Coherent states of quantum spacetimes for black holes and de Sitter spacetime

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We provide a group theory approach to coherent states describing quantum space-time and its properties. This provides a relativistic framework for the metric of a Riemmanian space with bosonic and fermionic coordinates, its continuum and discrete states, and a kind of *"quantum optics"* for the space-time. New results of this paper are: (i) The space-time is described as a physical coherent state of the complete covering of the SL(2C) group, e.g., the metaplectic group Mp(n). (ii) (The discrete structure arises from its two irreducible *even* $(2n)$ and *odd* $(2n+1)$ representations, $(n = 1, 2, 3\ldots)$, spanning the complete Hilbert space $\mathcal{H} = \mathcal{H}_{\mathrm{odd}} \oplus \mathcal{H}_{\mathrm{even}}$. Such a global or *complete* covering guarantees the CPT symmetry and unitarity. Large n yields the classical and continuum manifold, as it must be. (iii) The coherent and squeezed states and Wigner functions of quantum-space-time for black holes and de Sitter, and (iv) for the quantum space-imaginary time (instantons), black holes in particular. They encompass the semiclassical space-time behavior plus high quantum phase oscillations, and notably account for the classical–quantum gravity duality and trans-Planckian domain. The Planck scale consistently corresponds to the coherent state eigenvalue $\alpha = 0$ (and to the $n = 0$ level in the discrete representation). It is remarkable the power of coherent states in describing both continuum and discrete space-time. The quantum space-time description is *regular*, there is no any space-time singularity here, as it must be.

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I. INTRODUCTION AND RESULTS

Quantum space-time is a key concept both for quantum theory on its own and for a quantum gravity theory. Coherent states are a fundamental part in quantum physics with multiple theoretical and practical realizations from mathematical physics to quantum optics and wave packet experiments, see for example Refs. [1–7] and references therein. In this paper, within a group theory approach, we construct generalized coherent states to describe quantum space-time.

We describe quantum space-time as arising from a mapping $P(G, \mathcal{M})$ between the quantum phase space manifold of a group $\mathcal{G}$ and the real space-time manifold $\mathcal{M}$. The metric g_{ab} on the phase space group manifold determines the space-time metric of $\mathcal{M}$ after identification of one component of the momentum P operator with the time T. The signature of the metric depends on the compact or noncompact nature of the group, but in the most cases of physical interest, the real space-time signature and its hyperbolic structure require non compact groups.

A group theory approach, a quantum algebra, reveals a key part in the quantum space-time description in order to obtain the line element associated to a discrete quantum structure of the space-time. Such an emergent metric is obtained here from a Riemmanian phase space and described as a physical coherent state of the underlying covering of the group SL (2C): Interestingly, it appears necessary to consider the *complete covering* of the symplectic group, that is the Metaplectic group $Mp(n)$, its spectrum for all n leading in particular for very large n the continuum space-time. This approach allows us to construct here coherent states of the coset type for the quantum space-time and describe with them coherent de Sitter and black hole states.

This quantum description is based on the phase space of a relativistic particle in the superspace with bosonic and fermionic coordinates, allowing to conserve at the quantum level the square root forms of the geometrical operators (e.g., the Hamiltonian or Lagrangian). The discrete space-time structure arises from the basic states of the Metaplectic representation with one interesting feature to remark here: The decomposition of the SO(2,1) group into two

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irreducible representations span *even* $|2n\rangle$ and *odd* $|2n+1\rangle$ states, $(n = 1, 2, 3\ldots)$, respectively, whose totality is covered by the Metapletic group. In the Metaplectic representation the general or *complete* states must be the sum of the *two* kind of states: even and odd n states spanning respectively the two Hilbert sectors $\mathcal{H}_{1/4}$ and $\mathcal{H}_{3/4}$, whose complete covering is $\mathcal{H}_{1/4} \oplus \mathcal{H}_{3/4}$. This yields the relativistic quantum space-time metric with discrete structure. For increasing number of levels n, the metric solution goes to the continuum and to a classical manifold as it must be. Such a global or *complete* covering with the sum of the two sectors, even and odd states to have the complete Hilbert space reflects the CPT symmetry and unitarity of the description.

As we know, the metaplectic group $M_p(2)$ acts irreducibly on each of the subspaces $\mathcal{H}_{1/4}$, $\mathcal{H}_{3/4}$ (even and odd) by which the total Hilbert space (namely $\mathcal{H}$) is divided according to the Casimir operator:

$$K^2 = K_3^2 - K_1^2 - K_2^2 = k(k-1) = -\frac{3}{16}\mathbb{I}$$

giving precisely the values $k = 1/4, 3/4$. Then,

$$\mathcal{H}_{1/4} = \mathrm{Span}\{|n\ \mathrm{even}\rangle : n = 0, 2, 4, 6, \ldots\} \quad (1.1)$$

$$\mathcal{H}_{3/4} = \mathrm{Span}\{|n\ \mathrm{odd}\rangle : n = 1, 3, 5, 7, \ldots.\} \quad (1.2)$$

Based on the highest eigenvalue of the number operator $T_3|n\rangle = -\frac{1}{2}(n + \frac{1}{2})|n\rangle$ occurring in $\mathcal{H} \equiv \mathcal{H}_{1/4} \oplus \mathcal{H}_{3/4}$, the two unitary irreducible representations (UIR) $\mathcal{D}$ of $Mp(2)$ are denoted as:

$$(\mathrm{UIR}) \quad \text{restricted to}\, \mathcal{H}_{1/4} \to \mathcal{D}_{1/4} \in Mp(2) \quad (1.3)$$

$$(\mathrm{UIR}) \quad \text{restricted to}\, \mathcal{H}_{3/4} \to \mathcal{D}_{3/4} \in Mp(2) \quad (1.4)$$

One of the clear examples of the group theory approach presented here is the quantum space-time derived from the phase space of the harmonic oscillator (Refs. [8–10]), and the mapping $(X, P) \to (X, T)$, in the case of the inverted (imaginary frequency) oscillator, or alternatively $\to (X, iT)$, in the normal (real frequency) oscillator. The inverted oscillator in its different representations does appear in a variety of interesting physical situations from particle physics to black holes and modern cosmology as inflation and today dark energy, e.g., Refs. [11–23].

The group theory framework presented here to describe quantum space-time and its coherent states allows to correlate and extend the approaches of Refs. [8–10,24–26] to obtain new results. Novel results of this paper are

(1) the generalization of the quantum light-cone to include fermionic coordinates,
(2) the construction of coherent and squeezed states of quantum space-time, their properties and interpretation, their continuum and discrete representations, and for both de Sitter and black hole space-times.
(3) The coherent states for the quantum space-imaginary time instantons, for black holes in particular.
(4) We find that coherent states encompass the space-time behavior in the semiclassical and classical de Sitter and black hole regions, exhibit high quantum phase oscillations of the space-time, and account for the classical–quantum gravity duality and the trans-Planckian scales.

(i) It is remarkable the power of coherent states in describing both continuum and discrete space-time, *even* in the Planckian and trans-Planckian domains:
(ii) The Planck scale consistently corresponds here to the continuum coherent state eigenvalue $\alpha = 0$, (and to the fundamental state $n = 0$ in the discrete representation). Higher values of α in the quantum gravity (trans-Planckian) domain account for the smaller and sub-Planckian sizes and higher excitations.
(iii) One of the new features, for the space-imaginary time instantons is the emergence of a *maximum eigenvalue* α characterizing the coherent states due to the minimal nonzero quantum radius because of the minimal quantum uncertainty $\Delta X \Delta T = \hbar/2$, in particular in the central and *regular* black hole quantum region. The coherent state instanton remarkably accounts for this quantum gravity feature and determines the radius being

$$\mathcal{R}_0(l_P, t_P)^2 = \frac{1}{\sqrt{\pi}}\left[\frac{1}{l_P} + \frac{l_P}{\hbar}\right],$$

l_P being the Planck length. The origin is flurried or smoothed within this constant and bounded curvature region.
(iv) In the quantum space-time description, there is no any space-time singularity as it must be. The consistent description by coherent states of such quantum scales does appear here as a result of the classical–quantum gravity duality across the Planck scale, and reflected here in the double covering of the SL(2C) group or Metaplectic symmetry.

This paper is organized as follows: In Sec. II we construct the generalized or coset group coherent states and squeezed states and the corresponding Wigner quasi-probality functions. In Sec. III, we describe the Mp(n) general group approach including bosonic and fermionic coordinates, in particular Mp(2) and the geometrical interpretation of high quantum oscillatory effects in this context. Section IV describes the Mp (n) associated relativistic wave equation, the complete Hilbert space and the discrete representations. The physical states, the Mp(2) squeezed vacuum and the direct sum of the both odd and even states, necessary to uncover the complete space-time are discussed in this section. In Sec. V we find the coherent states for the de Sitter and black hole space-times, their properties and interpretation. Section VI deals with the coherent states of quantum (imaginary time) instantons, for black holes in particular and its new effects. In Sec. VII provides a discussion in the context of our results and other references, and Sec. VIII summarizes our remarks and conclusions.

II. QUANTUM COHERENT STATES

We construct first coherent states within a group theory approach of the Klauder-Perelomov type (Refs. [2,4]) or coset group coherent states and then in sections and we describe them in terms of the Metaplectic Mp(n) group and the associated relativistic wave equation. For this purpose we define the coset generators as the generalized displacement operators by means of the creation and annihilation operators a and a^+. These operators are analogous to those corresponding to quadratic Hamiltonians but the change of sign for the generalized coordinate introduces the imaginary frequency into the definition, (by analogy to the generalized *inverted* oscillator), namely

$$a = \left(\frac{iz}{2\hbar|z|}\right)^{1/2}\left(\sqrt{m\omega}q + \frac{p}{\sqrt{m\omega}}\right),$$
$$a^+ = \left(\frac{iz^*}{2\hbar|z|}\right)^{1/2}\left(\sqrt{m\omega}q - \frac{p}{\sqrt{m\omega}}\right) \tag{2.1}$$

Precisely, the change in the character of the frequency introduces the global phase factor $e^{i\pi/4}$.

Consequently, the general displacement operator $D(\alpha)$ for any general complex parameter α and z takes the following form

$$D(\alpha)S(z) = \exp\left(\alpha a^+ - \alpha^* a\right) \times \exp\frac{1}{2}\left(z a^{+2} - z^* a^2\right)$$
$$= \exp\left[\left(\frac{i}{2\hbar|z|}\right)^{1/2}\left(\sqrt{m\omega}d(\alpha,z)q - \frac{p}{\sqrt{m\omega}}d(\alpha,z)\right)\right]\exp\left[\frac{-i|z|}{2\hbar}(qp+pq)\right], \tag{2.2}$$

$$d(\alpha,z) \equiv \left(\alpha\sqrt{z^*} - \alpha^*\sqrt{z}\right) \tag{2.3}$$

The displacement operator $D(\alpha)$ is a unitary operator. In the coordinate representation $p = -i\hbar\partial_q$, the displacement operator takes the form

$$D(\alpha)S(z) = \exp\left[\left(\frac{i}{2\hbar|z|}\right)^{1/2}\left(\sqrt{m\omega}d(\alpha,z)q + \frac{i\hbar}{\sqrt{m\omega}}d(\alpha,z)\frac{d}{dq}\right)\right]\exp\left[-|z|\left(q\frac{d}{dq} + \frac{1}{2}\right)\right], \tag{2.4}$$

The operator $D(\alpha)$ Eq. (2.4) acts on the vacuum of the inverted oscillator, namely

$$\langle q|0\rangle_{\text{inv-osc}} = \left(\frac{im\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{im\omega}{2\hbar}q^2\right) \tag{2.5}$$

Therefore, we obtain the generalized coherent states with the following form:

$$\psi_{\alpha,z}(q) = \left(\frac{im\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{1}{2}\left(|z| + |\alpha|^2 e^{-2|z|} + \frac{e^{-|z|}}{|z|}\left(\alpha^2 z^*\cosh|z| - \alpha^{*2} z\sinh|z|\right)\right)\right]\cdot \tag{2.6}$$

$$\exp\left[-\frac{im\omega}{2\hbar}q^2 + \sqrt{\frac{2im\omega}{|z|\hbar}}\left(\alpha\sqrt{z^*}\cosh|z| - \alpha^*\sqrt{z}\sinh|z|\right)e^{-|z|}q\right] \tag{2.7}$$

These states are squeezed because the quantum uncertainty in space and momentum coordinates is not equally distributed in the both directions. In particular, we can test it putting $z = 0$, and we obtain the coherent state for the inverted harmonic oscillator:

$$\psi_\alpha(q)_{\text{inv-osc}} = \left(\frac{im\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}|\alpha|^2} \exp\left(-\frac{im\omega}{2\hbar}q^2 + \sqrt{\frac{m\omega}{2\hbar}}(1+i)\alpha q\right) \tag{2.8}$$

It is convenient to consider this type of coherent states as being based in a Lie group G with a unitary, irreducible representation T acting on some Hilbert space $\mathcal{H}$. If we take a fixed vector ψ_0 of $\mathcal{H}$, we define the coherent state system $\{T, \psi_0\}$ to be the set of vectors $\psi \in \mathcal{H}$ such that $\psi = T(g)\psi_0$ for some $g \in G$. Then, generalized coherent states are defined as the states $|\psi\rangle$ corresponding to these vectors in $\mathcal{H}$.

