f) = \log|f| + \Phi_1, \quad (3.21)$$

where we also included the dilaton. It may be that eventually x can be traded for f , but we will not explore

this here. Still, the above makes clear that we have a solution if we know $M(f)$.

Without $M(f)$ we cannot say much about the specifics of each solution, but we can still analyze some global aspects of them: the first thing to notice is that, from our starting point in (2.44), since $f(M)$ has a power series expansion in M , its inverse $M(f)$, at least perturbatively, is expected to have the following expansion

$$M(f) = -\frac{1}{2}f \left[1 + \frac{\epsilon}{2}f^2 + \mathcal{O}(\epsilon^2 f^4)\right] = -\frac{1}{2}f[1 + h(f^2)], \quad (3.22)$$

where we introduced the function h as follows

$$h(\xi) \equiv \frac{\epsilon}{2}\xi + \mathcal{O}(\epsilon^2 \xi^2). \quad (3.23)$$

Here $M(f) = -\frac{f}{2}$, obtained for $h(\xi) = 0$, corresponds to the standard two-derivative case, for which we recover the exterior BH solution of (2.16), as we will show in Sec. III C. The expansion (3.22) implies the following two properties for generic $M(f)$

$$M(f=0) = 0, \quad M(-f) = -M(f), \quad (3.24)$$

and

$$h(\xi=0) = 0, \quad h'(\xi=0) = \text{finite}. \quad (3.25)$$

These observations have the important consequence that $f=0$ corresponds to the faraway region in all solutions. To see this we study the behavior of (3.19) and (3.20) near $f=0$ for generic $M(f)$. Using (3.22), (3.24), and $P(f=0) = 1$, we get

$$\begin{aligned} \Phi(0) &= -\infty, & dx &\simeq \pm \frac{df}{f}, \\ m(f) &\simeq m(0)e^{\mp \frac{1}{2}f}, & R(0) &= 0. \end{aligned} \quad (3.26)$$

From the second relation we infer $x \simeq \pm \ln f + \text{const}$ and hence that $f=0$ corresponds to an asymptotic region with infinite x . From the third relation we see that we can choose the integration constant such that $m(0) = 1$. Using this together with $\Phi(0) = -\infty$ in (2.4) we obtain $\phi(0) = -\infty$. All these results are compatible with the interpretation as a faraway region.

This makes $f=0$ the end of the spacetime, which implies that solutions with $f < 0$ and with $f > 0$ should be treated separately, not as two regions of the same exterior solution. On top of this distinction, we have the two branches of the square root of (3.11) corresponding to the \pm choices. While this seems to suggest we have four different solutions, all of them should be physically equivalent in string theory. More precisely, the negative

and positive regions of f are related by T duality since $M \rightarrow -M$ and (3.22) imply

$$f \rightarrow \hat{f} = -f. \quad (3.27)$$

Moreover, solutions with different signs of the square root (3.11) are connected via a trivial sign flip of the x coordinate. More precisely, the solution with minus sign and $f < 0$ is identical to the solution with plus sign and $f > 0$, upon changing $x \rightarrow -x$. Their T -dual solutions can be obtained by (3.27).

Interior solution. In order to get the interior solutions in the Gasperini-Veneziano parametrization, we repeat the procedure we just performed for the exterior case, but this time using the Eq. (2.48). Without going into details, the solutions read

$$\frac{d\tilde{\Phi}}{dx}(\tilde{f}) = \pm \sqrt{\tilde{P}(\tilde{f})}, \quad \tilde{\Phi}(\tilde{f}) = \log|\tilde{f}| + \Phi_1, \quad (3.28a)$$

$$dx = \pm \frac{d\tilde{f}}{\tilde{f} \sqrt{\tilde{P}(\tilde{f})}}, \quad (3.28b)$$

$$\frac{\tilde{m}(x_2)}{\tilde{m}(x_1)} = \frac{\tilde{m}_{\pm}(\tilde{f}_2)}{\tilde{m}_{\pm}(\tilde{f}_1)} = \exp\left(\pm \int_{\tilde{f}_1}^{\tilde{f}_2} \frac{d\tilde{f}}{\tilde{f}} \frac{\tilde{M}(\tilde{f})}{\sqrt{\tilde{P}(\tilde{f})}}\right), \quad (3.28c)$$

$$\tilde{R}_{\pm}(\tilde{f}) = 2\left(\tilde{M}^2 + \frac{d\tilde{M}}{d\tilde{f}} \frac{d\tilde{f}}{dx}\right) = 2\left(\tilde{M}^2 \pm \tilde{f} \frac{d\tilde{M}}{d\tilde{f}} \sqrt{\tilde{P}(\tilde{f})}\right). \quad (3.28d)$$

Here the argument of the square roots, called $\tilde{P}(\tilde{f})$, is defined as follows

$$\tilde{P}(\tilde{f}) \equiv -1 + \int_0^{\tilde{f}} \tilde{M}(\tilde{f}') d\tilde{f}'. \quad (3.29)$$

Since $\tilde{P}(\tilde{f} = 0) = -1 < 0$, this time the solution excludes the point $\tilde{f} = 0$. Moreover, in general it is not even guaranteed that the interior solution exist at all. There would be no interior solution if $\tilde{P}(\tilde{f})$ is negative for all \tilde{f} .

Using (2.54) we can determine the function $\tilde{M}(\tilde{f})$ for the interior region from the function $M(f)$ for the exterior region given in (3.22):

$$\tilde{M}(\tilde{f}) = \frac{1}{2}\tilde{f} \left[1 - \frac{\epsilon}{2}\tilde{f}^2 + \mathcal{O}(\epsilon^2\tilde{f}^4) \right] = \frac{1}{2}\tilde{f}[1 + h(-\tilde{f}^2)], \quad (3.30)$$

with the same $h(\xi)$ defined in (3.23).

The range of validity of the interior solution is very different from the one of the exterior solution. From now on we assume there is always at least one positive branch point

$f_0 > 0$ such that $\tilde{P}(f_0) = 0$ and $\tilde{P}(\tilde{f}) > 0$ for some interval $\tilde{f} \in (f_0, f_1)$, where $f_1 > f_0$ can be another branch point or the point at infinity. If this is the case, due to the even parity of $\tilde{P}(\tilde{f})$, the interior solution exists at least for an interval $\tilde{f} \in (-f_1, -f_0) \cup (f_0, f_1)$. However, since for the interior solution T duality also acts as

$$\tilde{f} \rightarrow \hat{\tilde{f}} = -\tilde{f}, \quad (3.31)$$

both negative and positive regions are related via a duality transformation. We are imposing to have at least one branch point for positive f . With multiple such branch points it is possible to have multiple separated domains within $\tilde{f} > 0$ such that $\tilde{P}(\tilde{f})$ is positive in each of those disconnected regions.

C. The standard black hole

Here we work out the presentation of the standard two-derivative BH solution of Sec. II A in the Gasperini-Veneziano parametrization. The experience gained here will prove essential for the later generalizations.

Exterior solution. The $M(f)$ ansatz for the exterior region of the two-derivative theory corresponds to the limit $\epsilon \rightarrow 0$ in (3.22):

$$M(f) = -\frac{1}{2}f. \quad (3.32)$$

Associated to M we have $P(f)$ given in (3.20):

$$P(f) = 1 + \frac{1}{4}f^2 > 0 \quad \forall f \in \mathbb{R}, \quad (3.33)$$

which implies that all f are in principle allowed. The duality-invariant dilaton does not depend on the specific ansatz for $M(f)$, so it is still given by (3.14). Using (3.33) in (3.11), its x derivative is given by

$$\frac{d\Phi}{dx} = \sqrt{1 + \frac{1}{4}f^2}, \quad (3.34)$$

where we choose the plus sign to fit conventions where $x = 0^-$ is the horizon and the exterior region is $x < 0$, with the AF region at $x \rightarrow -\infty$. Moreover, to stick to this convention, we need to choose $f \geq 0$. The relation between x and f can be obtained from (3.13), which for this ansatz takes the form

$$dx = \frac{df}{f} \frac{1}{\sqrt{1 + \frac{1}{4}f^2}} = -d \text{ArcSinh} \frac{2}{f}. \quad (3.35)$$

Integrating,

$$x(f) = -\text{ArcSinh} \frac{2}{f} \rightarrow \frac{f}{2} = -\frac{1}{\sinh x}, \quad (3.36)$$

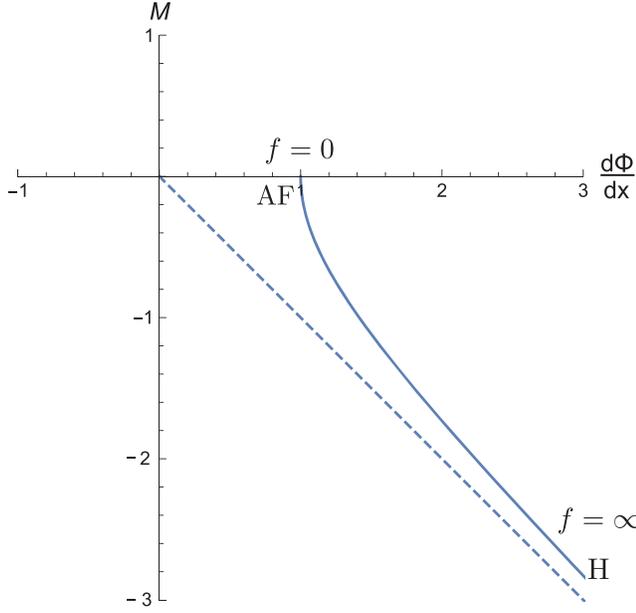


FIG. 3. Black hole exterior. The solid line is the plot of the dilaton derivative $d\Phi/dx$ and M as a function of $f \in [0, \infty)$. The point on the horizontal axis is $f = 0$ and represents the AF region. As $f \rightarrow \infty$ we reach the horizon and the curve asymptotes to the dashed line $|M| = |\frac{d\Phi}{dx}|$.

where we choose the integration constant to be zero. The $x = 0^-$ horizon corresponds to $f = \infty$ and the AF region $x = -\infty$ corresponds to $f = 0^+$ (see Fig. 3):

$$\begin{aligned} \text{AF region: } x &= -\infty, & f &= 0^+, \\ \text{Horizon: } x &= 0, & f &= +\infty. \end{aligned} \quad (3.37)$$

Increasing x corresponds to increasing f .

