Hamiltonian renormalization. VII. Free fermions and doubler free kernels

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The Hamiltonian renormalization program motivated by constructive quantum field theory and Osterwalder-Schrader reconstruction that was recently launched for bosonic field theories is extended to fermions. As fermion quantization is not in terms of measures, the scheme has to be mildly modified accordingly. We exemplify the scheme for free fermions for both compact and noncompact spatial topologies, respectively (i.e., with and without IR cutoff) and demonstrate that the convenient Dirichlet or Shannon coarse-graining kernels recently advertised in a companion paper lead to a manifestly doubler free flow.

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I. INTRODUCTION

The Hamiltonian or canonical approach to quantum gravity [1] aims at implementing the constraints as operators on a Hilbert space. In the classical theory, the constraints generate the Einstein equations via the Hamiltonian equations of motion [2]. They underlie the numerical implementation of the initial value formulation of Einstein's equations, e.g., in black hole merger and gravitational wave template codes [3].

The mathematically sound construction of canonical quantum gravity is a hard problem because the constraints are nonpolynomial expressions in the elementary fields and in that sense much more nonlinear than even the most complicated interacting quantum field theory (QFT) on Minkowski space such as QCD whose Hamiltonian is still polynomial in gluon and quark fields. As the theory is nonrenormalizable and thus believed to exist only nonperturbatively, the loop quantum gravity (LQG) approach has systematically developed such a nonperturbative program [4]. LQG derives its name from the fact that it uses a connection rather than metric-based formulation; hence, it is phrased in the language of Yang-Mills type gauge fields and thus benefits from the nonperturbative technology introduced for such theories, specifically gauge invariant Wilson loop variables [5].

The current status of LQG can be described as follows: While the quantum constraints can indeed be implemented in a Hilbert space representation [6] of the canonical (anti)commutation and adjointness relations as densely defined operators [7] and while its commutator algebra is mathematically consistent in the sense that it closes, it closes with the wrong structure "functions." The inverted commas refer to the fact that the classical constraints do not form a Lie Poisson algebra because for a Lie algebra it is required that one has structure constants. By contrast, here we have nontrivial structure functions in the classical theory that are dictated by the fundamental hypersurface deformation algebra [8], and in the quantum theory they become operators themselves and are not simply constant multiples of the identity operator. We therefore call them structure operators.

The most important missing step in LQG is therefore to correct those structure operators. It is for this reason that more recently Hamiltonian renormalization techniques were considered [9]. There one actually works with a one-parameter family of gauge fixed versions of the theory [10] so that the constraints no longer appear and are traded for a Hamiltonian which drives that one-parameter evolution. The reason for doing this is twofold: On the one hand, working with the gauge fixed version means solving the constraints classically and saves the work to determine the quantum kernel and Hilbert space structure on it. On the other hand, the techniques of [9] were derived from the Osterwalder-Schrader reconstruction [11], which deals with theories whose dynamics is driven by an actual Hamiltonian rather than constraints (however, see Ref. [12]). Still, that Hamiltonian uniquely descends from the constraints, and therefore its quantization implicitly depends on the quantization of the constraints. Therefore, the quantum constraints and their structure operators are implicitly also present in the gauge fixed version. In addition, in [13] we have shown that the techniques of [9] can be "abused" also for constrained quantum theories in the sense that the renormalization steps to

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be carried out can be performed independently for all constraints "as if they were actual Hamiltonians," even if the corresponding operators are not bounded from below. In that sense the methods of [9] complement those of [14] where the correction of the structure operators is approached by exploiting the spatial diffeomorphism invariance of the classical theory in an even more nonlinear fashion than it was already done in [7].

The program of [9] rests on the following observation: In quantizing an interacting classical field theory one cannot proceed directly but rather has to introduce at least an UV cutoff M where we may think of M^{-1} as a spatial resolution. Introducing M produces quantization ambiguities that are encoded in a set of parameters depending on M. Almost all points in that set do not define consistent theories where a consistent theory is defined to be one in which the theory at resolution M is the same as the theory at higher resolution $M' > M$ after "integrating out" the extra degrees of freedom. Renormalization introduces a flow on these parameters whose fixed or critical points define consistent theories. In this way, the correct structure operators or algebra of constraints referred to above are also believed to be found, either explicitly or implicitly. In [13] we have shown that this is what actually happens for the much simpler case of two-dimensional (2D) parametrized field theory [15] whose quantum hypersurface deformation algebra coincides with the Virasoro algebra. One of the lessons learned from this is that the quantum constraint algebra *must not close* at any finite resolution even if *the continuum algebra closes with the correct structure operators*. In other words, it is physically correct that the finite resolution constraints are "anomalous" while the actual continuum theory is anomaly free. The "anomalous" terms just reflect a discretization artifact that decays to zero as we increase the resolution. The other lesson learned is that while there is a substantial amount of freedom in the choice of the renormalization flow, for a general theory the coarse-graining kernel should have sufficient smoothness properties. A systematic classification of these choices of flow is made possible using multiresolution analysis [16] known from wavelet theory [17].

In [18] we have further tested [9] for free bosons (scalars and vector fields). Theories with fermions were not considered so far. In this paper we close this gap; see also [19] for a closely related formulation.

The architecture of this paper is as follows:

In Sec. II we briefly recall the bosonic theory from [9].

In Sec. III we adapt the bosonic theory to the fermionic setting.

In Sec. IV we test the fermionic Hamiltonian renormalization theory for free Dirac, Majorana, or Weyl fermions both with and without IR cutoff using the Dirichlet-Shannon kernel as a coarse-graining scheme and confirm a manifestly doubler free spectrum at each resolution M at the fixed point. The Nielsen-Ninomiya theorem [20] is evaded because the finite resolution Hamiltonians are spatially nonlocal as it is usually the case when one "blocks from the continuum," i.e., computes the "perfect Hamiltonian." A similar observation was made in the context of QCD in the Euclidean action approach [21]. Indeed, if one uses the local but discontinuous Haar kernel to define the renormalization flow, the usual doubler troubled pole structure of the Feynman propagator is found while for the smooth but only quasilocal Dirichlet-Shannon kernel a doubler free pole structure results.

In Sec. V we summarize and conclude.

II. REVIEW OF HAMILTONIAN RENORMALIZATION FOR BOSONS

To be specific we will consider the theory either with IR cutoff so that space is a d-torus T^d or without IR cutoff so that space is d-dimensional Euclidean space $\mathbb{R}^d$, and it will be sufficient to consider one coordinate direction as both spaces are Cartesian products. Thus $X = [0, 1)$ or $X = \mathbb{R}$ in what follows.

Thus, for simplicity we consider a bosonic, scalar quantum field Φ (operator valued distribution) with conjugate momentum Π on X with canonical commutation and adjointness relations (in natural units $\hbar = 1$)

$$[\Pi(x),\Phi(y)] = i\delta(x,y), \quad \Phi(x)^* = \Phi(x), \quad \Pi^*(x) = \Pi(x), \tag{2.1}$$

where

$$\delta(x,y) = \sum_{n\in\mathbb{Z}} e_n(x)e_n(y)^*, \qquad e_n(x) = e^{2\pi inx} \tag{2.2}$$

is the periodic δ distribution on the torus or

$$\delta(x,y) = \int_{\mathbb{R}} \frac{dk}{2\pi} e_k(x)e_k(y)^*, \qquad e_k(x) = e^{ikx} \tag{2.3}$$

on the real line, respectively. It is customary to work with the bounded Weyl operators

$$w[f,g] = \exp(i[\Phi(f)+\Pi(g)]), \quad \Phi(f) = \int_X dx f(x)\Phi(x),$$
$$\Pi(g) = \int_X dx g(x)\Pi(x) \tag{2.4}$$

with $f, g \in L = L_2(X, dx)$ test functions or smearing functions usually with some additional properties such as differentiability or even smoothness. For tensor fields of higher degree a similar procedure can be followed (see Ref. [9]).