We can see that in the definition of the coherent state of the Klauder-Perelomov type, the general displacement operator contains, in the exponential representation of the coset, a linear part in the annihilation and creation operators and another quadratic part corresponding to the "squeezed" sector, e.g., see Eq. (2.2): The latter belongs in this representation to $Mp(2)$. Therefore, at least for this purely squeezed part in the a^2, a^{+2} representation the *complete* vacuum state is

$$(j + ka^+)\langle q|0\rangle_{\text{inv-osc}}$$

where (j, k) are constants determined by the normalization of states and the boundary conditions. This is a consequence of the action of the metaplectic group which increases the spectrum of physical states: $\Delta n = \pm 2$, because the complete states are spanned by both: the $\mathcal{H}_{1/4}$ states (eg. even $(2n)$ states), and the $\mathcal{H}_{3/4}$ states (e.g., odd $(2n+1)$ states). Thus, the lowest level $(n = 0)$ is in $\mathcal{H}_{1/4}$, while in $\mathcal{H}_{3/4}$ it is $n = 1$.

Consequently, being the complete vacuum under the action of an element of $Mp(2)$, Eq. (2.5) would take as a wave function the precise form:

$$\psi_{\text{vacuum}}|_{Mp(2)} \to \left(\frac{im\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{im\omega}{2\hbar}q^2}\left(1 + e^{i\pi/4}\sqrt{\frac{m\omega}{4\hbar}}q\right) \tag{2.9}$$

A. Momentum representation

Analogously to the case of the representation of coordinates, the generalized coherent states are calculated in the same way but taking into account that in the moment representation p remains the same but $q = i\hbar\partial_p$ in all the operators, from those of annihilation and creation, namely

$$\begin{aligned} a_p &= \left(\frac{iz}{2\hbar|z|}\right)^{1/2}\left(\sqrt{m\omega}i\hbar\partial_p + \frac{p}{\sqrt{m\omega}}\right), \\ a_p^+ &= \left(\frac{iz^*}{2\hbar|z|}\right)^{1/2}\left(-\sqrt{m\omega}i\hbar\partial_p + \frac{p}{\sqrt{m\omega}}\right) \end{aligned} \tag{2.10}$$

as well as in the operators of displacement $D(\alpha)S(z)$ Eq. (2.4) acts on the vacuum of the inverted oscillator in the momentum representation, namely

$$\langle p|0\rangle_{\text{inv-osc}} = \left(\frac{i}{\pi\hbar m\omega}\right)^{1/4} \exp\left(-\frac{ip^2}{2\hbar m\omega}\right). \tag{2.11}$$

Therefore, we obtain the generalized coherent states with the following form:

$$\begin{aligned} \psi_{\alpha,z}(p) = \left(\frac{i}{\pi\hbar m\omega}\right)^{1/4} \exp\left[-\frac{1}{2}\left(|z| + |\alpha|^2 e^{-2|z|} + \frac{e^{-|z|}}{|z|}(\alpha^2 z^* \cosh|z| - \alpha^{*2} z \sinh|z|)\right)\right]\cdot \\ \times \exp\left[-\frac{ip^2}{2\hbar m\omega} + \sqrt{\frac{2i}{|z|\hbar m\omega}}(\alpha\sqrt{z^*}\cosh|z| - \alpha^*\sqrt{z}\sinh|z|)e^{-|z|}p\right]. \end{aligned}$$

These states are squeezed because the quantum uncertainty in the space and momentum coordinates is not equally distributed. In particular, we can test it putting $z = 0$, and we obtain the coherent state for the inverted harmonic oscillator in the p representation:

$$\psi_\alpha(p)_{\text{inv-osc}} = \left(\frac{i}{\pi\hbar m\omega}\right)^{1/4} e^{-\frac{1}{2}|\alpha|^2} \exp\left(-\frac{ip^2}{2\hbar m\omega} + \sqrt{\frac{1}{2\hbar m\omega}}(1+i)\alpha p\right).$$

B. Wigner function quasiprobability

As we have made mention, the inverted Hamiltonian is formally obtainable from the standard harmonic oscillator by the change $\omega \to \pm i\omega$ and it corresponds to the Hamiltonian of the harmonic oscillator with purely imaginary frequency. Therefore, this replacement transforms the eigenfunctions of the harmonic oscillator into generalized eigenvectors of the inverted harmonic oscillator, which, from the spectral point of view, leads us to a discrete purely imaginary spectrum: $E_{\mathrm{inv-osc}} = \pm i E_{h-\mathrm{osc.}} = i\hbar\omega(n+1/2)$. Notice that the replacement $\omega \to \pm i\omega$ generates in the fiducial or fundamental states of the inverted oscillator the following forms:

$$\langle q|0\rangle_{\mathrm{inv-osc}} = \left(\frac{im\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{im\omega}{2\hbar}q^2\right) \quad \text{and}$$
$$\widetilde{\langle q|0\rangle}_{\mathrm{inv-osc}} = \left(\frac{-im\omega}{\pi\hbar}\right)^{1/4} \exp\left(\frac{im\omega}{2\hbar}q^2\right).$$

Consequently, in this approach and in order to have functions to be truly L^2, one must consider $\langle \tilde{\varphi}^*_{\mathrm{inv-osc.}}|\varphi_{\mathrm{inv-osc}}\rangle$ to take the square norms, (e.g., a biorthonormalization condition). Note that according to these symmetries, both for the oscillator states and for the coherent states obtained here, it is fulfilled:

$$\tilde{\varphi}^*_{\mathrm{inv-osc}}(q) = \varphi(q) \qquad \text{for the inverted oscillator}$$
$$\tilde{\psi}^*_{\alpha}(q) = \psi_{\alpha}(q) \qquad \text{for coherent states}$$

and this is a consequence of the underlying symmetry of $Mp(2)$ since the generator $T_1 = \frac{1}{4}(qp+pq) = \frac{i}{4}(a^{+2}-a^2)$ is the one that produces the rotation or mapping on the states of the harmonic oscillator, e.g.,

$$\varphi_{\mathrm{inv-osc}}(q) = e^{-\frac{\pi}{2}T_1}\varphi_{HO}(q)$$
$$\tilde{\varphi}_{\mathrm{inv-osc}}(q) = e^{\frac{\pi}{2}T_1}\varphi_{HO}(q)$$

Taking this fact and symmetries into account, the Wigner function quasiprobability will be defined as

$$W(q,p) = \int dv e^{-i\frac{pv}{\hbar}} \tilde{\psi}^*_{\alpha}\left(q-\frac{v}{2}\right)\psi_{\alpha}\left(q+\frac{v}{2}\right)$$

Obtaining explicitly:

$$W(q,p) = \exp\left[-\left(\frac{m\omega}{\hbar}q^2 + \frac{p^2}{\hbar m\omega} - 2\sqrt{\frac{m\omega}{\hbar}}\alpha q + |\alpha|^2\right)\right]$$

Similarly, in the momentum representation, the Wigner function will be defined as

$$W(q,p) = \int du e^{\frac{iqw}{\hbar}} \tilde{\psi}^*_{\alpha,0}\left(p-\frac{u}{2}\right)\psi_{\alpha,0}\left(p+\frac{u}{2}\right)$$

Explicitly:

$$W(q,p) = \exp\left[-\left(\frac{m\omega}{\hbar}q^2 - \frac{p^2}{\hbar m\omega} - 2\sqrt{\frac{m\omega}{\hbar}}\alpha p + |\alpha|^2\right)\right]$$

We can also consider the Wigner function for the pure squeezed case, namely

$$\psi_z(q) = \left(\frac{im\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}|z|} \exp\left(-\frac{im\omega}{2\hbar}q^2\right) \tag{2.12}$$

Again, the Wigner function in this case will be defined as

$$W_{sq}(q,p) = \int dv e^{-i\frac{pv}{\hbar}} \tilde{\psi}^*_{z}\left(q-\frac{v}{2}\right)\psi_{z}\left(q+\frac{v}{2}\right)$$

with the result:

$$W_{sq}(q,p) = e^{-|z|} \exp\left[-\left(\frac{m\omega}{\hbar}q^2 - \frac{p^2}{\hbar m\omega}\right)\right] \tag{2.13}$$

which shows a purely Gaussian behavior without the α-shift tail.

As is it known the spectrum of the inverted oscillator gives rise to complex generalized eigenvalues because from the point of view of the potential this is unbounded from below. Here we saw that the fact of considering the mapping given by the T_1 generator of the group Mp(2) allows to treat in equal footing the respective solutions of the inverted oscillator and the standard harmonic one. It is worth mentioning that using quasi-Hermiticity techniques [27] the problem can be analogously solved by considering a scaling operator proportional to T_1.

On the other hand, solutions for the singular case (x-coordinates) lead to solutions of the parabolic cylinder type. These solutions only reflect the symmetries of the metaplectic vacuum, due that they are factorized, by means of hypergeometric type functions, in an *even* part and an *odd* part corresponding precisely to the two irreducible sectors of $Mp(2)$. It is useful to remember that the generators of $Mp(2)$ are the following ones

$$\begin{aligned} T_1 &= \frac{1}{4}(qp+pq) = \frac{i}{4}(a^{+2}-a^2), \\ T_2 &= \frac{1}{4}(p^2-q^2) = -\frac{1}{4}(a^{+2}+a^2), \\ T_3 &= -\frac{1}{4}(p^2+q^2) = -\frac{1}{4}(a^+a+aa^+). \end{aligned} \tag{2.14}$$

With the following commutation relations,

$$[T_3,T_1] = iT_2, \qquad [T_3,T_2] = -iT_1, \qquad [T_1,T_2] = -iT_3$$

being (q,p), alternatively (a,a^+), the variables of the standard harmonic oscillator, as usual.

In the following section we describe the generalized coherent states in terms of the *metaplectic group* MP(n), its metric constructed in phase space and its associated relativistic particle field equation.

III. METAPLECTIC GROUP $MP(n)$, ALGEBRAIC INTERPRETATION OF THE METRIC AND THE SQUARE ROOT HAMILTONIAN

One of the basis of the dynamical description is the Hamiltonian or Lagrangian of the square root type, that is, a nonlocal and nonlinear operator in principle. This is because the *invariance under reparametrizations* as a Lagrangian and as an associated Hamiltonian, generates the correct physical spectrum. The essential guidelines of our approach here are based on the items specifically described in the sequel:

(i) The elementary distance function (positive square root of the line element) is taken as the fundamental geometric object of the space-time-matter structure, the geometric Lagrangian (functional action) of the theory.

(ii) From (i) the geometric Hamiltonian is obtained in the usual way: this will be the fundamental classical-quantum operator.

(iii) This universal Hamiltonian (square root Hamiltonian) contains a zero moment P_0 characteristic of the complete phase space at the maximum level, from the point of view of the physical states. The inclusion of a P_0 prevents the arbitrary nullification of the Hamiltonian, a fact that occurs in the proper time system in which the evolution coincides with the time coordinate: in this case time "disappears" from the dynamic equations.

The method that we use to preliminarily expand the phase space to determine later here (via Hamilton equations) the physical role of P_0 is the Lanczos method [28], which is the most geometrically consistent and mathematically simplest.

(iv) The Hamiltonian, when rewritten in differential form, defines a new relativistic wave equation of second order and degree $1/2$. This can be reinterpreted as a Dirac-Sudarshan type equation of positive energies and internal variables (e.g., oscillator type variables), having a para-Bose or para-Fermi interpretation of the solution-states of the system.

