Hamiltonian renormalization. VI. Parametrized field theory on the cylinder

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(Received 13 April 2023; accepted 7 November 2023; published 15 December 2023)

Hamiltonian renormalization, as defined within this series of works, was derived from covariant Wilson renormalization via Osterwalder-Schrader reconstruction. As such it directly applies to quantum field theory (QFT) with a true (physical) Hamiltonian bounded from below. The validity of the scheme was positively tested for free QFT in any dimension with or without Abelian gauge symmetries of Yang-Mills type. The aim of this Hamiltonian renormalization scheme is to remove quantization ambiguities of Hamiltonians in interacting QFT that remain even after UV and IR regulators are removed as it happens in highly nonlinear QFT such as quantum gravity. Also, while not derived for that case, the renormalization flow formulas can without change also be applied to QFT without a single true Hamiltonian but rather an infinite number of Hamiltonian constraints. In that case a number of interesting questions arise: (1) Does the flow reach the correct fixed point also for an infinite number of "Hamiltonians" simultaneously? (2) As the constraints are labeled by test functions, which in the presence of a regulator are typically regularized (discretized and of compact support), how do those test functions react to the flow? (3) Does the quantum constraint algebra, which in the presence of a regulator is expected to be anomalous, close at the fixed point? These questions should ultimately be addressed in quantum gravity. Before one considers this interacting, constrained QFT, it is well-motivated to consider a free, constrained QFT where the fixed point is explicitly known. In this paper we therefore address the case of parametrized field theory for which the quantum constraint algebra coincides simultaneously with the hypersurface deformation algebra of quantum gravity (or any other generally covariant theory) and the Virasoro algebra of free, closed, bosonic string theory or other conformal field theories to which the results of this paper apply verbatim. The central result of our investigation is that the finite resolution (discretized) constraint algebras *typically do not close*, that there is not necessarily anything wrong with that, and that anomaly freeness of the continuum algebra is encoded in the convergence behavior of the renormalization flow.

DOI: 10.1103/PhysRevD.108.125006

I. INTRODUCTION

Interacting quantum field theory (QFT) typically have to be constructed: One first defines a regulated theory (with both UV and IR regulators present) and then tries to remove the regulator, thereby renormalizing the bare parameters (i.e., redefining them in terms of measured parameters and regulators). That procedure of *constructive QFT*, even if successful in the sense that the unregulated, nonperturbative theory is well-defined, may yet be ambiguous; i.e., it may keep a memory of which regularization procedure was applied. We will refer to such ambiguities as *quantization ambiguities*. One expects this problem the more likely to occur the more nonlinear the theory is. An extreme case is quantum gravity whose Einstein-Hilbert action depends nonpolynomially on the metric field.

Such ambiguities are not severe if they can be encoded by a finite number of (so-called relevant) parameters. They could be fixed by a finite number of experiments and thus lead to a predictive theory. However, if that parameter space is infinite dimensional, the theory is not predictive. To make it predictive, the number of free parameters must be downsized to a finite dimensional manifold. To achieve this, one imposes a restriction on the family of regulated theories: they must qualify, at the finite regulator, as the coarse-grained versions of a continuum theory at a resolution defined by that regulator. For instance, a Euclidean QFT may be defined by a family of measures μ_r where r denotes the regulator. The measure μ_r knows how to integrate functionals of the Euclidean quantum field smeared with test functions that are restricted up to resolution r. Thus, in order to produce unambiguous results, for any finer resolution $r' < r$ we must have $[\mu_{r'}]_r = \mu_r$; i.e., since the quantum field tested at resolution r can be written in terms of the quantum field at resolution r', we can use $\mu_{r'}$ instead of μ_r to integrate functions restricted to resolution r.