In order to determine the metric we take the plus sign in (3.19c) and use (3.32) and (3.33):

$$\frac{m(f_2)}{m(f_1)} = \exp\left(-\frac{1}{2} \int_{f_1}^{f_2} \frac{df}{\sqrt{1 + \frac{f^2}{4}}}\right). \quad (3.38)$$

We can integrate this analytically, but before doing so it is instructive to consider the ratio of the scale factor at the horizon $f = \infty$ divided by the value at the AF region about $f = 0$:

$$\frac{m(f = \infty)}{m(f = 0)} = \exp\left(-\frac{1}{2} \int_0^\infty \frac{df}{\sqrt{1 + \frac{f^2}{4}}}\right) = 0, \quad (3.39)$$

due to the logarithmic divergence at the top limit of integration, and confirming the vanishing of the metric at the horizon: $m(f = \infty) = 0$. Since the scale of m is not detected by the equations of motion, we can set $m(f = 0) = 1$. Setting $f_2 = f$ and $f_1 = 0$ in (3.38) we get

$$\begin{aligned} m(f) &= \exp\left(-\frac{1}{2} \int_0^f \frac{df'}{\sqrt{1 + \frac{f'^2}{4}}}\right) = \exp\left(-\text{ArcSinh} \frac{f}{2}\right) \\ &= -\frac{f}{2} + \sqrt{1 + \frac{f^2}{4}}. \end{aligned} \quad (3.40)$$

This indeed gives $m(0) = 1$, and $m(f) \simeq 1/f$ for large f , so that $m(\infty) = 0$. We now compute the curvature from (3.18), using the top sign and finding

$$R(f) = f \left(-\frac{f}{2} + \sqrt{1 + \frac{f^2}{4}}\right) = fm(f). \quad (3.41)$$

Note that $R(0) = 0$, as befits the AF region, and $R(f) \simeq 1$ for $f \rightarrow \infty$, which is the value of the curvature at the horizon.

At this point we have $M(f)$, $\Phi(f)$, $m(f)$ and $R(f)$, and so the full f -parametrized solution is determined. As a consistency check one may use the relation between f and x to recover the BH solution in the familiar form reviewed in Sec. II B. For the metric, for example, we have, using (3.36), and recalling that x is negative,

$$\begin{aligned} m(x) &= \frac{1}{\sinh x} + \sqrt{1 + \frac{1}{\sinh^2 x}} = \frac{1}{\sinh x} \left(1 - \sqrt{1 + \sinh^2 x}\right) \\ &= -\tanh \frac{x}{2}. \end{aligned} \quad (3.42)$$

For the dilaton we have, since $f > 0$,

$$\Phi = \log f + c = \log\left(\frac{-2}{\sinh x}\right) + c = -\log |\sinh x| + c'. \quad (3.43)$$

Interior solution. The BH interior solution of (2.31) can also be recovered in the \tilde{f} parametrization, using (3.28). The ansatz for $\tilde{M}(\tilde{f})$ is obtained by using the formula (2.54) on the ansatz for the exterior (3.32):

$$\tilde{M}(\tilde{f}) = \frac{1}{2} \tilde{f}. \quad (3.44)$$

The argument of the square root this time takes the form

$$\tilde{P}(\tilde{f}) = -1 + \frac{1}{4} \tilde{f}^2, \quad (3.45)$$

and so the square root has branch cuts on the real line with $|\tilde{f}| > 2$, with branch points at $\tilde{f} = \pm 2$. Therefore, the interior solution is valid for the interval $\tilde{f} \in (-\infty, -2) \cup (2, \infty)$, where negative and positive regions are related via T duality (3.31). From now on we choose $\tilde{f} \geq 2$.

In order to describe all of the internal region in the \tilde{f} parametrization we need to cover the range $\tilde{f} \geq 2$ twice.

We begin from $\tilde{f} = \infty$ to $\tilde{f} = 2$ traveling under the branch cut, and then we return from $\tilde{f} = 2$ to $\tilde{f} = \infty$ over the cut. The difference between paths is given by the choice of \pm sign in front of the square root in the solution (3.28). The square root is assumed to take negative values below the cut ($-$ sign) and positive values above the cut ($+$ sign).

When going under the cut, the dilaton derivative is given by

$$\frac{d\tilde{\Phi}}{dx}(\tilde{f}) = -\sqrt{-1 + \frac{1}{4}\tilde{f}^2}. \quad (3.46)$$

The relation between x and \tilde{f} in (3.28b) takes the form

$$dx = -\frac{d\tilde{f}}{\tilde{f}\sqrt{-1 + \frac{1}{4}\tilde{f}^2}} = -\frac{d\tilde{f}}{|\tilde{f}|\sqrt{-1 + \frac{1}{4}\tilde{f}^2}} = d \operatorname{arccsc} \frac{\tilde{f}}{2}, \quad (3.47)$$

$$\tilde{f} \geq 2.$$

Integrating this equation and setting the integration constant to zero we get

$$x(\tilde{f}) = \operatorname{arccsc} \frac{\tilde{f}}{2} \rightarrow \frac{\tilde{f}(x)}{2} = \csc x = \frac{1}{\sin x}. \quad (3.48)$$

In this parametrization as \tilde{f} decreases x increases. Indeed, we have $x(\tilde{f} = \infty) = 0$ and $x(\tilde{f} = 2) = \frac{\pi}{2}$.

$$\tilde{f} \in [\infty, 2]_- \rightarrow x \in \left[0, \frac{\pi}{2}\right], \quad (3.49)$$

with the minus subscript indicating that we are traveling under the cut. As we will see, $\tilde{f} = \infty$ ($x = 0$) corresponds to the position of the horizon, while $\tilde{f} = 2$ ($x = \frac{\pi}{2}$) is just an intermediate point in the interior solution.

For the metric we need (3.28c) with the minus sign. Choosing the lower boundary to be $\tilde{f}_1 = 2$ and leaving the upper boundary arbitrary we get

$$\begin{aligned} \tilde{m}_-(\tilde{f}) &= \tilde{m}_-(2) \exp\left(-\frac{1}{2} \int_2^{\tilde{f}} \frac{df}{\sqrt{-1 + \frac{1}{4}f^2}}\right) \\ &= \tilde{m}_-(2) \exp\left(-\operatorname{arccosh} \frac{\tilde{f}}{2}\right), \end{aligned} \quad (3.50)$$

and since $\operatorname{arccosh}(1) = 0$ we can take $\tilde{m}_-(2) = 1$. Since $\operatorname{arccosh} y = \log(y + \sqrt{-1 + y^2})$, we obtain

$$\tilde{m}_-(\tilde{f}) = \frac{\tilde{f}}{2} - \sqrt{-1 + \frac{\tilde{f}^2}{4}} = \frac{\tilde{f}}{2} \left(1 - \sqrt{-\frac{4}{\tilde{f}^2} + 1}\right), \quad (3.51)$$

where for the second equality we used that \tilde{f} is positive. For the curvature one can show with (3.28d) that

$$\tilde{R}_-(\tilde{f}) = \tilde{f} \left(\frac{\tilde{f}}{2} - \sqrt{-1 + \frac{\tilde{f}^2}{4}}\right) = \tilde{f} \tilde{m}_-(\tilde{f}). \quad (3.52)$$

As anticipated, $\tilde{m}_-(\tilde{f})$ and $\tilde{R}_-(\tilde{f})$ are consistent with $\tilde{f} = \infty$ being the location of the horizon since

$$\lim_{\tilde{f} \rightarrow \infty} \tilde{m}_-(\tilde{f}) = 0, \quad \lim_{\tilde{f} \rightarrow \infty} \tilde{R}_-(\tilde{f}) = 1, \quad (3.53)$$

as can be checked with the expansion around $\tilde{f} \rightarrow \infty$ of the second equality in (3.51). On the other hand, $\tilde{f} = 2$ is just an intermediate point in the interior solution such that

$$\tilde{m}_-(2) = 1, \quad \tilde{R}_-(2) = 2. \quad (3.54)$$

The other half of the solution is recovered by going over the cut. In this situation, the dilaton derivative (3.28a) and the relation between x and \tilde{f} from (3.28b) have the opposite sign as in (3.46) and (3.47). For the latter we have

$$dx = -d \operatorname{arccsc} \frac{\tilde{f}}{2} = d \operatorname{arccos} \frac{2}{\tilde{f}} \Rightarrow x(\tilde{f}) = \operatorname{arccos} \frac{2}{\tilde{f}} + c_0, \quad (3.55)$$

where after integration we picked a nontrivial integration constant c_0 . This is necessary since at $\tilde{f} = 2$ we already had $x = \frac{\pi}{2}$ from under the cut, so by continuity we must have

$$\frac{\pi}{2} = \operatorname{arccos} 1 + c_0 = c_0 \rightarrow c_0 = \frac{\pi}{2}. \quad (3.56)$$

Therefore,

$$x(\tilde{f}) - \frac{\pi}{2} = \operatorname{arccos} \frac{2}{\tilde{f}} \rightarrow \frac{2}{\tilde{f}(x)} = \cos\left(x - \frac{\pi}{2}\right) = \sin x. \quad (3.57)$$

Indeed we have $x(\tilde{f} = 2) = \frac{\pi}{2}$ and $x(\tilde{f} = \infty) = \pi$, since x grows with \tilde{f} . So we have

$$\tilde{f} \in [2, \infty)_+ \rightarrow x \in \left[\frac{\pi}{2}, \pi\right], \quad (3.58)$$

where the subscript $+$ indicates going on top of the cut. It is interesting to note that the relation $\tilde{f}(x)$ is the same for under the cut, cf. (3.48), and over the cut, cf. (3.57). The former covers $x \in (0, \frac{\pi}{2})$ and the latter the remaining half of the interior solution $x \in (\frac{\pi}{2}, \pi)$.

This time, taking the plus sign choice in (3.28c) and (3.28d), we obtain the expressions for the metric and curvature over the cut:

$$\tilde{m}_+(\tilde{f}) = \frac{\tilde{f}}{2} + \sqrt{-1 + \frac{\tilde{f}^2}{4}}, \quad \tilde{R}_+(\tilde{f}) = \tilde{f} \tilde{m}_+(\tilde{f}). \quad (3.59)$$

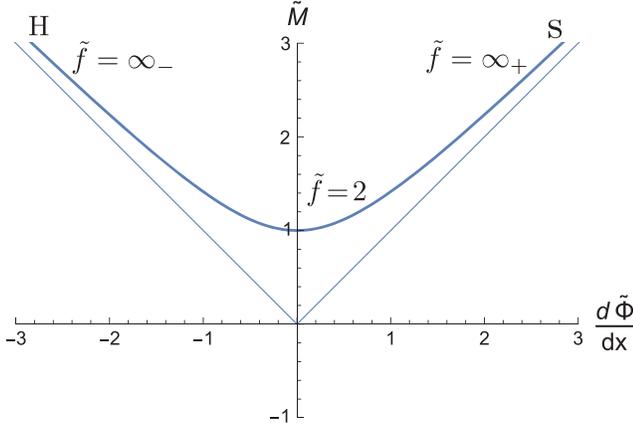


FIG. 4. Black hole interior. The solid line plots the relation between the dilaton spatial derivative and \tilde{M} . The horizon H is on the upper left, the singularity S on the upper right (both at $\tilde{f} = \infty$, below and above the cut). The branch point at $\tilde{f} = 2$ corresponds to the point with minimal $\tilde{M} = 1$. The \tilde{f} contour is that shown in Fig. 1(b).