(v) The spectrum will be formed by states that are bilinear in fundamental functions, which in the case of $Mp(2)$ are $f_{1/4}$ and $f_{3/4}$ having a spin weight $s = 1/4$ and $3/4$ supported and connected by a vector representation of the generators of $Mp(n)$, or those covered by this, e.g., $SU(p, n-p)$, $SL(nR)$, etc. A characteristic physical state of $Mp(2)$ is of the form $\Phi_\mu = \langle s|L_\mu|s'\rangle$ with $(s, s' = 1/4, 3/4)$, and L_μ being the vector representation of one of the generators of $Mp(2)$.

A. Mp(2), SU(1,1), and Sp(2)

Following on the items (i) to (v) above, we use as a base the line element in a $N = 1$ superspace with differential forms. Consequently, we extend our manifold to include fermionic coordinates. Geometrically, we take as starting point the functional action that describes the world line (measure on a superspace) of the superparticle as follows:

$$S = (x, \theta, \bar{\theta}) = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{\overset{\circ}{\omega}_\mu \overset{\circ}{\omega}^\mu + \gamma \dot{\theta}^\alpha \dot{\theta}_\alpha - \gamma^* \dot{\bar{\theta}}^{\dot{\alpha}} \dot{\bar{\theta}}_{\dot{\alpha}}} \tag{3.1}$$

where $\overset{\circ}{\omega}_\mu = \dot{x}_\mu - i(\dot{\theta}\sigma_\mu \bar{\theta} - \theta\sigma_\mu \dot{\bar{\theta}})$, and the dot indicates derivative with respect to the parameter τ as usual; the complex constant γ allows generality to characterize the states and describing limiting cases.

The above Lagrangian is constructed considering the line element (e.g., the measure, positive square root of the interval) of the nondegenerated supermetric introduced in [24]

$$ds^2 = \omega^\mu \omega_\mu + \gamma \omega^\alpha \omega_\alpha - \gamma^* \omega^{\dot{\alpha}} \omega_{\dot{\alpha}},$$

where the bosonic term and the Majorana bispinor compose a superspace $(1, 3|1)$, with coordinates $(t, x^i, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, and where the Cartan forms of the supersymmetry group are described by: $\omega_\mu = dx_\mu - i(d\theta\sigma_\mu \bar{\theta} - \theta\sigma_\mu d\bar{\theta})$, $\omega^\alpha = d\theta^\alpha$, $\omega^{\dot{\alpha}} = d\theta^{\dot{\alpha}}$ (obeying evident supertranslational invariance).

The *generalized momenta* from the geometric Lagrangian are computed in the usual way:

$$\mathcal{P}_\mu = \partial L / \partial x^\mu = (m^2 / L) \overset{\circ}{\omega}_\mu \tag{3.2}$$

$$\mathcal{P}_\alpha = \partial L / \partial \dot{\theta}^\alpha = i\mathcal{P}_\mu (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} + (m^2 \gamma / L) \dot{\theta}_\alpha \tag{3.3}$$

$$\mathcal{P}_{\dot{\alpha}} = \partial L / \partial \dot{\bar{\theta}}^{\dot{\alpha}} = i\mathcal{P}_\mu \theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} - (m^2 \gamma / L) \dot{\bar{\theta}}_{\dot{\alpha}}. \tag{3.4}$$

We write them in a *canonical form*

$$\Pi_\alpha = \mathcal{P}_\alpha + i\mathcal{P}_\mu (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \tag{3.5}$$

$$\Pi_{\dot{\alpha}} = \mathcal{P}_{\dot{\alpha}} - i\mathcal{P}_\mu \theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \tag{3.6}$$

($\mathcal{P}_\alpha$ and $\mathcal{P}_\mu$ being defined from the Lagrangian, as usual). Then, we start with the equation (which will become the wave equation):

$$\mathcal{S}[\Psi] = \mathcal{H}_s\Psi = \sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{\gamma}\Pi^\alpha\Pi_\alpha - \frac{1}{\gamma^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)}\Psi \tag{3.7}$$

As we have extended our manifold to include fermionic coordinates, it is natural to extend also the concept of a point particle trajectory to the superspace. To do this, we take the coordinates $x(\tau)$, $\theta^\alpha(\tau)$ and $\bar{\theta}^{\dot{\alpha}}(\tau)$ depending on the evolution parameter τ.

Consequently, there exist an algebraic interpretation of the pseudo-differential operator (square root) in the case of an underlying Metaplectic group structure $\mathrm{Mp}(n)$:

$$\sqrt{\mathcal{F}}|\Psi\rangle \equiv \sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{\gamma}\Pi^\alpha\Pi_\alpha - \frac{1}{\gamma^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)}|\Psi\rangle = 0 \tag{3.8}$$

$$\{[\mathcal{F}]^\alpha_\beta(\Psi L_\alpha)\}\Psi^\beta \equiv \left\{\left[m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{\gamma}\Pi^\alpha\Pi_\alpha - \frac{1}{\gamma^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)\right]^\alpha_\beta(\Psi L_\alpha)\right\}\Psi^\beta = 0 \tag{3.9}$$

Then, both structures can be identified, e.g.,:

$$\sqrt{\mathcal{F}}|\Psi\rangle \leftrightarrow \{[\mathcal{F}]^\alpha_\beta(\Psi L_\alpha)\}\Psi^\beta, \tag{3.10}$$

being the state Ψ the square root of a spinor Φ (where the "square root" Hamiltonian acts) such that it can be bilinearly defined as $\Phi = \Psi L_\alpha \Psi$.

The operability of the pseudodifferential "square root" Hamiltonian can be clearly interpreted if it acts on the square root of the physical states. In the case of the Metaplectic group, the square root of a spinor certainly exist [29–32] making the identification Eqs. (3.8)–(3.9) fully consistent both from the relativistic and group theoretical viewpoints.

Is also possible to describe a complete multiplet spanning spins from $(0, 1/2, 1, 3/2, 2)$. This is so because with the fundamental states and the allowed vectorial generators, the tower of states is finite and the states involved are *all physical*, as it must be from the physical viewpoint.

The choice of Eq. (3.1) as a functional action in superspace is justified because from the point of view of symmetries, it contains the largest symmetry algebra of the harmonic oscillator with 3 quadratic generators in a and a^+ (B_0: even sector) and the two generators in the B_1: odd sector, describing the superalgebra $Osp(1/2, R)$ with its 5 generators.

It is notable that in the general case, $Sp(2m)$ can be embedded somehow in a larger algebra as $Sp(2m) + R^{2m}$ admitting an Hermitian structure with respect to which it becomes the orthosymplectic superalgebra $Osp(2m, 1)$. Consequently the metaplectic representation of $Sp(2m)$ extends to an irreducible representation (IR) of $Osp(2m, 1)$ which can be realized in terms of the space H of all holomorphic functions $h: C^m \to C / \int |h(z)|^2 e^{-|z|^2} d\lambda(z) < \infty$ with $\lambda(z)$ the Lebesgue measure on C^m. The restriction of the $Mp(n)$ representation to $Sp(2m)$, implies that the two irreducible sectors are supported by the subspaces $H^\pm$ of H, where H^+ and H^- are the spans (closed) of the set of functions $z^n \equiv (z_1^{n_1}, \ldots, z_m^{n_m})$ with $n_\theta \in Z$, $|n| = \sum n_\theta$, *even* and *odd* respectively.

B. Geometrical spinorial SL(2C) description of the Zitterbewegung

Let us briefly analyze in an algebraic description, the origin of the quantum relativistic effects as the prolonged highly oscillations effect or so called "Zitterbewegung." There are two types of states: the basic (nonobservable) states and the observable physical states. The basic states are coherent states corresponding to the double covering of the $SL(2C)$, e.g., the metaplectic group [32,33] responsible for projecting the symmetries of the 6 dimensional $Mp(4)$ group space to the 4 dimensional space-time by means of a bilinear combination of the $Mp(4)$ generators. The supermultiplet solution for the geometric Lagrangian is given by

$$g_{ab}(0,\lambda) = \langle\psi_\lambda(t)|L_{ab}|\psi_\lambda(t)\rangle$$
$$g_{ab}(0,\lambda) = \exp[A]\exp[\xi\varrho(t)]\chi_f\langle\psi_\lambda(0)|([c]c)_{ab}|\psi_\lambda(0)\rangle,$$
$$A(t) = -\left(\frac{m}{|\gamma|}\right)^2 t^2 + c_1 t + c_2, \qquad (c_1, c_2) \in C \tag{3.11}$$

where we have written the corresponding indices for the simplest supermetric state solution, being L_{ab} the corresponding generators $\in Mp(n)$, and χ_f coming from the odd generators of the big covering group of the symmetries of the specific model. Considering for simplicity the "square" solution for the three compactified dimensions (spin λ fixed, $\xi \equiv -(\bar{\xi}^{\dot{\alpha}} - \xi^\alpha)$), the exponential even fermionic part is given by:

$$\varrho(t) \equiv \mathring{\phi}_\alpha[(\alpha e^{i\omega t/2} + \beta e^{-i\omega t/2}) - (\sigma^0)^\alpha_{\dot{\alpha}}(\alpha e^{i\omega t/2} - \beta e^{-i\omega t/2})] \tag{3.12}$$

$$+ \frac{2i}{\omega}[(\sigma^0)_\alpha{}^{\dot{\beta}}\bar{Z}_{\dot{\beta}} + (\sigma^0)^\alpha{}_{\dot{\alpha}}Z_\alpha] \tag{3.13}$$

$\mathring{\phi}_\alpha, Z_\alpha, \bar{Z}_{\dot{\beta}}$ being constant spinors, and α and $\beta\mathbb{C}$-numbers (the constant $c_1 \in \mathbb{C}$ due to the obvious physical reasons and the chirality restoration of the superfield solution. By consistency, (and as in the string case), two geometric-physical options are related to the orientability of the superspace trajectory: $\alpha = \pm\beta$. We take without loss of generality $\alpha = +\beta$ then, exactly, there are two possibilities:

(i) The compact case which is associated to the small mass limit (or $|\gamma| \gg 1$):

$$\varrho(t) = \begin{pmatrix} \mathring{\phi}_\alpha \cos(\omega t/2) + \frac{2}{\omega}Z_\alpha \\ -\mathring{\bar{\phi}}_{\dot{\alpha}} \sin(\omega t/2) - \frac{2}{\omega}\bar{Z}_{\dot{\alpha}} \end{pmatrix} \tag{3.14}$$

(ii) And the noncompact case, which can be associated to the imaginary frequency ($\omega \to i\omega$ generalized inverted oscillators) case:

$$\varrho(t) = \begin{pmatrix} \mathring{\phi} \cosh(\omega t/2) + \frac{2}{\omega}Z_\alpha \\ -\mathring{\bar{\phi}}_{\dot{\alpha}} \sinh(\omega t/2) - \frac{2}{\omega}\bar{Z}_{\dot{\alpha}} \end{pmatrix} \tag{3.15}$$

Obviously (in both cases), this solution represents a *Majorana fermion* where the $\mathbb{C}$ (or *hypercomplex*) symmetry wherever the case) is inside the constant spinors.

The spinorial even part of the superfield solution in the exponent becomes:

$$\xi\varrho(t) = \theta^\alpha\left(\mathring{\phi}_\alpha \cos(\omega t/2) + \frac{2}{\omega}Z_\alpha\right) - \bar{\theta}^{\dot{\alpha}}\left(-\mathring{\bar{\phi}}_{\dot{\alpha}} \sin(\omega t/2) - \frac{2}{\omega}\bar{Z}_{\dot{\alpha}}\right) \tag{3.16}$$

We easily see that in the above expression there appear a type of continuous oscillation between the chiral and antichiral part of the bispinor $\varrho(t)$, or *Zitterbewegung* as shown qualitatively in Fig. 1 for suitable values of the group parameters. This oscillation reflects in our context the underlying chiral and antichiral quantum structure of the spacetime. Thus, the physical meaning of such a relativistic oscillation (Zitterbewegung) does appear here as an underlying geometrical supersymmetric effect, namely a kind of duality between supersymmetric and relativistic effects.

In the next section we provide more details about how the quantum dynamics and space-time structure emerge from this principle of symmetry.

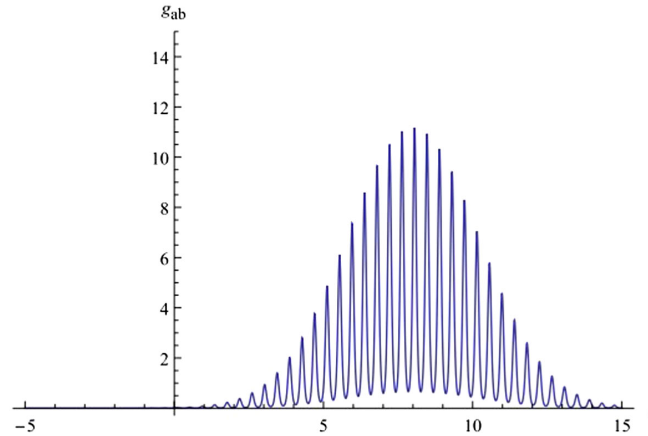

FIG. 1. Oscillation between the chiral and antichiral part of the bispinor $\varrho(t)$, or *Zitterbewegung,* for suitable values of the group parameters. This oscillation reflects the underlying chiral and antichiral quantum structure of the spacetime.