This so-called *cylindrical consistency* is basically the idea of Wilson renormalization [1]. A cylindrically

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consistent family of measures μ_r in turn defines a continuum measure μ that can integrate the quantum field at any resolution under rather mild assumptions [2]. From a practical viewpoint, the cylindrical family is then sufficient because in reality one never considers physical processes at infinite resolution, and thus the explicit construction of μ is not needed. Now in constructive QFT one typically starts with an initial family $\{\mu_r^{(0)}\}_{r\in R}$ where R is the regulator manifold. It typically comes with an in principle infinite number of parameters $p\in P$ that enter via the discretization freedom of the classical theory (action) that one starts from (e.g., next neighbor, next to next neighbor, … , terms in the Laplacian). Even if the limit $\mu^{(0)} := \lim_{r\to 0}\mu_r^{(0)}$ exists as a measure, it will typically retain a nontrivial dependence on *all* "directions" of the parameter manifold P. Therefore, it is natural to *improve* the initial family and define a sequence of families $n\in\mathbb{N}_0\mapsto\{\mu_r^{(n)}\}_{r\in R}$ by $\mu_r^{(n+1)} := [\mu_{\kappa(r)}^{(n)}]_r$ where $\kappa(r) < r$ maps to a finer resolution. This defines a flow of measure families which may have a fixed point $\{\mu_r^*\}_{r\in R}$ which by construction is consistent at least with respect to the *coarse grainings* $\kappa(r)\to r$. Experience shows that this usually also makes the fixed point family consistent with respect to all pairs $r' < r$. In the course of this process, it may happen that all but finite (relevant) directions in P have been fixed to a fixed value. In that case we say that the QFT has been nonperturbatively renormalized to a predictive QFT.

These ideas were first formulated in quantum statitistical field theory (i.e., Euclidean field theory [3]) using path integral methods. Using Osterwalder-Schrader (OS) reconstruction one can also translate them into the Hamiltonian language [4] (see also [5] for closely related earlier Hamiltonian renormalization schemes and references therein). The validity of [4] has been tested in free field theories without [6] and with [7] Abelian gauge symmetry of the Yang-Mills type. The motivation for [4] is actually its application in Hamiltonian quantum gravity, specifically in its loop quantum gravity (LQG) incarnation [8]. Since the classical Einstein-Hilbert action is nonpolynomial in the metric field, the quantization ambiguity problem is expected to be especially severe in this case. Indeed, quantum gravity is not perturbatively renormalizable, which motivates the nonperturbative path integral renormalization program known as asymptotic safety [9]. In the Hamiltonian setting, while it is possible to rigorously define the Hamiltonian constraint operators [10], they suffer from quantization ambiguities so that a Hamiltonian renormalization thereof is well-motivated [11,12]. See also [11] for a comparison with other renomalization schemes that are applied to general relativity.

At first it may look strange why an OS motivated Hamiltonian renormalization scheme should apply at all to quantum gravity: OS reconstruction delivers a Hamiltonian operator H bounded from below on a Hilbert space $\mathcal{H}$ and a ground state $\Omega\in\mathcal{H}$. However, canonical quantum gravity does not come with a Hamiltonian but rather an infinite number of Hamiltonian constraints $C(N)$ on a Hilbert space $\mathcal{H}'$ where N are test functions (called lapse functions). For no choice of N are these bounded from below, and rather than the spectrum of H on $\mathcal{H}$ one is interested in the joint kernel of the $C(N)$ on $\mathcal{H}'$ defining the physical Hilbert space $\mathcal{H}$ which does not coincide with $\mathcal{H}$ and is typically not a subspace thereof (typically it is a space of distributions on a dense subspace of $\mathcal{H}'$). However, on the one hand, it is possible to cast quantum gravity into the framework of an ordinary quantum Hamiltonian system by using Hamiltonian constraint gauge fixings [13]. In this reduced phase space approach one then retains a physical Hamiltonian directly on the physical Hilbert space $\mathcal{H}$.