The intermediate point $\tilde{f} = 2$ ($x = \frac{\pi}{2}$) gives again (3.54), but this time the boundary $\tilde{f} = \infty$ ($x = \pi$) corresponds to the singularity, where

$$\lim_{\tilde{f} \rightarrow \infty} \tilde{m}_+(\tilde{f}) = \infty, \quad \lim_{\tilde{f} \rightarrow \infty} \tilde{R}_+(\tilde{f}) = \infty. \quad (3.60)$$

As a consistency check one may again verify that the solution in the form (2.31) follows from the relation between \tilde{f} and x . A plot of the dilaton derivative $d\tilde{\Phi}/dx$ and the metric derivative \tilde{M} , both as a function of \tilde{f} helps visualize the solution (Fig. 4). As we go from the horizon to the singularity the dilaton derivative first decreases and then increases. The metric \tilde{m} goes from zero to infinity, while \tilde{M} first decreases and then increases.

IV. α' CORRECTED BLACK HOLE SOLUTIONS

In the first part of this section we present a class of functions $M(f)$ that lead to BH solutions in the conventional sense. These solutions have an exterior and an interior region. The exterior includes in f space the point zero, which can be identified with the AF region, and the point at infinity, which can be identified with a horizon. The interior region begins at a horizon and ends in a singularity, that is not removed in this class of solutions. Both regions, we believe, form a single solution because the curvature on both regions attain the same value at the horizon. We make no claims that this is the most general class of M s that lead to conventional BH solutions.

We then consider a cubic ansatz for $M(f)$, the simplest deformation of the expression $M(f) = -\frac{1}{2}f$ valid in the two-derivative theory. We do not find black hole solutions here. Instead we find situations where the exterior and

interior solutions do not seem to form a single solution, situations with naked singularities, and situations where only the exterior solution exists.

A. A family of BH solutions

This family of solutions is parametrized by a function $h(\xi)$ via the relation

$$M(f) = -\frac{f}{2}(1 + h(f^2)), \quad (4.1)$$

where we take

$$h(0) = 0, \quad h'(0) < \infty, \quad (4.2)$$

conditions that allow making $f = 0$ the far away region of the black hole. We also require that the correction to $M = -f/2$ implied by h is “small.” For this we impose

$$|h(\xi)| \leq 1, \quad \forall \xi \in \mathbb{R}. \quad (4.3)$$

This implies, in particular, that $M(f) \leq 0$ for all $f > 0$. To get a horizon we need conditions on the behavior of h for large argument:

$$\lim_{\xi \rightarrow \infty} h(\xi) = 0, \quad \int_0^{\infty} h(\xi) d\xi = \alpha < \infty, \quad \lim_{\xi \rightarrow \infty} \xi h'(\xi) = 0. \quad (4.4)$$

The first condition implies that $M(f) \simeq -f/2$ for very large f . The second condition strengthens the bound $|h(\xi)| \leq 1$ by requiring that the full integral over positive arguments be finite. The third condition follows from the first for a regular function at infinity.

For the interior of the black hole, we need conditions on $h(\xi)$ for large negative ξ . At the cost of some generality, we simply demand that h be an odd function of its argument:

$$h(-\xi) = -h(\xi). \quad (4.5)$$

The interior solution demands one further constraint. Let $f_0 > 0$ be the value of f where the function $\tilde{P}(f)$, defined in (3.29), first goes from negative to positive, so that $\tilde{P}(f_0) = 0$. We will demand that

$$h(f_0^2) < 1, \quad (4.6)$$

in particular, $h(f^2)$ is not equal to one at this point.

Exterior solution. For the exterior solution, we have Eq. (3.19). Picking the plus sign and inserting the function $M(f)$ as given in (4.1), the equations for the metric and curvature reduce to

$$\frac{m(f_2)}{m(f_1)} = \exp\left(\int_{f_1}^{f_2} df I_e(f)\right), \quad I_e(f) \equiv -\frac{1 + h(f^2)}{2\sqrt{P(f)}}, \quad (4.7a)$$

$$R(f) = -\frac{f}{2} \left[f(1 + h(f^2))^2 - 2 \left(1 + h(f^2) + 2f^2 \frac{dh(f^2)}{df^2} \right) \sqrt{P(f)} \right], \quad (4.7b)$$

with the subscript e for “exterior” and

$$P(f) = 1 + \frac{1}{4}f^2 + \frac{1}{4} \int_0^{f^2} d\xi h(\xi), \quad (4.8)$$

which determines the location of branches. From the bound (4.3) on $h(\xi)$, we infer that

$$-f^2 \leq \int_0^{f^2} d\xi h(\xi) \leq f^2, \quad (4.9)$$

and we conclude that

$$P(f) \geq 1 \quad \forall f \in \mathbb{R}, \quad (4.10)$$

showing that there are no branch cuts in the f plane and we can work with

$$f \in (0, \infty). \quad (4.11)$$

We now consider the behavior of the integrand I_e . Using the vanishing of $h(\xi = 0)$ and the first two conditions in (4.4), we conclude

$$\begin{aligned} \lim_{f \rightarrow 0} I_e(f) &= -\frac{1}{2}, \\ \lim_{f \rightarrow \infty} I_e(f) &\simeq -\frac{1}{f} \Rightarrow \int_0^f df' I_e(f') \simeq -\log f, \quad \text{as } f \rightarrow \infty. \end{aligned} \quad (4.12)$$

For the ratio of metric values, these limits imply

$$\frac{m(\infty)}{m(0)} = \exp\left(\int_0^\infty df I_e(f)\right) = \exp(-\infty) = 0, \quad (4.13)$$

which allows us to pick the integration constant such that

$$m(0) = 1, \quad m(\infty) = 0, \quad (4.14)$$

the former consistent with $f = 0$ being the faraway region and the latter consistent with $f = \infty$ being the horizon.

Additionally, as required, the curvature goes to zero for $f \rightarrow 0$ and is finite for $f \rightarrow \infty$. While the former is a general feature of solutions coming from the classification, the behavior at the horizon is a consequence of (4.4), which implies

$$\lim_{f \rightarrow \infty} R(f) \simeq -\frac{f^2}{2} \left[1 - \sqrt{1 + \frac{4 + \alpha}{f^2}} \right] \simeq 1 + \frac{\alpha}{4}. \quad (4.15)$$

Let us now show that the behavior of the dilaton is as in the standard two-derivative black hole: the duality invariant dilaton diverges at the horizon but the scalar dilaton is finite. For this recall that

$$\Phi = \log f + \Phi_1, \quad (4.16)$$

where Φ_1 is a constant, and we can take the logarithm without concern as $f \in (0, \infty)$. The asymptotically flat region is $f \sim 0$ and corresponds to $\Phi \rightarrow -\infty$ which is weak coupling. Using (2.4) and that $m \simeq 1$ in this region, the scalar dilaton also indicates weak coupling. The horizon is obtained as $f \rightarrow \infty$ and this means $\Phi \rightarrow \infty$ so we need to consider the metric contribution. We can obtain $\log m(f)$ by taking the log of (4.7a), and picking the upper bound of the integration to be $f_2 = f$:

$$\log m(f) = \log m(f_1) + \int_{f_1}^f df' I_e(f'). \quad (4.17)$$

For large f , using the last relation in (4.12) we find that

$$\log m \simeq -\log f + c_2, \quad \text{as } f \rightarrow \infty, \quad (4.18)$$

where c_2 is a constant. Therefore, at the horizon x_h , which corresponds to $f \rightarrow \infty$, from (2.4) we have

$$\phi(x_h) \simeq \frac{1}{2} (\log f + \Phi_1 - \log f + c_2) \simeq c, \quad (4.19)$$

a constant, as we wanted to confirm.

Interior solution. For the interior solution, we use (2.54) to find that the ansatz (4.1) gives

$$\tilde{M}(\tilde{f}) = -iM(-i\tilde{f}) = \frac{\tilde{f}}{2} (1 + h(-\tilde{f}^2)) = \frac{\tilde{f}}{2} (1 - h(\tilde{f}^2)), \quad (4.20)$$

where the last equality follows because h is odd. The expressions for the metric ratios in (3.28c) and the curvature in (3.28d), become

$$\frac{\tilde{m}_\pm(\tilde{f}_2)}{\tilde{m}_\pm(\tilde{f}_1)} = \exp\left(\pm \int_{\tilde{f}_1}^{\tilde{f}_2} df I_i(\tilde{f})\right), \quad I_i(\tilde{f}) \equiv \frac{1 - h(\tilde{f}^2)}{2\sqrt{\tilde{P}(\tilde{f})}}, \quad (4.21a)$$

$$\begin{aligned} \tilde{R}_\pm(\tilde{f}) &= \frac{\tilde{f}}{2} \left[\tilde{f}(1 - h(\tilde{f}^2))^2 \right. \\ &\quad \left. \pm 2 \left(1 - h(\tilde{f}^2) - 2\tilde{f}^2 \frac{dh(\tilde{f}^2)}{d\tilde{f}^2} \right) \sqrt{\tilde{P}(\tilde{f})} \right], \end{aligned} \quad (4.21b)$$

with i for ‘‘interior.’’ Regarding branch cuts, this time the range of \tilde{f} depends on the roots of the function

$$\tilde{P}(\tilde{f}) = -1 + \frac{1}{4}\tilde{f}^2 - \frac{1}{4}\int_0^{\tilde{f}^2} d\xi h(\xi). \quad (4.22)$$

This function picks the following values at the boundaries

$$\begin{aligned} \tilde{P}(0) &= -1 < 0, \\ \lim_{\tilde{f} \rightarrow \pm\infty} \tilde{P}(\tilde{f}) &= \lim_{\tilde{f} \rightarrow \pm\infty} \left(-1 - \frac{1}{4}\alpha + \frac{1}{4}\tilde{f}^2 \right) = \infty, \end{aligned} \quad (4.23)$$

where we used the second equation of (4.4). This shows that $\tilde{P}(\tilde{f})$ must change sign as \tilde{f} grows from zero. Let us call f_0 the value where $\tilde{P}(f_0) = 0$ a branch point. Notice also that

$$\frac{d\tilde{P}}{d\tilde{f}} = \frac{1}{2}\tilde{f}(1 - h(\tilde{f}^2)) \geq 0, \quad (4.24)$$

since $|h| \leq 1$. Note, however, that $h(f_0^2) < 1$ by assumption so that

$$\left. \frac{d\tilde{P}}{d\tilde{f}} \right|_{f_0} > 0. \quad (4.25)$$

Once $\tilde{P}(\tilde{f})$ becomes positive it will remain positive all the way since its derivative is never negative. Therefore, the range of validity of the solution is taken to be

$$\tilde{f} \in (f_0, \infty). \quad (4.26)$$

Using (4.9) one can show that $f_0 > \sqrt{2}$. For the standard BH we had $f_0 = 2$.

