Renormalization, wavelets, and the Dirichlet-Shannon kernels

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In constructive quantum field theory (CQFT) it is customary to first regularize the theory at finite UV and IR cutoff. Then one first removes the UV cutoff using renormalization techniques applied to families of CQFTs labeled by finite UV resolutions and then takes the thermodynamic limit. Alternatively, one may try to work directly without IR cutoff. More recently, wavelets have been proposed to define the renormalization flow of CQFTs which is natural as they come accompanied with a multiresolution analysis. However, wavelets so far have been mostly studied in the noncompact case. Practically useful wavelets that display compact support and some degree of smoothness can be constructed on the real line using Fourier space techniques but explicit formulas as functions of position are rarely available. Compactly supported wavelets can be periodized by summing over period translates keeping orthogonality properties but still yielding to rather complicated expressions which generically lose their smoothness and position locality properties. It transpires that a direct approach to wavelets in the compact case is desirable. In this contribution we show that the Dirichlet-Shannon kernels serve as a natural scaling function to define generalized orthonormal wavelet bases on tori or copies of real lines, respectively. These generalized wavelets are smooth, are simple explicitly computable functions, display quasilocal properties close to the Haar wavelet, and have compact momentum support. Accordingly they have a built-in cutoff in both position and momentum, making them very useful for renormalization applications.

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I. INTRODUCTION

Examples of rigorously defined interacting Wightman quantum field theories in four-dimensional Minkowski space are still not available. As the usual perturbative approach is mathematically ill-defined (Haag's theorem) [1], the nonperturbative constructive QFT (CQFT) program was proposed [2] which had spectacular success in two and three spacetime dimensions [3]. CQFT has both a rigorous path integral (Euclidean) and Hamiltonian formulation which are connected by Osterwalder-Schrader reconstruction. The systematic construction of interacting QFT via the CQFT approach is to consider a family of theories labeled by a UV and IR cutoff. This could be a lattice spacing M^{-1} and a toroidal radius R, respectively. Following the Wilsonian approach to renormalization one first constructs a renormalization flow defined by integrating out degrees of freedom at higher scales $M' > M$ to define an effective theory at scales M. Fixed points of this flow define consistent continuum theories at finite IR cutoff R in the sense that the continuum theory which corresponds to infinite resolution $M \to \infty$ analyzed at resolution M coincides with the effective theory for the fixed point family. In a nontrivial second step one then tries to take the thermodynamic limit $R \to \infty$.

In this work we are mainly but not only concerned with the renormalization process at finite IR cutoff, thus we consider QFTs at fixed finite R. It is convenient to study the QFT compactified on a d-torus T^d, and after trivial rescalings of the coordinates we can restrict to the unit torus $T^d = [0,1)^d$. As renormalization for all d directions is then done independently, we can restrict the considerations that follow to $d = 1$ as far as the coarse-graining (or renormalization) maps acting on the "one particle Hilbert space" (denoted L below) of the QFT is concerned. Note that this does not imply that this factorization also happens for the coarse-graining maps of the QFT Hilbert space (denoted $\mathcal{H}$ below). While we explain the general context of these notions in a later section for completeness, in this paper we are just interested in the behavior of L under coarse graining.

To define renormalization, one has to specify what one means by "the theory at resolution M." As quantum fields are operator valued distributions, it is necessary to smear them with test functions; thus, one can introduce the finite

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resolution scale M by a suitable space of test functions V_M labeled by M. The label set $\mathcal{M}$ from which the labels M are taken is supposed to be equipped with some partial order $\leq$ so that $M \leq M'$ distinguishes between lower (M) and higher (M') resolution. Then for $M \leq M'$ the spaces V_M and $V_{M'}$ are supposed to be nested, i.e., $V_M \subset V_{M'}$, which means that the quantum field degrees of freedom at scale M can be written as functions of the quantum field degrees of freedom at scale M'. This enables one to integrate out the extra degrees of freedom smeared by the functions in $V_{M'} - V_M$ and thus defines a renormalization flow.

It is clear that the details of the flow depend on the choice of the spaces $\{V_M\}_{M\in\mathcal{M}}$. However, the possible fixed point theories should not because the degrees of freedom can be smeared with any test functions and thus give "cylindrically" consistent effective theories labeled by the respective choice of test functions in V_M. As the above nested structure suggests, a systematic and "economic" approach to a suitable choice of the spaces V_M are multi-resolution analyses (MRAs). Here the spaces V_M are constructed from a single "scaling" test function ϕ whose rescaling by M and translates provide an orthonormal basis of V_M with respect to an inner product on the space V of all test functions which thus carries a Hilbert space structure. The spaces V_M are, however, by construction not mutually orthogonal but rather nested. To provide an orthonormal basis of V one can thus construct a sequence of spaces inductively defined by $W_{M_0} \coloneqq V_{M_0}$ and W_{M_n} is the orthogonal complement of V_{M_n} in $V_{M_{n+1}}$. Here $n \mapsto M_n$ is a divergent, linearly ordered sequence in $\mathcal{M}$, i.e., $M_n < M_{n+1}$. This provides an orthogonal decomposition $V = \oplus_{n=-\infty}^{\infty} W_{M_n}$ in the noncompact case and $V = \oplus_{n=0}^{\infty} W_{M_n}$ in the compact case. In discrete wavelet theory one often uses the sequence of scales $M_n = 2^n M_0$, $M_0 = 1$. A "mother wavelet" ψ is now a very special test function, namely its rescaling by M_n and translates generate an orthonormal basis of W_n and thus in turn of all of V. Given certain conditions on the scaling function ϕ, the mother wavelet ψ can be constructed from ϕ by Fourier analysis.

Wavelet theory is an active research field of mathematics, mathematical physics and signal processing [4]. In contrast to the "plane wave" basis used in the Fourier transform, wavelets by construction also display some notion of position space locality. One distinguishes between discrete wavelets (with discrete labels) and continuous wavelets (with continuous labels). The ones that naturally fit into the renormalization language developed above are the discrete ones. Historically the first discrete wavelet was the Haar wavelet [5] on the real axis whose scaling function is a step function. It is the only wavelet on the real axis known to date, whose dependence on position x is known in closed form and which is of compact support. If one is content with only quasicompact support, then the Shannon wavelet [6] which decays only slowly at infinity is an option if one is interested in explicitly known position space dependence (the scaling function is basically the sinc function). If manifest compact support is more important and in addition some degree of smoothness is required (the Haar wavelet is not even continuous), then one is led to the Daubechies [7] and Meyer [8] wavelets. It is well known that there is no "Schwartz" wavelet on the real axis, i.e., a wavelet that belongs to the space of Schwartz functions (smooth of rapid decrease) [8].

These well-studied examples on the real axis generalize immediately to $\mathbb{R}^d$ using the tensor product. To obtain wavelets on compact spaces such as tori or spheres one can consider compactly supported children wavelets $\psi_{n,m}(x) = 2^{-n/2}\psi(2^n x - m), m, n \in \mathbb{Z}$ on the real axis and periodize them by $[\pi \cdot \psi]_{n,m}(x) = \sum_{l\in\mathbb{Z}} \psi_{n,m}(x-l)$ which confines n to $\mathbb{N}_0$ and $m = 0, 1, 2, \ldots, 2^n - 1$. This yields an orthonormal basis (ONB) of $L = L_2([0,1), dx)$ of periodic functions thanks to the compact support of ψ but there are several drawbacks:

(1) Even if ψ has compact support on $\mathbb{R}$, the support of $\pi \cdot \psi$ on [0, 1) may not even be quasilocal (i.e., the function is not peaked).
(2) The complicated coordinate expression of ψ propagates to $\pi \cdot \psi$.
(3) Generically, the periodized wavelet is again discontinuous.

It transpires that a more direct approach to wavelets on compact spaces such as intervals or circles is desirable which does not rely at all on the theory of the infinite line (see [9] for such real line based approaches which retain smoothness properties but are technically very involved). There are constructions available in the literature [10] that use wavelet bases of finite order; i.e., there is a maximal resolution $M_{\max}$ allowed. However, these are neither peaked in position nor do they span all of L_2; they just span $V_{M_{\max}}$.

We will understand MRA, scaling functions, and wavelets in a generalized sense, which is inspired by minimal requirements that these should satisfy for purposes of renormalization. These are the following:

(I) A nested sequence of subspaces $V_M \subset V_{M'}, M \leq M'$ whose span is dense in the Hilbert space L of test functions. This allows one to consider arbitrarily high resolutions and coarse grainings between different resolutions.
(II) A real valued orthonormal basis of test functions of V_M which are obtained from a fixed finite set of "mother" (scaling) functions ϕ by rescaling and translation. Real valuedness is important because these functions are used for discretizations of quantum fields in the CQFT approach, and we do not want to change their adjointness relations.
(III) An orthonormal basis of $W_{M'} = V_M^{\perp}$ which is the orthogonal complement of V_M in $V_{M'}, M \leq M'$ which is obtained from a fixed finite set of "mother" wavelet functions ψ, which are directly related to the

scaling functions ϕ, by rescaling and translation. The fact that the coarse-graining maps built from the ONB of the V_M or W_M are based on a few scaling functions or wavelets makes the renormalization procedure systematic, economic, and tractable.

(IV) The ONB should display at least peakedness in position space in order that it can be used for discretization of quantum fields on the lattice defined by the UV regulator.

In this paper we show that such a generalized MRA on $T^1 = S^1 = [0, 1)$ can be constructed based on the Dirichlet kernel [11] as a scaling function. It has the following features:

(1) Real valuedness.
(2) Smoothness.
(3) Compact momentum support.
(4) It is a simple trigonometric polynomial which can be explicitly summed to obtain a simple position space expression which is related to the Shannon scaling function.
(5) Peakedness (quasilocality) in position space.
(6) Its rescalings and translates generate an MRA.
(7) There are two associated mother wavelets whose rescalings and translates generate an ONB of $L_2([0, 1), dx)$.
(8) Being smooth and reflection symmetric, it has an infinite number of vanishing trigonometric moments (the moments must be defined using trigonometric rather than proper polynomials as the latter are not periodic).
(9) The wavelet basis can be considered as a smoothened version of the Haar wavelet on the torus with improved features for purposes of renormalization: not only field operators can be systematically discretized but also their derivatives (these are ill-defined in the discontinuous Haar case).