IV. RELATIVISTIC WAVE EQUATION AND THE COMPLETE HILBERT SPACE

The importance of illustrating with this model based on the simplest N = 1 supergroup is that it has a formal equivalence with known cases containing the Poincare group, generally coming from symmetry breaking models with minimum group manifolds SO(1,4), SO(2,3) as characteristic examples, $SUSY_{N=1} \sim SO(1,4)$. Recalling the geometric Lagrangian constructed from the line element from the Maurer Cartan forms induced via pullback (e.g., nonlinear realization for example) of a fundamental symmetry group:

$$S = \int_{\tau_1}^{\tau_2} d\tau L(x) = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{\omega_{AB}\omega^{AB}} \tag{4.1}$$

$A, B = 0, \ldots, 5$. The line element is based on the Cartan forms of the symmetry group, for which it is induced and reflected in the geometric Lagrangian. Consequently, for SO(1,4), for example, we have 10 that agree with the number of generators of the group, as it must be, the indices of the forms run from 0 to 4. If by some process, the symmetry is preferably dynamically broken, the Cartan forms from the point of view of the algebra, are divided into the 6 generators of SO(1,3) plus 4 generators of the Cartan forms, namely $\to \omega_{AB} \to \omega_{\mu\nu}, \omega_{\mu 4} \sim \sqrt{\lambda}\theta_\mu$ (Poincaré—tetrad fields), $\mu, \nu = 0, \ldots, 3$

$$S = \int_{\tau_1}^{\tau_2} d\tau L(x) = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{\omega_{\mu\nu}\omega^{\mu\nu} + \omega_{\mu 4}\omega^{\mu 4}}$$

with $\omega_{\mu 4}\omega^{\mu 4} = \lambda\theta_\mu\theta^\mu$. Following on the arguments given in the precedent paragraphs, we are going to see how wave equations for physical states emerge from the very spacetime structure.

A. Mp (2)—Coherent basic and bilinear states

Now we will demonstrate how the sector of the metaplectic group becomes determinant in the problem of determining the geometric structure and symmetries of the interplay between physical states and spacetime. To this end, we know from the so-called positive energy equations, that these types of equations should emerge. We introduce the transformation (evolution-type ansatz)

$$\Phi_\gamma(t) = e^{[B(t)+p_i x^i+\xi\varrho(t)]}\Phi_\gamma(0)$$

Note that, in contrast to the case where only $\sigma^0 = I_2$ comes into play, here we include the parameters p_i in order to generate the complete and less trivial matrix structure. Consequently,

$$\left[|\gamma|^2(\partial_0^2 - \partial_i^2) + \frac{\Delta_+ - \Delta_-}{4} + m^2\right]^{1/2}|\Psi\rangle = 0 \tag{4.2}$$

$$\left\{\left[|\gamma|^2(\partial_0^2 - \partial_i^2) + \frac{\Delta_+ - \Delta_-}{4} + m^2\right]^\alpha_\beta |\Phi_\alpha\rangle\right\}^{1/2} = 0 \tag{4.3}$$

where $\Delta_\pm \equiv [\partial_\eta \mp \partial_\xi \pm i\sigma^\mu\partial_\mu(\eta\pm\xi)]^2$ and we consider the equivalence at the level of operators between the square root on the basic state of the metaplectic $|\Psi\rangle$ defined as an independent coherent state in each even or odd irreducible sector, and the radicand on the bilinear $\Phi_\alpha = \langle\Psi|L_\alpha|\Psi\rangle$ written in the ket usual form: $|\Phi_\alpha\rangle = \binom{a}{a^+}_\alpha|\Psi\rangle$. Consequently, the sector B_0 (Bose) generates the system

$$|\gamma|^2(\ddot{B} + \dot{B}^2 - p_i^2) + m^2 = 0$$

where the function B is determined by

$$B = \ln[c_2\cos b(t)], \qquad b(t) = \sqrt{\frac{m^2}{|\gamma|^2} - p_i^2}(t - t_0)$$

It is important to notice that in the general case $B = \ln[c_2\cos b(t) + c_2'\sin b(t)]$ we take without losing generality $c_2' = 0$ because we concentrate on the Mp(2) part. It is easy to see that if $c_2' = ic_2$, the solution for B is proportional to $\sqrt{\frac{m^2}{|\gamma|^2} - p_i^2}(t - t_0)$ and also to the Gaussian resolvent packet with the factor $(\frac{m^2}{|\gamma|^2} + p_i^2)$ instead of just $\frac{m^2}{|\gamma|^2}$.

The sector B_1 (Fermi N = 1) gives us the equation

$$|\gamma|^2\xi(\ddot{\rho} + 2\cdot\rho\dot{B}) = 0$$

with a general solution of the form

$$\varrho(t) = \frac{1}{c_2^2\sqrt{\frac{m^2}{|\gamma|^2} - p_i^2}}\mathring{\phi}_\alpha[\alpha\tan b(t) - \beta(\sigma^0)^\alpha{}_{\dot{\alpha}}\sec b(t)] \tag{4.4}$$

The two parts are not independent (in chiral and antichiral zones). Therefore, the equation reduces finally to:

$$\begin{pmatrix} -ip_z - \dot{B} & -ip_x - p_y \\ -ip_x + p_y & ip_z - \dot{B}\end{pmatrix}^\alpha_\beta |\Phi_\alpha\rangle = 0 \tag{4.5}$$

Knowing that (i) $|\Phi_\alpha\rangle = L_\alpha|\Psi\rangle$ is the generator in vector representation based on annihilation and creation operators, and that (ii) It transforms as a spinor under the group SO(1,2), SU(1,1), and Mp(2) (with the respective mappings between them), it is shown that $|\Psi\rangle$ is the coherent state formed by two separate *even* and *odd* coherent states of the considered metaplectic group. We explicitly have

$$\begin{pmatrix} -ip_z - \dot{B} & -ip_x - p_y \\ -ip_x + p_y & ip_z - \dot{B}\end{pmatrix}^\alpha_\beta \begin{pmatrix} a \\ a+ \end{pmatrix}_\alpha |\Psi\rangle = 0 \tag{4.6}$$

which have exactly the same appearance as the equations of the type of internal variables and positive energies of Majorana and Dirac for example. This is easily seen by introducing the choice of parameters: $p_z = -i\epsilon$, $p_x = 0$, $p_y = p$:

$$\begin{pmatrix} \epsilon + \dot{B} & p \\ -p & -\epsilon + \dot{B}\end{pmatrix}^\alpha_\beta \begin{pmatrix} a \\ a+ \end{pmatrix}_\alpha |\Psi\rangle = 0 \tag{4.7}$$

Notice that $\dot{B}$ would take the formal role of "mass" and the transformations are just of the squeezed type.

B. The Mp(2) squeezed vacuum and physical states

The displacement operator in the case of the vacuum squeezed is an element of Mp(2) written in the respective variables of the canonical annihilation and creation operators.

$$S(\xi) = \exp\frac{1}{2}(\xi^* a^2 - \xi a^{+2}) \in Mp(2). \tag{4.8}$$

Seeing Eqs. (4.6) and (4.7) the relationship is shown directly:

$$\begin{pmatrix} a \\ a^+ \end{pmatrix} \to S(\xi)\begin{pmatrix} a \\ a^+ \end{pmatrix} S^{-1}(\xi) = \begin{pmatrix} \lambda & \mu \\ \mu^* & \lambda^* \end{pmatrix}\begin{pmatrix} a \\ a^+ \end{pmatrix} \tag{4.9}$$

From Eqs. (4.8) and (4.9), we see that the dynamics of these "square root" fields of Φ_γ, in the particular representation that we are interested in, is determined by considering these fields as coherent states in the sense that they are eigenstates of a^2 via the action of the Mp(2) group that is of the type:

$$|\Psi_{1/4}(0,\xi,q)\rangle = \sum_{k=0}^{+\infty} f_{2k}(0,\xi)|2k\rangle = \sum_{k=0}^{+\infty} f_{2k}(0,\xi)\frac{(a^{\dagger})^{2k}}{\sqrt{(2k)!}}|0\rangle$$
$$|\Psi_{3/4}(0,\xi,q)\rangle = \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)|2k+1\rangle = \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)\frac{(a^{\dagger})^{2k+1}}{\sqrt{(2k+1)!}}|0\rangle \quad (4.10)$$

For simplicity, we will take all normalization and fermionic dependence or possible fermionic realization, into the functions $f(\xi)$. Explicitly, at $t=0$, the states are

$$|\Psi_{1/4}(0,\xi,q)\rangle = f(\xi)|\alpha_+\rangle$$
$$|\Psi_{3/4}(0,\xi,q)\rangle = f(\xi)|\alpha_-\rangle \quad (4.11)$$

where $|\alpha_\pm\rangle$ are the CS basic states in the subspaces $\lambda = \frac{1}{4}$ and $\lambda = \frac{3}{4}$ of the full Hilbert space. In other words, the action of an element of $Mp(2)$ keeps them invariant (coherent), ensuring the irreducibility of such subspace, e.g.,:

$$\mathcal{H} \sim \begin{pmatrix} \mathcal{H}_{1/4} & \\ & \mathcal{H}_{3/4} \end{pmatrix}$$

Consequently, the two symmetric and antisymmetric combinations $(\pm)$ of the two sets of states $(1/4, 3/4)$ will span *all* the Hilbert space: $\mathcal{H}$:

$$|\Psi_\pm\rangle = |\Psi_{1/4}\rangle \pm |\Psi_{3/4}\rangle, \qquad |\pm\rangle = |+\rangle \pm |-\rangle \quad (4.12)$$

And the general bilinear states are of the type:

$$\langle\pm|L_\alpha|\mp\rangle \quad \text{and} \quad \langle\pm|\mathbb{L}_\alpha|\pm\rangle$$

where:

$$L_\alpha = \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}, \qquad \mathbb{L}_\alpha = \begin{pmatrix} a^2 \\ (a^+)^2 \end{pmatrix}_\alpha;$$
$$|\Psi_{1/4}\rangle = |+\rangle, \quad |\Psi_{3/4}\rangle = |-\rangle$$

For example, we have for the states with the explicit form:

$$\Phi_\alpha(t,\lambda) = \langle\Psi_\lambda(t)|\mathbb{L}_\alpha|\Psi_\lambda(t)\rangle$$
$$= e^{A(t)}e^{\xi\varrho(t)}\langle\Psi_\lambda(0)|\begin{pmatrix} a^2 \\ (a^+)^2 \end{pmatrix}_\alpha|\Psi_\lambda(0)\rangle \quad (4.13)$$

$$\Phi_\alpha(t,\lambda) = e^{A(t)}e^{\xi\varrho(t)}|f(\xi)|^2\begin{pmatrix} \alpha_\lambda^2 \\ \alpha_\lambda^{*2} \end{pmatrix}_\alpha \quad (4.14)$$

λ being the helicity label or the spanned subspace, e.g., $(\pm)$, and $A(t)$ is given by

$$A = -\left(\frac{m}{|\gamma|}\right)^2 t^2 + c_1 t + c_2; \qquad (c_1, c_2) \in \mathbb{C} \quad (4.15)$$

The square root solution takes the following form

$$\Psi_\lambda(t) = e^{\frac{1}{2}A(t)}e^{\frac{\xi\varrho(t)}{2}}|f(\xi)|\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_\lambda \quad (4.16)$$

where $\lambda = (1/4, 3/4)$. Notice the difference with the case of the Heisenberg-Weyl realization for the states Ψ:

$$|\Psi\rangle = \frac{f(\xi)}{2}(|\alpha_+\rangle + |\alpha_-\rangle) = f(\xi)|\alpha\rangle \quad (4.17)$$

where, the linear combination of the states $|\alpha_+\rangle$ and $|\alpha_-\rangle$ span now the full Hilbert space, being for this CS basis $\lambda = \frac{1}{2}$. The "square" states at $t=0$ are

$$\Phi_\alpha(0) = \langle\Psi(0)|L_\alpha|\Psi(0)\rangle = f^*(\xi)f(\xi)\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_\alpha. \quad (4.18)$$

The square state and the obtained square root state at time t are

$$\Phi_\gamma(t) = e^A e^{\xi\varrho(t)}|f(\xi)|^2\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_\alpha,$$
$$\Psi(t) = e^{\frac{1}{2}A}e^{\frac{\xi\varrho(t)}{2}}|f(\xi)|\begin{pmatrix} \alpha^{1/2} \\ \alpha^{*1/2} \end{pmatrix}. \quad (4.19)$$

Let us discuss the obtained results:

(i) We can see that the algebra, carrying the topological information of the group manifold, is "mapped" over the spinors solutions through the eigenvalues α and α^* from the dynamical viewpoint. The constants in the exponential functions of the Gaussian type in the solutions come from the action of a unitary operator over the respective coherent basic states in each Irreducible representation.