On the other hand, it turns out that the Hamiltonian renormalization flow, while derived from the OS renormalization scheme, can formally be applied also to more than one operator and in particular also those that are not bounded from below, certainly but not necessarily when they share a common ground state Ω. This observation allows the attractive perspective to monitor the fate of the commutator algebra of the $C(N)$ during the renormalization process that is not possible in the reduced phase approach where the $C(N)$ are solved classically. Classically we have the closed hypersurface deformation algebra [14] $\{C(M), C(N)\} = C(f(M,N))$ where $f(M,N)$ are new test functions which in more than two spacetime dimensions or with density weight different from two also depend on the metric. This fact makes it especially difficult to turn this into an *anomaly-free* constraint operator commutator $[C(M), C(N)] = i\hbar$ "$C(f(M,N))$" because of the ordering problem involved in $C(f(M,N))$ [15]. Indeed, the development of [10] can be interpreted as saying that $[C(M), C(N)] = i\hbar C(\tilde{f}(M,N))$ closes with the correct ordering [i.e., the kernel of the $C(f(M,N))$ is contained in that of the $C(N)$] but with the wrong "structure functions," that is, the operators $\tilde{f}(M,N)$ do not qualify as the quantization of $f(M,N)$. To improve on this state of affairs, one may modify the quantization of the $C(N)$ without resorting to renormalization methods, an ambitious very interesting program that is now in motion [16] and to which the developments of the current paper may be viewed as complementary; see especially the parametrized field theory application of that program [17] (and also [18] where qualitatively similar results were obtained without changing the notion of convergence of regulated operators as defined in [10]).

More in detail, the Hamiltonian renormalization flow works with a family of triples $(\mathcal{H}_r, H_r, \Omega_r)$ where $\mathcal{H}_r$ is a Hilbert space, H_r is a self-adjoint operator on $\mathcal{H}_r$ (bounded from below if coming from an OS measure), and Ω_r is a ground state of H_r, i.e., $H_r\Omega_r = 0$. The regulator labels r belong to partially ordered and directed set R. Given isometric embeddings $J_{rr'}: \mathcal{H}_r \to \mathcal{H}_{r'}; r < r'$ to be constructed subject to the consistency condition $J_{r'r''}J_{rr'} = J_{rr''}; r < r' < r''$ and that ensure $J_{rr'}\Omega_r = \Omega_{r'}$, one defines the inductive limit Hilbert space $\mathcal{H}$ by a standard construction [19]. Moreover, at the fixed point, the H_r form a consistently defined family of quadratic forms $H_r = J^\dagger_{rr'} H_{r'} J_{rr'}, r < r'$ defining a continuum form H. That form may or may not define a self-adjoint operator on $\mathcal{H}$ and in particular is in general not to be confused with the inductive limit of the H_r which is not granted to exist.

In extending this framework to more than one (in field theory, even an infinite number of) operators, we face several new questions:

(1) We start with an initial family $C^{(0)}_r(N)$ of operators on an initial family of Hilbert spaces $\mathcal{H}^{(0)}_r$, one for each resolution scale r and one for each continuum smearing function N. The origin of r typically comes from a discretization of the continuum field ϕ and conjugate momentum π in terms of coarse-grained variables ϕ_r, π_r and substituting them for ϕ, π in the expression for $C(N)$. Does this automatically induce a discretization $N^{(0)}_r$ of N as well? If not, should one supply one by hand or leave N in its continuum form?

(2) Is it possible or necessary to find a common zero eigenvector $\Omega^{(0)}_r \in \mathcal{H}^{(0)}_r$ of the $C^{(0)}_r(N)$ or $C^{(0)}_r(N^{(0)}_r)$ independent of N or $N^{(0)}_r$? This is far from trivial: while the classical continuum constraints form a closed Poisson algebra of real functions, there is no reason to take it for granted that the algebra of the discretized $C^{(0)}_r(N)$ or $C^{(0)}_r(N^{(0)}_r)$ closes under taking commutators. In fact, this is most likely not the case because typically the classical constraint algebra rests on the validity of the Leibniz rule for partial derivatives. However, discretized derivatives do not obey the Leibniz rule [20]. Thus, not only can these constraints not be simultaneously diagonalized, it may even be that their joint kernel just consists of the zero vector. In that case, we have to assume that there exists at least a cyclic vector $\Omega^{(0)}_r$ for the algebra of operators under consideration in the common dense domain of all constraints.