The locality features of this MRA is, of course, not surprising because it is well-known that the rescalings of the Dirichlet kernel provide smooth approximants of the periodic δ-distribution. However, to the best of our knowledge, the usefulness of the Dirichlet kernel for purposes of renormalization and its relations to MRAs on S^1 have not been highlighted before. In that respect, the purpose of the present paper is to assemble available knowledge about the analytic properties of the Dirichlet kernel together with MRA and renormalization framework. In tandem, we show that the Shannon kernel on the real line, which we study from the above generalized point of view, has very similar properties.

The architecture of this article is as follows:

In Sec. II, for the benefit of the unfamiliar reader we give a minimal account on MRAs and wavelets. This has the only purpose of preparing for the next section and will be far from complete.

In Sec. III we briefly recall what we mean by Hamiltonian renormalization in the language of [12] which has been applied and tested in [13] for free field theories without constraints and in [14] with constraints whose algebra is isomorphic to that of quantum gravity. We exhibit how coarse-graining or blocking maps that define renormalization flows are naturally generated by MRA structures. In particular, we show that the renormalization flow in the works [13] is simply based on the Haar scaling function, which the authors of [13] were not aware of. Rather, the blocking kernels used there were obtained by rather independent arguments, specifically lattice gauge theory technology [15]. For earlier uses of MRA structures in the CQFT program see, e.g., [16] and references therein. In [14] it became obvious that the renormalization flow should be driven by kernels that display at least a minimal amount of smoothness which therefore directly motivated the present work. The impact of the choice of kernel on the physical properties of the fixed point theory was emphasized before in [17].

In Sec. IV we define the Dirichlet and Shannon kernel, recall some of its analytical properties, and demonstrate how it generates a generalized MRA and an associated orthornormal mother wavelet pair. We highlight in what sense the corresponding blocking kernels can be considered as smooth versions of the Haar blocking kernel, which makes it well adapted to discretization of continuum QFT in the CQFT approach.

In Sec. V we showcase how the Dirichlet renormalization flow tremendously simplifies the Haar flow of [13] while not changing the fixed point theory. This is due to the translation invariance of both the Shannon and Dirichlet kernels that is not shared by the Haar kernel.

In Sec. VI we summarize and conclude.

II. GENERALIZED MULTIRESOLUTION ANALYSIS

We consider first the torus $X = T^1$ to define a generalized MRA and after that explain where the definition has to be modified for the real line $X = \mathbb{R}$.

We consider the torus T^1 as $\mathbb{R}/\mathbb{Z}$, i.e., as the interval [0, 1) with boundary points identified. By $L := L_2([0,1), dx)$ we denote the square integrable periodic functions on [0, 1). It has the ONB

$$e_n(x) := e^{2\pi i n x}, \qquad n \in \mathbb{Z}. \tag{2.1}$$

We consider a subset $\mathcal{M} \subset \mathbb{N}$, $1 \in \mathcal{M}$ equipped with a partial order $\leq$, i.e., an antisymmetric, reflexive, and transitive relation on $\mathcal{M}$ with respect to which it is also directed; i.e., for any $M_1, M_2 \in \mathcal{M}$ we find $M_3 \in \mathcal{M}$ such that $M_1, M_2 \leq M_3$. We require that for pairs $M \leq M' \in \mathcal{M}$ there is a scale factor number $s(M, M') \in \mathbb{N}$, and we define for $f \in L$, $s \in \mathbb{N}$ the dilatated function $(D_s f)(x) = f(sx)$ which is again one-periodic. Furthermore, for any $M \in \mathcal{M}$ we require that there exists a dimension number $d(M) \in \mathbb{N}$, and for any $f \in L$, $d > 0$ we define the translation $(T_d f)(x) = f(x - d)$.

Definition 2.1. I. A generalized MRA of $L = L_2([0,1), dx)$ subordinate to $\mathcal{M}$, s, d is an assignment $\mathcal{M} \ni M \mapsto V_M$ (principal translation invariant subspaces) where V_M is a closed, finite dimensional subspace of L of dimension $d(M)$ such that

(i) $V_1 = \mathbb{C}$.
(ii) If $M \le M'$, then $V_M \subset V_{M'}$.
(iii) $\cup_{M \in \mathcal{M}} V_M$ is dense in L.
(iv) If $f \in V_M$, $M \le M'$, then $D_{s(M,M')} f \in V_{M'}$.
(v) There exists a fixed finite set of scaling functions $\phi \in L$ such that an ONB χ_m^M of V_M is obtained as a fixed set of rational functions of their dilatations $D_{d(M)}\phi$ and translations $T^m_{1/d(M)}\phi, m = 0, 1, \ldots, d(M) - 1$ or combinations thereof.

II. A wavelet subordinate to a generalized MRA is a fixed finite set of functions $\psi \in L$ which are algebraic functions of the scaling functions ϕ such that a fixed set of rational functions of their dilatations $D_{d(M)}\psi$ and translations $T^m_{1/d(M)}\psi, m = 0, 1, \ldots, d(M) - 1$ or combinations thereof provides an ONB of W_M where $W_M = V_M^\perp$ is the orthogonal complement of V_M in $V_{M'(M)}$ and where $M'(M) \ge M$ is a fixed resolution higher than M.

A couple of remarks are in order:

(1) In the usual wavelet literature on the real line one considers mostly the set $\mathcal{M} = \{2^N, N \in \mathbb{Z}\}$ of integer powers of two with the usual linear order $\le$ on real numbers. The reason why in our case positive powers are sufficient is that negative powers would produce a lattice spacing larger than the lattice itself and thus maps us out of the space of one-periodic functions.
(2) The reason why we consider more general partial orders is because we allow more positive integers than positive powers of two, and we wish that for $M \le M'$ the lattice defined by the points $m/d(M), m = 0, 1, \ldots, d(M) - 1$ is a sublattice of the lattice defined by the points $m'/d(M')$, $m' = 0, 1, \ldots, d(M') - 1$.
(3) On the real line the spaces V_M are all infinite dimensional, i.e., $d(2^N) = \infty$ for all $N \in \mathbb{N}$, and instead of $V_1 = \mathbb{C}$ we have $\cap_N V_{2^N} = \{0\}$.
(4) On the real line the function one usually restricts s to $2^N, 2^{N+k}$ in which case it takes the value $s = 2^k$.
(5) On the real line the functions

$$\chi_m^N(x) := 2^{N/2}\phi(2^N x - m), \qquad m \in \mathbb{Z}, \tag{2.2}$$

are an orthonormal basis for V_{2^N} if χ_m^0 is an ONB of V_{2^0}. We had to modify this for two reasons: First, the space V_{2^0} is only one-dimensional on S^1 while infinite dimensional on $\mathbb{R}$ and thus cannot serve to build a basis for the higher dimensional spaces V_M. Second, the integer shifts of an one-periodic function are trivial. Therefore we disentangled the simultaneous rescaling and shifting performed on the function ϕ in (2.2) and allowed pure dilatations and pure shifts or combinations thereof in order to assemble χ_m^M as a rational function (i.e., a fraction of polynomials) of those. In the standard case (2.2) we only need one such function ϕ, and the rational aggregate formed from it is just the function itself multiplied by a constant. While our more general construction of the χ_m^M is more complicated than in the standard case, it keeps the spirit of building the basis χ_m^M of "children" functions from a few "mother" functions ϕ. We restrict to rational functions in order to keep the expressions involved manageable and because in the examples we have constructed, rational functions appear to be sufficient.
(6) On the real line the relation between scaling function ϕ and wavelet ψ is less direct: it starts with a function m_0 in Fourier space subject to a support (Cohen's) condition and a normalization condition on its modulus squared. Then one defines the Fourier transform of ϕ as an infinite product of dilatations of m_0, and the Fourier transform of ψ is a product of m_0, the Fourier transform of ϕ, and a phase factor depending on momentum. Only in rare cases can one solve the Fourier integral in closed form to obtain an explicit position space expression. Our definition is again motivated by the essential idea that the wavelet basis should arise from a few mother wavelets that are computable by a concrete formula from the scaling functions. In contrast to the noncompact case we do not provide a procedure for how to obtain ψ from ϕ but we allow for more complicated (algebraic rather than rational) relations between those, again because also in the standard case the relation is more complicated and because in the examples we have constructed an algebraic relation appears to be sufficient. The algebraic rather than rational functions issue can, however, be avoided if one increases the number of mother wavelets. Thus, this difference is not essential.

We now spell out how the definition needs to be or can be modified in the noncompact case $X = \mathbb{R}^1$. First of all, the label set $\mathcal{M}$ can be generalized to a subset of the positive rationals $\mathbb{Q}_+$ equipped with some partial order with respect to which it is directed. For $M \le M'$ the number $s(M, M')$ is still required to be a positive natural. The function $d(M)$ is supposed to take values in $\mathbb{Q}_+$ and no longer has the interpretation of a dimension.

Definition 2.2. A generalized MRA and wavelet of $L = L_2(\mathbb{R}, dx)$ subordinate to $\mathcal{M}$, s, d is identical to the generalized MRA and wavelet for the case $L = L_2([0,1), dx)$ with the following modifications:

I.1: $\cap_{M \in \mathcal{M}} = \{0\}$.

I.v and II.: The translations $T^m_{1/d(M)}$ are now labeled by $m \in \mathbb{Z}$ and are not confined to $0, \ldots, d(M)$.

It will be helpful to test the definition against a well-known example which can be used in both the noncompact and the compact cases. This is the Haar wavelet. Its mother scaling function is given by

$$\phi(x) := \chi_{[0,1)}(x), \tag{2.3}$$

where $\chi_{[a,b)}$ denotes the characteristic function of the clopen (i.e. left closed, right open) interval $[a, b)$.