(ii) The $Osp(1/2,\mathbb{R})$ supergroup allows a metaplectic representation containing the complete superalgebra in functions of a single complex variable z exactly coinciding with the example treated here: it contains $SU(1,1)$ as subgroup which can lead or explain the fermionic factors of the type $[\exp(\frac{\xi\varrho(t)}{2})] \times |f(\xi)|$ in the solutions.

(iii) The $K_\pm$ and K_0 generators operate over the Bose states (B_0 sector). The B_1 sector of the algebra given by a and a^+ operates over the fermionic part. In this case, the coherent and squeezed states that can be constructed are eigenstates of the displacement and squeezed operators respectively (as in the standard case) but they cannot minimize simultaneously the dispersion of the quadratic Casimir operator, such that they are not minimum uncertainty states. This is so because the only states which minimize the Schrodinger uncertainty relation are those obtained by applying the displacement or squeezed operator on the lowest normalized state.

(iv) Geometrically, in the description of any physical system through SU(1,1) coherent states (CS) or squeezed states (SS), the orbits will appear as the intersections of constant-energy surfaces with one sheet of a two sheeted hyperboloid—the curved phase space of $SU(1,1)$ or Lobachevsky plane—in the space of averaged algebra generators. The group containing $SU(1,1)$ as subgroup linear and bilinear functions of the algebra generators, can factorize operators as the Hamiltonian or the Casimir operator (when averaged with respect to the group CS or SS): this defines corresponding curves in the averaged algebra space. If the exact dynamics is confined to the $SU(1,1)$ hyperboloid, the validity of the Ehrenfest's theorem for the coherent or squeezed states implies that it necessarily coincides with the variational motion that derives from the Euler-Lagrange equations for the Lagrangian

$$\mathcal{L} = \langle z| i\frac{\hat{\partial}}{\partial t} - \hat{H}|z\rangle,$$

that will be different if $|z\rangle = |\alpha\rangle$ or $|z\rangle = |\alpha_\pm\rangle$, as it is evident.

C. Discrete representation

Equation (4.16) describes a standard coherent state (eigenstate of the operator (a) as a linear combination of two states belonging to $\mathcal{H}_{1/4}$ and $\mathcal{H}_{3/4}$ respectively, (which are two independent coherent states as eigenstates of (a^2). The corresponding metaplectic vacuum as fiducial vector of the physical system is:

$$|z_0\rangle_{Mp(2)} = \mathcal{M}(1 + \mathcal{M}^2 a^+)|0\rangle \tag{4.20}$$

$$\mathcal{M} \equiv [|m^2 - \epsilon^2| + p^2 \mathrm{sign}(\epsilon^2 - m^2)]^{1/4} \tag{4.21}$$

Notice that this vacuum is not singular at $m \to \epsilon$ but is analytically continued into the complex plane where it is defined. Then, the solution for Eq. (4.7) is the following:

$$|\Psi\rangle_{Mp(2)} \equiv S(t, A, p, \epsilon)|z_0\rangle_{Mp(2)} \tag{4.22}$$

$$\begin{aligned} |\Psi\rangle_{Mp(2)} &= \left[1 + \frac{p^2 \mathrm{sign}\mathcal{E}}{|\mathcal{E}|}\right]^{1/4} e^{\frac{p/2}{(m+\epsilon)}(a^+)^2}\left[1 + \left(1 + \frac{p^2 \mathrm{sign}\mathcal{E}}{|\mathcal{E}|}\right)^{1/2} a^+\right]|0\rangle \\ \mathcal{E} &\equiv \epsilon^2 - (\dot{A})^2 \end{aligned} \tag{4.23}$$

being $S(t, A, p, \epsilon) \in Mp(2)$ the operator Eq. (4.8) for the set of parameters and functions in Eq. (4.7). The total solution of the system Eqs. (4.2) and (4.3) for these parameters being $\mathcal{G}|\Psi\rangle_{Mp(2)}$ with $\mathcal{G} \equiv e^{(A+\xi\rho)(t,m,p,\epsilon)} e^{(py - i\epsilon z)}$.

The Bargmann representation of $\mathcal{H}$ associates an entire analytic function $f(z)$ of a complex variable z, with each vector $|\varphi\rangle \in \mathcal{H}$ in the following manner:

$$|\varphi\rangle \in \mathcal{H} \to f(z) = \sum_{n=0}^{\infty} \langle n|\varphi\rangle \frac{z^n}{\sqrt{n!}} \tag{4.24}$$

$$\langle\varphi|\varphi\rangle \equiv \|\varphi\|^2 = \sum_{n=0}^{\infty} |\langle n|\varphi\rangle|^2 = \int \frac{d^2 z}{\pi} e^{-|z|^2} |f(z)|^2 \tag{4.25}$$

where the integration is over the entire complex plane. The above association can be compactly written in terms of the normalized coherent states. Consequently:

(i) The $\mathcal{H}_{1/4}$ states occupy the sector even of the full Hilbert space $\mathcal{H}$ and we describe them as:

$$f^{(+)}(z, \omega) = (1 - |\omega|^2)^{1/4} e^{\omega z^2/2} = (1 - |\omega|^2)^{1/4} \sum_{n=0,1,2,\ldots} \frac{(\omega/2)^{2n}}{(2n)!} z^{2n} \tag{4.26}$$

Then, in the vector representation we have:

$$|\Psi^{(+)}(\omega)\rangle = (1-|\omega|^2)^{1/4} \sum_{n=0,1,2,\ldots} \frac{(\omega/2)^{2n}}{\sqrt{(2n)!}} |2n\rangle. \tag{4.27}$$

Consequently, the $\mathcal{H}_{1/4}$ (or *even*) number representation is obtained as:

$$\langle 2n|\Psi^{(+)}(\omega)\rangle = (1-|\omega|^2)^{1/4} \frac{(\omega/2)^{2n}}{\sqrt{(2n)!}}, \qquad \langle 2n+1|\Psi^{(+)}(\omega)\rangle \equiv 0. \tag{4.28}$$

(ii) The $\mathcal{H}_{3/4}$ states occupy the odd sector of the full Hilbert space $\mathcal{H}$ and we similarly describe them as for $\mathcal{H}_{1/4}$:

$$f^{(-)}(z,\omega) = (1-|\omega|^2)^{3/4} z e^{\omega z^2/2} = (1-|\omega|^2)^{3/4} \sum_{n=0,1,2,\ldots} \frac{(\omega/2)^{2n+1}}{(2n+1)!} z^{2n+1} \tag{4.29}$$

and the vector representation is

$$|\Psi^{(-)}(\omega)\rangle = (1-|\omega|^2)^{3/4} \sum_{n=0,1,2,\ldots} \frac{(\omega/2)^{2n+1}}{\sqrt{(2n+1)!}} |2n+1\rangle \tag{4.30}$$

The $\mathcal{H}_{3/4}$ (or *odd*) number representation is consequently:

$$\langle 2n+1|\Psi^{(-)}(\omega)\rangle = (1-|\omega|^2)^{3/4} \frac{(\omega/2)^{2n+1}}{\sqrt{(2n+1)!}}, \qquad \langle 2n|\Psi^{(-)}(\omega)\rangle \equiv 0 \tag{4.31}$$

(iii) The full Hilbert space, defined by the direct sum $\mathcal{H} = \mathcal{H}_{1/4} \oplus \mathcal{H}_{3/4}$, is the following:

$$f(z,\omega) = f^{(+)}(z,\omega) + f^{(-)}(z,\omega) \tag{4.32}$$

$$f(z,\omega) = (1-|\omega|^2)^{1/4} \sum_{n=0,1,2,\ldots} \frac{(\omega/2)^{2n}}{(2n)!} z^{2n} \left[1 + \frac{(1-|\omega|^2)^{1/2}}{(2n+1)} z\right] \tag{4.33}$$

Then, in complete analogy with their *even* and *odd* subspaces, the corresponding states are

$$\Psi(\omega) = \Psi^{(+)}(\omega) + \Psi^{(-)}(\omega) \tag{4.34}$$

$$\Psi(\omega) = (1-|\omega|^2)^{1/4} \sum_{n=0,1,2,\ldots} \frac{(\omega/2)^{2n}}{\sqrt{(2n)!}} \left[1 + \frac{(1-|\omega|^2)^{1/2}}{(2n+1)} a^+\right] |2n\rangle \tag{4.35}$$

$$\langle n|\Psi(\omega)\rangle = \begin{cases} (1-|\omega|^2)^{1/4} \frac{(\omega/2)^{2n}}{(2n)!} \sqrt{2n!} & \text{even states} \\ (1-|\omega|^2)^{3/4} \frac{(\omega/2)^{2n+1}}{(2n+1)!} \sqrt{(2n+1)!} & \text{odd states} \end{cases} \tag{4.36}$$

where the link between the physical observables and the group parameters is given by the following expression (measure):

$$\left(1 + \frac{p^2 \mathrm{sign}(\epsilon^2 - m^2)}{|m^2 - \epsilon^2|}\right)^{1/4} \to (1-|\omega|^2)^{1/4} \tag{4.37}$$

Figures 2 and 3 display the discrete spectra in the number representation of the coherent states in $\mathcal{H}_{1/4}$ (even n) and $\mathcal{H}_{3/4}$ (odd n).

The limit $\epsilon \to m$:

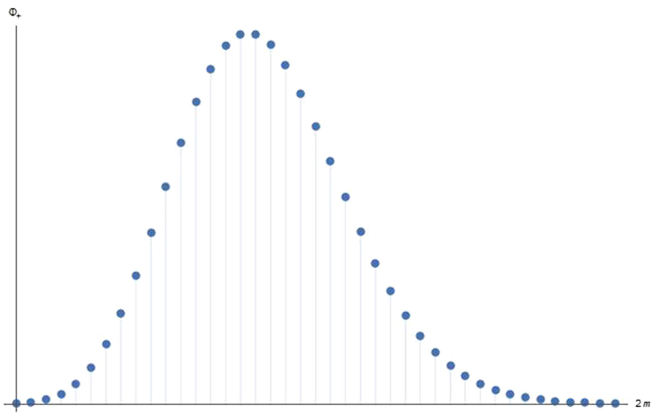

FIG. 2. The $\mathcal{H}_{1/4}$ discrete (number representation) states occupy the even sector of the full Hilbert space $\mathcal{H}$. This irreducible representation of the Mp (2) group is not dense (in a topological sense) but it contains the ground state $|0\rangle$.

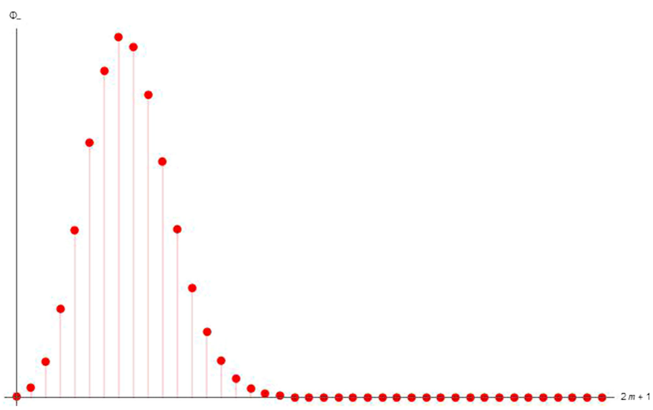

FIG. 3. The $\mathcal{H}_{3/4}$ discrete (number representation) states occupy the sector odd of the full Hilbert space $\mathcal{H}$. This Irreducible representation of Mp (2) is not dense (in a topological sense) but its lower or fundamental state is the first excited state $|1\rangle$.