Since the space L enters the stage naturally, we use multiresolution analysis (MRA) language [16] familiar from wavelet theory [17] to define a renormalization group flow. MRAs serve as a powerful organizing principle to define renormalization flows in terms of coarse-graining kernels, and while the choice of the kernel should intuitively not have much influence on the fixed point or continuum theory (at least in the presence of universality) the examples of [13,22] show that generic features such as smoothness can have an impact.

In the most general sense an MRA is a nested sequence of Hilbert subspaces $V_M \subset L$ indexed by $M \in \mathcal{M}$ where $\mathcal{M}$ is partially ordered and directed by $\leq$. That is, one has $V_M \subset V_{M'}$ for $M \leq M'$ and $\cup_{M\in\mathcal{M}} V_M$ is dense in L. Pick an orthonormal basis (ONB) $d(M)^{1/2}\chi_m^M$ for V_M where m is from a countably finite (infinite) index set Z_M for $X = [0,1)$ $(X = \mathbb{R})$, respectively, and $d(M)$ is a finite number. In the case that $X = [0,1)$ typically Z_M is the lattice $x_m^M, m/d(M)$ and $d(M) = \dim(V_M)$ the number of points in it. Let $L_M = l_2(Z_M)$ be the Hilbert space of square summable sequences indexed by Z_M with inner product

$$\langle f_M, g_M\rangle \coloneqq d(M)^{-1}\sum_{m\in Z_M} f_M^*(m) g_M(m). \tag{2.5}$$

This scalar product offers the interpretation of $f_M(m) \coloneqq f(x_m^M), x_m^M \coloneqq \frac{m}{d(M)}$, and similar for g_M as the discretized values of some functions $f, g \in L$ in which case (2.5) is the Riemann sum approximant of $\langle f, g\rangle_L$. It is for this reason that we did not normalize the χ_m^M.

What follows works for any such choice of ONB indexed by M. However, to reduce the amount of arbitrariness and to give additional structure to MRAs one requires, in both wavelet theory and renormalization, in addition that the ONBs descend from a few mother scaling functions ϕ by dilatations depending on M and translations depending on m. In wavelet theory on the real line one is rather specific about the concrete descendance. In particular, there is only one mother scaling function, the χ_m^M and ϕ are linearly related, $\mathcal{M}$ just consists of the powers $M = 2^N, N \in \mathbb{Z}$, and $\chi_m^M = \phi(Mx - m)$. As advertised in [16] we allow a more general descendance and thus accept a finite, fixed number of mother scaling functions and that the χ_m^M are dilatations and translations of a rational function of those mother scaling functions. This keeps the central idea of providing minimal structure to an MRA while increasing flexibility.

The spaces V_M and L_M are in bijection via

$$I_M\colon L_M \to L, \qquad f_M \mapsto \sum_m f_M(m)\chi_m^M. \tag{2.6}$$

Note that (2.6) has a range in $V_M \subset L$ only. Its adjoint $I_M^\dagger\colon L \to L_M$ is defined by

$$\langle I_M^\dagger f, f_M\rangle_{L_M} \coloneqq \langle f, I_M f_M\rangle_L \tag{2.7}$$

so that

$$(I_M^\dagger f)(m) = d(M)\langle \chi_M, f\rangle_L. \tag{2.8}$$

Clearly

$$(I_M^\dagger I_M f_M)(m) = d(M)\langle \chi_m^M, I_M f_M\rangle_L = f_M(m). \tag{2.9}$$

i.e., $I_M^\dagger I_M = 1_{L_M}$ while

$$(I_M I_M^\dagger f)(x) = d(M)\sum_m \chi_m^M(x)\langle \chi_m^M, f_M\rangle_L = (p_M f)(x) \tag{2.10}$$

is the projection $P_M\colon L \mapsto V_M$.

Given $M \leq M'$ we define the coarse-graining map

$$I_{MM'} \coloneqq I_{M'}^\dagger I_M\colon L_M \mapsto L_{M'}. \tag{2.11}$$

It obeys

$$I_{M'} I_{MM'} = p_{M'} I_M = I_M \tag{2.12}$$

because I_M has range in $V_M \subset V_{M'}$ for $M \leq M'$. This is the place where the MRA property of the nested set of subspaces V_M was important. Next for $M_1 \leq M_2 \leq M_3$ we have

$$I_{M_2M_3} I_{M_1M_2} = I_{M_3}^\dagger p_{M_2} I_{M_1} = I_{M_3}^\dagger I_{M_1} = I_{M_1M_3} \tag{2.13}$$

for the same reason. This is called the condition of cylindrical consistency which is crucial for the renormalization group flow.

To see the importance of (2.13) we consider a probability measure ν on the space $\mathcal{F}$ of field configurations Φ which defines a Hilbert space $\mathcal{H} = L_2(\mathcal{F}, d\nu)$ and a representations space for the Weyl algebra $\mathfrak{A}$ generated from the Weyl elements (2.4). We set $w[f] \coloneqq w[f, g = 0]$ and define the generating functional of moments of ν by

$$\nu(f) \coloneqq \nu(w[f]). \tag{2.14}$$

If we restrict f to V_M we obtain an effective measure on the space of discretized quantum fields $\Phi_M = I_M^\dagger \Phi$ via

$$w[I_M f_M] = w_M[f_M] = e^{i\Phi_M(f_M)}, \quad \Phi_M(f_M) = \langle f_M, \Phi_M\rangle_{L_M} \tag{2.15}$$

and

$$\nu_M(f_M) \coloneqq \nu(w[I_M f_M]) = \nu_M(w_m[f_M]). \tag{2.16}$$

The measures ν_M on the spaces $\mathcal{F}_M$ of fields Φ_M are consistently defined by construction

$$\nu_{M'}(I_{MM'} f_M) = \nu_M(f_M) \tag{2.17}$$

for any $M < M'$ since the ν_M descend from a continuum measure. Conversely, given a family of measures ν_M satisfying (2.17) a continuum measure ν can be constructed known as the projective limit of the ν_M under mild technical assumptions [23]. To see the importance of (2.13) for this to be the case, suppose we write $f \in L$ in two equivalent ways $f = I_{M_1} f_M = I_{M_2} g_{M_2}$, and then we should have $\nu_{M_1}(f_{M_1}) = \nu_{M_2}(g_{M_2})$. Now while M_1 and M_2 may not be in relation, as $\mathcal{M}$ is directed we find $M_1, M_2 \leq M_3$. Applying $I^\dagger_{M_3}$ we conclude $I_{M_1 M_3} f_{M_1} = I_{M_2 M_3} g_{M_2}$; thus due to (2.17) indeed

$$\nu_{M_1}(f_1) = \nu_{M_3}(I_{M_1 M_3} f_{M_1}) = \nu_{M_3}(I_{M_2 M_3} g_{M_2}) = \nu_{M_2}(g_{M_2}). \tag{2.18}$$

In constructive quantum field theory the task is to construct a representation of the Weyl algebra $\mathfrak{A}$ with additional properties such as allowing for the implementation of a Hamiltonian operator $H = H[\Phi, \Pi]$ which imposes severe restrictions on the Hilbert space representation. One may start with discretized Hamiltonians

$$H^{(0)}_M[\Phi_M, \Pi_M] \coloneqq H[p_M \Phi, p_M \Pi] \tag{2.19}$$

on $\mathcal{H}^{(0)}_M \coloneqq L_2(\mathcal{F}_M, \nu^{(0)}_M)$ where $\nu^{(0)}_M$ is any probability measure to begin with, for instance, a Gaussian measure or a measure constructed from the ground state $\Omega^{(0)}_M$ of the Hamiltonian $H^{(0)}_M$. The point of using the IR cutoff is that there are only finitely many, namely $d(M)$ degrees of freedom Φ_M, Π_M which are conjugate

$$[\Pi_M(m), \Phi(m')] = i d(M) \delta(m, m'), \qquad \Phi_M(m)^* = \Phi_M(m), \qquad \Pi^*_M(m) = \Pi_M(m) \tag{2.20}$$

so that construction of $\nu^{(0)}_M$ does not pose any problems. In the case that there is no IR cutoff it is significantly harder to show that the theories even at finite UV cutoff exist. Assuming this to be the case, one fixes for each $M \in \mathcal{M}$ an element $M \leq M'(M) \in \mathcal{M}$ and defines isometric injections

$$J^{(n+1)}_{MM'(M)} \colon \mathcal{H}^{(n+1)}_M \to \mathcal{H}^{(n)}_{M'(M)}, \qquad \mathcal{H}^{(n)}_M \coloneqq L_2(\mathcal{F}_M, d\nu^{(n)}_M) \tag{2.21}$$

via

$$\nu^{(n+1)}_M(f_M) \coloneqq \nu^{(n)}_{M'(M)}(I_{MM'(M)} f_M), \tag{2.22}$$

and with these the flow of Hamiltonians

$$H^{(n+1)}_M \coloneqq J^\dagger_{MM'(M)} H^{(n)}_{M'(M)} J_{MM'(M)}. \tag{2.23}$$

The isometry of the injections relies on the assumption that the span of the $w_M[f_M]$ is dense in $\mathcal{H}^{(0)}_M$ which is typically the case.