Consider first the noncompact case. Then the functions $\chi^M_m(x)$ are for the set $\mathcal{M} = \{2^N, N \in \mathbb{Z}\}$ with $M := 2^N$ and $m \in \mathbb{Z}$,

$$\chi^M_m(x) := M^{1/2}\chi_{[0,1)}(Mx - m), \tag{2.4}$$

which have support in $[x^M_m, x^M_{m+1}), x^M_m := \frac{m}{M}$, and $d(M) = M$. They are indeed just simple dilatations of translations of the mother scaling functions. As these partition the real line into intervals of length M^{-1}, they are orthogonal at fixed M with respect to the standard inner product on $L = L_2(\mathbb{R}, dx)$,

$$\langle \chi^M_m, \chi^M_{m'} \rangle = \delta_{m,m'}. \tag{2.5}$$

Their span at fixed M defines a dense subset of the subspace V_M of L. As

$$\chi^M_m = 2^{-1/2}[\chi^{2M}_{2m} + \chi^{2M}_{2m+1}] \tag{2.6}$$

obviously $V_M \subset V_{2M}$. That $\cup_M V_M$ is dense in L follows, for instance, from the way the Lebesgue measure is constructed as a Borel measure. Thus, indeed, we obtain an MRA on the real line using (2.3). The orthogonal decomposition $V_{2M} = V_M \oplus W_M$ can be done by direct methods in this case: Obviously, we have to assemble an ONB of W_M from the ONB of V_{2M} because W_M is a subspace thereof and every basis function of W_M must be orthogonal to each of the basis functions of V_M. In view of (2.6) this leads to the natural choice

$$\begin{aligned}\psi^M_m(x) &= 2^{-1/2}[\chi^{2M}_{2m}(x) - \chi^{2M}_{2m+1}(x)] \\ &= (2M)^{1/2}\psi(2Mx - 2m),\end{aligned} \tag{2.7}$$

where

$$\psi(x) = 2^{-1/2}[\chi_{[0,1)}(x) - \chi_{[0,1)}(x-1)] \tag{2.8}$$

is the corresponding mother wavelet. It is a linear (and therefore rational) aggregate of mother scaling functions. Finally, as the intersection of the spaces $V_M, M = 2^N$, $N \geq N_0$ coincides with the space of square integrable functions that can be expanded in the basis χ^M_m which are piecewise constant on intervals of length M^{-1}, we see that for $N_0 \to \infty$ only $\{0\}$ results as there is no non-vanishing, square integrable, constant function on $\mathbb{R}$. It follows

$$\langle \psi^M_m, \psi^{M'}_{m'} \rangle = \delta^{M,M'}\delta_{m,m'}, \tag{2.9}$$

i.e., the Haar wavelets form an orthonormal basis of L.

Now consider the compact case. First of all, $\chi_{[0,1)}(x) \equiv 1$ for $x \in [0, 1) = T^1$ so indeed $V_1 = \mathbb{C}$. Otherwise we may use the same functions (2.6) but with $x \in [0, 1)$ and m restricted to $0, 1, \ldots, M-1$ to define an M-dimensional subspace V_M of $L = L_2([0, 1), dx)$ consisting of mutually orthonormal, periodic functions on [0, 1) exploiting the fact that $\chi^M_m(x)$ drops to zero outside of its support. From here on the construction of ψ follows exactly the same steps as in the noncompact case, just obeying the respective finite ranges $x \in [0, 1)$ and $m \in \{0, 1, \ldots, M-1\}$ and $M = 2^N \geq 1$ for ψ^M_m. In particular, we see that the example of the Haar wavelet fits into the definition (2.6) for an MRA on the torus with $d(M) = M = 2^N$, $s(M, 2^k M) = 2^k$, and $M'(M) = 2$.

III. HAMILTONIAN RENORMALIZATION

In this section we will combine the renormalization technology from [12] with the MRA framework developed in the previous section.

Again it will be sufficient to consider one coordinate direction as T^d and $\mathbb{R}^d$ are Cartesian products. Thus for simplicity we consider a bosonic, scalar quantum field Φ (operator valued distribution) with conjugate momentum Π on [0, 1) with canonical commutation and adjointness relations (in natural units $\hbar = 1$)

$$[\Pi(x), \Phi(y)] = i\delta(x, y), \quad \Phi(x)^* = \Phi(x), \quad \Pi^*(x) = \Pi(x), \tag{3.1}$$

where

$$\delta(x, y) = \sum_{n \in \mathbb{Z}} e_n(x)e_n(y)^*, \qquad e_n(x) = e^{2\pi i n x} \tag{3.2}$$

is the periodic δ distribution on the torus or the standard δ distribution on the real line, respectively. It is customary to work with the bounded Weyl operators for $X = [0, 1)$ or $X = \mathbb{R}$,

$$w[f, g] = \exp(i[\Phi(f) + \Pi(g)]), \qquad \Phi(f) = \int_X dx\, f(x)\Phi(x),$$
$$\Pi(g) = \int_X dx\, g(x)\Pi(x) \tag{3.3}$$

with $f, g \in L = L_2(X, dx)$ test functions or smearing functions usually with some additional properties such as differentiability or even smoothness. For more complicated tensor fields or spinor fields a similar procedure can be followed (see, e.g., the last two references in [13]).

Since the space L enters the stage naturally we use MRA ideas to define a renormalization group flow. Suppose that $M \mapsto V_M \subset$ with $M \in \mathcal{M}$ defines an MRA with orthogonal basis $\chi^M_m, m \in \mathbb{Z}_M = \{0, \ldots, d(M) - 1\}$ for $X = [0, 1)$ and $\mathbb{Z}_M = \mathbb{Z}$ for $X = \mathbb{R}$ of V_M normalized according to $\|\chi^M_m\|^2 = d(M)^{-1}$. The reason why we do not normalize the χ^M_m (we could) will become obvious in a moment. In tandem with V_M we define the space $L_M \coloneqq l_2(\mathbb{Z}_M)$ of square integrable sequences with $d(M)$ entries and with inner product

$$\langle f_M, g_M \rangle \coloneqq d(M)^{-1} \sum_{m \in \mathbb{Z}_M} f^*_M(m) g_M(m). \tag{3.4}$$

This scalar product offers the interpretation of $f_M(m) \coloneqq f(x^M_m), x^M_m \coloneqq \frac{m}{d(M)}$, and similar for g_M as the discretized values of some functions $f, g \in L$ in which case (3.5) is the Riemann sum approximant of $\langle f, g \rangle_L$. It is for this reason that we did not normalize the χ^M_m.

The spaces V_M and L_M are in bijection via

$$I_M\colon L_M \to L, f_M \mapsto \sum_m f_M(m) \chi^M_m. \tag{3.5}$$

Note that (3.5) has range in $V_M \subset L$ only. Its adjoint $I^\dagger_M\colon L \to L_M$ is defined by

$$\langle I^\dagger_M f, f_M \rangle_{L_M} \coloneqq \langle f, I_M f_M \rangle_L \tag{3.6}$$

so that

$$(I^\dagger_M f)(m) = d(M) \langle \chi^M_m, f \rangle_L. \tag{3.7}$$

Clearly

$$(I^\dagger_M I_M f_M)(m) = d(M) \langle \chi^M_m, I_M f_M \rangle_L = f_M(m), \tag{3.8}$$

i.e., $I^\dagger_M I_M = 1_{L_M}$, while

$$(I_M I^\dagger_M f)(x) = d(M) \sum_m \chi^M_m(x) \langle \chi^M_m, f_M \rangle_L = (p_M f)(x) \tag{3.9}$$

is the projection $p_M\colon L \mapsto V_M$.

Given $M \leq M'$ we define the coarse-graining map

$$I_{MM'} \coloneqq I^\dagger_{M'} I_M\colon L_M \mapsto L_{M'}. \tag{3.10}$$

It obeys

$$I_{M'} I_{MM'} = p_{M'} I_M = I_M \tag{3.11}$$

because I_M has range in $V_M \subset V_{M'}$ for $M \leq M'$. This is the place where the MRA property of the nested set of subspaces V_M was important. Next for $M_1 \leq M_2 \leq M_3$ we have

$$I_{M_2 M_3} I_{M_1 M_2} = I^\dagger_{M_3} p_{M_2} I_{M_1} = I^\dagger_{M_3} I_{M_1} = I_{M_1 M_3} \tag{3.12}$$

for the same reason. This is called the condition of cylindrical consistency, which is crucial for the renormalization group flow.

To see the importance of (3.12) we consider a probability measure ν on the space $\mathcal{F}$ of field configurations Φ, which defines a Hilbert space $\mathcal{H} = L_2(\mathcal{F}, d\nu)$ and a representations space for the Weyl algebra $\mathfrak{A}$ generated from the Weyl elements (3.3). We set $w[f] \coloneqq w[f, g = 0]$ and define the generating functional of moments of ν by

$$\nu(f) \coloneqq \nu(w[f]) \coloneqq \int_{\mathcal{F}} d\nu(\Phi) \exp(i\Phi(f)). \tag{3.13}$$

If we restrict f to V_M, we obtain an effective measure on the space of discretized quantum fields $\Phi_M = I^\dagger_M \Phi$ via

$$w[I_M f_M] = w_M[f_M] = e^{i\Phi_M(f_M)}, \quad \Phi_M(f_M) = \langle f_M, \Phi_M \rangle_{L_M} \tag{3.14}$$

and measures

$$\nu_M(f_M) \coloneqq \nu(w[I_M f_M]) = \nu_M(w_M[f_M]). \tag{3.15}$$

The measures ν_M with support on the spaces $\mathcal{F}_M$ of fields Φ_M are consistently defined by construction

$$\nu_{M'}(I_{MM'} f_M) = \nu_M(f_M) \tag{3.16}$$

for any $M \leq M'$ since the ν_M descend from a continuum measure. Conversely, given a family of measures ν_M satisfying (3.16), a continuum measure ν can be constructed known as the projective limit of the ν_M under mild technical assumptions [18]. To see the importance of (3.12) for this to be the case, suppose we write $f \in L$ in two equivalent ways $f = I_{M_1} f_M = I_{M_2} g_{M_2}$, and then we should have $\nu_{M_1}(f_{M_1}) = \nu_{M_2}(g_{M_2})$. Now while M_1 and M_2 may not be in relation, as $\mathcal{M}$ is directed we find $M_1, M_2 \leq M_3$. Applying $I^\dagger_{M_3}$ we conclude $I_{M_1 M_3} f_{M_1} = I_{M_2 M_3} g_{M_2}$; thus, due to (3.16), indeed,

$$\nu_{M_1}(f_1) = \nu_{M_3}(I_{M_1 M_3} f_{M_1}) = \nu_{M_3}(I_{M_2 M_3} g_{M_2}) = \nu_{M_2}(g_{M_2}). \tag{3.17}$$

In CQFT the task is to construct a representation of $\mathfrak{A}$ with additional properties such as allowing for the implementation of a Hamiltonian operator $H = H[\Phi, \Pi]$ which imposes severe restrictions on the Hilbert space representation. One may start with discretized Hamiltonians (H is the classical Hamiltonian function)

$$H^{(0)}_M[\Phi_M, \Pi_M] := H[p_M\Phi, p_M\Pi] \tag{3.18}$$

on $\mathcal{H}^{(0)}_M := L_2(\mathcal{F}_M, \nu^{(0)}_M)$ where $\nu^{(0)}_M$ is any probability measure to begin with, for instance, a Gaussian measure or a measure constructed from the ground state $\Omega^{(0)}_M$ of the Hamiltonian $H^{(0)}_M$. The definition (3.18) is incomplete without some ordering prescription, and we assume that such a prescription has been chosen.