This is precisely the limit $|\omega|^2 \to 1$, which from the point of view of the Metaplectic analysis corresponds to the edge of the complex disc. As we could easily see, the state solutions span the full spectrum corresponding to $\mathcal{H}$. What happens is that in the limit $\epsilon \to m$ the density of states corresponding to $\mathcal{H}_{1/4}$ is greater than that of the odd states belonging to $\mathcal{H}_{3/4}$. It is for this reason that the states belonging to $\mathcal{H}_{1/4}$, will survive in this limit.

V. QUANTUM SPACE-TIME: DE SITTER AND BLACK HOLE COHERENT STATES

A. Quantum space-time

We restrict in the sequel to the purely bosonic space-time and consider the (X, T) quantum space and time dimensions which are relevant to the quantum space-time structure. The remaining spatial transverse dimensions $X_\perp$ are not considered here as fully quantum noncommuting coordinates. Notice that although the transverse spatial dimensions $\perp$ have zero commutators they can fluctuate. This corresponds to quantize the two-dimensional space-time surface which is relevant to determine the light-cone structure. This is enough for considering the novel features arising in the global quantum space-time and the *quantum light cone*.

The relevant quantum space-time (X, T) structure is described essentially by a quantum inverted oscillator type algebra with discrete hyperbolic levels $(X^2 - T^2)_n = (2n+1), n = 0, 1, 2, \ldots$. The zero point energy $(n = 0)$ being the Planck energy level. The truly quantum gravity (trans-Planckian) vacuum in the quantum space time is delimitated by the four quantum hyperbolae $X^2 - T^2 = \pm 1$ (in Planck units) of the Planck scale $(n = 0)$ level. This is precisely a constant curvature de Sitter vacuum.

The de Sitter space-time can be described as a (*inverted*, i.e., with imaginary frequency) harmonic oscillator, the *oscillator constant and length* being [9,34]:

$$\kappa_{\rm osc} = H^2, \qquad H = \sqrt{\frac{(8\pi G\Lambda)}{3}} = c/l_{\rm osc} \qquad (5.1)$$

The *oscillator length* $l_{\rm osc}$ is classically the Hubble radius, the Hubble constant $H = \kappa$ being the surface gravity, as the black hole surface gravity is the inverse of (twice) the black hole radius.

Interestingly, the description of de Sitter space-time as an (inverted, classical, and quantum) harmonic oscillator derives from three results:

(i) From the Einstein Equations on the one hand, [9–11,35],

(ii) From the de Sitter geometrical description on the other hand: an hyperboloid embedded in flat Minkowski space-time with one more spatial dimension:

$$-T^2 + X^2 + X_i^2 + Z^2 = L_{QG}^2 \qquad (5.2)$$

$$L_{QG} = (L_Q + L_G) = l_P\left(\frac{H}{h_P} + \frac{h_P}{H}\right), \qquad (5.3)$$

L_{QG} is the complete length allowing to describe both the classical, semiclassical and quantum (trans-Planckian) gravity domains, l_P the constant Planck length:

$$L_Q = l_P^2/L_G, \qquad l_P = (2G\hbar/c^3)^{1/2}, \qquad h_P = c/l_P \qquad (5.4)$$

(iii) From the hyperbolic quantum space-time structure which delimitates a purely quantum trans-Planckian central region of constant curvature, [8–10].

In the anti–de Sitter space-time, the description is the same but with $-T^2 + X^2 + X_i^2 + Z^2 = -L_{QG}^2$, and therefore anti–de Sitter background is associated to a real frequency (noninverted) harmonic oscillator. Also, the propagation of fields and linearized perturbations in the de Sitter vacuum all satisfy equations which are like the *inverted* oscillator equations, [12,13,36], or the normal oscillator equations in anti–de Sitter space-time.

In the (Schwarzschild) black hole space-time: (quantum interior constant curvature vacuum; semiclassical and classical exterior regions), the physical magnitudes as the oscillator constant H^2 and the typical oscillator length l_{osc} are related to the black hole mass M:

$$H = c/l_{\text{osc}} = h_P\left(\frac{m_P}{M}\right), \quad \Lambda = \lambda_P\left(\frac{m_P}{M}\right)^2, \quad \lambda_P = 3h_P^2/c^4 \tag{5.5}$$

Classical space-time regions or regimes are described by the low values of Λ and of the gravitational density ρ_G, and the large classical gravitational sizes $L_G \gg l_P$:

$$L_G = l_P\sqrt{\frac{\lambda_P}{\Lambda}} = l_P\left(\frac{M}{m_P}\right) \tag{5.6}$$

Truly Quantum gravitational regimes, eg in the trans-Planckian domain of very small sub-Planckian sizes, very high quantum density ρ_Q and very high vacuum values Λ_Q:

$$L_Q = l_P\sqrt{\frac{\Lambda}{\lambda_P}} = l_P\left(\frac{m_P}{M}\right), \qquad \Lambda_Q = \frac{\lambda_P^2}{\Lambda} \tag{5.7}$$

Consistently, the *high* value of the classical/semiclassical gravitational entropy S_G is equal (in Planck units) to such high Λ_Q value. This is clearly elucidated by the following classical–quantum gravity duality relations in this context:

$$\frac{\rho_G}{\rho_P} = \left(\frac{l_P}{L_G}\right)^2 = \left(\frac{m_P}{M}\right)^2 = \left(\frac{S_Q}{s_P}\right) \tag{5.8}$$

$$\frac{\rho_Q}{\rho_P} = \left(\frac{l_P}{\Lambda}\right) = \left(\frac{M}{m_P}\right)^2 = \left(\frac{S_G}{s_P}\right)$$
$$\rho_P = 3h_P^2/8\pi G, \qquad s_P = \pi\kappa_B \tag{5.9}$$

The last right-hand side (rhs) of Eqs. (5.8) and (5.9) show the link to the gravitational *entropy*: quantum gravitational entropy S_Q and classical/semiclassical S_G entropy. (This last is the Bekenstein-Hawking-Gibbons entropy [37–39]). Lower case magnitudes with subscript P denote the corresponding Planck scale fundamental constant magnitudes.

The external BH region is precisely a *classical gravity dilute vacuum*, which $(\Lambda, \rho_G)_{\text{BH}}$ values in the present universe cannot be larger than the observed very low values of (Λ, ρ_G) Refs. [14–22]. Their quantum duals provide an upper bound to the high values (Λ_Q, ρ_Q) in the quantum central BH vacuum region as determined by Eqs. (5.8) and (5.9).

B. QST deS and BH. Minimal uncertainty and Mp(2) vacuum

For the quantum space-time (QST) de Sitter states, the oscillator parameters entering in the coherent states and their representations Sec. II are the following:

As discussed above, de Sitter space-time is described by an *inverted* oscillator with oscillator length $l_{\text{osc}} = c/H$, and the generic quantum coherent states built in Sec. II, have in particular the inverted oscillator length $l_{\text{osc}} = \sqrt{\hbar/m\omega}$.

The de Sitter quantum space-time coherent states are described by the states Eqs. (2.2)–(2.4), Eq. (2.6) with the corresponding oscillator constant given by:

$$l_{\text{osc}dS}^{-2} = \left(\frac{m\omega}{\hbar}\right)_{dS} = H^2 = \frac{\Lambda}{3} \tag{5.10}$$

In the (Schwarzschild) black hole space-time, the physical magnitudes as the oscillator constant and the oscillator length are related to the black hole mass M:

$$l_{\text{osc}BH}^{-2} = \left(\frac{m\omega}{\hbar}\right)_{\text{BH}} = l_P^{-2}\left(\frac{m_P}{M}\right)^2$$
$$l_P = (2G\hbar/c^3)^{1/2}, \qquad h_P = c/l_P \tag{5.11}$$

The complete length L_{QG} in Eq. (5.2) covers both the classical, semiclassical and quantum (trans- Planckian) gravity domains. Quantum space-time derives from the quantum non commutative space and momentum (phase space) operators with the mapping of momentum into time, Refs. [8–10,40]. As a consequence, quantum space-time described by coherent states have minimal and equally distributed uncertainty: $\Delta X \Delta T = \hbar/2$

$$(\Delta X)^2 = \left(\frac{\hbar}{2m\omega}\right), \qquad (\Delta T)^2 = \left(\frac{\hbar m\omega}{2}\right). \tag{5.12}$$

Therefore, coherent states of quantum de Sitter space-time have the spatial and temporal uncertainty:

$$(\Delta X)^2{}_{deS} = \left(\frac{\hbar}{2m\omega}\right)_{deS} = \frac{\hbar}{2H^2} \tag{5.13}$$

$$(\Delta T)^2_{deS} = \frac{\hbar H^2}{2}. \tag{5.14}$$

And for the Black Hole coherent states, the quantum uncertainty in space and time is:

$$(\Delta X)^2{}_{\text{BH}} = \left(\frac{\hbar}{2m\omega}\right)_{\text{BH}} = \frac{l_P^2}{2}\left(\frac{M}{m_P}\right)^2 \tag{5.15}$$

$$(\Delta T)^2{}_{\rm BH} = \frac{t_P^2}{2}\left(\frac{m_P}{M}\right)^2 \qquad (5.16)$$

(l_P and t_P being the Planck length and time). The de Sitter and black hole coherent states derive from the explicit expressions Eqs. (2.2)–(2.4), Eq. (2.6) with the respective (deS) and (BH) physical magnitudes given by Eqs. (5.10) and (5.11). In particular, the quantum metaplectic Mp (2) vacuum is given by:

$$(\psi_{\rm vacuum}|_{MP(2)})_{deS} = \left(\frac{i}{\pi}\right)^{1/4}\sqrt{H}e^{-\frac{i}{2}(HX)^2}\left[\frac{HX}{2\sqrt{2}}(1+i) + \frac{iHX}{2\sqrt{2}}\right]$$

$$(\psi_{\rm vacuum}|_{MP(2)})_{\rm BH} = \left(\frac{i}{\pi}\right)^{1/4}\sqrt{2\mathcal{K}}e^{-\frac{i}{2}(2\mathcal{K}X)^2}\left[\frac{\mathcal{K}X}{\sqrt{2}}(1+i) + \frac{i\mathcal{K}X}{\sqrt{2}}\right] \qquad (5.17)$$

where:

$$\mathcal{K} = 1/(2R_{\rm BH}) = 1/(4GM) \qquad (5.18)$$

Both vacuum states are expressed in terms of the surface gravity (H or $\mathcal{K}$) respectively, or similarly in terms of the de Sitter or BH radius. Both states are *totally regular*, as it must be for quantum space-time. For $X \gg R_{\rm BH}$, ($R_{\rm BH}$ being the BH radius), and asymptotically for very large X, the quantum coherent state consistently encompasses the quantum space classicalization, as such exterior BH regions are semiclassical and classical. We discuss below the excited (α) states.

C. Continuum and discrete deS and BH coherent states

Quantum space-time de Sitter and black hole coherent states follow from Eqs. (2.2)–(2.4) and Eq. (2.6) with the physical magnitudes and uncertainty relations Eqs. (5.13)–(5.16). The quantum space-time *deS* coherent states have the following expressions:

$$\psi_\alpha(X)_{deS} = \left(\frac{i}{\pi}\right)^{1/4}\sqrt{H}e^{-\frac{1}{2}|\alpha|^2} \times \exp\left[\frac{\alpha HX}{\sqrt{2}}(1+i) - \frac{iH^2X^2}{2}\right] \qquad (5.19)$$

$$\psi_\alpha(T)_{deS} = \left(\frac{i}{\pi}\right)^{1/4}\frac{1}{\sqrt{\hbar H}}e^{-\frac{1}{2}|\alpha|^2} \times \exp\left[\frac{\alpha T}{\sqrt{2}\hbar H}(1+i) - \frac{iT^2}{2\hbar^2H^2}\right]. \qquad (5.20)$$

A similar coherent state expression holds for the BH space-time with the corresponding BH factor $2\mathcal{K}$ instead of H, being $\mathcal{K}$ the surface gravity Eq. (5.18).

α is the complex constant number, eigenvalue of the displacement operator $D(\alpha)$, which characterizes the coherent state excitations (displacement from the vacuum), and their continuum spectrum.

(i) The quantum space-time coherent states Eqs. (5.19) and (5.20) clearly display an exponential de Sitter expansion term $\alpha T/(\sqrt{2}\hbar H)$ plus a phase (linear and quadratic) in $[(\alpha T/(\sqrt{2}\hbar H)]$, which is simply $T/(2\sqrt{2}\pi T_H$, T_H being the Hawking de Sitter temperature. The quantum space-time exhibits an accelerated expansion plus quantum oscillations of the same sign (linear term), and of different sign (quadratic term). The presence of these oscillations is a new feature of quantum space-time.