(3) Given that $\Omega^{(0)}_r$ can be found, one can proceed as in the case of just one Hamiltonian operator and construct a sequence of families of Hilbert spaces $\mathcal{H}^{(n)}_r$ and isometric injections $J^{(n)}_{rr'}: \mathcal{H}^{(n+1)}_r \to \mathcal{H}^{(n)}_{r'}$ for $r' < r$ such that $J^{(n)}_{rr'}\Omega^{(n+1)}_r = \Omega^{(n)}_{r'}$. The isometry requirement translates into flow equations for the Hilbert space measures $\nu^{(n)}_r$ underlying $\mathcal{H}^{(n)}_r = L_2(Q_r, d\nu^{(n)}_r)$ where Q_r is a flow invariant model configuration space. Assuming that a fixed point $J_{rr'}$ of this flow of isometric injections can be found (equivalently, a cylindrically consistent measure family ν_r), one can construct a continuum Hilbert space $\mathcal{H}$ as the inductive limit of the $\mathcal{H}_r = L_2(Q_r, d\nu_r)$. In tandem, one constructs a flow of families' quadratic forms $C^{(n+1)}_r(N) \coloneqq [J^{(n)}_{rr'}]^\dagger C^{(n)}_{r'}(N) J^{(n)}_{rr'}$, one for each N. Can one arrange that all of them flow into a fixed point whatever choice of N is made? Or should one rather also let the discretized smearing functions flow according to $C^{(n+1)}_r(N^{(n+1)}_r) \coloneqq [J^{(n)}_{rr'}]^\dagger C^{(n)}_{r'}(N^{(n)}{}_{r'}) J^{(n)}_{rr'}$?

(4) Suppose that a simultaneous fixed point family $C_r(N)$ or $C_r(N_r)$ can be found. Then by construction $C_r(N) = J^\dagger_r C(N) J_r$ or $C_r(N_r) = J^\dagger_r C(N) J_r$ where $J_r: \mathcal{H}_r \to \mathcal{H}$ is the isometric embedding granted to exist by the inductive limit construction. Is it true that $C(N)$ is no longer plagued by an infinite number of quantization ambiguities? Is it true that the algebra of commutators of $C(N)$ is nonanomalous? Note that it is not clear that the commutators can even be computed because $C(N)$ is just a quadratic form.

(5) Assuming that these questions can be answered in the affirmative, how does one recognize anomaly freeness at finite resolution? Note that the $C_r(N)$ will most certainly not close under forming commutators even if the $C(N)$ do because [11]

$$[C_r(M), C_r(N)] = J^\dagger_r [C(M) P_r C(N) - C(N) P_r C(M)] J_r, \tag{1.1}$$

where $P_r \coloneqq J_r J^\dagger_r$ is a projection in $\mathcal{H}$. *It is therefore generically not expected that the finite resolution projections of the constraints form a closed algebra.* However, given closure in the continuum, we may rewrite (1.1) as

$$[C_r(M), C_r(N)] = i\hbar C_r(f(M,N)) - J^\dagger_r [C(M)(1_{\mathcal{H}} - P_r) C(N) - C(N)(1_{\mathcal{H}} - P_r) C(M)] J_r, \tag{1.2}$$

and the anomalous term naively vanishes as r is removed and P_r becomes $1_{\mathcal{H}}$. This, when supplied by a suitable operator topology of convergence, may serve as a practical guide toward proving anomaly freeness even if one cannot determine the continuum operator $C(N)$ in closed form.

It would be very interesting to find necessary and sufficient conditions under which the above questions can be answered in the affirmative. In this paper we confine ourselves to the much easier task to illustrate and work out the catalog of questions and answers for the case of parametrized massless Klein-Gordon field theory in $1+1$ spacetime dimensions.