We now investigate the condition for the horizon. We have from (4.21a)

$$\frac{\tilde{m}_{\pm}(\infty)}{\tilde{m}_{\pm}(f_0)} = \exp\left(\pm \int_{f_0}^{\infty} d\tilde{f} I_i(\tilde{f})\right). \quad (4.27)$$

We need the right-hand side to be infinite for the plus sign; this makes $\tilde{m}(\infty)$ over the cut infinite, assuming $\tilde{m}(f_0)$ is finite. This is as desired. For the minus sign the right-hand side should be zero, which makes $\tilde{m}(\infty)$ below the cut equal to zero—this is as desired as it corresponds to the horizon. To show this, we consider the integral of I_i on the right-hand side. At the branch point f_0 there is an integrable square-root singularity: $\tilde{P}(f_0) = 0$ and the derivative $\tilde{P}'(f_0)$ is nonzero, see (4.24). We thus need only focus on the behavior of I_i for large \tilde{f} . For the $\tilde{f} \rightarrow \infty$ limit, using (4.4) we find

$$I_i(\tilde{f}) \simeq \frac{1}{\tilde{f}} \quad \text{as } \tilde{f} \rightarrow \infty \Rightarrow \int_{f_0}^{\tilde{f}} d\tilde{f}' I_i(\tilde{f}') \simeq \ln \tilde{f}, \quad f \rightarrow \infty. \quad (4.28)$$

Finally, inserting this into (4.27) we get $\frac{\tilde{m}_{\pm}(\infty)}{\tilde{m}_{\pm}(f_0)} = e^{\pm\infty}$, from where we can choose as integration constant a finite value of \tilde{m} at the branch point $\tilde{m}_+(f_0) = \tilde{m}_-(f_0) = \tilde{m}_0$, so that we have, as desired

$$\tilde{m}_-(\infty) = 0, \quad \text{and} \quad \tilde{m}_+(\infty) = \infty. \quad (4.29)$$

The curvature at the branch point f_0 is easily evaluated since $\tilde{P}(f_0)$ vanishes:

$$\tilde{R}_-(f_0) = \tilde{R}_+(f_0) = 2\tilde{M}^2(f_0) = \frac{1}{2}f_0^2(1 - h(f_0^2))^2. \quad (4.30)$$

For the infinite limit we have, from (4.21b),

$$\tilde{R}_{\pm}(\tilde{f}) \simeq \frac{1}{2}\tilde{f}^2 \left[1 \pm \sqrt{1 - \frac{4 + \alpha}{\tilde{f}^2}} \right], \quad \tilde{f} \rightarrow \infty. \quad (4.31)$$

As a result we find

$$\lim_{\tilde{f} \rightarrow \infty} \tilde{R}_-(\tilde{f}) = 1 + \frac{\alpha}{4}, \quad \lim_{\tilde{f} \rightarrow \infty} \tilde{R}_+(\tilde{f}) = \infty. \quad (4.32)$$

The first is the curvature at the horizon as computed from the interior. As required, it coincides with the curvature at the horizon computed from the exterior in (4.15). The second result is the infinite curvature at the singularity.

For the duality-invariant dilaton in the interior region we have

$$\tilde{\Phi} = \log \tilde{f} + \Phi_1, \quad (4.33)$$

which is well defined for $\tilde{f} \in (f_0, \infty)$. $\tilde{\Phi}$ does not differentiate between lower or upper branches. It diverges at $\tilde{f} \rightarrow \infty$ but it is finite for $\tilde{f} = f_0$. For the scalar dilaton we use

$$\tilde{\phi}_{\pm}(\tilde{f}) = \frac{1}{2}(\tilde{\Phi}(\tilde{f}) + \log \tilde{m}_{\pm}(\tilde{f})), \quad (4.34)$$

which does distinguish branches through \tilde{m}_{\pm} . While it is clear that $\tilde{\phi}_+(f_0) = \tilde{\phi}_-(f_0) = \text{finite}$, for $\tilde{f} \rightarrow \infty$ under and over the cut we need $\log \tilde{m}_{\pm}$. To this end, as we did for the exterior region, we integrate the metric ratio in (4.21a)

$$\log \tilde{m}_{\pm}(f) = \log \tilde{m}_{\pm}(f_1) \pm \int_{f_1}^{\tilde{f}} d\tilde{f}' I_i(\tilde{f}'). \quad (4.35)$$

Using large \tilde{f} value of the integral from (4.28), we find

$$\log \tilde{m}_{\pm}(f) \simeq \pm \log \tilde{f} + c_{\pm}, \quad \tilde{f} \rightarrow \infty, \quad (4.36)$$

where c_{\pm} are constants. Plugging this expansion back into (4.34), together with (4.33) we find that for $\tilde{f} \rightarrow \infty$

$$\tilde{\phi}_-(\tilde{f}) \simeq \tilde{c}, \quad \tilde{\phi}_+(\tilde{f}) \simeq \log \tilde{f}. \quad (4.37)$$

The first one is consistent with the horizon interpretation, where we need $\tilde{c} = c$ with c in (4.19), in order to match exterior and interior regions. The behavior of $\tilde{\phi}_+$ close to infinity signals, once more, the presence of a singularity there.

To sum up, the general ansatz (4.1) parametrized with a function $h(\xi)$ satisfying (4.3)–(4.5) lead to BH solutions. There are infinitely many functions satisfying these conditions. For example

$$h(\xi) = \xi \exp(-\xi^2), \quad (4.38)$$

which satisfies $\int_0^\infty h(\xi) d\xi = \frac{1}{2}$ and $|h(\xi)| \leq \frac{1}{\sqrt{2e}} < 1$.

The way the more general solutions are parametrized in terms of f mimics the standard BH. The exterior corresponds to $f \in (0, \infty)$ with $f = 0$ the faraway region and $f = \infty$ the horizon. The interior region is covered by two patches: $\tilde{f} \in (f_0, \infty)_-$ and $\tilde{f} \in (f_0, \infty)_+$, where the first goes under the branch cut [picking the minus sign in (4.21)] and the second over it [with the plus sign in (4.21)]. We begin under the branch at $\tilde{f} = \infty$ (the horizon) until $\tilde{f} = f_0$, the branch point, where all quantities are finite. From there, we return to $\tilde{f} = \infty$ over the branch cut. This time, the metric and curvature diverge at this point, which is identified as the singularity.

B. Cubic ansatz for $M(f)$

The class of solutions introduced above assume a function $h(\xi)$ that vanishes at zero and at infinity, among other conditions. Here we assume that $h(\xi)$ is just linear in ξ , leading to a *cubic* ansatz for $M(f)$:

$$M(f) = -\frac{1}{2}f + cf^3, \quad h(\xi) = -2c\xi, \quad (4.39)$$

with $c \neq 0$, otherwise we are back to the two-derivative BH. This $h(\xi)$ does not satisfy the conditions in (4.3) and (4.4), and so the physics is very different. We will find that the solutions do not seem to have a BH interpretation due to the presence of naked singularities and disconnected regions. This shows that arbitrary α' corrections can easily eliminate BH solutions. We were also motivated to consider this case because some of the pre-big bang solutions of [10] were constructed with a cubic ansatz for $M(f)$.

Without going into the calculations which are straightforward, we proceed to enumerate the main characteristics of these solutions. To this end, we need the branch structures of exterior and interior solutions, which can be inferred, respectively, from

$$P(f) = 1 + \frac{1}{4}f^2 - \frac{c}{4}f^4, \quad \text{and} \quad \tilde{P}(\tilde{f}) = -1 + \frac{1}{4}\tilde{f}^2 + \frac{c}{4}\tilde{f}^4. \quad (4.40)$$

The analysis is different depending on the sign of c . For simplicity we give the results for $f > 0$, but similar results

hold for the negative region, which is connected to the positive one via T duality.

(1) $c > 0$:

The exterior region is parametrized by $f \in (0, f_0)$, where

$$f_0^2 = \frac{\sqrt{1+16c}+1}{2c} \simeq \frac{1}{c} + \mathcal{O}(\sqrt{c}), \quad c \rightarrow 0. \quad (4.41)$$

For positive f , the function $P(f)$ only vanishes at f_0 . Since $f_0 \sim 1/\sqrt{c}$, the appearance of a branch cut for the exterior solution is nonperturbative in c . As it should be, $f = 0$ still corresponds to the far away region. This time, the boundary at f_0 is a point with nonvanishing metric and *finite* curvature.

The interior region exists for $|\tilde{f}| \geq \tilde{f}_0$, with $\tilde{f}_0^2 = \frac{\sqrt{1+16c}-1}{2c}$. As in the standard BH and the class of deformations studied above, we cover the solution by the two patches $\tilde{f} \in (\tilde{f}_0, \infty)_- \cup (\tilde{f}_0, \infty)_+$, corresponding to going under the cut and then coming back over the cut. At one end, $\tilde{m}_- = 0$ but $\tilde{R}_- = \infty$. At the other end $\tilde{m}_+ = \infty$ and $\tilde{R}_+ = \infty$.

The absence of a horizon in the exterior and the presence of singularities in the interior makes this solution incompatible with a black hole. Moreover, the two geometries do not connect. Each one is a separate solution.

(2) $c < 0$:

In this situation, $P(f)$ is always positive and therefore the exterior solution is valid for $f \in (0, \infty)$ as in the two-derivative case. As usual, $f \sim 0$ is the AF region, but $f \rightarrow \infty$ is not a horizon but a point with vanishing metric with infinite curvature. This region then corresponds to a naked singularity.

The interior region has now two branch points, \tilde{f}_0 and \tilde{f}_1 . The point \tilde{f}_0 is the same as for the $c > 0$ case, and \tilde{f}_1 is given from $\tilde{f}_1^2 = \frac{-\sqrt{1+16c}-1}{2c}$. The interior region only exists for $c > -\frac{1}{16}$ and $\tilde{f} \in (\tilde{f}_0, \tilde{f}_1)$. Since this \tilde{f} interval is finite, is easy to see that \tilde{m}_\pm and \tilde{R}_\pm are regular everywhere but the metric never vanishes. For $c < -\frac{1}{16}$, there is no interior solution, just an exterior one.