(ii) The continuum (α)-coherent states Eqs. (5.19) and (5.20) describe semi-classical, (or semiquantum), space-time regimes and in agreement with the space-time described by quantum oscillators. Quantum discrete space-time becomes more and more continuous for large n in agreement with its description by continuum coherent states. Consistently, the continuum coherent states are characterized by the Hawking temperature which is a semiclassical (or semiquantum) temperature.

(iii) We see that:

$$|\Psi_\alpha(X)_{deS}|^2 - |\Psi_\alpha(T)_{deS}|^2 = \frac{1}{\sqrt{\pi}}\exp^{-|\alpha|^2}\left|H\exp[c(\alpha)HX] - \frac{1}{\hbar H}\exp\left[c(\alpha)\frac{T}{\hbar H}\right]\right|$$
$$c(\alpha) = \sqrt{2}({\rm Re}\alpha - {\rm Im}\alpha) \qquad (5.21)$$

which reflects the quantum hyperbolic space-time structure. Let us define:

$$\mathcal{R}_\alpha(X,T)^2 \equiv |\Psi_\alpha(X)_{deS}|^2 - |\Psi_\alpha(T)_{deS}|^2$$

(i) For $\alpha = 0$:

$$\mathcal{R}_0(X,T)^2 = \frac{1}{\sqrt{\pi}}\left|H - \frac{1}{\hbar H}\right| \qquad (5.22)$$

which can be also expressed in terms of the quantum uncertainties:

$$\mathcal{R}_0(X,T)^2 = \sqrt{\frac{2}{\pi\hbar}}\left|\Delta T - \frac{\Delta X}{\hbar}\right| \qquad (5.23)$$

(ii) For:

$$\mathcal{R}_\alpha(X,T)^2 = 0 \to |\Psi_\alpha(X)_{deS}| = \pm|\Psi_\alpha(T)_{deS}| \qquad (5.24)$$

Clearly, $\alpha = 0$, corresponds to $H = 1/\sqrt{\hbar}$, that is the Planck scale. We see the power of coherent states in describing

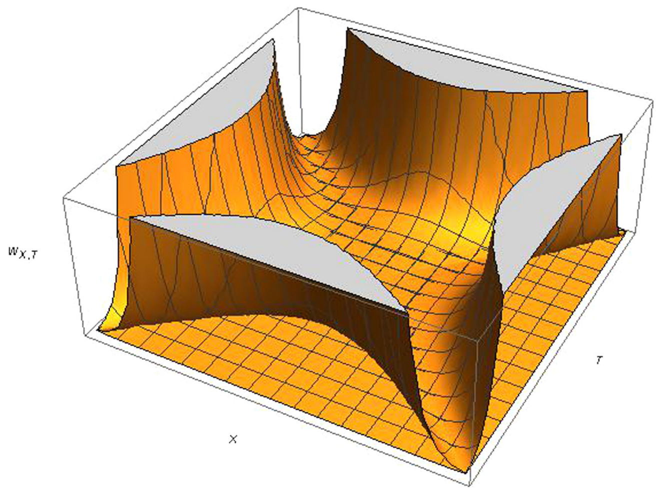

FIG. 4. The squeezed quasi-probability Wigner function $W(X,T)$ of quantum space-time. $W(X,T)$ clearly shows the hyperbolic light-cone space-time structure and with symmetric form. For the coherent states, $W_\alpha(X,T)$ endows the hyperbolic structure but with a linear α tail deformation in X or T.

space-time and even accounting for the Planck scale, at which:

$$\Delta X = \hbar/\sqrt{2}, \quad \Delta T = 1/\sqrt{2} \quad \text{(Planck scale)} \tag{5.25}$$

Obviously, for coherent states it satisfies $\Delta X \Delta T = \hbar/2$.

For the squeezed states, particularly interesting is the Wigner quasiprobability function, which have here the following expression:

$$W_{sq}(XT)_{deS} = \exp\left[-\left(H^2X^2 - \frac{T^2}{\hbar^2 H^2}\right)\right] \tag{5.26}$$

This clearly shows the hyperbolic structure of quantum space-time. The characteristic light-cone structure is manifest here because there is no any α-deformation in this case.

Figure 4 displays the space-time squeezed state Wigner function and its light-cone hyperbolic structure.

(iii) The discrete quantum space-time (Planckian and trans-Planckian) regimes are described by discrete states, e.g., the discrete coherent states of Sec. IV C. The discrete spectrum of these states describes the different quantum space-time excitation levels, the less excited (fundamental, $n = 0$) level corresponding to the Planck scale, (the *crossing* or transition scale). Interestingly, as seen in Secs. IV and IV C the metaplectic group states with its *both* sectors and discrete representations, $|2n\rangle$ and $|2n+1\rangle$, *even* and *odd* states, fully *cover* the *complete* Hilbert space $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_{(+)} \oplus \mathcal{H}_{(-)} \tag{5.27}$$

The $(\pm)$ symmetric and antisymmetric sum of the two kind (*even* and *odd*) states provides the *complete* covering of the Hilbert space and of the space-time mapped from it:

$$\Psi(n) = \Psi^{(+)}(2n) + \Psi^{(-)}(2n+1) \tag{5.28}$$

where $\Psi^{(+)}$ and $\Psi^{(-)}$ are obtained from Eqs. (4.34)–(4.36). For de Sitter space, both sets of states are given by:

$$\Psi^{(+)}(2n)_{deS} = |1 - H^4|^{1/4} \frac{(H^2/2)^{2n}}{\sqrt{(2n)!}} \tag{5.29}$$

$$\Psi^{(-)}(2n+1)_{deS} = |1 - H^4|^{3/4} \frac{(H^2/2)^{2n+1}}{\sqrt{(2n+1)!}} \tag{5.30}$$

where we take into account that in the fully quantum trans-Planckian de Sitter phase [9]: the quantum H is $H > 1$ and thus the analytic covering in this phase. In addition, the quantum discrete levels of H are [9]: $H_{Qn} = \sqrt{2n}$ (*even* levels), and $H_{Qn} = \sqrt{2n+1}$, (*odd* levels), which leads to:

$$\Psi^{(+)}(2n)_{deS} = |1 - 4n^2|^{1/4} \frac{(2n)^{2n}}{2^{2n}\sqrt{(2n)!}} \tag{5.31}$$

$$\Psi^{(-)}(2n+1)_{deS} = [4n(n+1)]^{3/4} \frac{(2n+1)^{2n+1}}{2^{2n+1}\sqrt{(2n+1)!}} \tag{5.32}$$

It is worth mentioning that independently of this Mp(n) coherent state framework, we obtained in Refs. [8,9,40], similar discrete levels in terms of the global cart X, or the local ones x constructed from the global (complete) classical–quantum duality including gravity [40]. In such levels, the two kind of sectors and their global $(\pm)$ covering do appear, which reflects some kind of relation between the $Mp(n)$ symmetry and classical–quantum duality:

$$X_n = \sqrt{2n+1}, \quad \text{or} \quad x_{n\pm} = [X_n \pm \sqrt{X_n^2 - 1}], \quad n = 0, 1, 2, \ldots. \tag{5.33}$$

The condition $X_n^2 \geq 1$ simply corresponds to the whole spectrum $n \geq 0$:

$$x_{n\pm} = [\sqrt{2n+1} \pm \sqrt{2n}]$$
$$x_{n=0}(+) = x_{n=0}(-) = 1: \text{ Planck scale}, \tag{5.34}$$

which complete *all* the levels. The $(\pm)$ branches consistently reflect:

(i) The classical–quantum duality properties of the *global* space-time.

(ii) The two $\sqrt{(2n+1)}$ and $\sqrt{2n}$, *even* and *odd* (local) sectors. Each symmetric or antisymmetric sum is necessary to cover the whole manifold. The corresponding $(\pm)$ global states are complete, *CPT* and unitary, the levels $n = 0, 1, 2, \ldots$, cover the whole Hilbert space $\mathcal{H} = \mathcal{H}_{(+)} \oplus \mathcal{H}_{(-)}$ and all space-time regimes.

(iii) The total n states range over *all* scales from the lowest excited levels to the highest excited ones covering the two dual branches (+) and (−) or Hilbert space sectors.

VI. IMAGINARY TIME. COHERENT STATES OF QUANTUM GRAVITATIONAL INSTANTONS

Taking imaginary time $T = i\mathcal{T}$, $t = i\tau$, yields to the elliptic (or circular) structure of space-time and of the phase space, e.g., this corresponds in particular to the normal (non inverted) oscillator description. That is to say, quantum space-imaginary time instantons correspond to the real frequency quantum oscillators of phase space. They describe in particular, quantum tunneling effects between different states or different vacua, or different phase (space) regions. Besides being saddle points in an euclidean quantum gravity path integral, they can describe thermal features if the imaginary time endows periodicity.

In the classical (nonquantum) BH space-time, the identification $T = i\mathcal{T}$, $t = i\tau$, transforms the hyperbolic space-time structure into a circular structure: The classical horizon $X = \pm T$ collapses to the origin $X = \pm\mathcal{T} = 0$. In the classical (nonquantum) BH *instanton*, the interior *is cutted*, no horizon, and no central curvature singularity, does appear: The classical BH instanton is *regular* but *not complete*: The interior BH region is not covered by the classical instanton.

In the complete quantum BH space-time, the quantum hyperbole $(X^2 - T^2 = l_P^2)$ replace the characteristic lines due to the nonzero $[X, T]$ commutators, and in the corresponding quantum BH instanton the horizon does not collapse to the origin but to the Planck scale circle $(X^2 + \mathcal{T}^2 = l_P^2)$. The complete *quantum* BH instanton includes the usual classical/semiclassical BH instanton for radius larger than the Planck length, plus a *new central* highly dense *quantum core* of Planck length radius and high constant and *finite* curvature corresponding to the *black-hole interior*, Ref. [10] which is *absent* in the classical BH instanton.

Particularly interesting here is the Wigner quasiprobability function for the squeezed states, which for the BH have the following expression:

$$W_{sq}(X\mathcal{T})_{\mathrm{BH}} = 2\exp\left[-\left(4\mathcal{K}^2X^2 + \frac{\mathcal{T}^2}{4\hbar^2\mathcal{K}^2}\right)\right] \tag{6.1}$$

where the BH oscillatory space-time parameters are expressed in terms of the BH surface gravity $\mathcal{K}$ Eq. (5.18). $W_{sq}(X\mathcal{T})$ clearly shows the circular structure of the quantum space- imaginary time instantons. The circular structure is manifest here without deformation because the α tail present for the coherent states is absent in this case.

The coherent states for the quantum gravitational instanton, here we elucidate for the BH, follow similar expressions as Eqs. (5.19) and (5.20) but with the BH factor $2\mathcal{K}$:

$$\Psi_\alpha(X)_{\mathrm{BH}} = \left(\frac{i}{\pi}\right)^{1/4}\sqrt{2\mathcal{K}}e^{-\frac{1}{2}|\alpha|^2}\exp\left[\sqrt{2}\alpha\mathcal{K}X(1+i) - 2i\mathcal{K}^2X^2\right] \tag{6.2}$$

$$\Psi_\alpha(\mathcal{T})_{\mathrm{BH}} = \left(\frac{i}{\pi}\right)^{1/4}\sqrt{\frac{1}{2\hbar\mathcal{K}}}e^{-\frac{1}{2}|\alpha|^2}\exp\left[-\frac{\alpha\mathcal{T}}{2\sqrt{2}\hbar\mathcal{K}}(1-i) + \frac{i\mathcal{T}^2}{8\hbar^2\mathcal{K}^2}\right] \tag{6.3}$$

Therefore:

$$|\Psi_\alpha(X)_{\mathrm{BH}}|^2 + |\Psi_\alpha(\mathcal{T})_{\mathrm{BH}}|^2 = \frac{e^{-|\alpha|^2}}{\sqrt{\pi}}\left[2\mathcal{K}\exp\left[2c(\alpha)\mathcal{K}X\right] + \frac{1}{2\hbar\mathcal{K}}\exp\left[c(\alpha)\frac{\mathcal{T}}{2\hbar\mathcal{K}}\right]\right]$$
$$c(\alpha) = \sqrt{2}(\mathrm{Re}\,\alpha - \mathrm{Im}\,\alpha) \tag{6.4}$$

which reflects the quantum elliptic (circular) structure of the space-imaginary time instanton. We define:

$$\mathcal{R}_\alpha(X,\mathcal{T})^2 \equiv |\Psi_\alpha(X)_{\mathrm{BH}}|^2 + |\Psi_\alpha(\mathcal{T})_{\mathrm{BH}}|^2 \tag{6.5}$$

We see that: For $\alpha = 0$:

$$\mathcal{R}_0(X,\mathcal{T})^2 = \frac{1}{\sqrt{\pi}}\left[2\mathcal{K} + \frac{1}{2\hbar\mathcal{K}}\right] \tag{6.6}$$

For $(X,\mathcal{T}) \to 0$:

$$\mathcal{R}_\alpha(0)^2 = e^{-|\alpha|^2}\mathcal{R}_0^2 \tag{6.7}$$

The origin is blurred or erased within a quantum circular core of radius $\mathcal{R}_\alpha(0)$. This confirms with a coherent state approach, the *regular* (nonsingular) quantum internal BH region obtained in Ref. [10] by using quantum Schwarzschild-Kruskal coordinates.