The architecture of this paper is as follows:

In Sec. II we briefly review $(1+1)$-dimensional parametrised field theory (PFT) following the notation of [18]. We treat both the classical and the quantum theories.

In Sec. III we specialize the general framework of [4,11] to PFT. We choose as the regulator space a nested system of square lattices. Here we learn the first important lesson from the present work: The constraint operators are ill-defined on the dense domain of finite resolution subspaces generated by the discretized Weyl algebra unless the test functions that enter that Weyl algebra and that define the renormalization flow display at least a minimal amount of smoothness. This issue did not arise in the works [6] because there the renormalization could be phrased in terms of the covariance of the Gaussian measure which decays sufficiently fast at infinity in momentum space even when smeared against the discontinuous test functions used. However, in PFT we also need inverse powers of that covariance. This observation triggered the work [21] where we generalize [11] in a natural way to a generalized multiresolution analysis (MRA) based renormalization flows of which there are even smooth candidates, thus removing the aforementioned obstacle. In fact, Ref. [6] turns out to be a special case of [21] as [6] is based on the so-called Haar MRA. On the other hand, as the convergence to the continuum via sequences of discontinuous or smooth functions should not affect the continuum fixed point theory, we also offer an equivalent solution to the just mentioned smoothness problem within the Haar MRA class based on zeta function regularization which is a common tool in conformal field theories (CFT) such as PFT.

In Sec. IV we show that there exists a well-motivated discretization of the PFT constraints. Clearly, because of the central term in the Virasoro algebra there does not exist a single vector in the joint point kernel of all constraints, not even in the continuum. However, there does exist a preferred cyclic vector in the common dense domain of all constraints which serves as a substitute, both in the continuum and at finite resolution. We can then proceed in a similar manner to Refs. [6,7] and compute the flow and fixed point of the corresponding Hilbert space measures.

In Sec. V we compute the flow of the constraint operators. We show that the first option of leaving the smearing functions N untouched (not discretized by hand) does not induce a canonical discretization of the smearing functions of the constraints. On the other hand, using the coarse-graining map that is used to compute the flow of measures, vacua, and constraints to discretize their smearing functions by hand does lead to a cylindrically consistent system (under change of resolution) of constraints.

In Secs. VI and VII we compute the algebra of constraints at finite resolution and illustrate the behavior of (1.1) and (1.2). It is at this point that we learn the second most important lesson from the present work when trying to show that the discrete algebra converges to the continuum algebra in the weak operator topology:

- (i) When working with nondiscretized constraint smearing functions, there is just one correction to the continuum algebra at the finite resolution indicated in (1.1) and (1.2). However, when additionally discretizing the constraint smearing function by hand, an additional correction arises.
- (ii) Convergence to zero of the first correction requires a minimal amount of smoothness of the test functions of the Weyl algebra for reasons similar to those mentioned before concerning the domain of definition of the constraints.
- (iii) Convergence to zero of the second correction requires sufficient smoothness of the discretized smearing function N of the constraints, which is, of course, not surprising because the Virasoro algebra depends on third order (Schwartzian) derivatives of those smearing functions.

We establish convergence using for instance the Dirichlet flow of [21] rather than the Haar flow of [6].

In Sec. VIII we summarize and conclude our findings for this model which presents the next logical step in the research program started in [4,6,7,11].

The most important lessons learned from the present work are as follows:

- (A) Finite resolution constraints *typically do not close*.
- (B) This is *no problem at all; in fact, it would be physically wrong:* It just displays the mathematical fact that the constraints typically are not block diagonal with respect to different resolution Hilbert subspaces. If they would be, they would "know about the dynamics"; however, the finite resolution projections we use are entirely kinematical. The failure to close is *no anomaly but a finite resolution artifact*.
- (C) Whether the *continuum algebra* closes, i.e., is free of anomalies, *can be checked using finite resolution analysis*: The finite resolution artifact should converge to zero. This is of practical importance because in more complicated theories one will hopefully be able to construct the theory at finite resolution but perhaps computing the infinite resolution (continuum) theory may be too hard but also unnecessary as measurements always have finite resolution.