This defines a sequence or flow (indexed by n) of families (indexed by M) of theories $\mathcal{H}^{(n)}_M$ and $H^{(n)}_M$. At a critical or fixed point of this flow the consistency condition (2.17) is satisfied [at first in the linearly ordered sets of $\mathcal{M}(M) \coloneqq \{(M')^N(M), N \in \mathbb{N}_0\}$ and then usually for all of $\mathcal{M}$ by universality], and one obtains a consistent family $(\mathcal{H}_M, H_M)$. This family defines a continuum theory $(\mathcal{H}, H)$ as one obtains inductive limit isometric injections $J_M \colon \mathcal{H}_M \mapsto \mathcal{H}$ such that $J_{M'} J_{MM'} = J_M, M \leq M'$, thanks to the fixed point identity $J_{M_2 M_3} J_{M_1 M_2} = J_{M_1 M_3}$, $M_1 \leq M_2 \leq M_3$, and such that

$$H_M = J^\dagger_M H J_M \tag{2.24}$$

is a consistent family of quadratic forms $H_M = J^\dagger_{MM'} H_{M'} J_{MM'}, M \leq M'$.

We conclude this section by noting that wavelet theory actually also seeks to decompose the spaces as $V_{M'} = V_M \oplus W_M$ where W_M is the orthogonal complement of V_M in $V_{M'}, M \leq M'$, to provide an ONB for the W_M and to require that this basis descends from a mother wavelet ψ concretely related to the scaling function in the same specific way as outlined above for the scaling function. For the purpose of renormalization this additional structure is not essential, and thus we will not go into further details. We remark, however, that in [16] we also generalized the notion of wavelets in the same way as for the scaling function which again keeps the central idea of structurizing the MRA and showed that the Dirichlet and Shannon kernels are nontrivial realizations of that more general definition.

III. HAMILTONIAN RENORMALIZATION FOR FERMIONS

To distinguish the bosonic field Φ from the previous section from the present fermionic field we use the notation ξ_B for a chiral (or Weyl) fermion where $B = 1, 2$ transforms in one of the two fundamental representations of $SL(2, \mathbb{C})$. Its Majorana conjugate $\epsilon \xi^*$ with $\epsilon = i\sigma_2$ (Pauli matrix) of opposite chirality then transforms in the dual fundamental representation. We have the fundamental simultaneous canonical anticommutation relations (CAR)

$$[\xi_B(x), \xi_C(y)^*]_+ \coloneqq \xi_B(x)\xi_C(y)^* + \xi_C(y)^* \xi_B(x) = \delta_{BC} \delta(x, y) \tag{3.1}$$

with all other anticommutators vanishing. Dirac fermions and Majorana fermions can be considered as usual by using direct sum $SL(2, \mathbb{C})$ representations of independent Weyl fermions of opposite chirality or of the direct sum of a fermion with its Majorana conjugate. It will be sufficient to consider a single Weyl fermion species ξ_B for what follows.

The measure theoretic language given for bosons of the previous section cannot apply because of several reasons: i. the "Weyl elements" $w[f] \coloneqq \exp(i\langle f, \xi\rangle_L^\dagger)$ are not mutually commuting; ii. the $w[f]\Omega$ are not dense in the Fock space $\mathcal{H}$ defined by $\langle f, \xi\rangle\Omega = 0$ because, in fact, $w[f] = 1_{\mathcal{H}} + i\langle f, \xi\rangle^\dagger$ due to nilpotency; and iii. $w[f]$ is not unitary. To avoid this one can formally work with Berezin "integrals" [24] and anticommuting smearing fields f but then we cannot immediately transfer the functional analytic properties of the commuting test functions from the bosonic theory, and apart from serving as a compact organising tool, anticommuting smearing functions do not have any advantage over what we say below.

One of the motivations to work with Weyl elements rather than say $\Phi(f), \Pi(f)$ in the bosonic case is that the Weyl elements are bounded operators. However, the operators $\xi(f) \coloneqq \langle f, \xi\rangle_L, \xi(f)^\dagger$ are already bounded by $||f||_L$ as follows from the CAR:

$$[\xi(f), \xi(f)^*]_+ = \|f\|_L^2 1_{\mathcal{H}}$$
$$\Rightarrow \|\xi(f)\psi\|_{\mathcal{H}}^2, \|\xi(f)\psi\|_{\mathcal{H}}^2 \leq \|f\|_L^2\|\psi\|^2. \tag{3.2}$$

The derivation of the renormalization scheme given in [9], in fact, covers both the bosonic and the fermionic cases, but the practical implementation for bosons used measures [18]. We thus adapt the bosonic renormalization scheme by reformulating it in an equivalent way which then extends to the fermionic case:

Given cyclic vectors $\Omega_M^{(n)}$ for the algebra generated by the annihilation operator

$$\xi_M(f_M) \coloneqq \langle I_M f_M, \xi\rangle_L = \langle f_M, I_M^\dagger \xi\rangle_{L_M} = \frac{1}{d(M)} \sum_{B,m\in Z_M} [f_M^B(m)]^* \xi_{M,B}(m) \tag{3.3}$$

and their adjoints (perhaps the vacua of the Hamiltonians $H_M^{(n)}$), we define the flow of isometric injections [e.g., for $M' = M'(M)$]

$$J_{MM'}^{(n+1)}\Omega_M^{(n+1)} \coloneqq \Omega_{M'}^{(n)},$$
$$J_{MM'}^{(n+1)}\Xi_M(F_{M,1})\cdots\Xi_M(F_{M,N})\Omega_M^{(n+1)} \coloneqq \Xi_{M'}(I_{MM'}F_{M,1})^*\cdots\Xi_{M'}(I_{MM'}F_{M,N})^*\Omega_{M'}^{(n)},$$

starting from an initial vector $\Omega_M^{(0)}$ defined below. Note that $\xi_M = d(M)I_M^\dagger\xi$ preserve the CAR in the sense that

$$[\xi_M(m), [\xi_M(m')]^*]_+ = d(M)\delta_{mm'} \tag{3.4}$$

and $\Xi(F) = \sum_B[\langle f_B, \xi_B\rangle_L + \langle\tilde{f}_B, \xi_B\rangle^*]$ where we have collected four independent smearing functions $f_B, \tilde{f}_B$, $B = 1, 2$ into one symbol F. The same notation was used in (3.4) for the M dependent quantities. With these we define the flow of Hamiltonian quadratic forms as

$$H_M^{(n+1)} \coloneqq [J_{MM'}^{(n+1)}]^\dagger H_{M'}^{(n)} J_{MM'}^{(n+1)}. \tag{3.5}$$

These formulas are even simpler than in the bosonic case because there is no fermionic Gaussian measure and corresponding covariance to consider. However, as in the bosonic case, one has to give initial data for this flow. This can be done, e.g., by defining

$$H_M^{(0)}[\xi_M, \xi_M^*] \coloneqq\, :H[p_M\xi, (p_M\xi)^*]: \tag{3.6}$$

where $(p_M\xi)_B \coloneqq I_M I_M^\dagger \xi_B$, H is the classical Hamiltonian, and $:.:$ denotes normal ordering with respect to a Fock space $\mathcal{H}_M^{(0)}$ with cyclic Fock vacuum $\Omega_M^{(0)}$ annihilated by $A_{B,M}^{(0)}$ assembled from $\xi_{M,B}$ and $\xi_{M,B}^*$ as suggested by the form of $H[p_M\xi, (p_M\xi)^*]$. As in the bosonic case, the fields $\xi_{M,B}$ do not depend on the sequence label n while the annihilators $A_{M,B}^{(n)}$ do as one obtains them from the $\xi_{M,B}$ using extra discretized structure that depends on M, typically lattice derivatives and more complicated aggregates made from those (Dirac-Weyl operators, Laplacians, etc.).