At the Planck scale: $(X, \mathcal{T}) \to (l_P, t_P)$, $\mathcal{K} \to \kappa_P = 1/(2l_P)$:

$$\mathcal{R}_\alpha(l_P, t_P)^2 = e^{[c(\alpha)-|\alpha|^2]} \frac{1}{\sqrt{\pi}} \left[\frac{1}{l_P} + \frac{l_P}{\hbar}\right] \tag{6.8}$$

As clearly seen, $\alpha = 0$ corresponds to the Planck scale (the onset scale in the trans-Planckian domain). Consistently, the values $\alpha \neq 0$, $(0 < \alpha < \infty)$, imply smaller sub-Planckian radii and more excited states, entering deeper in the quantum trans-Planckian region; α can be very high but it is bounded, as the quantum radius cannot be zero because of the quantum uncertainty, the notion of a *maximum* value $\alpha_{\max}$ does appear related here to a minimal radius $\mathcal{R}_{\min}$ due to the quantum uncertainty:

$$\mathcal{R}_{\min} = \mathcal{R}_{\alpha-\max}(0) = e^{-|\alpha_{\max}|^2/2}\mathcal{R}_0 \tag{6.9}$$

$\mathcal{R}_0(X, \mathcal{T})$ Eq. (6.6) can be expressed in terms of the quantum uncertainties here:

$$\Delta X = \frac{1}{\sqrt{2}2\mathcal{K}}, \qquad \Delta\mathcal{T} = \sqrt{2}\hbar\mathcal{K}$$

$$\mathcal{R}_0(X, T)^2 = \sqrt{\frac{2}{\pi}}\frac{1}{\hbar}[\Delta T + \Delta X] \tag{6.10}$$

which is always nonzero because here $\Delta X \Delta T = \hbar/2$, (minimal uncertainty). $\alpha_{\max}$ is thus given by:

$$\alpha_{\max}^2 = 2\log\left[\frac{1}{\pi}\left(\frac{1}{\Delta X} + \frac{2\Delta X}{\hbar}\right)\right] \tag{6.11}$$

Consistently, at the Planck scale, we have:

$$(\Delta X)_P = l_P/\sqrt{2}, \qquad (\Delta T)_P = \hbar/(\sqrt{2}l_P)$$

$$\mathcal{R}_0(l_P, t_P)^2 = \frac{1}{\sqrt{\pi}}\left[\frac{1}{l_P} + \frac{l_P}{\hbar}\right] \tag{6.12}$$

which coincides with Eq. (6.8) for $\alpha = 0$, as it must be. It is remarkable how coherent states account for a consistent quantum space-time description even at the Planckian and trans-Planckian scales.

Finally, the total discrete n-states are given by the sum of the *even* and *odd* states. The BH n-states are similar to expressions Eq. (5.29) for deS space-time but, as we have seen, the instanton corresponds to the normal (noninverted) oscillator and then we have:

$$\Psi(n) = \Psi^{(+)}(2n) + \Psi^{(-)}(2n+1) \tag{6.13}$$

where:

$$\Psi^{(+)}(2n)_{\mathrm{BH}} = |1 + \mathcal{K}^4|^{1/4} \frac{(\mathcal{K}^2)^{2n}}{\sqrt{(2n)!}} \tag{6.14}$$

$$\Psi^{(-)}(2n+1)_{\mathrm{BH}} = |1 + \mathcal{K}^4|^{3/4} \frac{(\mathcal{K}^2)^{2n+1}}{\sqrt{(2n+1)!}} \tag{6.15}$$

In addition, we take into account the quantum $\mathcal{K}$ discrete levels, [9,10]: $\mathcal{K}_{Qn} = \sqrt{2n}$ (*even* levels), and $\mathcal{K}_{Qn} = \sqrt{2n+1}$, (*odd* levels), which yields:

$$\Psi^{(+)}(2n)_{\mathrm{BH}} = |1 + (2n)^2|^{1/4} \frac{(2n)^{2n}}{2^{2n}\sqrt{(2n)!}} \tag{6.16}$$

$$\Psi^{(-)}(2n+1)_{\mathrm{BH}} = [1 + (2n+1)^2]^{3/4} \frac{(2n+1)^{2n+1}}{2^{2n+1}\sqrt{(2n+1)!}} \tag{6.17}$$

which completes all the states. The total covering is given by the sum of both $(\pm)$ states which cover the full Hilbert $Mp(2)$ space $\mathcal{H}_{1/4} \oplus \mathcal{H}_{3/4}$. This also shows that when considered in its full quantum discrete phase, quantum gravity must be a theory of pure numbers.

VII. DISCUSSION

It is interesting to discuss in this context the work by Ford Ref. [41] in which in a perturbative approach, quantum metric fluctuations can act as a regulator of the ultraviolet divergences of quantum fields. Metric fluctuations, as those due to gravitons in a quantum vacuum state, can modify the behavior of Green functions near the light cone, smearing it (and for instance in the one-loop electron self-energy). In other words, gravitons in a quantum squeezed state could regulate ultraviolet divergences.

Our approach here is *nonperturbative*, the light cone is fully quantum, the singularity is smeared out because of the nonzero commutators $[X, T$, or $P]$ or their quantum uncertainties, and a whole quantum region of finite curvature does appear: Thus, Ford perturbative proposal of smearing the light cone is fully realized in our approach nonperturbatively within a whole quantum space-time description (and quantum light cone). Our transverse spatial directions (or higher dimensions) to the lightcone are commutative but can quantum fluctuate.

Other points of comparison between our work here and Ref. [41] are the following:

(1) The line element in our case has in itself a quantum structure (group valued manifold) consequently giving rise to all the symmetries of both space-time and physically admissible states. The shift metric in Ref. [41] is a perturbation that makes the resulting line element remain as the standard one plus the shift.

Compared to our line element in the super-Minkowski case (e.g., unperturbed metric), let us notice that the shift looks like the part of our line element that contains the complex (γ) coefficients: the

B_1 part of our ds^2 (Sec. III .A), for instance Eq. (3.11), and Sec. IV.

From the point of view of the obtained solutions in both approaches: the same comparison is reflected in the Gaussian part of our solutions with the role of the complex parameters of the B1 part of our line element in the super-Minkowski case similar to the role of σ_1 in Eq. (10) of Ref. [41].

(2) The shift in the interval (distance between events) of Ref. [41] plays a similar role to the "fermionic" part (B_1) of the line element described by us, not only from the point of view of the line element, but from the point of view of the Gaussian part of our solutions and the Green functions constructed by Ford in Ref. [41]: both the square norm of the metric coefficients of the fermionic part of our line element and the shift σ_1 in the case of Ref. [41], locate the Gaussian function.

Nevertheless, from a conceptual point of view, they are different because in our case the coefficients are not perturbations.

(3) In our approach the solution states are states with a certain spin content, in particular the case of spin 2 that correspond to the graviton field, and does not have singularity nor dynamic problems as we saw throughout the work.

(4) We could conclude that the Ford approach could be introduced or combined in our proposal but the action of the perturbation is screened by the metric coefficients of the fermionic part of the line element, given that these are not perturbations. On the other hand, as a quantum result, the zitterbewegung does appear in our approach, which could approximately resemble to the fluctuations described in detail in Ford Ref. [41].

VIII. REMARKS AND CONCLUSIONS

We have presented a nonperturbative group theory approach to describe quantum space-time and its states with new results both for quantum theory in its own and quantum gravity.

The results here provide further support to, and are consistent with, the idea that a quantum theory of gravity must be *a finite theory*, (in the Wilson-Kadanoff sense), which is more than a renormalizable theory, as discussed in Ref. [10]. And that a ultimate quantum theory of gravity must be a theory of *pure numbers*.

(i) We constructed here coherent and squeezed states of quantum space-time, in its continuum and discrete representations, and for both, de Sitter and black hole space-times. They are naturally expressed in terms of the surface gravity.

(ii) We found that coherent and squeezed states of quantum space-time encompass the space-time behavior in the semiclassical and classical de Sitter and black hole regions, they also exhibit high quantum phase space-time oscillations and they account consistently the Planckian and trans-Planckian scales.

(iii) We found the coherent states for the quantum space-imaginary time instantons, black holes in particular, covering the whole *complete* manifold including the quantum central region, absent in the classical black hole instanton.

(iv) The metaplectic group, the *complete* covering of SL(2C), plays an important role in providing the phase space symmetry of the coherent states and the complete Hilbert space whose two irreducible sectors span both the *odd* and *even* states, and whose total (symmetric or antisymmetric) covering guarantee CPT symmetry and unitarity.

The results presented here confirm that classical–quantum duality extended to gravity is a key part to understand the quantum gravity physical magnitudes, in particular the mass, and the space-time structure: classical–quantum gravity duality through the Planck scale (the crossing scale).

(i) It is remarkable the power of coherent states in describing both continuum and discrete space-time, *even* in the Planckian and trans-Planckian scales:

(ii) The continuum coherent state eigenvalue $\alpha = 0$, (and the fundamental state $n = 0$ in the discrete representation), consistently correspond here to the Planck scale. Higher values of α consistently account for the smaller and sub-Planckian sizes and higher excitations in the quantum gravity domain.

(iii) We find a *maximum eigenvalue* α characterizing the coherent states due to the minimal nonzero quantum space radius because of the minimal quantum uncertainty $\Delta X \Delta T = \hbar/2$, in particular in the central and *regular* black hole quantum region.

(iv) In the quantum space-time description, there is no any space-time singularity as it must be. The consistent description of such quantum scales by coherent states does appear as a result of the space and time quantum uncertainties $(\Delta X \Delta T)$ and the classical–quantum gravity duality.

A new support to CPT and unitarity of the quantum gravity theory does appear here through the metaplectic group description. In particular, recall that in semiclassical gravity, QFT in curved space-times and its backreaction effects, the necessity of considering *complete* or CPT invariant states does appear in requiring unitarity of the theory, most investigated in the context of the identification of space-time ("IST"), [42–47]. See also [48] for a recent account without IST. We have not used any identification of space-time ("IST") here, but our results here, as those in [9,10,40], support CPT and IST in the full quantum theory.

In semiclassical gravity, the symmetric (or antisymmetric) QFT provides a CPT symmetry of the theory. In the euclidean (imaginary time) manifold, the different causally disconnected regions became automatically identified. The quantum instanton contains in addition the central regular constant curvature region of Planck scale radius not covered by the classical instanton. The coherent state instanton remarkably accounts for this quantum gravity feature and determines the radius being

$$\mathcal{R}_0(l_P, t_P)^2 = \frac{1}{\sqrt{\pi}}\left[\frac{1}{l_P} + \frac{l_P}{\hbar}\right],$$

l_P being the Planck length. The origin is flurried or smoothed within this constant and bounded curvature region.

The results of this paper confirm that the quantum de Sitter vacuum and the quantum interior region of the black hole are both of the same nature: totally regular without any curvature singularity and of constant curvature. These results provide too a quantum space-time support to the effective or phenomenological models, [49–51], describing the BH interior as a de Sitter core of bounded curvature and being totally regular.

The results of this paper are expected to provide new insights to explore the quantum space-time structure and its signals, being from black holes, the gravitational wave domaine and the high energy domain, or the de Sitter primordial phases (inflation and before inflation), cosmological structures and the late de Sitter cosmological vacuum (today dark energy), Refs. [52–61]. The classical–quantum gravity duality allows that signals in the quantum gravity (trans-Planckian) domaine do appear as low energy effects in the semiclassical/classical universe today.

Notice too that the quantum gravity regions, the black hole interiors for instance, are present in all black holes of *all* masses, including the most macroscopic and astrophysical black holes.

Interestingly, the results of this paper can also provide with the coherent states of quantum space-time a *quantum optics* of the space-time and its tests, or find analogous of it, in the wave packet type and laboratory experiments.

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