II. BRIEF REVIEW OF PFT

This section mainly serves to introduce our notation and follows [18]. See [18] for more information and references therein. See also [22] for more details on the quantization of

PFT using classically equivalent constraints for which the quantum anomaly is formally a coboundary so that it can be (formally—i.e., modulo showing the existence of corresponding Hilbert space representations) absorbed into a noncentral quantum correction of the constraints. See [23] for renormalization of closely related (fermionic) CFTs.

A. Classical theory

The spacetime is the infinite cylinder $Z_R = \mathbb{R}\times C_R$ where C_R is the circle of radius R with Minkowski metric $\eta = \mathrm{diag}(-1,1)$ and Cartesian coordinates $T := X^0 \in \mathbb{R}$, $X := X^1 \in [0, 2\pi R)$. We introduce another cylinder Z of unit radius $Z = \mathbb{R}\times S^1$ with coordinates $(x^0 = t, x^1 = x)$ and consider the diffeomorphism $\varphi: Z \to Z_R; (t,x) \mapsto T(t,x), X(t,x)$ upon which T, X become fields on Z. Note that T is periodic $T(t,x+1) = T(t,x)$ while X is an angular variable $X(t,x+1) = X(t,x) + 2\pi R$.

The action of the massless Klein-Gordon field ϕ on Z_R,

$$S = -\frac{1}{2}\int_{Z_R} d^2X \eta^{AB}\phi_{,A}\phi_{,B}, \tag{2.1}$$

is pulled back by the above diffeomorphism and yields via $\phi = \varphi^*\Phi$ the PFT action

$$S = -\frac{1}{2}\int_{Z} d^2x |\det(g)|^{1/2} g^{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta}; \qquad g = \phi^*\eta, \tag{2.2}$$

which by construction is invariant under reparametrizations (diffeomorphisms) of Z. It is thus an example of a generally covariant field theory, and thus its canonical formulation in terms of Hamiltonian C and spatial diffeomorphism constraints D must yield a representation of the abstract hypersurface deformation algebra of the one parameter family of hypersurfaces $t \mapsto \Sigma_t = \varphi(t,[0,1))$ discovered in [14]. Using standard methods one finds

$$H = PX' + YT' + \frac{1}{2}[\Pi^2 + (\Phi')^2], \quad D = PT' + YX' + \Pi\Phi', \tag{2.3}$$

where $\dot{(.)} = \partial_t(.), (.)' = \partial_x(.)$, and (P, Y, Π) are the momenta conjugate to (T, X, Φ), respectively; i.e., the nontrivial equal t Poisson brackets are

$$\{P(u), T(v)\} = \{Y(u), X(v)\} = \{\Pi(u), \Phi(v)\} = \delta(u,v) \tag{2.4}$$

with the δ distribution on S^1

$$\delta(u,v) = \sum_{n\in\mathbb{Z}} e^{i2\pi(u-v)n}. \tag{2.5}$$

One quickly verifies the hypersurface deformation algebra $\mathfrak{h}$ relations

$$\{D(f), D(g)\} = D([f,g]), \quad \{D(f), H(g)\} = H([f,g]),$$
$$\{H(f), H(g)\} = D([f,g]); \quad [f,g] := f'g - fg', \tag{2.6}$$

where f and g are periodic, real valued smearing functions on S^1 and, e.g., $D(f) = \int_{S^1} dx f D$. Geometrically, C and D are scalar densities of weight two, and f and g are scalar densities of weight minus one, which is why $[f,g]$ is independent of the spatial metric $q = g_{xx}$, an effect that can happen only in one spatial dimension.