The point of using an IR cutoff, that is, the compact space $X = [0, 1)$, is that there are only finitely many, namely $d(M)$ degrees of freedom Φ_M and Π_M, which are conjugate

$$[\Pi_M(m), \Phi(m')] = id(M)\delta(m, m'), \qquad \Phi_M(m)^* = \Phi_M(m), \qquad \Pi^*_M(m) = \Pi_M(m), \tag{3.19}$$

so that the construction of $\nu^{(0)}_M$ does not pose any problems. For $X = \mathbb{R}$ (3.19) still holds, but now the existence of $\nu^{(0)}_M$ is not granted and requires further analysis. Assuming this one fixes for each $M \in \mathcal{M}$ an element $M \leq M'(M) \in \mathcal{M}$ and defines isometric injections

$$J^{(n)}_{MM'(M)}\colon \mathcal{H}^{(n+1)}_M \to \mathcal{H}^{(n)}_{M'(M)}, \qquad J^{(n)}_{MM'(M)} w_M[f_M]\Omega^{(n+1)}_M := w_{M'(M)}[I_{MM'(M)} f_M]\Omega^{(n)}_{M'(M)},$$
$$\nu^{(n)}_M(w_M(f_M)) =: \langle \Omega^{(n)}_M, w_M(f_M)\Omega^{(n)}_M\rangle_{\mathcal{H}^{(n)}}, \qquad {}^{(n)}_M := L_2(\mathcal{F}_M, d\nu^{(n)}_M) \tag{3.20}$$

via

$$\nu^{(n+1)}_M(f_M) := \nu^{(n)}_{M'(M)}(I_{MM'(M)} f_M), \tag{3.21}$$

and with these the flow of Hamiltonians

$$H^{(n+1)}_M := J^\dagger_{MM'(M)} H^{(n)}_{M'(M)} J_{MM'(M).} \tag{3.22}$$

The isometry of the injections relies on the assumption that the span of the $w_M[f_M]\Omega^{(0)}_M$ is dense in $\mathcal{H}^{(0)}_M$, which is typically the case. Here we have written the measures ν_M interpreted as positive linear functionals on the Abelian C^*-algebra $\mathfrak{A}_M$ generated by the $w_M[f_M]$ as expectation value functionals with respect to cyclic vectors $\Omega^{(n)}_M$ in $\mathcal{H}^{(n)}_M$ following the Gel'fand, Naimark, Segal (GNS) construction [1]. These details will not concern us for what follows.

This defines a sequence or flow (indexed by n) of families (indexed by M) of theories $(\mathcal{H}^{(n)}_M, H^{(n)}_M, \Omega^{(n)}_M)$. At a critical or fixed point of this flow the consistency condition (3.16) is satisfied [at first in the linearly ordered sets of $\mathcal{M}(M) := \{(M')^N(M), N \in \mathbb{N}_0\}$ and then usually for all of $\mathcal{M}$ by universality], and one obtains a consistent family $(\mathcal{H}_M, H_M, \Omega_M)$. This family defines a continuum theory $(\mathcal{H}, H)$ as one obtains inductive limit isometric injections $J_M\colon \mathcal{H}_M \mapsto \mathcal{H}$ such that $J_{M'}J_{MM'} = J_M, M \leq M'$ thanks to the fixed point identity $J_{M_2M_3}J_{M_1M_2} = J_{M_1M_3}$, $M_1 \leq M_2 \leq M_3$ and such that

$$H_M = J^\dagger_M H J_M \tag{3.23}$$

is a consistent family of quadratic forms $H_M = J^\dagger_{MM'} H_{M'} J_{MM'}, M \leq M'$.

The conclusion of the present section is that the MRA framework fits quite naturally with the construction of the Hamiltonian renormalization flow. All that is needed is, in fact, the nested structure of the V_M; it is strictly speaking not necessary to have mother scaling functions or mother wavelets. However, to reduce the arbitrariness in the nested structure or choice of coarse-graining maps, the requirement that the nesting descends from a (finite number of) scaling function(s) is very useful. In fact, in [13] the authors used, without being aware of it, the MRA based on the Haar scaling function. While this works, it makes the formalism unnecessarily complicated because the Haar scaling function is not even continuous and thus the discretization prescription (3.18) is ill-defined as it stands as soon as H depends on derivatives of Φ and Π which is typically the case. Thus in [13] one had to use an additional prescription to define those discrete derivatives, which increases the discretization ambiguity that one actually wants to avoid. It is for this reason that we try to base an MRA on scaling functions with additional smoothness properties while keeping sufficient spatial locality such that formula (3.18) provides a suitable spatial discretization. We will show in the next section that one possibility is based on the Shannon and Dirichlet kernels.

IV. RESOLUTIONS OF THE IDENTITY MRAS AND THE SHANNON-DIRICHLET KERNELS

In the previous section we have shown that a nested structure of subspaces $V_M \subset L$ with $V_M \subset V_{M'}, M \leq M'$, $M, M' \in \mathcal{M} \subset \mathbb{N}$ whose span is dense in L leads to a useful renormalization flow in CQFT for any choice of orthonormal basis $\chi^M_m, m = 0, 1, \ldots, d(M) = \dim(V_M)$. This uses only part of the definition of an MRA: It was not specified how that basis of V_M is to be generated; in particular, it was not required that the χ_m^M descend from one or several fixed mother scaling functions. To systemize this choice the concept of mother functions and therefore the full definition of an MRA appear natural.

A. Torus

We begin with the compact case. First of all, one may pick an ONB e_n of L which for the sake of definiteness we label by $n \in \mathbb{Z}$ (if one prefers $n \in \mathbb{N}_0$ set $b_{2n} := [e_n + e_{-n}]/\sqrt{2}; n \geq 0, b_{2n+1} := [e_n - e_{-n}]/\sqrt{2}, n > 0$). This could be the eigenbasis of a self-adjoint operator on L with a pure point spectrum. For reasons explained in Sec. III, we want the e_n to have at least some degree of differentiability. Then for any odd integer M we may consider the $d(M) = M$ dimensional subspaces V_M of L spanned by the functions $e_n, |n| \leq (M-1)/2$. Picking $\mathcal{M} \subset \mathbb{N}$ as the odd naturals equipped with the usual ordering relation $\leq$ on the naturals, one obtains trivially a nested structure of Hilbert spaces.

However, this is still too general and not useful for our renormalization intentions. This is because the $e_n(x)$ typically fail to be localized with respect to x, because the spectral label n has in general nothing to do with the points $x_m^M = \frac{m}{M}$ of the lattice of [0, 1) at which we wish to localize and discretize our quantum fields Φ and Π. Thus we need to connect the label n to the lattice label m in such a way that the resulting orthogonal basis functions χ_m^M display some form of peakedness in position space around the points x_m^M.

To do this, we use the following notation: Let for M odd $\mathbb{Z}_M := \{0, 1, 2, \ldots, M-1\}$ and $\hat{\mathbb{Z}}_M := \{-\frac{M-1}{2}, -\frac{M-1}{2} + 1, \ldots, \frac{M-1}{2}\}$ and $d(M) := M$. Pick any unitary $M \times M$ matrix with entries $e_n^M(m), n \in \hat{\mathbb{Z}}_M, m \in \mathbb{Z}_M$, and consider

$$\chi_m^M(x) := M^{-1/2} \sum_{n \in \hat{\mathbb{Z}}_M} e_n(x)[e_n^M(m)]^*. \tag{4.1}$$

Then by construction (i.e., unitarity)

$$\langle \chi_m^M, \chi_{m'}^M \rangle_L = M^{-1} \delta_{m,m'}. \tag{4.2}$$

The question now arises whether it is possible to pick that unitary matrix in such a way that $\chi_m^M(x)$: 1. is real valued (so that they can be used to define Weyl operators); 2. is localized around x_m^M; and 3. such that the χ_m^M descend from some mother scaling functions ϕ in the generalized sense of Sec. II.

We will not give an exhaustive answer about the maximal freedom there is in doing so but rather show that there is at least one example that satisfies all three criteria. Moreover, the resulting χ_m^M will not only be smooth (thus, its Fourier coefficients decay rapidly at infinity) but even trigonometric polynomials (i.e., finite linear combinations of the eigenbasis of the momentum operator $-id/dx$ on L; hence, the Fourier coefficients are of compact support).