When both exist, the exterior and the interior solutions do not combine. They are separate solutions, and we have no black hole.⁵

V. HORIZONS AND SINGULARITIES

Having gained experience on the constraints of black hole solutions in the previous section, we now want to

⁵The construction of [14] was in this class of negative c theories, so while there seems to be a regular interior solution, the exterior solution does not work out.

discuss some general lessons. We show that for dilaton-gravity theories in two-dimensional time-independent backgrounds T duality implies a constraint on the construction of regular black holes (geometries with a horizon but no curvature singularity). We will prove that T duality maps a geometry with a regular horizon to one with a curvature singularity. This shows that the observation of Giveon [9] for the two-derivative black hole is general. For a solution that is expected to be self-dual, this means one cannot have a horizon and no singularity. Moreover, we explain, as a corollary, how the usual branch structure of the interior solution implies self-duality and thus a singularity. In the next section we will give black hole solutions that have an unusual branch structure and that are *not* self-dual so that the singularity *can* be avoided.

To prove that a horizon implies a singularity in the dual geometry we begin with a configuration where, without loss of generality, we can take $x = 0$ as the position of the horizon, the point where the metric $m(x)$ vanishes. A regular configuration is one where the scalar curvature is finite everywhere. Since the conclusions should be independent of any gauge choice, we pick $n = 1$ for simplicity.

Demanding $R = -2m''/m$ (2.13) to be finite everywhere implies, in particular, $m''(x)/m(x)$ must be finite at the horizon. Then, from $\hat{R} = -R - 4M^2$, (2.9), we see that regularity of the dual curvature \hat{R} additionally forces $M = m'/m$ to be finite everywhere, in particular at the horizon. Together with the horizon condition we obtain

$$\lim_{x \rightarrow 0} m(x) = 0, \quad \lim_{x \rightarrow 0} \frac{m''(x)}{m(x)} = \text{finite}, \quad \lim_{x \rightarrow 0} \frac{m'(x)}{m(x)} = \text{finite}. \quad (5.1)$$

To gain intuition, we now show that (5.1) cannot be satisfied for analytic metric components. If $m(x)$ is analytic, then it admits a regular Taylor expansion around $x = 0$, and the location of the horizon is

$$m = a_1 x + \frac{1}{2} a_2 x^2 + \frac{1}{3} a_3 x^3 + \dots, \quad (5.2)$$

where the first condition in (5.1) rules out the presence of a constant term. Plugging (5.2) into the second condition of (5.1) we find

$$\lim_{x \rightarrow 0} \frac{m''(x)}{m(x)} \simeq \frac{a_2}{a_1} \frac{1}{x} + \dots, \quad (5.3)$$

which is finite only if $a_2 = 0$. Finally, with $a_2 = 0$ the third condition on (5.1) now reads

$$\lim_{x \rightarrow 0} \frac{m'(x)}{m(x)} \simeq \frac{1}{x} + \dots, \quad (5.4)$$

which diverges regardless the coefficients of the Taylor expansion. This logarithmic derivative diverges at the

position of the horizon. As claimed the conditions cannot be satisfied.

A. A proof that a horizon implies a curvature singularity

Having considered the case when $m(x)$ is analytic we now build a general proof. We define the neighborhood $N_\epsilon = (0, \epsilon)$, and the neighborhood $\bar{N}_\epsilon = [0, \epsilon)$ where we include the point 0. We assume that there is an ϵ sufficiently small such that the following hold:

- (1) The function $m(x)$ is continuous in \bar{N}_ϵ and vanishing at $x = 0$. This is an isolated zero: the neighborhood \bar{N}_ϵ does not contain another zero of m .
- (2) The curvature $R(x)$ in \bar{N}_ϵ is finite.

These are the natural conditions satisfied by a conventional horizon.

Claim: The dual curvature \hat{R} cannot be finite in \bar{N}_ϵ .

Proof. Since the curvature R and the dual curvature \hat{R} are related as follows

$$\hat{R} = -R - 4 \left(\frac{m'}{m} \right)^2, \quad (5.5)$$

and R is assumed finite in \bar{N}_ϵ this means that the finiteness condition

$$\frac{m'(x)}{m(x)} < \infty, \quad \forall x \in \bar{N}_\epsilon, \quad (5.6)$$

cannot hold. Let us assume it holds and derive a contradiction.

Since the sign of $m(x)$ is immaterial to the definition of the metric component g_{00} , we can change it at will so that we can require $m(x) > 0$ for $x \in N_\epsilon$. This follows from the continuity of m and the isolated zero. Given the positivity of $m(x)$ in \bar{N}_ϵ we can write $m(x)$ in terms of its logarithm $g(x)$:

$$m(x) = e^{g(x)}, \quad \text{with } m(0) = 0 \rightarrow g(0) = -\infty. \quad (5.7)$$

Since $m(x) \neq 0$ in N_ϵ , the function $g(x)$ is finite in N_ϵ . Taking a derivative of m we have

$$\frac{m'(x)}{m(x)} = g'(x). \quad (5.8)$$

Given the hypothesis (5.6) we conclude that $g'(x) < \infty$ in \bar{N}_ϵ . Now consider the relation

$$g(0) = g(x) - \int_0^x g'(u) du, \quad \forall x \in N_\epsilon, \quad (5.9)$$

and examine the right-hand side above. For $x \in N_\epsilon$, the function $g(x)$ is finite. Moreover, since g' is finite in \bar{N}_ϵ and x is finite, the integral is also finite. But with both terms on

the right-hand side finite, we cannot have $g(0) = -\infty$ as required. This contradiction means that (5.6) does not hold; we need g' to diverge in order for the integral above to give an infinite $g(0)$. This is what we wanted to show. The proof does not tell us where is \hat{R} infinite, but the intuition of the analytic case indicates that it is expected to be at $x = 0$. ■

The result above is reasonable: In one dimension, T -duality maps $m(x) \rightarrow 1/m(x)$, which makes a vanishing metric unavoidably dual to a divergent one. All we had to prove was that such dual metric has divergent curvature.

This result actually gives a simple proof that the interior solution based on the contour extending below and above the branch cut in f space must lead to a singularity. The main observation here is simple. Recall that \tilde{R} is given in (2.26) and its dual \hat{R} is obtained by letting $\tilde{M} \rightarrow -\tilde{M}$,

$$\begin{aligned}\tilde{R} &= 2\tilde{M}^2 + 2D\tilde{M}, \\ \hat{R} &= 2\tilde{M}^2 - 2D\tilde{M}.\end{aligned}\quad (5.10)$$

Compare with Eq. (3.28d) for the curvatures above (+) and below (-) the cut:

$$\tilde{R}_{\pm}(\tilde{f}) = 2\tilde{M}^2 \pm 2\tilde{f} \frac{d\tilde{M}}{d\tilde{f}} \sqrt{\tilde{P}(\tilde{f})}.\quad (5.11)$$

Since $\tilde{M}(\tilde{f})$ and $\tilde{P}(\tilde{f})$ are functions without branches, each term in the above curvature formula takes the same value above or below the cut; the only difference being that they enter with different sign combinations for the curvature and for its dual. It thus follows that we can identify $\tilde{R} = \tilde{R}_+$ and $\hat{R} = \tilde{R}_-$. The curvatures above and below the cut are related by T duality! Since the \tilde{f} contour defining the x domain extends both above and below the cut, this means that having a horizon below the cut (as we conventionally set it up) will imply a curvature singularity above the cut. The interior solution is self-dual: T duality, supplemented with the time reparametrization that exchanges the values of $x(\tilde{f})$ for points immediately above and below the cut, leaves the fields invariant.

VI. REGULAR BLACK HOLES

In this section we modify the analysis of Sec. IV A where we found black hole solutions with conventional properties. We will show that with a couple of extra constraints on the function $M(f)$ we can obtain *regular* black holes: solutions where the exterior, as usual, has an asymptotically flat region and a horizon, and the interior has no singularity in the metric, the curvature, or the dilaton. As in Sec. IV A we parametrize M in terms of a function h

$$M(f) = -\frac{1}{2}f[1 + h(f^2)],\quad (6.1)$$

with $h(\xi)$ satisfying the same properties as before: $h(0) = 0$, $h'(0)$ is finite, $|h(\xi)| \leq 1$ for all ξ , $h(\infty) = 0$, $\xi h'(\xi)|_{\xi=\infty} = 0$, the parity condition $h(-\xi) = -h(\xi)$ and $\int_0^\infty h(\xi)d\xi = \alpha < \infty$. If this is the case, the interior solution is parametrized by

$$\tilde{M}(\tilde{f}) = \frac{1}{2}\tilde{f}[1 - h(\tilde{f}^2)].\quad (6.2)$$

There was one additional condition on $h(\xi)$ (recall that $\xi = f^2$): it is not supposed to reach the value of one at the point $f_0 > 0$ where the function $\tilde{P}(f)$ first goes from negative to positive. We will explicitly relax this condition; this will be essential for the interior solution. Letting $\xi_0 = f_0^2$, we demand that

$$\begin{aligned}h(\xi_0) &= 1, & h'(\xi_0) &= 0, & h''(\xi_0) &< 0, \\ \text{and } \int_0^{\xi_0} d\xi h(\xi) &= \xi_0 - 4,\end{aligned}\quad (6.3)$$

We will confirm that with these conditions $\tilde{P}(f_0) = 0$. This fact and the motivation for the above set of conditions will be explained below.

If we call \mathcal{T} the space of all possible α' corrections [the space of all $M(f)$ s], and \mathcal{S} the subspace of Sec. IV A leading to conventional 2D black holes, the regular black holes arise from the boundary of \mathcal{S} . This is clear because the inequality $h(f_0^2) < 1$, needed for conventional black holes, is saturated for regular black holes. They do not cover the full boundary, because we require additional conditions.

We will now examine the equations of motion in the f parametrization, both in the exterior and the interior, finding solutions. We will see that in the interior the noninvertibility of $\tilde{M}(\tilde{f})$ plays a central role. We will also explain that the solution for the regular black hole has an interior that is a cosmology that at late times is asymptotic to Minkowski space with a constant dilaton. This behavior suggests strongly that this is not a solution in string theory.