IV. HAMILTONIAN RENORMALIZATION OF FREE FERMIONS AND FERMION DOUBLING

In this section we will concretely choose the renormalization structure as follows (see Ref. [16] for more details): Z_M will be the lattice of points x_m^M with $m \in \mathbb{Z}$ if $X = \mathbb{R}$ and $m \in \mathbb{Z}_M \coloneqq \{0, 1, 2, \ldots, M-1\}$ if $X = [0, 1)$ respectively, and $d(M) = M$. The set $\mathcal{M}$ consists of the odd naturals with partial order $M \leq M'$ iff $M'/M \in \mathbb{N}$. The renormalization sequence will be constructed using $M'(M) = 3M$ for simplicity. The MRAs are based on the Shannon [24] and Dirichlet [25] kernels, respectively, that is,

$$\chi_m^M(x) = \begin{cases} \frac{\sin(M\pi(x-x_m^M))}{M\pi(x-x_m^M)} & X = \mathbb{R} \\ \frac{\sin(M\pi(x-x_m^M))}{M\sin(\pi(x-x_m^M))} & X = [0,1) \end{cases}. \tag{4.1}$$

Their span is dense in V_M, and they are mutually orthogonal with norm M^{-1}. The Dirichlet kernel is 1-periodic as it should be. Both have maximal value 1 at $x = x_m^M$, are symmetric about this point, and (slowly) decay away from it, thus displaying some position space locality. They are real valued and smooth and have compact momentum support $k\in[-\pi M,\pi M]$ and $k=2\pi n, n\in\hat{\mathbb{Z}}_M=\{-\frac{M-1}{2}, -\frac{M-1}{2}+1,\ldots,\frac{M-1}{2}\}$, respectively.

Recall the following facts about the topologies of position space and momentum space via the Fourier transform where we denote by M the spatial resolution of the lattice x^M_m with either $m\in\mathbb{Z}$ or $m\in\mathbb{Z}_M=\{0,1,2,\ldots,M-1\}$ where for M odd we set $\hat{\mathbb{Z}}_M=\{-\frac{M-1}{2},\ldots,\frac{M-1}{2}\}$ [c: compact, nc: noncompact, d: discrete, nd: nondiscrete (continuous)]:

$$\begin{array}{ccc} \text{space}-\text{topology} & \text{momentum}-\text{topology} & \text{Fourier}-\text{function} \\ \text{nc, nd: } \mathbb{R} & \text{nc, nd: } \mathbb{R} & e_k(x)=e^{ikx}, \\ \text{nc, d: } \frac{1}{M}\cdot\mathbb{Z} & \text{c, nd: } [-M\pi, M\pi) & e^M_k(m)=e^{ikx^M_m}, \\ \text{c, nd: } [0,1) & \text{nc, d: } \mathbb{Z} & e_n(x)=e^{2\pi inx}, \\ \text{c, d: } \frac{1}{M}\cdot\mathbb{Z}_M & \text{c, d: } \hat{\mathbb{Z}}_M & e^M_n(m)=e^{2\pi inx^M_m}. \end{array} \tag{4.2}$$

Accordingly, in the noncompact and compact cases, respectively, the space of Schwartz test functions is a suitable subspace of $L=L_2(\mathbb{R},dx)$ and $L=L_2([0,1),dx)$, respectively, which have momentum support in $2\pi\mathbb{R}$ and $2\pi\cdot\mathbb{Z}$, respectively. Upon discretizing space into cells of width $1/M$ the momentum support $\mathbb{R}$ and $\mathbb{Z}$, respectively, gets confined to the Brillouin zones $[-\pi M,\pi M)$ and $\hat{\mathbb{Z}}_M$, respectively.

The corresponding completeness relations or resolutions of the identity read

$$\begin{aligned} \delta_{\mathbb{R}}(x,x') &= \int_{\mathbb{R}} \frac{dk}{2\pi} e_k(x-x'), \\ M\delta_{\mathbb{Z}}(m,m') &= \int_{-\pi M}^{\pi M} \frac{dk}{2\pi} e^M_k(m-m'), \\ \delta_{[0,1)}(x,x') &= \sum_{n\in\mathbb{Z}} e_n(x-x'), \\ M\delta_{\mathbb{Z}_M}(m,m') &= \sum_{n\in\mathbb{Z}_M} e^M_n(m-m'). \end{aligned} \tag{4.3}$$

While the first and third relations in (4.3) define the δ distribution on $\mathbb{R}$ and [0, 1), respectively, the second and fourth relations in (4.3) are the restrictions to the lattice of the regular functions

$$\begin{aligned} \delta_{\mathbb{R},M}(x) &= \int_{-\pi M}^{\pi M} \frac{dk}{2\pi} e_k(x) = \frac{\sin(\pi Mx)}{\pi x}, \\ \delta_{[0,1),M}(x) &= \sum_{n\in\mathbb{Z}_M} e_n(x) = \frac{\sin(\pi Mx)}{\sin(\pi x)}, \end{aligned} \tag{4.4}$$

which we recognize as the Shannon (sinc) and Dirichlet kernels, respectively. After dividing and dilating them by M and translating them by m, we obtain precisely the functions (4.1). These kernels can be considered as regularizations of the aforementioned δ distributions in the sense that the momentum integral $k\in\mathbb{R}$ or momentum sum $n\in\mathbb{Z}$ has been confined to $|k|<\pi M$ and $|n|<\frac{M-1}{2}$, respectively. Both are real valued, smooth, strongly peaked at $x=0$ and have compact momentum support. The Shannon kernel like the Dirichlet kernel is an L_2 function but it is not of rapid decay with respect to position.

The simplest possible action for fermions is the massless, chiral theory in 2D Minkowski space

$$S=i\int_{\mathbb{R}} dt \int_X dx \bar{\xi}\partial\!\!\!/\xi. \tag{4.5}$$

Here $X=\mathbb{R}$ or $X=[0,1)$. The 2D Clifford algebra with signature $(-1,+1)$ is generated by $\gamma^0=\epsilon=i\sigma_2, \gamma^1=\sigma_1$, where $\sigma_1,\sigma_2,\sigma_3=\epsilon\sigma_1$ are the Pauli matrices. Then $\partial\!\!\!/=\gamma^\mu\partial_\mu, x^0=t, x^1=x$, and $\bar{\xi}=(\xi^*)^T\gamma^0$. Due to $([\gamma^0\gamma^\mu]^*)^T=\gamma^0\gamma^\mu$ the action is real valued. Generalizations to higher dimensions, massive theories, with more species or higher spin are immediate and just require the corresponding Clifford algebras.