We pick for $m \in \mathbb{Z}_M, n \in \hat{\mathbb{Z}}_M$

$$e_n(x) := e^{2\pi i n x}, \quad e_n^M(m) := e_n(x_m^M), \quad x_m^M := \frac{m}{M}. \tag{4.3}$$

Then, indeed,

$$\frac{1}{M} \sum_{m \in \mathbb{Z}_M} e_n^M(m)[e_{n'}^M(m)]^* = \delta_{n,n'}, \qquad \sum_{n \in \hat{\mathbb{Z}}_M} e_n^M(m)[e_n^M(m')]^* = M\delta(m, m'), \tag{4.4}$$

and we have, using $e_n^* = e_{-n}, e_n e_{n'} = e_{n+n'}$,

$$\chi_m^M(x) = M^{-1} \sum_{n \in \hat{\mathbb{Z}}_M} e_n(x)[e_n^M(m)]^* = \sum_{|n| \leq \frac{M-1}{2}} e_n(x - x_m^M) \tag{4.5}$$

from which the real valuedness of χ_m^M is manifest. Also, clearly χ_m^M is smooth being a trigonometric polynomial of order $(M-1)/2$ and thus has compact momentum support rather than having only a rapid momentum decrease. Furthermore, the geometric series (4.5) can be explicitly summed to yield the explicit expression

$$M\chi_m^M(x) = \frac{\sin(\pi M[x - x_m^M])}{\sin(\pi[x - x_m^M])}, \tag{4.6}$$

which is a rational function of dilatations and translations of the sin function. Thus, if we define the *mother scaling* function to be

$$\phi(x) := \sin(\pi x), \tag{4.7}$$

then

$$M\chi_m^M(x) = \frac{[D_M T_{1/M}^m \phi](x)}{[T_{1/M}^m \phi](x)}, \tag{4.8}$$

which is precisely of the form required in Definition 2.1 if we remember that $d(M) = M$. To complete the definition we must decide on the choice of $\mathcal{M}$ and its partial order. We pick $\mathcal{M}$ to be the odd naturals and define $M \leq M'$ iff $\frac{M'}{M}$ is a (necessarily odd) integer. This partial order is motivated by the requirement that the lattice labeled by M should be a sublattice of the lattice labeled by M'. With this partial order, $\mathcal{M}$ is directed as given M_1, M_2, and we may pick $M_3 = M_1 M_2$ (or more economically the smallest common multiple) to achieve $M_1, M_2 \leq M_3$.

Finally, we note that $M\chi_m^M(x) = \delta_M(x - y), y = x_m^M$ is the restriction to our lattice points x_m^M of the *Dirichlet kernel*

$$\delta_M(x-y) = \sum_{|n|\leq\frac{M-1}{2}} e_n(x)[e_n(y)]^*, \tag{4.9}$$

which is an approximant to the δ-distribution on [0, 1) cut off at momentum $(M-1)/2$. This makes it plausible to be strongly peaked at $x = x_m^M$ as M grows large. To investigate this, we perform some elementary analysis on the function δ_M. Having period 1 and being symmetric around $x = 0$ it will be sufficient to investigate the interval $x \in [0, \frac{1}{2})$ of the function

$$\delta_M(x) = \frac{\sin(\pi M x)}{\sin(\pi x)}, \tag{4.10}$$

whose denominator vanishes only at $x = 0$ in $[-1/2, 1/2)$. However, δ_M is smooth at $x = 0$ with $\delta_M(0) = M$. Close to $x = 0$ it becomes the sinc function $M\mathrm{sinc}(\pi M x)$, which is the scaling function of the Shannon wavelet on the real line. Besides $x = 0$ the numerator vanishes at the zeros $z_m^M = \frac{m}{M}, m = 1, \ldots, \frac{M-1}{2}$. To compute its extrema between those zeros we take the derivative

$$[\delta_M]'(x) = \frac{\pi}{\sin^2(\pi x)}[M\cos(\pi M x)\sin(\pi x) - \sin(\pi M x)\cos(\pi x)]. \tag{4.11}$$

It vanishes at $x = 0$ as the numerator $\propto x^3$ while the denominator $\propto x^2$ there. It also vanishes at $x = 1/2$ due to the cosines and because M is odd. For $0 < x < 1/2$ both $\cos(\pi x)$ and $\sin(\pi x)$ are nonvanishing, and since a zero of $\cos(\pi M x)$ is an extremum of $\sin(\pi M x)$, the vanishing of (4.11) for $0 < x < 1/2$ yields the transcendental equation

$$\tan(My) = M\tan(y); \qquad 0 < y = \pi x < \frac{\pi}{2}. \tag{4.12}$$

This equation has $1 + \frac{M-1}{2}$ solutions $y_m^M, m = 0, \ldots, \frac{M-1}{2}$ with $0 = y_0^M < z_1^M < y_2^M < \cdots < z_{[M-1]/2}^M < y_{[M-1]/2}^m < \frac{1}{2}$. To see this, note that the right-hand side is positive, strictly monotonously increases, and diverges at $y = \frac{\pi}{2}+$. The left-hand side runs through one positive fundamental branch of the tan function between $y = 0, \pi/(2M)$, and $(M-1)/2$ full (negative and positive) fundamental branches between $y = (2k-1)/(2M)\pi, (2k+1)/(2M)\pi, k = 1, \ldots, (M-1)/2$. Since $\tan(My)$ is strictly monotonously increasing but at a faster rate than $M\tan(y)$ in each of those full branches, we get one solution. The solution $y_k^M, k = 1, \ldots, \frac{M-1}{2}$ lies very close to $\frac{2k+1}{2M}\pi$, the larger k [because $\tan(y)$ is monotonously increasing] and the larger M (since $[\tan(My) - M\tan(y)]' = M[\tan^2(My) - \tan^2(y)] > 0$ for $y > 0$ we have $\tan(My) > M\tan(y)$ for $0 < y < \pi/(2M)$ and $d/dM[\tan(My) - M\tan(y) = 1 + \tan^2(My) - \tan(y) > 1 + \tan^2(y) - \tan(y) = [\tan(y) + 1/2]^2 + 3/4 > 0)$. Therefore we can construct them iteratively by setting $y_k^M =: \frac{2k+1}{2M}\pi - \Delta_k^M$ with $\Delta_k^M \leq \frac{\pi}{M}$ and writing (4.12) as

$$\tan(M\Delta_k^M) = \frac{1}{M}\cot(y_k^M), \tag{4.13}$$

which grants that $\Delta_k^M < \pi/(2M)$. We can solve (4.13) by reinserting it into itself. To lowest order in $1/M$,

$$\Delta_k^M = \frac{1}{M}\arctan\left(\frac{\cot((2k+1)/(2M)\pi}{M}\right). \tag{4.14}$$

The value of δ_M at y_k^M can be seen from

$$[\delta_M(x = y_k^M/\pi)]^2 = \frac{M^2}{1 + [M^2 - 1]\sin^2(y_k^M)}. \tag{4.15}$$

For $k = 0$ we get $\delta_M^2(0) = M^2$, while for $k > 0$ we get $\delta_M^2(y_k^M/\pi) = O(1)$. Thus the maximum at $x = 0$ exceeds the other extrema by at least an order of M.

Accordingly one can visualize δ_M roughly as a smoothened version of a symmetric triangle of height M and width $2/M$ between the first zeros $z_1^M = \pm 1/M$ of δ_M. Outside that interval, which has a volume smaller than 1, the function is relatively bounded as $O(1/M)$ compared to its maximum. Thus the central triangle has area $O(1)$. If we compare to the Haar scaling function $M\chi_m^M(x) = M\chi_{[-\frac{1}{2M}, \frac{1}{2M})}(x - m)$, we see that basically the rectangle of height M and width $1/M$ has been replaced by that triangle, except for subdominant contributions to the triangle that are the price to pay for having a smooth kernel.

That price, however, is well worth paying for: We want to use δ_M to discretize functions f on the lattice $x_m^M = \frac{m}{M}$ by

$$f_M(m) \coloneqq [I_M^\dagger f](m) = M\langle\chi_m^M, f\rangle L. \tag{4.16}$$

This formula is well-defined for both the Haar and the Dirichlet scaling functions. However, what about derivatives? Using the same formula we would get $[f']_M(m) = \langle\chi_m^M, f'\rangle$, which is still well-defined but one would like to relate this to some sort of discrete derivative of f_M. In the case of the Haar scaling function one can do the integral and obtains $[f']_M(m) = M[f(x_{m+1}^M) - f(x_m^M)]$, which is a possible definition of the discrete derivative; however, the function values $f(x_m^M)$ are *not linearly* related to the values $f_M(m)$ in (4.16). In particular, if we replace f by the quantum field, this definition of derivative would map us out of the space of already discretized fields $\Phi_M(m)$.

By contrast, in the case of the Dirichlet kernel, as the χ_m^M are smooth we can integrate by parts (no boundary terms occur because all functions involved are periodic) to obtain

$$[f']_M(m) = -\langle [\chi_m^M]', f\rangle_L. \tag{4.17}$$

The functions χ_m^M are in V_M which is spanned by the $e_n, |n| \le (M-1)/2$. As $e_n' = 2\pi i n e_n$ the functions $[\chi_m^M]'$ are still in V_M and thus can be expressed as linear combinations of the $\chi_{m'}^M$. It follows that (4.17) defines a linear map on the sequence $m \mapsto f_M(m) = \langle \chi_m^M, f\rangle_L$,

$$[\partial_M f_M](m) \coloneqq [f']_M(m) = \sum_{\tilde{m}\in\mathbb{Z}_M} \partial_M(m,\tilde{m}) f_M(\tilde{m}). \tag{4.18}$$

Without working it out explicitly, we can already determine the dominant contribution of the matrix ∂_M: Since χ_m^M is steepest of inclination $\pm M$ close to $x_{m\mp 1}^M$, we know without further calculation that $\partial_M(m,\tilde{m})$ will be approximated by $cM[\delta_{\tilde{m},m+1} - \delta_{\tilde{m},m-1}]$ where c is a numerical constant of order unity. By construction, this is an antisymmetric matrix as being related to the derivative of a symmetric kernel.

Accordingly, the formula (4.16) can be *universally* used to discretize fields, their momenta, and their arbitrarily high derivatives as they appear in the classical Hamiltonian without introducing extra structure, thereby downsizing the discretization ambiguities. All of the renormalization program can therefore be based on a *single* input, namely the MRA based on a scaling function with suitable properties that we listed in the Introduction.

We complete the analysis by computing the mother wavelets corresponding to the Dirichlet scaling function. This requires, for each $M \in \mathcal{M}$, the specification of a fixed element $M'(M) \in \mathcal{M}$ such that $M'(M) > M$. We pick the simplest choice $M'(M) = 3M$. We will content ourselves with considering the linearly ordered subset $\mathcal{M}_3; = \{3^N, N \in \mathbb{N}_0\}$, which is what one always does in discrete wavelet analysis (with 2 replaced by 3). Note that for renormalization the essential structure is the MRA and mother scaling function while the mother wavelet function is a convenient but not essential additional structure.