A. Regular BH solutions

Exterior solution. The additional conditions (6.3) preserve all good features of the exterior BH solution. In particular, for the exterior region f_0 has no special meaning, it is just a finite intermediate point in the solution's domain. The object that determines the latter is the argument of the square root:

$$P(f) = 1 + \frac{1}{4}f^2 + \frac{1}{4} \int_0^{f^2} d\xi h(\xi),\quad (6.4)$$

and the domain $f \in (0, \infty)$ that we use remains unchanged since, as a consequence of the bound $|h(\xi)| \leq 1$, we have

$P(f) > 0$ for such domain. The analysis of the exterior solution in Sec. IV A remains unchanged, and the results for the metric, curvature and dilaton derived there hold here as well.

Interior solution. For the interior solution, the role of f_0 is crucial to get a regular solution. The idea is that the conditions (6.3) render the position $x(f_0)$ infinitely far away. Therefore, after going from the horizon ($\tilde{f} = \infty$) to $\tilde{f} = f_0$ under the cut, we *do not need* to go back to $\tilde{f} = \infty$ on the top of the cut, where the singularity always was. Now, we can just stop at f_0 because in terms of x this is the end of spacetime.

We begin with the function $\tilde{P}(\tilde{f})$ appearing in the interior formulas under a square root:

$$\tilde{P}(\tilde{f}) = -1 + \int_0^{\tilde{f}} \tilde{M}(\tilde{f}') d\tilde{f}' = -1 + \frac{1}{4}\tilde{f}^2 - \frac{1}{4} \int_0^{\tilde{f}^2} d\xi h(\xi). \quad (6.5)$$

We then have

$$\begin{aligned} \tilde{P}(\tilde{f} = 0) &= -1, & \tilde{P}(f_0) &= 0, \\ \tilde{P}(\tilde{f}) &\simeq \frac{1}{4}\tilde{f}^2 & \text{for } \tilde{f} &\rightarrow \infty. \end{aligned} \quad (6.6)$$

The first equality is manifest, the second equality follows from the last condition in (6.3), and the last one from the convergence of $\int_0^\infty h(\xi) d\xi$. To understand the nature of the interior solution we must consider derivatives of $\tilde{P}(\tilde{f})$. The first derivative coincides with $\tilde{M}(\tilde{f})$:

$$\tilde{P}'(\tilde{f}) = \tilde{M}(\tilde{f}) = \frac{\tilde{f}}{2} [1 - h(\tilde{f}^2)] \geq 0 \quad \forall \tilde{f} \geq 0, \quad (6.7)$$

where the inequality follows from $|h| \leq 1$. We then have the following values

$$\begin{aligned} \tilde{P}'(0) &= \tilde{M}(0) = 0, & \tilde{P}'(f_0) &= \tilde{M}(f_0) = 0, \\ \tilde{P}'(\tilde{f}) &= \tilde{M}(\tilde{f}) \simeq \frac{1}{2}\tilde{f} & \text{for } \tilde{f} &\rightarrow \infty. \end{aligned} \quad (6.8)$$

The first equality is manifest, the second follows from $h(\xi_0) = 1$, the third from the vanishing of h for large argument. The second derivative is given by

$$\tilde{P}''(\tilde{f}) = \tilde{M}'(\tilde{f}) = \frac{1}{2}(1 - h(\xi)) - \xi h'(\xi), \quad (6.9)$$

using $\xi = \tilde{f}^2$. We obtain

$$\tilde{P}''(\tilde{f} = 0) = \tilde{M}'(\tilde{f} = 0) = \frac{1}{2}, \quad \tilde{P}''(f_0) = \tilde{M}'(f_0) = 0. \quad (6.10)$$

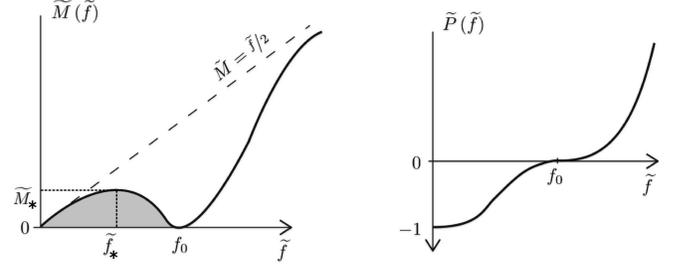


FIG. 5. Left: sketch of $\tilde{M}(\tilde{f})$. The shaded region must have unit area, the point \tilde{f}_* indicates the position of the first local maximum $\tilde{M}(\tilde{f}_*) = \tilde{M}_*$, the point f_0 is a minimum, and for large \tilde{f} the curve approaches the line $\tilde{f}/2$. Right: sketch of $\tilde{P}(\tilde{f})$. This function changes sign at f_0 , which is an inflection point.

The second equality holds because $h(\xi_0) = 1$ and $h'(\xi_0) = 0$. For the third derivative we have

$$\tilde{P}'''(\tilde{f}) = \tilde{M}''(\tilde{f}) = -3\tilde{f}h'(\xi) - 2\tilde{f}^3h''(\xi). \quad (6.11)$$

The special values here are

$$\begin{aligned} \tilde{P}'''(\tilde{f} = 0) &= \tilde{M}''(\tilde{f} = 0) = 0, \\ \tilde{P}'''(f_0) &= \tilde{M}''(f_0) = -2f_0^3h''(\xi_0) > 0, \end{aligned} \quad (6.12)$$

the second relation following from the assumption $h''(\xi_0) < 0$.

All these results can be represented collectively in the two plots in Fig. 5, where we show $\tilde{M}(\tilde{f})$ and $\tilde{P}(\tilde{f})$.

The plot of \tilde{M} (left) shows a function that begins at zero with positive slope and remains strictly positive until f_0 where $\tilde{M}(f_0) = \tilde{M}'(f_0) = 0$ and $\tilde{M}''(f_0) > 0$, which makes f_0 a minimum. The area under the curve in the interval $(0, f_0)$ is exactly one because $\tilde{P}(f_0) = -1 + \int_0^{f_0} d\tilde{f} \tilde{M}(\tilde{f}) = 0$. For $\tilde{f} > f_0$ the function remains always positive [see (6.7)] and approaches infinity exactly as the standard BH.

Let us now turn to the graph of $\tilde{P}(\tilde{f})$. This function equals -1 for $\tilde{f} = 0$, the minimum of $\tilde{P}(\tilde{f})$. The function then increases with \tilde{f} until f_0 , where it vanishes together with its first and second derivative. Since the third derivative is nonzero, in fact, $\tilde{P}'''(f_0) > 0$, we see that f_0 is an inflection point. Afterwards, $\tilde{P}(\tilde{f})$ keeps increasing monotonically, as can be seen from (6.7), and at infinity it behaves as $\tilde{f}^2/4$.

Note now that with $h(\xi)$ as discussed, the domain for the interior solution is $\tilde{f} \in (f_0, \infty)$. With the function $\tilde{P}(\tilde{f})$, its first derivative, and its second derivative all vanishing at f_0 , we see that \tilde{P} near f_0 is given by

$$\tilde{P}(\tilde{f}) \simeq \frac{1}{6}\tilde{P}'''(f_0)(\tilde{f} - f_0)^3 = -\frac{1}{3}f_0^3h''(f_0^2)(\tilde{f} - f_0)^3 > 0 \quad \text{for } \tilde{f} \sim f_0. \quad (6.13)$$

This behavior will allow us to regularize the BH's interior.

We first confirm the presence of the horizon in this interior solution (below the cut). Consider the third equation in (3.28), picking the minus sign and integrating over the whole range of \tilde{f} :

$$\frac{\tilde{m}(\infty)}{\tilde{m}(f_0)} = \exp\left(-\int_{f_0}^{\infty} d\tilde{f} \frac{1-h(\tilde{f}^2)}{2\sqrt{\tilde{P}(\tilde{f})}}\right). \quad (6.14)$$

The integral is convergent at the lower limit since the integrand in fact vanishes there (the numerator vanishes like $(\tilde{f} - f_0)^2$ and the denominator vanishes like $(\tilde{f} - f_0)^{3/2}$. At the upper limit we get a logarithmic divergence since $\sqrt{\tilde{P}} \sim \tilde{f}$ [see (6.6)] and the numerator approaches a constant. Thus the integral is infinite and the right-hand side of (6.14) becomes zero. Since $\tilde{m}(f_0)$ is finite we can choose

$$\tilde{m}(f_0) = 1, \quad \tilde{m}(\infty) = 0. \quad (6.15)$$

This is the horizon at $\tilde{f} = \infty$.

For the curvature, we can use (5.11) with the lower sign, as we are working below the cut,

$$\tilde{R}_-(\tilde{f}) = 2\left[(\tilde{P}'(\tilde{f}))^2 - \tilde{f}\tilde{P}''(\tilde{f})\sqrt{\tilde{P}(\tilde{f})}\right], \quad (6.16)$$

where we rewrote the dependence on \tilde{M} in terms of derivatives of \tilde{P} . The curvature vanishes at $\tilde{f} = f_0$, because \tilde{P} , \tilde{P}' and \tilde{P}'' all vanish at f_0 . For $\tilde{f} \rightarrow \infty$ the behavior of \tilde{P} is the same as we had in Sec. IV A, and therefore the curvature computation we did there applies here as well. The relevant result is given by the first expression in (4.32). In summary, we have

$$\tilde{R}_-(f_0) = 0, \quad \tilde{R}_-(\infty) = 1 + \frac{\alpha}{4}. \quad (6.17)$$

For the dilaton we have, as usual, $\tilde{\Phi}(\tilde{f}) = \log \tilde{f} + \Phi_1$, which is finite for f_0 and diverges for $\tilde{f} \rightarrow \infty$. The scalar dilaton is given by $\tilde{\phi} = \frac{1}{2}(\tilde{\Phi} + \log \tilde{m})$. For $\tilde{f} = f_0$ we get $\tilde{\phi}(f_0) = \frac{1}{2}\tilde{\Phi}(f_0)$ since $\tilde{m}(f_0) = 1$. The scalar dilaton, as opposed to the duality invariant dilaton $\tilde{\Phi}$, has no divergence as $\tilde{f} \rightarrow \infty$. The argument follows identically as the one used in Sec. IV A to arrive at (4.37), which in this case reads

$$\tilde{\phi}(\tilde{f}) \simeq \tilde{c} \quad \text{for } \tilde{f} \rightarrow \infty.$$

Finally, the relation between x and \tilde{f} is very different from that in the family of singular black holes: the point halfway from the horizon to the singularity is moved to infinite distance. This is the branch point at f_0 . Using the behavior (6.13) of $\tilde{P}(\tilde{f})$ close to f_0 , the differential equation (3.13) relating x and \tilde{f} takes the form

$$dx = -d\tilde{f} \frac{1}{\tilde{f}\sqrt{\tilde{P}(\tilde{f})}} \simeq -d\tilde{f} \sqrt{\frac{3}{f_0^5|h''(\xi_0)|}}(\tilde{f} - f_0)^{-\frac{3}{2}}$$

for $\tilde{f} \sim f_0$. (6.18)

This integral diverges as $\tilde{f} \rightarrow f_0$. On the other hand, for $\tilde{f} \rightarrow \infty$ we get a finite integral: here $\tilde{P}(\tilde{f}) \simeq \frac{1}{2}\tilde{f}$ and therefore $dx \simeq -2d\tilde{f}/\tilde{f}^2$, which is integrable as $\tilde{f} \rightarrow \infty$. This allows us to define $x(\tilde{f})$ with the following values at the boundaries

$$x(f_0) = \infty, \quad x(\infty) = 0. \quad (6.19)$$

These results make the interior region to be a regular cosmology, identifying x with a time coordinate τ and t with a spatial coordinate w , as required by the signature of the metric

$$ds^2 = \tilde{m}^2(x)dt^2 - dx^2 = -d\tau^2 + \tilde{m}^2(\tau)dw^2. \quad (6.20)$$

This cosmology begins at time $\tau = 0$ with finite curvature $\tilde{R}(\tau = 0) = 1 + \frac{\alpha}{4}$ and evolves as $\tau \rightarrow \infty$ to a geometry with zero curvature and hence to flat space. In this cosmology there is no big-bang, and time begins at the position of the horizon.