Then $i[\xi^A]^*, A=1,2$ is canonically conjugate to ξ^A which results in the nonvanishing CAR

$$[\xi^A(x),(\xi^B)^*(y)]_+=\delta^{AB}\delta(x,y), \tag{4.6}$$

and the Hamiltonian is

$$H=-i\int_X dx\{[\xi^*]^T\sigma_3\xi'\}(x) \tag{4.7}$$

with $\xi'=\partial\xi/\partial x$ which is linear in spatial derivatives. Indeed, the Dirac-Weyl equation $\partial\!\!\!/\xi=0$ is reproduced by the Heisenberg equation of (4.7)

$$i\dot{\xi}=[H,\xi]=i\sigma_3\xi' \Leftrightarrow \epsilon\dot{\xi}-\epsilon\sigma_3\xi'=\partial\!\!\!/\xi=0. \tag{4.8}$$

As (4.7) is indefinite as it stands, we introduce the self-adjoint projections on $L=L_2(X,dx)$ with $s=\pm1$,

$$\begin{aligned} &Q_s=\frac{1}{2}\left[1_L+is\frac{\partial}{\omega}\right]Q, \qquad Q=1_L-1\langle 1\cdots\rangle_L/\|1\|^2_L, \\ &\omega=\sqrt{-\partial^2}, \qquad i\partial Q_s=s\omega Q_s, \end{aligned} \tag{4.9}$$

where 1 is just the constant function equal to unity, 1_L is the identity operator on the Hilbert space L, and ω is the positive square root of minus the Laplacian. Note $Q = 1_L$ for $X = \mathbb{R}$. Using the identity $\omega[Q_+ - Q_-] = i\partial$ and the definition of the Pauli matrix, we then rewrite the Hamiltonian as

$$\begin{aligned} -H &= \langle \xi_1, [Q_+ - Q_-]\omega\xi_1\rangle_L - \langle \xi_2, [Q_+ - Q_-]\omega\xi_2\rangle_L \\ &= \langle Q_+\xi_1, \omega Q_+\xi_1\rangle_L - \langle Q_-\xi_1, \omega Q_-\xi_1\rangle_L \\ &\quad - \langle Q_+\xi_2, \omega Q_+\xi_2\rangle_L + \langle Q_-\xi_2, \omega Q_-\xi_2\rangle_L. \end{aligned} \tag{4.10}$$

Thus we declare

$$\begin{aligned} A_{1,+} &:= (Q_+\xi_1)^*, \qquad A_{1,-} := Q_-\xi_1, \\ A_{2,-} &:= (Q_-\xi_2)^*, \qquad A_{2,+} := Q_+\xi_2 \end{aligned} \tag{4.11}$$

as annihilators and obtain the normal ordered, positive semidefinite Hamiltonian

$$:H := \sum_{B=1,2;\sigma=\pm} \int_X dx A^*_{B,\sigma}\omega A_{B,\sigma}, \tag{4.12}$$

where the $A_{B,\sigma}$ obey the CAR

$$[A_{B,s}(x), [A_{B',s'}(x')]^*]_+ = \delta_{BB'}\delta_{ss'}Q_s(x,x'), \tag{4.13}$$

where $Q_s(x,x')$ is the integral kernel $(Q_s f)(x) = \int_X dx' Q_s(x,x')f(x')$. Note that the zero modes of ξ_B do not contribute to H so we have to quantize them without guidance from the form of the Hamiltonian. With $Q^\perp = 1_L - Q$ we define $A_{B,0} := Q^\perp \xi_B$ as the annihilation operator which is nonvanishing only for $X = [0,1)$.

From this perspective, the problem of the fermion doublers on the lattice $\frac{1}{M}\mathbb{Z}$ or $\frac{1}{M}\mathbb{Z}_M$ for $X = \mathbb{R}$ and $X = [0,1)$, respectively, is encoded in the way one discretizes the partial derivative ∂ that appears in the projections Q_s (in Hamiltonian renormalization the time variable and time derivatives are kept continuous). For scalar theories, ∂ appears only quadratically in the Laplacian $\Delta = -\partial^2$ while for fermions it appears linearly. This problem is therefore not only present for fermions but for all theories in which besides the Laplacian also the partial derivatives themselves are involved in the quantization process. One such example is the parametrized field theory which shares many features with string theory [11].

Alternatively, this problem shows up in the discretization of the two-point functions of the theory (as the theory is free, the two-point function determines all higher N-point functions). To compute them from the current Hamiltonian setting we use the CAR to compute the Heisenberg time evolution of the annihilators (from now on normal ordering is being understood)

$$A_{B,\sigma}(t,x) = e^{-itH}A_{B,\sigma}(x)e^{itH} = [e^{it\omega}A_{B,\sigma}](x), \tag{4.14}$$

where $Q_\sigma A_{B,\sigma} = A_{B,\sigma}$ was used. Then the nonvanishing two-point functions before discretization are for the case $= \mathbb{R}$ using the definitions in (4.9)

$$\begin{aligned} \langle \Omega, \xi_B(s,x)\xi_C(t,y)^*\Omega\rangle &= \langle \Omega, ([Q_+ + Q_- + Q^\perp]\xi_B)(s,x)([Q_+ + Q_- + Q^\perp]\xi_C](t,y)^*\Omega\rangle \\ &= \langle \Omega, \{\delta_{B,1}[A^*_{1,+} + A_{1,-} + A_{1,0}] + \delta_{B,2}[A_{2,+} + A^*_{2,-} + A2,0]\}(s,x) \\ &\quad \times \{\delta_{C,1}[A_{1,+} + A^*_{1,-} + A^*_{1,0}] + \delta_{C,2}[A^*_{2,+} + A_{2,-} + A^*_{2,0}]\}(t,y)\Omega\rangle \\ &= \langle \Omega, \{\delta_{B,1}[A_{1,-} + A_{1,0}] + \delta_{B,2}[A_{2,+} + A_{2,0}]\}(s,x)\{\delta_{C,1}[A^*_{1,-} + A^*_{1,0}] + \delta_{C,2}[A^*_{2,+} + A^*_{2,0}\}(t,y)\Omega\rangle \\ &= e^{is\omega_x - it\omega_y}\{\delta_{1,B}\delta_{1,C}[Q_-(x,y) + Q^\perp](x,y) + \delta_{2,B}\delta_{2,C}Q_+(x,y) + Q^\perp](x,y)\} \\ &= \frac{1}{2}e^{is\omega_x - it\omega_y}\left\{\delta_{BC}(1 + Q^\perp) - i[\sigma_3]_{BC}\frac{\partial_x}{\omega_x}\right\}\delta(x,y) \\ &= \frac{\delta_{BC}}{2\|1\|^2} + \int \frac{dk}{2\pi 2\omega(k)} e^{i[\omega(k)(s-t) - k(x-y)]}[\omega(k)1_2 - k\sigma_3]_{BC} \\ &= \frac{\delta_{BC}}{2\|1\|^2} + \int \frac{dk}{2\pi 2\omega(k)} e^{-iK\cdot(X-Y)}[K^0(1 + Q^\perp)1_2 - K^1\sigma_3]_{BC} \\ &= \frac{\delta_{BC}}{2\|1\|^2} - i[1_2(1 + Q^\perp)\partial_{X^0} + \sigma_3\partial_{X^1}]_{BC}\int \frac{dk}{2\pi 2\omega(k)} e^{-iK\cdot(X-Y)} \\ &= \frac{\delta_{BC}}{2\|1\|^2} + i([\epsilon(1 + Q^\perp)\partial_{X^0} + \sigma_1\partial_{X^1}]\epsilon)_{BC}\Delta_+(x-y) \\ &= \frac{\delta_{BC}}{2\|1\|^2} + i[\partial\!\!\!/_X\epsilon]_{BC}\Delta_+(X-Y) \end{aligned} \tag{4.15}$$

with $K^0 \coloneqq \omega(k) = |k|, K^1 = k$ and $X^0 = s, X^1 = x, Y^0 = t, Y^1 = y$ and $K \cdot X = -K^0 X^0 + K^1 X^1$. Here Δ_+ is the Wightman two-point function of a free massless Klein-Gordon field in 2D Minkowski space:

$$\Delta_+(X-Y) = \int \frac{dk}{2\pi 2\omega(k)} e^{-iK\cdot(X-Y)}. \tag{4.16}$$