We have with $\delta^N \coloneqq \delta_{M=3^N}$ the delta distribution approximation on the lattice with 3^M points, and since $(3^{N+1}-1)/2 = 3^N + (3^N-1)/2$,

$$\begin{aligned}\delta^{N+1} = &\sum_{|n|\le\frac{3^{N+1}-1}{2}} e_n = (e_{3^N} + 1 + e_{-3^N}) \\ &\times \sum_{|n|\le\frac{3^N-1}{2}} e_n = (e_{3^N} + 1 + e_{-3^N})\delta^N,\end{aligned} \tag{4.19}$$

which displays a self-similar structure

$$\delta^N = \prod_{k=1}^{N-1} \gamma^k, \qquad \gamma^k = e_{3^k} + 1 + e_{-3^k} \tag{4.20}$$

and nicely illustrates how the Dirichlet kernel δ^N is built from the 3^N basis vectors $e_n, |n| \le (3^N-1)/2$. This also makes it easy to give an explicit parametrization of the orthogonal complement W_{3^N} of V_{3^N} in $V_{3^{N+1}}$, which has twice the dimension of V_{3^N}, which is $d(3^N) = 3^N$: it is given by the functions $e_{n\pm 3^N}, |n| \le (3^N-1)/2$. Let us therefore define $e_n^\sigma \coloneqq e_{n+\sigma 3^N}, |n| \le (3^N-1)/2, \sigma \in \{0,\pm 1\}$. Then

$$\langle e_n^\sigma, e_{\tilde{n}}^{\tilde{\sigma}}\rangle_L = \delta_{\sigma,\tilde{\sigma}}\delta_{n,\tilde{n}}. \tag{4.21}$$

We are thus led to consider two kernels $\delta_\pm^N$ in addition to $\delta^N \coloneqq \delta_0^N$, which are also real valued,

$$\delta_+^N(x) = \cos(3^N 2\pi x)\delta^N(x), \qquad \delta_-^N(x) = \sin(3^N 2\pi x)\delta^N(x) \tag{4.22}$$

and set for $\sigma = \pm 1$,

$$\psi_{\sigma,m}^N(x) \coloneqq \delta_\sigma^N(x - x_m^N). \tag{4.23}$$

We have with $M = 3^N$

$$\begin{aligned}\langle \psi_{+,m}^N, \psi_{-,\tilde{m}}^N\rangle &= \frac{1}{4i}\langle [e_{3^N} e_{-3^N}(x_m^M) + e_{-3^N} e_{3^N}(x_m^M)]\chi_m^M, [e_{3^N} e_{-3^N}(x_m^M) - e_{-3^N} e_{3^N}(x_m^M)]\chi_{\tilde{m}}^M\rangle \\ &= \frac{1}{4i}[e_{3^N}(x_m^M - x_{\tilde{m}}^M) - e_{-3^N}(x_m^M - x_{\tilde{m}}^M)]\langle \chi_m^M, \chi_{\tilde{m}}^M\rangle \\ &= 0,\end{aligned} \tag{4.24}$$

where we used that $e_{3^N}\chi_n^M$ contains only modes $(3^N+1)/2 \le 3^N + n \le (3^{N+1}-1)/2$ while $e_{-3^N}\chi_n^M$ contains only modes $-(3^N+1)/2 \ge -3^N + n \ge -(3^{N+1}-1)/2$ which lie in mutually disjoint sets. On the other hand,

$$\begin{aligned}\langle \psi_{\pm,m}^N, \psi_{\pm,\tilde{m}}^N\rangle &= \frac{1}{4}\langle [e_{3^N} e_{-3^N}(x_m^M) \pm e_{-3^N} e_{3^N}(x_m^M)]\chi_m^M, [e_{3^N} e_{-3^N}(x_m^M) \pm e_{-3^N} e_{3^N}(x_m^M)]\chi_{\tilde{m}}^M\rangle \\ &= \frac{1}{4}[e_{3^N}(x_m^M - x_{\tilde{m}}^M) + e_{-3^N}(x_m^M - x_{\tilde{m}}^M)]\langle \chi_m^M, \chi_{\tilde{m}}^M\rangle \\ &= \frac{1}{2}3^N\delta_{m,\tilde{m}}.\end{aligned} \tag{4.25}$$

Accordingly, the $\psi^N_{\pm,n}$, $|n| \le (3^N - 1)/2$ are an ONB for W_{3^N} up to normalization, and as N varies they provide an ONB of L. We now relate them to the scaling function using $M = 3^N$, $y^M_m = x - x^M_m$, $m \in \mathbb{Z}_M$ and determine the mother wavelets. We have with normalization $2M = 23^N$ (the dimension of W_N)

$$\begin{aligned}
\psi^N_{+,m}(x) &= 2\frac{\cos(2\pi M y^M_m)\sin(\pi M y^M_m)}{\sin(\pi y^M_m)} = 2\frac{\sin(\pi M y^M_m) - 2\sin^3(\pi M y^M_m)}{\sin(\pi y^M_m)}\\
&= \frac{\sin(3\pi M y^M_m) - \sin(\pi M y^M_m)}{\sin(\pi y^M_m)},\\
\psi^N_{-,m}(x) &= 2\frac{\sin(2\pi M y^M_m)\sin(\pi M y^M_m)}{\sin(\pi y^M_m)} = 4\frac{\cos(\pi M y^M_m)\sin^2(\pi M y^M_m)}{\sin(\pi y^M_m)}\\
&= 2\frac{-\cos(3\pi M y^M_m) + \cos(\pi M y^M_m)}{\sin(\pi y^M_m)}.
\end{aligned} \tag{4.26}$$

Thus, we may define two mother wavelets $\psi_+(x) = \sin(\pi x) = \phi(x)$ and $\psi_-(x) = \cos(\pi x) = \pm\sqrt{1 - \phi(x)}$ algebraically related to the mother scaling function ϕ and can write (4.26) as translations and rescalings of rational functions of those

$$\begin{aligned}
\psi^N_{+,m}(x) &= \frac{([D_{3M} - D_M]T^m_{1/M}\psi_+)(x)}{(T^m_{1/M}\psi_+)(x)},\\
\psi^N_{-,m}(x) &= -\frac{([D_{3M} - D_M]T^m_{1/M}\psi_-)(x)}{(T^m_{1/M}\psi_+)(x)}.
\end{aligned} \tag{4.27}$$

B. Noncompact case

Recall the following facts about the topologies of position space and momentum space via the Fourier transform where we denote by M the spatial resolution of the lattice x^M_m with either $m \in \mathbb{Z}$ or $m \in \mathbb{Z}_M = \{0, 1, 2, \ldots, M-1\}$ where for M odd we set $\hat{\mathbb{Z}}_M = \{-\frac{M-1}{2}, \ldots, \frac{M-1}{2}\}$ [c: compact, nc: noncompact, d: discrete, and nd: nondiscrete (continuous)]:

$$\begin{array}{ccc}
\text{space – topology} & \text{momentum – topology} & \text{Fourier – kernel}\\
\text{nc, nd: } \mathbb{R} & \text{nc, nd: } \mathbb{R} & e_k(x) = e^{ikx},\\
\text{nc, d: } \frac{1}{M}\cdot\mathbb{Z} & \text{c, nd: } [-M\pi, M\pi) & e^M_k(m) = e^{ikx^M_m},\\
\text{c, nd: } [0,1) & \text{nc, d: } \mathbb{Z} & e_n(x) = e^{2\pi i n x},\\
\text{c, d: } \frac{1}{M}\cdot\mathbb{Z}_M & \text{c, d: } \hat{\mathbb{Z}}_M & e^M_n(m) = e^{2\pi i n x^M_m}.
\end{array} \tag{4.28}$$

Accordingly, in the noncompact and compact cases, respectively, the space of Schwartz test functions is a suitable subspace of $L = L_2(\mathbb{R}, dx)$ and $L = L_2([0,1), dx)$, respectively, which have momentum support in $2\pi\mathbb{R}$ and $2\pi\cdot\mathbb{Z}$, respectively. Upon discretizing space into cells of width $1/M$ the momentum support $\mathbb{R}$ and $\mathbb{Z}$, respectively, gets confined to the Brillouin zones $[-\pi M, \pi M)$ and $\hat{\mathbb{Z}}_M$, respectively.

The corresponding completeness relations or resolutions of the identity read

$$\begin{aligned}
\delta_{\mathbb{R}}(x, x') &= \int_{\mathbb{R}} \frac{dk}{2\pi} e_k(x - x'),\\
M\delta_{\mathbb{Z}} m, m' &= \int_{-\pi M}^{\pi M} \frac{dk}{2\pi} e^M_k(m - m'),\\
\delta_{[0,1)}(x, x') &= \sum_{n\in\mathbb{Z}} e_n(x - x'),\\
M\delta_{\mathbb{Z}_M} m, m' &= \sum_{n\in\mathbb{Z}_M} e^M_n(m - m').
\end{aligned} \tag{4.29}$$

While the first and third relations in (4.29) define the δ distribution on $\mathbb{R}$ and [0, 1), respectively, the second and fourth relations in (4.29) are the restrictions to the lattice of the regular functions

$$\delta_{\mathbb{R},M}(x) = \int_{-\pi M}^{\pi M} \frac{dk}{2\pi} e_k(x) = \frac{\sin(\pi M x)}{\pi x},$$
$$\delta_{[0,1),M}(x) = \sum_{n\in\mathbb{Z}_M} e_n(x) = \frac{\sin(\pi M x)}{\sin(\pi x)}, \tag{4.30}$$

which we recognize as the Shannon (sinc) and Dirichlet kernels, respectively. These kernels can be considered as regularizations of the aforementioned δ distributions in the sense that the momentum integral $k \in \mathbb{R}$ or momentum sum $n \in \mathbb{Z}$ has been confined to $|k| < \pi M$ and $|n| < \frac{M-1}{2}$, respectively. Both are real valued, smooth, strongly peaked at $x = 0$ and have compact momentum support. The Shannon kernel like the Dirichlet kernel is an L_2 function but it is not of rapid decay with respect to position.

In the previous subsection we already have explored the Dirichlet kernel and proved it to be both very useful for renormalization and for defining a generalized MRA, respectively. In this section we will show that analogous properties hold for the Shannon kernel.