We now perform a local analysis to see how the solution approaches the flat region to compare it with standard cosmology. For clarity we continue to use x (rather than τ). We begin by integrating (6.18) to get the relation between x and \tilde{f} close to f_0

$$x(\tilde{f}) \simeq \sqrt{\frac{12}{f_0^5|h''(\xi_0)|}} \frac{1}{\sqrt{\tilde{f} - f_0}} \quad \text{for } \tilde{f} \sim f_0, \quad (6.21)$$

where we dropped an integration constant. Inverting this expression we get

$$\tilde{f}(x) \simeq f_0 + \frac{12}{f_0^5|h''(\xi_0)|} \frac{1}{x^2} \quad \text{for } x \rightarrow \infty. \quad (6.22)$$

With this relation we can express all quantities locally in terms of x . For $\tilde{M}(x)$ we use (6.2) to expand $\tilde{M}(\tilde{f})$ around $\tilde{f} \sim f_0$:

$$\tilde{M}(\tilde{f}) = \tilde{M}(\tilde{f}_0) + \tilde{M}'(f_0)(\tilde{f} - f_0) + \frac{1}{2}\tilde{M}''(f_0)(\tilde{f} - f_0)^2 + \mathcal{O}((\tilde{f} - f_0)^3). \quad (6.23)$$

Recalling that $\tilde{M}(f_0) = \tilde{M}'(f_0) = 0$ and using the value for the second derivative from (6.12) we get

$$\tilde{M}(\tilde{f}) \simeq f_0^3|h''(\xi_0)|(\tilde{f} - f_0)^2 \quad \text{for } \tilde{f} \sim f_0, \quad (6.24)$$

which in terms of x reads

$$\tilde{M}(x) \simeq \frac{144}{f_0^7 |h''(\xi_0)| x^4} \quad \text{for } x \rightarrow \infty. \quad (6.25)$$

Now $\tilde{m}(x)$ is given by

$$\begin{aligned} \tilde{m}(x) &= C \exp\left(\int^x dx' \tilde{M}(x')\right) \simeq C \exp\left(-\frac{48}{f_0^7 |h''(\xi_0)| x^3}\right) \\ &\simeq C \left(1 - \frac{48}{f_0^7 |h''(\xi_0)| x^3}\right), \end{aligned} \quad (6.26)$$

with C a constant of integration. For the curvature we use (2.27) to get

$$\tilde{R}(x) = 2(\tilde{M}^2(x) + \tilde{M}'(x)) \simeq 2\tilde{M}'(x) \simeq -\frac{1152}{f_0^7 |h''(\xi_0)| x^5}, \quad (6.27)$$

where we noted that $\tilde{M}^2 \sim x^{-8}$ and thus can be ignored to lowest order. Finally, the duality invariant dilaton is given by

$$\tilde{\Phi} = \log \tilde{f} + \Phi_1 \simeq \frac{1}{f_0} (\tilde{f} - f_0) + \log f_0 + \Phi_1 \quad \text{for } \tilde{f} \sim f_0, \quad (6.28)$$

which in terms of x reads

$$\tilde{\Phi}(x) \simeq \frac{12}{f_0^6 |h''(\xi_0)| x^2} + \log f_0 + \Phi_1. \quad (6.29)$$

The scalar dilaton is given by $\tilde{\phi}(x) \simeq \frac{1}{2} \tilde{\Phi}(x)$ since the contribution from the logarithm of \tilde{m} is of order x^{-3} and thus subleading.

Remark. It is important to note that the interior solution, for infinite time x , becomes flat Minkowski space with a *constant* dilaton. This may seem surprising, given the well-known fact that the ‘‘cosmological’’ term in the 2D action requires a linear dilaton when the spacetime becomes flat. Indeed, the black hole exterior in the far away region operates in this way, both in the two-derivative theory and in the case above: the metric approaches the flat metric and the dilaton profile approaches a linear function. To see what is happening in the interior consider the exterior and interior relations (3.10), (3.11) and (3.15), (3.28a):

$$\begin{aligned} \left(\frac{d\Phi}{dx}\right)^2 &= 1 - g(f) = P(f), \\ \left(\frac{d\tilde{\Phi}}{dx}\right)^2 &= -1 + \tilde{g}(\tilde{f}) = \tilde{P}(\tilde{f}). \end{aligned} \quad (6.30)$$

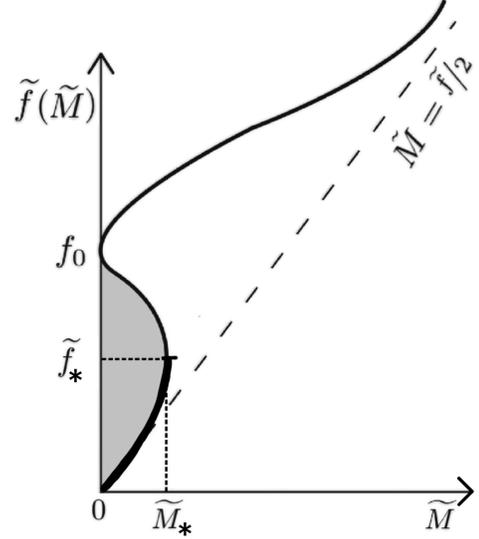


FIG. 6. Sketch of $\tilde{f}(\tilde{M})$, the inverse of $\tilde{M}(\tilde{f})$ in Fig. 5, making its multivaluedness evident. The bold curve is $\tilde{f}(\tilde{M})$ on the domain of expected convergence of its series representation.

Moreover, just as we have (3.7) in the exterior, there is a similar equation in the interior

$$\begin{aligned} \check{g}(M) &= g(f(M)), & M &\in [0, M_*], \\ \check{g}(\tilde{M}) &= \tilde{g}(\tilde{f}(\tilde{M})), & \tilde{M} &\in [0, \tilde{M}_*], \end{aligned} \quad (6.31)$$

where \tilde{M}_* is the value beyond which the inverse $\tilde{f}(\tilde{M})$ of $\tilde{M}(\tilde{f})$ is not expected to have a convergent series. Looking at Fig. 5, \tilde{M}_* is the maximum of $\tilde{M}(\tilde{f})$ attained for $\tilde{f}_* \in (0, f_0)$. The non-invertibility issue can be better appreciated in Fig. 6.

Consider first the exterior, in particular the AF region $f \sim 0$ and $M \sim 0$. Here $P(f=0) = 1$ [see (6.4)], and therefore $g(0) = 0$ [(6.30)]. This is consistent with the top equation in (6.31): perturbatively $\check{g}(M) = -M^2 + \dots$ which gives (left-hand side) $\check{g}(0) = 0$ consistent with (right-hand side) $g(f(0)) = g(0) = 0$. Indeed, this requires a rolling dilaton [(6.30)].

Consider now the interior, in particular the AF region $\tilde{f} \sim f_0$ and $\tilde{M} \sim 0$. Here $\tilde{P}(f_0) = 0$ (6.6), therefore $\tilde{g}(f_0) = 1$ (6.30), and there is no need for a rolling dilaton (6.30). The value $\tilde{g}(f_0) = 1$ may sound surprising given that $\tilde{M}(f_0) = 0$ and $\check{g}(\tilde{M} = 0) = 0$. There is no contradiction, however, with the bottom equation in (6.31): $\check{g}(\tilde{M}) = \tilde{M}^2 + \dots$ gives (left-hand side) $\check{g}(\tilde{M} = 0) = 0$ consistent with (right-hand side) $\tilde{g}(\tilde{f}(0)) = \tilde{g}(0) = 0$. Since both $\tilde{M}(0)$ and $\tilde{M}(f_0)$ are zero, the inverse function $\tilde{f}(\tilde{M})$ is necessarily multivalued. In the domain of definition of the perturbative expansion $\tilde{f}(0) = 0$, so this expansion cannot see what is happening for $\tilde{f} \sim f_0$.

The above discussion shows how this regular BH convincingly emerges from a suitable choice of $M(f)$. Still, the interior solution is approaching a background

defined by two-dimensional Minkowski space and a *fixed* dilaton. Such a background is associated with a $c = 2$ conformal field theory, not a $c = 26$ one. Since classical backgrounds of (bosonic) string field theory are $c = 26$ conformal field theories, one does not expect the regular BH solution to be a string theory solution. More likely, the choice of $M(f)$ leading to it defines α' corrections that do not occur in string theory.

B. A particular example

As a proof of the existence of a function with the properties mentioned above, we built a concrete example:

$$h(\xi) = \frac{16\xi_0^3 \xi}{(\xi^2 + 3\xi_0^2)^2}. \quad (6.32)$$

In here ξ_0 is a parameter that will be adjusted to make h satisfy all the requisite conditions. The derivatives of h are

$$h'(\xi) = \frac{48\xi_0^3(\xi_0^2 - \xi^2)}{(\xi^2 + 3\xi_0^2)^2}, \quad h''(\xi) = \frac{192\xi_0^3\xi(\xi^2 - 3\xi_0^2)}{(\xi^2 + 3\xi_0^2)^4}. \quad (6.33)$$

We see that consistent with conditions (4.2) we have $h(0) = 0$ and finite $h'(0)$. It is simple to check that, as required by (4.3), $h(\xi) \leq 1$ for all ξ , with $|h(\pm\xi_0)| = 1$. The integral of h is easily calculated

$$\int_0^\xi d\xi' h(\xi') = \frac{8\xi_0}{3} \frac{\xi^2}{\xi^2 + 3\xi_0^2}. \quad (6.34)$$

This implies that

$$\int_0^\infty d\xi' h(\xi') = \frac{8\xi_0}{3} = \alpha < \infty, \quad (6.35)$$

so that the finiteness condition expressed in the second relation in (4.4) is obeyed. The first and third condition in (4.4) (vanishing of h and $\xi h'$ as $\xi \rightarrow \infty$) are also satisfied. Note also that h is an odd function, as required.