A similar computation yields (X, Y and B, C and Q_+, Q_- switch and the contribution from $A_{B,0}$ is missing leading to $-\delta_{BC}$ in the final result)

$$\begin{aligned} &\langle \Omega, \xi_C(t,y)^* \xi_B(s,x)\Omega\rangle \\ &= -\frac{\delta_{BC}}{2\|1\|^2} + i\epsilon[\epsilon\partial_{Y^0} - \sigma_3\partial_{Y^1}]_{CB}\Delta_+(Y-X) \\ &= -\frac{\delta_{BC}}{2\|1\|^2} + i[\epsilon\not{\partial}_Y]_{CB}\Delta_+(Y-X). \end{aligned} \tag{4.17}$$

Using the conjugate spinor $\bar{\xi} = [\xi^*]^T\epsilon$ we may rewrite (4.16) and (4.17) as

$$\begin{aligned} \langle \Omega, \xi(X)\otimes\bar{\xi}(Y)\Omega\rangle &= \frac{\epsilon}{2\|1\|^2} + i\not{\partial}_X\Delta_+(X-Y), \\ \langle \Omega, \bar{\xi}(Y)\otimes\xi(X)\Omega\rangle &= -\frac{\epsilon}{2\|1\|^2} + i\not{\partial}_Y\Delta_+(Y-X), \end{aligned} \tag{4.18}$$

which gives the time ordered two-point function or Feynman propagator

$$\begin{aligned} D_F(X-Y) &\coloneqq \langle \Omega, T[\xi(X)\otimes\bar{\xi}(Y)]\Omega\rangle \\ &\coloneqq \theta(X^0-Y^0)\langle\Omega, \xi(X)\otimes\bar{\xi}(Y)\Omega\rangle \\ &\quad - \theta(Y^0-X^0)\langle\Omega,\bar{\xi}(Y)\otimes\xi(X)\Omega\rangle \\ &= \not{\partial}_X\Delta_F(X-Y), \end{aligned} \tag{4.19}$$

where

$$\Delta_F(X-Y) = -i\lim_{\epsilon\to 0+}\int\frac{d^2K}{(2\pi)^2}\frac{e^{-iK\cdot(X-Y)}}{-K\cdot K - i\epsilon} \tag{4.20}$$

is the Feynman propagator of the 2D massless Klein-Gordon field. We see that $\not{\partial}_X D_F(X-Y) = i\delta^{(2)}(X-Y)$ due to $\not{\partial}^2 = \Box$, i.e., $D_F = i\not{\partial}^{-1}$.

Turning to the discretization, in Hamiltonian renormalization one discretizes only x, ∂_x and confines only $|K^1| < \pi M$, while in the Euclidean approach one discretizes also t, ∂_t and confines $|K^0| < \pi M$. In any case we see that it is the projections Q_s that directly translate into $\not{\partial}$ which is linear in the derivatives. If the propagator is to keep the property to invert the Dirac-Weyl operator $\not{\partial}$, then we are forced to write the momentum expression of (4.19), say in the Hamiltonian approach, as

$$\frac{\epsilon K_0 + \sigma_1\lambda_M(K_1)}{K_0^2 - \lambda_M(K_1)^2 - i\epsilon}, \tag{4.21}$$

where $[\partial_M e_{K_1}](X^1) = i\lambda_M(K_1)e_{K_1}(X^1), X^1\in\mathbb{Z}/M, |K_1| \le \pi M$ defines the eigenvalues of the discrete derivative and indices are moved with the Minkowski metric.

The case $X = [0,1)$ is literally the same, just that we must sum over $k = K^1 = 2\pi n, n\in\mathbb{Z}$ rather than integrating over $K^1\in\mathbb{R}$ with measure $dK^1/(2\pi)$. Also the $Q^\perp$ contribution is now nontrivial but cancels in the Feynman propagator. That is, all expressions before discretization remain the same except that we must replace Δ_+, Δ_F by

$$\begin{aligned} \Delta_+(X-Y) &= \sum_{n\in\mathbb{Z}}\frac{1}{2\omega(n)}e^{-iK\cdot(X-Y)}, \qquad \omega(n) = 2\pi|n|, \qquad K_1 = 2\pi n, \\ \Delta_F(X-Y) &= -i\int\frac{dK^0}{2\pi}\sum_{n\in\mathbb{Z}}\frac{e^{-iK\cdot(X-Y)}}{-K\cdot K - i\epsilon}, \qquad K_1 = 2\pi n. \end{aligned} \tag{4.22}$$

These can now be discretized as for the case $X = \mathbb{R}$, i.e., by using (4.21) with $[\partial_M e_{K_1}](X^1) = i\lambda_M(K_1)e_{K_1}(X^1)$ where $X^1\in\mathbb{Z}_M/M, |K_1|\in\mathbb{Z}_M$.

Turning to the details of the discretization, in the so-called "naive" discretization one writes

$$(\partial_M f_M)(m) \coloneqq \frac{M}{2}[f_M(m+1) - f_M(m-1)] \tag{4.23}$$

for $f_M\in L_M$ the Hilbert space of square symmable sequences on the lattice. Using the Fourier functions $f_M(m) = e_k^M(m) = e_k(x_m^M)$ with $|k| < \pi M$ for $X = \mathbb{R}$ and $f_M(m) = e_n^M(m) = e_{2\pi n}(x_m^M)$ with $|n| \le \frac{M-1}{2}$ and $x_m^M = \frac{m}{M}$ with $m\in\mathbb{Z}$ or $m\in\mathbb{Z}_M$, respectively, we find the eigenvalues $\lambda_M(k)$ given by $iM\sin(\frac{k}{M})$ and $iM\sin(\frac{2\pi n}{M})$, respectively. These vanish in the allowed domain of k

and n, respectively, at $k=0, k=\pm\pi M$ and $n=0, n=\frac{M}{2}$ if M is even, otherwise only at $n=0$ with a corresponding doubler pole in the propagator when $K^0=0$. We see that there are no doublers in the compact case for lattices with odd numbers of points even with respect to the naive discretization of the discrete derivative. Still, even in the compact case and for odd M the eigenvalue $iM\sin(\pi\frac{M-1}{M}) = -iM\sin(\pi/M)$ for $n=\frac{M-1}{2}$ approaches $-i\pi$ for large M while most other eigenvalues are large of order M, and thus $n=\pm(M-1)/2$ can be considered as an "almost" doubler mode.

We now show that the spectrum of ∂_M is *doubler free* if we do not pick the naive discretization but rather the *natural discretization* provided by the maps I_M and $I_M^\dagger$ in terms of which the renormalization flow is defined. This discretization is defined by

$$\partial_M := I_M^\dagger \partial I_M \tag{4.24}$$

for both $X=\mathbb{R}$ and $X=[0,1)$ and is well defined whenever the MRA functions χ_m^M are at least C^1. Note that with this definition ∂_M is automatically antisymmetric since ∂ is. In fact, for the Haar flow [16] based on characteristic functions $\hat{\chi}_m^M$ of intervals $[m/M, (m+1)/M)$ partitioning X and which is not C^1 we formally find

$$\begin{aligned}\partial_M f_M(m) &= M\sum_{\tilde{m}}\langle\hat{\chi}_m^M, [\hat{\chi}_{\tilde{m}}^M]'\rangle_L f_M(m)\\ &= -M\sum_{\tilde{m}}\langle[\hat{\chi}_m^M]', \hat{\chi}_{\tilde{m}}^M\rangle_L f_M(m)\\ &= \frac{M}{2}[f_M(m+1) - f_M(m-1)],\end{aligned} \tag{4.25}$$

i.e., precisely the naive derivative where we have formally integrated by parts in between and used that $\hat{\chi}_m^M$ is of compact support for $X=\mathbb{R}$ and periodic for $X=[0,1)$, respectively. Thus, the Haar flow results in the naive discretization that yields the doubler troubled spectrum.

Note that the map $I_M\colon L_M \to L$ has a range in V_M and, in fact, $I_M^\dagger\colon L\to L_M$ restricts to the inverse as $I_M^\dagger I_M = 1_{LM}$; i.e., L_M and V_M are in bijection. Thus, if, in fact, ∂ preserves V_M, then the spectrum of ∂_M will simply coincide with that of ∂ except that k will be restricted from $\mathbb{R}$ to $[-\pi M, \pi M]$ and n from $\mathbb{Z}$ to $\hat{\mathbb{Z}}_M$. This is precisely what happens for both the Shannon and the Dirichlet kernels as we will now confirm.