We begin with the MRA structure of the Shannon kernel, which we denote by $\delta_M(x)$ for the rest of this section. As it involves the Fourier modes $|k| \le \pi M$ in complete analogy to the compact case, we consider the space V_M as the closure in $L = L_2(\mathbb{R}, dx)$ of the smooth functions with compact momentum support in $(-\pi M, \pi M)$. This obviously gives a nested structure of subspaces $V_M \subset V_{M'}$, in fact, for any positive real numbers $0 < M \le M'$ with the usual ordering relation, but due to the lattice context we restrict again to the odd positive integers $\mathcal{M}$ with $M \le M'$ if $M'/M \in \mathbb{N}$.

The analogy to the compact case suggests to consider the functions

$$\chi_m^M(x) \coloneqq \frac{1}{M} \delta_M(x - x_m^M) \in V_M. \tag{4.31}$$

We have

$$M^2 \langle \chi_m^M, \chi_{m'}^M \rangle L = \int_{-\pi M}^{\pi M} \frac{dk}{2\pi} \int_{-\pi M}^{\pi M} \frac{dk'}{2\pi} e_k(x_m^M) e_{-k'}(x_{m'}^M) \langle e_k, e_{k'} \rangle_L$$
$$= \int_{-\pi M}^{\pi M} \frac{dk}{2\pi} e_k(x_m^M) e_{-k}(x_{m'}^M)$$
$$= \delta_M(x_m^M - x_{m'}^M) = M \delta_{m,m'}, \tag{4.32}$$

which shows that the χ_m^M form an orthogonal system of functions in V_M.

Next let f belong to the dense subset of V_M consisting of smooth functions with compact momentum support in $(-\pi M, \pi M)$. We have

$$M \sum_{m\in\mathbb{Z}} \chi_m^M(x) \langle \chi_m^M, f \rangle_L = M^{-1} \int_{-\pi M}^{\pi M} \frac{dk}{2\pi} \hat{f}(k) \int_{-\pi M}^{\pi M} \frac{dk'}{2\pi} e_{k'}(x) \left[\sum_{m\in\mathbb{Z}} e^{i\frac{k-k'}{M} m} \right]$$
$$= M^{-1} \int_{-\pi M}^{\pi M} \frac{dk}{2\pi} \hat{f}(k) \int_{-\pi M}^{\pi M} \frac{dk'}{2\pi} e_{k'}(x) \left[2\pi \sum_{m\in\mathbb{Z}} \delta_{\mathbb{R}}((k-k')/M - 2\pi m) \right]$$
$$= \int_{-\pi M}^{\pi M} \frac{dk}{2\pi} \hat{f}(k) e_k(x) = f(x), \tag{4.33}$$

where the Fourier transform of f is

$$\hat{f}(k) = \int_{\mathbb{R}} dx e_{-k}(x) f(x) = \langle e_k, f \rangle_L \tag{4.34}$$

and where we used that for $|k|, |k'| < \pi M$ the condition $k - k' = 2\pi m$ has a solution only for $m = 0$ which is $k' = k$. It follows that the $\sqrt{M}\chi_m^M$ form an ONB of V_M. We may write them as

$$\sqrt{M}\chi_m^M = M^{1/2} D_M T_{1/M}^m \phi, \qquad \phi(x) = \mathrm{sinc}(\pi x), \tag{4.35}$$

demonstrating that the scaling function of this MRA is nothing but the sinc function. While this is well-known, we have rederived this here without any effort and from the regularization of the δ function perspective, which in turn is motivated by the desire to produce MRA ONB bases with locality features.

Next we turn to the underlying wavelet structure. Again we proceed in complete analogy to the compact case and consider the sequence $M_N = 3^N, N \in \mathbb{N}_0$ (we could also allow $N \in \mathbb{Z}$; however, for the purpose of renormalization one is interested in large M only). We thus have to decompose V_{3M} into V_M and its orthogonal complement W_M in V_{3M}. To do this note that

$$\chi_m^{3M}(x) = [e^{2\pi M(x - x_m^{3M})} + e^{-2\pi M(x - x_m^{3M})} + 1]$$
$$\times \int_{-\pi M}^{\pi M} \frac{dk}{2\pi} e_k(x - x_m^{3M}). \tag{4.36}$$

The integral that appears in (4.36) defines an element of V_M and thus can be decomposed into the χ_m^M. The functions that appear in the square brackets lie in the span of $1, \sin(2\pi M(x - x_m^M)), \cos(2\pi M(x - x^m - m)$. We conclude that the χ_m^{3M} can be decomposed into the functions $\chi_{\sigma,m'}^M$ with $\sigma = 0, \pm 1$ where $\chi_{0,m'}^M = \chi_{m'}^M$ and

$$\chi_{+,m}^M(x) = \cos(2\pi M(x - x_m^M))\chi_m^M(x),$$
$$\chi_{-,m}^M(x) = \sin(2\pi M(x - x_m^M))\chi_m^M(x). \qquad (4.37)$$

Using that $2\pi M + k > \pi M, -2\pi M + k < -\pi$ for $|k| < \pi M$ it is not difficult to see by a calculation completely analogous to that of Sec. IV that the $\chi_{\sigma,m}^M$ are mutually orthogonal. Thus W_M is spanned by the $\chi_{\pm,m}^M$, which can be written as

$$\chi_{\pm,m}^M = \pm(D_{3M} - D_M)T_{1/M}^m \psi_\pm, \qquad \psi_+(x) = \mathrm{sinc}(\pi x),$$
$$\psi_-(x) = \mathrm{cosinc}(\pi x), \qquad (4.38)$$

exhibiting the two mother wavelets. The fact that we can deal here with just linear aggregates of mother wavelets rather than rational or algebraic ones is due to the fact that in the noncompact case the denominator function πx scales under dilatation while in the compact case the denominator function $\sin(\pi x)$ does not. These wavelets are, of course, well-known in the literature (there only one mother wavelet is required because the MRA is based on powers of 2 rather than powers of 3 as considered here); however, the novel point here is, apart from using powers of 3 rather than 2, that we have constructed them here effortlessly, directly, by elementary means starting from the cutoff resolution of the identity point of view and without going through the complicated algorithm involving Cohen's condition. Note that (4.38) extends naturally from M being positive powers of 3 to negative powers.

The analysis (localization and height) of the extrema of the Shannon kernel is even simpler than for the Dirichlet kernel since it is basically the function $\mathrm{sinc}(y), y = \pi M x$. The absolute maximum is at $y = 0$ of height 1, and the other extrema have to obey $y = \tan(y)$ whose approximate solution is $y = \pi/2 + N\pi$ for large y with $N \in \mathbb{N}$ without loss of generality (w.l.g.). (we just consider $y \geq 0$ since the function is symmetric). They thus take the approximate value $(-1)^N/(\pi/2 + N\pi)$ and in contrast to the Dirichlet kernel decay as a consequence of large N and not because they are suppressed by an order of M which is, of course, the difference between the compact and noncompact situations.

As a final remark we note that the Shannon kernel is symmetric and thus has an infinite number of odd polynomial vanishing moments (which, of course, do not converge absolutely).

C. Translation invariant kernels and discretization of derivatives

We close this section with the following observation.

Theorem 4.1: (Theorem 6.1).Suppose that $\partial_M \coloneqq I_M^\dagger \partial I_M$ is the natural discrete derivative with respect to a coarse-graining kernel $I_M: L_M \to L$ and such that $[\partial, I_M I_M^\dagger] = 0$. Then for any measurable function f on $\mathbb{R}$ we have $I_M^\dagger f(i\partial) I_M = f(i\partial_M)$.

Proof. We have

$$\partial_M^N = I_M^\dagger(\partial[I_M I_M^\dagger])^{N-1}\partial I_M. \qquad (4.39)$$

While $I_M^\dagger I_M = 1_{L_M}$ by isometry, $p_M \coloneqq I_M I_M^\dagger$ is a projection in L (onto the subspace V_M of the MRA). Thus, if $[\partial, p_M] = 0$, we find $\partial_M^N = I_M^\dagger \partial^N I_M$. The claim then follows from the spectral theorem (functional calculus) since $i\partial_M$ is self-adjoint because $i\partial$ is. ∎

To see that both the Shannon and Dirichlet kernels satisfy the assumption of the theorem, it suffices to remark that they only depend on the difference $x - y$; i.e., they are translation invariant. More precisely, since the χ_m^M with $m \in \mathbb{Z}$ and $m \in \mathbb{Z}_M$, respectively, are an ONB of V_M just as are the $e_k, |k| \leq \pi M$ and $e_{2\pi n}, |n| \leq \frac{M-1}{2}$, respectively,

$$(p_M f)(x) = \sum_m \chi_m^M(x)\langle\chi_m^M, f\rangle = \begin{cases} \int_X dy\left[\int_{-\pi M}^{\pi M} \frac{dk}{2\pi} e_k(x-y)\right] f(y) & X = \mathbb{R} \\ \int_X dy\left[\sum_{|n|\leq\frac{M-1}{2}} e_{2\pi n}(x-y)\right] f(y) & X = [0,1) \end{cases}, \qquad (4.40)$$

and integration by parts does not lead to boundary terms due to the support properties of f or by periodicity, respectively.

Translation invariance of the Shannon and Dirichlet kernels, respectively, is, besides smoothness, another important difference from the Haar kernel

$$\sum_m \chi_m^M(x)\chi_m^M(y) = \sum_m \chi_{[\frac{m}{M},\frac{m+1}{M})}(x)\chi_{[\frac{m}{M},\frac{m+1}{M})}(y), \qquad (4.41)$$

which is not translation invariant. Therefore in this case the flows of, e.g., ω_M or ω_M^{-1}, are not simply related by $\omega_M = I_M^\dagger \omega I_M, \omega_M^{-1} = I_M^\dagger \omega^{-1} I_M$, and thus one must define ω_M as the inverse of the covariance ω_M^{-1}. Here $\omega \coloneqq +\sqrt{-\Delta}$ is the positive square root of the Laplacian $\Delta = \partial^2$ and $\omega_M = +\sqrt{-\Delta_M}, \Delta_M = \partial_M^2$ its discretization. As $M \to \infty$ this difference disappears but at finite M it is present and makes the study of the flow with respect to a nontranslation invariant kernel much more and unnecessarily involved.