All of the above conditions were already imposed in the previous section, but here we have the new conditions concerning the point ξ_0 , where h reaches the value of one. We indeed have, as required by the conditions listed in (6.3),

$$h(\xi_0) = 1, \quad h'(\xi_0) = 0, \quad h''(\xi_0) = -\frac{3}{2} \frac{1}{\xi_0^2} < 0. \quad (6.36)$$

The value of ξ_0 is determined by the integral constraint in (6.3)

$$\int_0^{\xi_0} d\xi' h(\xi') = \frac{2\xi_0}{3} = \xi_0 - 4 \rightarrow \xi_0 = 12. \quad (6.37)$$

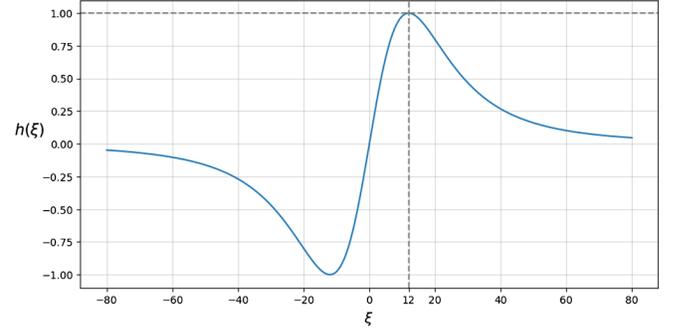


FIG. 7. Plot of $h(\xi) = \frac{16\xi_0^3 \xi}{(\xi^2 + 3\xi_0^2)^2}$ with $\xi_0 = 12$. The dotted lines indicate the position of the maximum.

We see that for this value of ξ_0 we have $\alpha = 32$. The function h has a maximum at ξ_0 , and being odd, a minimum at $-\xi_0$. It vanishes both for $\xi = 0$ and as $\xi \rightarrow \infty$. A plot of this function can be seen in Fig. 7. Figure 8 shows the plot of $d\tilde{\Phi}/dx$ and \tilde{M} as a function of \tilde{f} for this regular black hole (continuous line). Superposed in the figure we see the similar plot (Fig. 4) for the two-derivative black hole (dashed lines). The coordinate x for the interior as a function of \tilde{f} is given by the integral

$$x = \int_{\tilde{f}}^{\infty} \frac{df}{f \sqrt{P(f)}}, \quad (6.38)$$

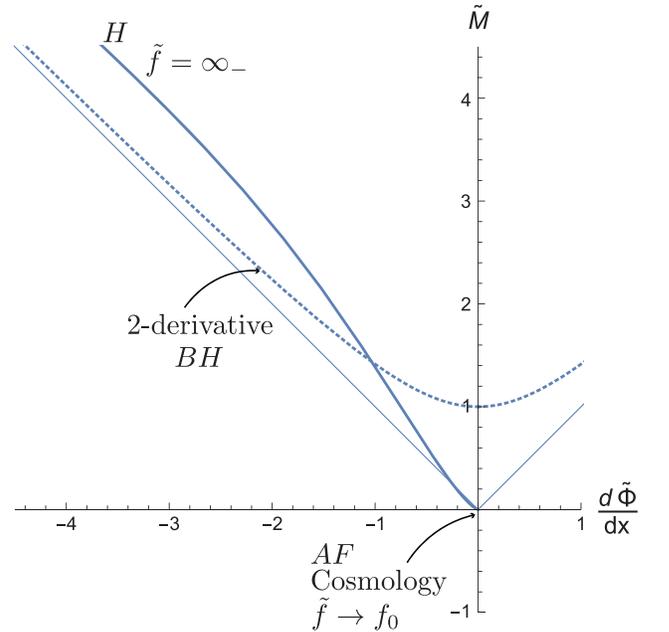


FIG. 8. Parametric plot for $\frac{d\tilde{\Phi}}{dx}$ and \tilde{M} both as functions of $\tilde{f} \in (12, \infty)$ for the interior of the black hole with resolved singularity (continuous line). Shown dashed is the analogous plot for the two-derivative black hole. The faint lines are asymptotes at 45° and 135° . The horizon is far on the 135° asymptote, and the late-time asymptotically Minkowski part of the cosmology is around the origin.

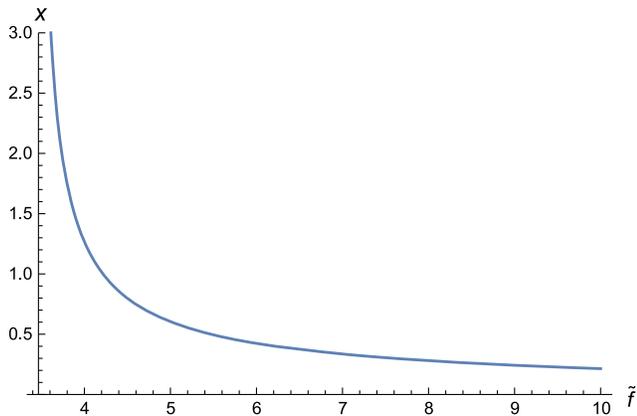


FIG. 9. The coordinate x as a function of the parameter $\tilde{f} \in (f_0, \infty)$ for the resolved BH interior. Here $f_0 = \sqrt{12} \sim 3.464$.

which correctly gives the horizon $x = 0$ for $\tilde{f} = \infty$ and $x \rightarrow \infty$ for $\tilde{f} \rightarrow f_0$. The integral above is best done numerically, and a plot of it is shown in Fig. 9.

VII. CONCLUDING REMARKS

In this paper we have considered the theory space of duality-invariant derivative corrections to the lowest order action that gives a solution identified as the string theory two-dimensional black hole. We describe the theory space by the choice of function $M(f)$, whose inverse is related to the usual perturbatively defined α' corrections. Using f as a parameter, and with square roots of nontrivial functions appearing in the equations, f space naturally becomes a space with branch points and branch cuts. In this f plane, the underlying structure of the solution is determined by the type of branch points which determine the lines or contours that parametrize the solution. We have argued that the f parametrization provides an extension of the original α' corrected theory that applies for situations where the series defining the corrections in the action does not converge. We have used a notation where f and \tilde{f} are the parameters for the exterior and interior solutions, respectively.

In fact, we saw that the black hole interior leads to a branch point at some real, positive f_0 with a cut going all the way to $\tilde{f} = \infty$. For the black hole interior with a singularity, the x coordinate reaches a finite value at the branch point f_0 , a point that can be reached in finite proper time. This means that the space cannot end there, and one must indeed return to $\tilde{f} = \infty$ over the cut. To avoid the singularity, while preserving a horizon, we altered the nature of the branch point f_0 . Now x reaches an infinite value as we approach f_0 , and it takes infinite proper time to get there. Thus we get a complete space without having to go over the branch. This yields a black hole with a horizon but a regular interior, a regular black hole. The main caveat is that such solution does not appear to be a string theory one, as discussed in Sec. VI A. It could be that while

general T -duality invariant α' corrections allow for regular black holes, such regular two-dimensional black holes do not arise in string theory.

Our work can be viewed as an exploration of the space \mathcal{T} of all possible α' corrections in two dimensions; the space of all possible $M(f)$. If we call \mathcal{S} the subspace we showed leads to conventional singular black holes, the regular black holes we identified appear on some subspace $\mathcal{R} \subset \partial\mathcal{S}$ of the boundary of \mathcal{S} . Note that we have not delineated the maximal space $\hat{\mathcal{S}} \supset \mathcal{S}$ that leads to conventional black holes. Of course, we would very much like to know where $D = 2$ string theory sits in \mathcal{T} . If the string theory black hole has a conventional singularity, then string theory would lie in $\hat{\mathcal{S}}$.

We have also discussed duality in the black hole solutions. The maximally extended geometry of the black hole of the two-derivative theory describes a self-dual solution where, as noted by Giveon, duality maps the horizon to the curvature singularity. We showed this is actually a general result. In particular, complete solutions that have horizons but no singularities are not self-dual. Their T -duals give an alternative description of the space.

A few issues could be investigated next:

- (1) The present work makes some progress but does not completely describe the type of situations that can arise throughout the theory space \mathcal{T} . We have seen some unusual solutions, and it would be of interest to know all that can happen. We identified a subspace $\mathcal{R} \subset \mathcal{T}$ leading to regular black holes, but there may be a larger subspace $\hat{\mathcal{R}}$ giving nonsingular black holes, perhaps containing some that could be expected to be string theory solutions. Since 2D string theory is strongly believed to have a black hole, the regions of \mathcal{T} for which no black hole solution exists could be thought of as the “swampland” in the theory space.
- (2) To better understand solutions, instead of using the $n = 1$ gauge, which results in disconnected f contours for the exterior and interior solutions, one may try for a better, more geometric understanding in the $nm = 1$ gauge, where maximal extensions may be more easily constructed.
- (3) It would indeed be important to determine the maximal extensions of the regular black holes obtained here and to display their Penrose diagrams. The challenge here is doing the needed work analytically.
- (4) To get hints if the 2D string theory black hole is regular, it may be of interest to do a more careful study of the Dijkgraaf, Verlinde, Verlinde ansatz [17] to understand its geometry and the behavior of the dilaton. In [19], the geometry is first extended to include a region with Euclidean signature, which is later removed, with completeness restored by gluing infinite number of copies with such excision. Of course, we still lack insight on how this configuration could be a solution of the α' corrected *string*

equations of motion. In [22] we showed that the ansatz, in its given form, does not fit the framework of our analysis.

- (5) Our work could help constrain the α' corrections of string theory. We have seen that if the string theory BH remains singular then the α' corrections are strongly constrained. Indeed for generic classes of corrections the BH presumably does not exist. At present the class of $M(f)$ s for which the BH has no singularity seems strongly restricted, but further investigation could change this.
- (6) As emphasized in the introduction, our search for a regular BH geometry is based on standard general relativity notions, where a singularity is probed by point particles and geodesics. Since in string theory spacetime should be probed by strings, the ultimate picture might be quite different. As shown here, the T -dual of a regular BH geometry necessarily features a curvature singularity, but it might be that when probed by winding modes this singularity is

harmless. This could be explored with a genuine double field theory [33,34] on 2D black hole backgrounds.

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