For the Shannon kernel in the case $X=\mathbb{R}$ we compute

$$\begin{aligned}(\partial_M f_M)(m) &= M\sum_{\tilde{m}\in\mathbb{Z}} f_M(\tilde{m})\langle\chi_m^M, \partial\chi_{\tilde{m}}^M\rangle_L(\partial_M f_M)(m)\\ &= M\sum_{\tilde{m}\in\mathbb{Z}} f_M(\tilde{m})\int_{\pi M}^{\pi M}\frac{dk}{2\pi}(ik)\langle\chi_m^M, e_k\rangle L\langle e_k, \chi_{\tilde{m}}^M\rangle_L\\ &= M\sum_{\tilde{m}\in\mathbb{Z}} f_M(\tilde{m})\int_{\pi M}^{\pi M}\frac{dk}{2\pi}(ik)e_k(x_m^M - x_{\tilde{m}}^M)\\ &= \sum_{\tilde{m}\in\mathbb{Z}} f_M(\tilde{m})[\partial_x\chi_{\tilde{m}}^M(x)]_{x=x_m^M}\\ &= \sum_{\tilde{m}\in\mathbb{Z}} f_M(\tilde{m})\left[\frac{y\cos(M\pi y) - (M\pi)^{-1}\sin(\pi M y)}{y^2}\right]_{y=x_m^M - x_{\tilde{m}}^M},\end{aligned} \tag{4.26}$$

which displays the nonlocal nature of the discrete derivative as all points $\tilde{m}\in\mathbb{Z}$ contribute. However, Eq. (4.26) vanishes at $m=\tilde{m}$ and takes the maximal value $\mp M$ at $m-\tilde{m}=\pm 1$, which shows that it approximates the naive derivative in the vicinity of m. On the other hand, for $f_M = e_k^M$ we find the exact eigenfunctions

$$(\partial_M e_k^M)(m) = M\int_{\pi M}^{\pi M}\frac{dq}{2\pi}(iq)e_q(x_m^M)\sum_{\tilde{m}\in\mathbb{Z}} e_{k-q}(x^M\tilde{m}) = ike_k^M(m) \tag{4.27}$$

with manifestly doubler free spectrum.

For the Dirichlet kernel in the case $X=[0,1)$ the computations are completely analogous:

$$\begin{aligned}(\partial_M f_M)(m) &= M\sum_{\tilde{m}\in\mathbb{Z}_M} f_M(\tilde{m})\langle\chi_m^M, \partial\chi_{\tilde{m}}^M\rangle_L(\partial_M f_M)(m)\\ &= M\sum_{\tilde{m}\in\mathbb{Z}_M} f_M(\tilde{m})\sum_{|n|\le\frac{M-1}{2}}(2\pi\, in)\langle\chi_m^M, e_{2\pi n}\rangle_L\langle e_{2\pi n}, \chi_{\tilde{m}}^M\rangle_L\end{aligned}$$

$$
\begin{aligned}
&= M \sum_{\tilde{m}\in\mathbb{Z}_M} f_M(\tilde{m}) \sum_{|n|\le\frac{M-1}{2}} (2\pi\, in) e_{2\pi n}(x^M_m - x^M_{\tilde{m}}) \\
&= \sum_{\tilde{m}\in\mathbb{Z}_M} f_M(\tilde{m}) [\partial_x \chi^M_{\tilde{m}}(x)]_{x=x^M_m} \\
&= \sum_{\tilde{m}\in\mathbb{Z}_M} f_M(\tilde{m}) \pi \left[\frac{\sin(\pi y)\cos(M\pi y) - M^{-1}\sin(\pi M y)\cos(\pi y)}{\sin^2(\pi y)}\right]_{y=x^M_m - x^M_{\tilde{m}}},
\end{aligned}
\tag{4.28}
$$

which displays the nonlocal nature of the discrete derivative as all points $\tilde{m}\in\mathbb{Z}_M$ contribute. However, Eq. (4.28) vanishes at $m=\tilde{m}$ and takes the maximal value $\mp M$ at $m-\tilde{m}=\pm 1$ that approximates the naive derivative in the vicinity of m. On the other hand, for $f_M = e^M_n$ we find the exact eigenfunctions

$$
(\partial_M e^M_n)(m) = M \sum_{\tilde{m}\in\mathbb{Z}_M} e^M_n M(\tilde{m}) \sum_{|\tilde{n}|\le\frac{M-1}{2}} (2\pi\, i\tilde{n}) e^M_{\tilde{n}}(m-\tilde{m}) = 2\pi\, in\, e^M_n(m)
\tag{4.29}
$$

with manifestly doubler free spectrum.

We now study the Shannon or Dirichlet flow of the (non)compact theory. We start with some initial discretization $\partial^{(0)}_M, \omega^{(0)}_M = \sqrt{-[\partial^{(0)}_M]^2}, Q^{(0)}_{M,s} = \frac{1}{2}[1_{L_M} + is\frac{\partial^{(0)}_M}{\omega^{(0)}_M}]$, which determines the annihilators in analogy to (4.11):

$$
\begin{aligned}
A^{(0)}_{M,1,+} &:= (Q^{(0)}_{M,+}\xi_{M,1})^*, \qquad & A^{(0)}_{M,1,-} &:= Q^{(0)}_{M,-}\xi_{M,1}, \\
A^{(0)}_{M,2,-} &:= (Q^{(0)}_{M,-}\xi_{M,2})^*, \qquad & A^{(0)}_{M,2,+} &:= Q^{(0)}_{M,+}\xi_{M,2},
\end{aligned}
\tag{4.30}
$$

the vacuum $\Omega^{(0)}_M$, the Fock space $\mathcal{H}^{(0)}_M$, and the initial Hamiltonian family

$$
H^{(0)}_M = \sum_{m\in\mathbb{Z}} \sum_{B,\sigma} [A^{(0)}_{M,B,\sigma}]^* \omega^{(0)}_M A^{(0)}_{M,B,\sigma},
\tag{4.31}
$$

and similar for the compact case with the restriction $m\in\mathbb{Z}_M$.

We can encode the flow (3.4) and (3.5) into a single quantity $\partial^{(n)}_M$ in terms of which we define analogously $\omega^{(n)}_M = \sqrt{-[\partial^{(n)}_M]^2}$ and $Q^{(n)}_{M,s} = \frac{1}{2}\left[1_{L_M} + is\frac{\partial^{(n)}_M}{\omega^{(n)}_M}\right]$ as well as

$$
\begin{aligned}
A^{(n)}_{M,1,+} &:= (Q^{(n)}_{M,+}\xi_{M,1})^*, \qquad & A^{(n)}_{M,1,-} &:= Q^{(n)}_{M,-}\xi_{M,1}, \\
A^{(n)}_{M,2,-} &:= (Q^{(n)}_{M,-}\xi_{M,2})^*, \qquad & A^{(n)}_{M,2,+} &:= Q^{(n)}_{M,+}\xi_{M,2},
\end{aligned}
\tag{4.32}
$$

and the initial Hamiltonian family

$$
H^{(n)}_M = \sum_{m\in\mathbb{Z}} \sum_{B,\sigma} [A^{(n)}_{M,B,\sigma}]^* \omega^{(n)}_M A^{(n)}_{M,B,\sigma},
\tag{4.33}
$$

and again for the compact case we just restrict to $m\in\mathbb{Z}_M$.