In [19] translation invariance of the Shannon and Dirichlet kernels will be exploited to show that the discrete fermion theories on the lattices labeled by M they define is manifestly doubler free. The Nielsen-Ninomiya theorem [20] is evaded because the kernels are merely peaked (quasilocal) but not local (compact support). This mechanism is similar to the nonlocality provided by perfect (blocked from the continuum) actions in the Euclidean path integral approach [21].

V. FREE SCALAR FIELD RENORMALIZATION WITH DIRICHLET FLOW

In this section we repeat some of the computations done in [13] in terms of the *Haar renormalization flow* but now using the Dirichlet kernel which may be called the *Dirichlet renormalization flow*. We content ourselves by blocking from the continuum.

The covariance of the Gaussian measure of a Klein-Gordon field on the cylinder with mass $p > 0$ is $C = (2\omega)^{-1}, \omega = \sqrt{-\Delta + p^2}$ where Δ is the Laplacian on T^1. For the massless case $p = 0$ let $Q^\perp = 1\langle 1, \rangle_L$ be the projection on the zero mode with orthogonal complement $Q = 1_L - Q^\perp$. In this case fix any number $\omega_0 > 0$ and set

$$C = Q^\perp (2\omega_0)^{-1} Q^\perp + Q(2\omega)^{-1} Q. \tag{5.1}$$

The first observation is that ($1 = 1_M$ the constant function equal to unity)

$$\begin{aligned} Q^\perp I_M f_M &= \sum_{m \in \mathbb{Z}_M} f_M(m) 1 \langle 1, \chi^M_m \rangle_L \\ &= \left[\frac{1}{M} \sum_{m \in \mathbb{Z}_M} f_M(m) \right] 1 \\ &= \langle 1_M, f_M \rangle_{L_M} 1_M =: Q^\perp_M f_M, \end{aligned} \tag{5.2}$$

that is, $Q^\perp I_M = Q^\perp_M$. We also set $Q_M := 1_{L_M} - Q^\perp_M$. Then

$$I^\dagger_M Q^\perp \omega_0^{-1} Q^\perp I_M = Q^\perp_M \omega_0^{-1} Q^\perp_M, \tag{5.3}$$

while

$$\begin{aligned} (I^\dagger_M Q\omega^{-1} Q I_M f_M)(m) &= M \langle \chi^M_m, Q\omega^{-1} Q I_M f_M \rangle_L \\ &= M \sum_{\hat{m} \in \mathbb{Z}_M} f_M(\hat{m}) \langle \chi^M_m, Q\omega^{-1} Q \chi^M_{\hat{m}} \rangle_L \\ &= \sum_{\hat{m} \in \mathbb{Z}_M} f_M(\hat{m}) \sum_{0<|n|<\frac{M-1}{2}} \omega(n)^{-1} e^M_n(\hat{m})^* \langle \chi^M_m, e_n \rangle_L \\ &= \frac{1}{M} \sum_{\hat{m} \in \mathbb{Z}_M} f_M(\hat{m}) \sum_{0<|n|<\frac{M-1}{2}} \omega(n)^{-1} e^M_n(m - \hat{m}) \\ &= \sum_{0<|n|<\frac{M-1}{2}} \omega(n)^{-1} e^M_n(m) \hat{f}_M(n) \\ &= [Q_M \omega_M Q_M f_M](m), \end{aligned} \tag{5.4}$$

where the Fourier transform of $f_M \in L_M$ is defined by

$$\hat{f}_M(n) = \langle e^M_n, f_M \rangle_{L_M}, \qquad f_M = \sum_{n \in \hat{\mathbb{Z}}_M} e^M_n \hat{f}_M(n). \tag{5.5}$$

This shows that we have simply $C_M(n)^{-1}/2 = \omega_M(n) = \omega(n) = 2\pi|n|$ for $0 < |n| \le (M-1)/2$ and $C_M(0)^{-1}/2 := \omega_0$. In the massive case simply $\omega_M(n) = \omega(n), 0 \le |n| \le (M-1)/2$. This should be contrasted with the rather complicated expression for $\omega_M(n)$ given for the Haar flow displayed in [13], which involves also all the $\omega(n), |n| > (M-1)/2$. While these are subdominant for large M, they are not decaying rapidly. This is caused by the discontinuity of the Haar scaling function.

These nonrapidly decaying terms can cause convergence problems that are artifacts of using discontinuous approximants to actually smooth functions $f \in L$. In [14] we encounter the operator on L,

$$K_M := Q[C^{-1} - I_M C^{-1}_M I^\dagger_M] Q \tag{5.6}$$

in a massless theory. We compute its action on $e_n, n \neq 0$ for the Dirichlet flow

$$\begin{aligned}\frac{1}{2}K_M e_n &= \omega(n)e_n - \sum_{\hat{n}\in\hat{\mathbb{Z}}_M} C_M^{-1}(\hat{n})(QI_M e_{\hat{n}}^M)\langle e_{\hat{n}}^M, I_M^\dagger e_n\rangle_{L_M} \\ &= \omega(n)e_n - \sum_{\hat{n}\in\hat{\mathbb{Z}}_M} C_M^{-1}(\hat{n})(QI_M e_{\hat{n}}^M)\langle I_M e_{\hat{n}}^M, e_n\rangle_L \\ &= \omega(n)e_n - \sum_{0\neq\hat{n}\in\hat{\mathbb{Z}}_M} \omega_M^{-1}(\hat{n})\delta_{n,\hat{n}}e_n \\ &= \omega(n)\theta\left(|n| - \frac{M-1}{2} + 1\right)e_n, \end{aligned} \tag{5.7}$$

where θ is the Heaviside step function and where we used for $|n| \leq (M-1)/2$

$$I_M e_n^M = \sum_{m\in\mathbb{Z}_M} \chi_m^M e_n^M(m) = \frac{1}{M}\sum_{|\hat{n}|\leq\frac{M-1}{2}} e_{\hat{n}} \sum_{m\in\mathbb{Z}_M} e_{n-\hat{n}}^M(m) = e_n. \tag{5.8}$$

Accordingly,

$$K_M f = \sum_{|n|>\frac{M-1}{2}} \omega(n)e_n\hat{f}(n), \tag{5.9}$$

where the Fourier transform

$$\hat{f}(n) = \langle e_n, f\rangle_L, \qquad f = \sum_{n\in\mathbb{Z}}\hat{f}_n e_n, \tag{5.10}$$

was used. For smooth f, $\hat{f}(n)$ is of rapid decay, and thus, we get the sup norm estimate

$$\begin{aligned}\|K_M f\|_\infty &\leq 2\pi \sum_{|n|>\frac{M-1}{2}} \frac{|n|}{n^4}|n^4\hat{f}(n)| \\ &\leq M^{-1}\left[\sum_{|n|>\frac{M-1}{2}} n^{-2}\right]\sup_{n\in\mathbb{Z}}|n^4\hat{f}(n)|, \end{aligned}\tag{5.11}$$

which decays as $M\to\infty$. Thus, since T^1 is compact and therefore $\|.\|_1 \leq \|.\|_2 \leq \|.\|_\infty$, we get convergence to zero with respect to all three norms. This is not the case with respect to the Haar kernel.

VI. CONCLUSION AND OUTLOOK

In this contribution we intended to achieve three goals:

(1) To show that a useful (generalized) MRA in the compact case can be obtained by direct methods not relying on periodization of noncompact MRAs if one is willing to accept that the associated ONB is created by rescalings and translations of rational rather than linear aggregates of mother scaling functions. Corresponding mother wavelets then also are to be understood in this generalized sense.

(2) To show that MRA is directly related to Hamiltonian renormalization and serves as a very useful organizational principle, thereby reducing the freedom that one has in choosing the renormalization flow.

(3) To show that the Dirichlet and Shannon kernel fits into the generalized MRA scheme and that its flow has much improved analytical properties as compared to the Haar flow, which not only tremendously simplifies many calculations but even makes computations possible that are otherwise plagued by singularities due to insufficient differentiability of the kernel; see [14] for an example.

There are many directions into which this work can be extended. We mention three of them:

(I) Instead of the Shannon and Dirichlet kernels one can choose other ones that also have promising properties. For instance, in the compact case the Fejer kernel [22] is the Cesaro average of the Dirichlet kernel, shares many properties of the Dirichlet kernel, and in addition is manifestly non-negative, which is not true for the Dirichlet kernel.

(II) A more general question is: Which kernels are optimal for which renormalization application? Which ones display as few as possible "nonlinearities" when one allows algebraic rather than linear aggregates of scaling functions to generate an MRA? We have seen that translation invariant kernels lead to major simplifications in the renormalization flow. See also [16] for the explicit use of Daubechies wavelets and the properties of the corresponding flow.

(III) While tori are particularly convenient, there may be other applications in which different compact topologies (e.g., spheres) are preferable. We expect that in this case the theory laid out here generalizes by substituting for the corresponding harmonic analysis (e.g., spherical harmonics $Y_{l,m}$ on S^2 rather than toroidal harmonics e_{n_1,n_2} on T^2).

(IV) In the noncompact case the Shannon kernel has a sharp cutoff in momentum space at $|k| = \pi M$ of the Fourier transform of the δ distribution. This causes it to have compact momentum transport but to decay only slowly in position space. If we turn the momentum cutoff function from a step function into a smooth function of compact support or of rapid decrease, then the corresponding kernel will also be of rapid decrease in position space and in that sense keep its locality. However, only if it is really of compact support rather than merely of rapid decrease does it define an MRA in the sense of having a nested structure of subspaces of L as otherwise the spaces V_M all coincide with L (consider, e.g., a Gaussian cutoff of width M). But even then are simple translates and dilatations of the resulting kernel not automatically orthogonal, and thus it is not a scaling function of the corresponding MRA.

In fact, it is well-known that no MRA in the strict sense of [4] exists with scaling functions of rapid decrease. In that sense the Shannon kernel performs better, being simultaneously smooth of some decay and a scaling function. This suggests that one picks the momentum cutoff not sharply (discontinuously) but also not smoothly [usually one uses mollifiers based on the smooth function $\exp(-x^{-2})$] so that the position space decay (locality) is improved while a scaling function results. For practical calculations it is important that this momentum cutoff function be analytically manageable. It would be interesting to have a scaling function with these properties at one's disposal.

Note added. This article is part of a series, related concepts occur in [14] and [19] in PRD and [13] in CQG.

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