To see that this is, indeed, possible we note that in the corresponding Fock spaces it is sufficient to check isometry on vectors of the form

$$
\begin{aligned}
\Psi^{(n)}_{M'}(I_{MM'}F_{M,1},\ldots,I_{MM'}F_{M,N}) &:= A^{(n)}_{M'}(I_{MM'}F_{M,1})^* \cdots A^{(n)}_{M'}(I_{MM'}F_{M,N})^* \Omega^{(n)}_{M'}, \\
A^{(n)}_M(F_M) &:= \sum_{B,\sigma} \langle F_{M,B,\sigma}, A^{(n)}_{M,B,\sigma}\rangle_{L_M}.
\end{aligned}
\tag{4.34}
$$

These give the inner products

$$
\begin{aligned}
&\langle \Psi^{(n)}_{M'}(I_{MM'}F_{M,1},\ldots,I_{MM'}F_{M,N}), \Psi^{(n)}_{M'}(I_{MM'}G_{M,1},\ldots,I_{MM'}G_{M,\tilde{N}})\rangle_{\mathcal{H}^{(n)}_{M'}} \\
&\quad = \delta_{N,\tilde{N}} \det([\langle Q^{(n)}_{M'} I_{MM'}F_{M,k}, Q^{(n)}_{M'} I_{MM'}G_{M,l}\rangle_{L^4_{M'}}]^N_{k,l=1}),
\end{aligned}
\tag{4.35}
$$

where

$$
\begin{aligned}
\langle Q^{(n)}_{M'} I_{MM'}F_M, Q^{(n)}_{M'} I_{MM'}G_M\rangle_{L^4_{M'}} &= \sum_{B,\sigma} \langle I_{MM'}F_{M,B,\sigma}, Q^{(n)}_{M'\sigma} I_{MM'}G_{M,B,\sigma}\rangle_{L_{M'}} \\
&= \sum_{B,\sigma} \langle F_{M,B,\sigma}, [I^\dagger_{MM'} Q^{(n)}_{M',\sigma} I_{MM'}] G_{M,B,\sigma}\rangle_{L_M}.
\end{aligned}
\tag{4.36}
$$

We used that, whatever $\partial_M^{(n)}$ is, the corresponding operators $Q_{M,s}^{(n)} Q_{M,s'}^{(n)} = \delta_{s,s'} Q_{M,s}^{(n)}$ are orthogonal projections and that the $B = 1, 2$ species anticommute. Comparing with

$$\langle \Psi_M^{(n+1)}(F_{M,1}, \ldots, F_{M,N}), \Psi_M^{(n+1)}(G_{M,1}, \ldots, G_{M,\tilde{N}}) \rangle_{\mathcal{H}_M^{(n+1)}}, \tag{4.37}$$

we obtain isometry iff

$$Q_{M,\sigma}^{(n+1)} = I_{MM'}^\dagger Q_{M',\sigma}^{(n)} I_{MM'}. \tag{4.38}$$

Similarly, since

$$[H_{M'}^{(n)}[A_{M'}^{(n)}(I_{MM'}F_M)]^*] = -[A_{M'}^{(n)}(\omega_{M'}^{(n)} Q_{M'}^{(n)} I_{MM'} F_M)]^*, \tag{4.39}$$

we get a match between the matrix elements of Hamiltonians iff

$$\omega_M^{(n+1)} = I_{MM'}^\dagger \omega_{M'}^{(n)} I_{MM'}, \tag{4.40}$$

where we used that by construction $[\omega_M^{(n)}, Q_{M,s}^{(n)}] = 0$.

We now ask under what conditions on the coarse-graining kernel I_M both (4.38) and (4.40) are implied by

$$\partial_M^{(n+1)} \coloneqq I_{MM'}^\dagger \partial_{M'}^{(n)} I_{MM'}. \tag{4.41}$$

Theorem 1. Suppose that $\partial_M^{(0)} \coloneqq I_M^\dagger \partial I_M$ is the natural discrete derivative with respect to a coarse-graining kernel $I_M\colon L_M \to L$ and such that $[\partial, I_M I_M^\dagger] = 0$. Then (4.41) implies both (4.38) and (4.40).

Proof. By (4.41) we have

$$\partial_M^{(1)} = [I_{MM'}]^\dagger \partial_{M'}^{(0)} I_{MM'} = I_M^\dagger \partial I_M = \partial_M^{(0)} \tag{4.42}$$

since by construction $I_M = I_{M'} I_{MM'}$. Thus by iteration $\partial_M^{(n)} = \partial_M^{(0)} = \partial_M$ is already fixed pointed, no matter what the coarse-graining maps I_M are as long as they descend from an MRA.

It follows that

$$\partial_M^N = I_M^\dagger (\partial [I_M I_M^\dagger])^{N-1} \partial I_M. \tag{4.43}$$

While $I_M^\dagger I_M = 1_{L_M}$ by isometry, $p_M \coloneqq I_M I_M^\dagger$ is a projection in L (onto the subspace V_M of the MRA). Thus, if $[\partial, p_M] = 0$, we find $\partial_M^N = I_M^\dagger \partial^N I_M$. The claim then follows from the spectral theorem (functional calculus). ∎

To see that both the Shannon and Dirichlet kernels satisfy the assumption of the theorem, it suffices to remark that they only depend on the difference $x - y$; i.e., they are translation invariant. Explicitly, since the χ_m^M with $m \in \mathbb{Z}$ and $m \in \mathbb{Z}_M$, respectively, are an ONB of V_M just as are the e_k, $|k| \le \pi M$ and $e_{2\pi n}$, $|n| \le \frac{M-1}{2}$, respectively,

$$(p_M f)(x) = \sum_m \chi_m^M(x) \langle \chi_m^M, f \rangle = \begin{cases} \int_X dy \left[\int_{-\pi M}^{\pi M} \frac{dk}{2\pi} e_k(x-y) \right] f(y) & X = \mathbb{R} \\ \int_X dy \left[\sum_{|n| \le \frac{M-1}{2}} e_{2\pi n}(x-y) \right] f(y) & X = [0,1) \end{cases}, \tag{4.44}$$

and integration by parts does not lead to boundary terms due to the support properties of f or by periodicity, respectively.

It follows that by using the natural discretization the free Weyl fermion theory is already at its fixed point and the fixed point family member at resolution M coincides with the continuum theory blocked from the continuum to resolution M; that is, by simply dropping the superscript $^{(n)}$ we have

$$J_M \Omega_M = \Omega, \qquad J_M A_M(F_{M,1})^* \cdots A_M(F_{M,N})^* \Omega_M, = A(I_M F_{M,1})^* \cdots A(I_M F_{M,N})^* \Omega, \qquad H_M = J_M^\dagger H J_M. \tag{4.45}$$

This would not hold using the Haar discretization, and more complicated theories require further analysis also in the presence of the Dirichlet-Shannon kernel.

Remark. Thus, the translation invariance of the Shannon and Dirichlet kernels, respectively, is, besides smoothness, another important difference with the Haar kernel [26]

$$\sum_m \chi_m^M(x) \chi_m^M(y) = \sum_m \chi_{[\frac{m}{M}, \frac{m+1}{M})}(x) \chi_{[\frac{m}{M}, \frac{m+1}{M})}(y), \tag{4.46}$$

which is not translation invariant. Therefore in this case the flows of ω_M or ω_M^{-1} are not simply related by $\omega_M = I_M^\dagger \omega I_M$ and $\omega_M^{-1} = I_M^\dagger \omega^{-1} I_M$, and thus one must define ω_M as the inverse of the covariance ω_M^{-1}. As $M \to \infty$ this difference disappears, but at finite M it is present and makes the study of the flow with respect to a nontranslation invariant kernel much more and unnecessarily involved.

V. CONCLUSION AND OUTLOOK

In this paper we have extended the definition of Hamiltonian renormalization in the sense of [9], which is motivated by quantum gravity from the bosonic to the fermionic case. The definition given in [9], in fact, covers both cases but the practical implementation for bosons was in terms of measures [18] that cannot be used for fermions. We have tested the scheme for massless 2D chiral fermion theories, the extension to the massive and higher dimensional case being immediate, just requiring the higher dimensional Clifford algebra. In particular, we showed that using the smooth local Shannon-Dirichlet kernel for renormalization and discretization results in simple flow, an easy computable fixed point theory that coincides with the known continuum theory and has manifestly doubler free spectrum even at finite resolution due to the inherent nonlocality with respect to the chosen finite resolution microscopes based on those kernels.

An immediate extension of the current paper that suggests itself is to apply the current framework to the known solvable 2D interacting fermion theories [27].

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