We note that the constraints depend only on the derivatives of X, T, Φ and thus do not contain information about their respective zero modes. We denote them by Φ_0, X_0, T_0. Also, since X is not periodic in contrast to Y, P, Π, T, Φ, the field X' has a phase space independent zero mode given by $2\pi R$. We thus write

$$X(x) = 2\pi R x + \tilde{X}(x), \tag{2.7}$$

where $\tilde{X}$ has the same zero mode as X and is still conjugate to P. We can thus write the constraints as

$$D = 2\pi R Y + \tilde{D}, \qquad H = 2\pi R P + \tilde{H}, \tag{2.8}$$

where $\tilde{D}$ and $\tilde{H}$ differ from D and H upon replacing X by $\tilde{X}$. The zero modes of Y, P, Π can be extracted as

$$Y_0 = Q_\perp \cdot Y := \int_0^1 dx Y(x), \qquad Q := 1_L - Q_\perp, \tag{2.9}$$

and similar for P_0, Π_0. Note Q is an orthogonal projection on $L := L_2([0,1), dx)$ extracting the nonzero modes of a function.

It is convenient to introduce the field combinations

$$X_\pm := \tilde{X} \pm T, \qquad P_\pm := \frac{1}{2}(Y \pm P), \qquad A_\pm := P_+ \pm X'_+,$$
$$B_\pm := P_- \pm X'_-, \qquad C_\pm := \Pi \pm \Phi' \tag{2.10}$$

in terms of which we can write the constraints as

$$D_\pm := \frac{1}{2}(\tilde{D} \pm \tilde{H}), \qquad \tilde{D}_+ = \frac{1}{4}[(A_+)^2 - (A_-)^2 + (C_+)^2],$$
$$\tilde{D}_- = \frac{1}{4}[(B_+)^2 - (B_-)^2 - (C_-)^2]. \tag{2.11}$$

One checks

$$\{A_\pm(u), A_\pm(v)\} = \pm 2\partial_v \delta(u,v) \{A_\pm(u), A_\mp(v)\} = 0 \tag{2.12}$$

and similar for B and C, all other brackets vanishing, so that

$$\{D_\pm(f), D_\pm(g)\} = D_\pm([f,g]), \qquad \{D_\pm(f), D_\mp(g)\} = 0. \tag{2.13}$$

The original variables can be recovered from $A_\pm, B_\pm, C_\pm$ except for the zero modes of the configuration variables

$$\Pi = \frac{1}{2}[C_+ + C_-], \quad Y = P_+ + P_- = \frac{1}{2}[A_+ + A_- + B_+ + B_-], \quad P = P_+ - P_- = \frac{1}{2}[A_+ + A_- - B_+ - B_-],$$
$$\Phi' = \frac{1}{2}[C_+ - C_-], \quad \tilde{X}' = X'_+ + X'_- = \frac{1}{2}[A_+ - A_- + B_+ - B_-], \quad T' = X'_+ - X'_- = \frac{1}{2}[A_+ - A_- - B_+ + B_-], \tag{2.14}$$

so that the zero modes of Y, P, Π but not those of $\tilde{X}, T, \Phi$ are available from $A_\pm, B_\pm, C_\pm$. For the original constraints we find

$$\tilde{D}_\pm = \frac{1}{2}[D \pm H] = D_\pm + 2\pi R P_\pm \tag{2.15}$$

with $P_+ = \frac{1}{2}[A_+ + A_-]$ and $P_- = \frac{1}{2}[B_+ + B_-]$. Therefore, also

$$\{\tilde{D}_\pm(f), \tilde{D}_\pm(g)\} = \tilde{D}_\pm([f,g]), \qquad \{\tilde{D}_\pm(f), \tilde{D}_\mp(g)\} = 0. \tag{2.16}$$

In what follows we will only consider the a lgebra of the $D_\pm(f)$. The algebra of the $\tilde{D}_\pm$ can be treated by identical methods.

B. Quantum theory

The classical system consists of three independent scalar fields X, T, Φ, which are coupled via the constraints that are only quadratic in the fields and their momenta. We thus use a Fock representation. In most approaches to PFT and also the closed bosonic string [24] one constructs a Fock space using the mode functions $e_n(x) \coloneqq \exp(i2\pi n x)$ which form an orthonormal basis of the "one particle Hilbert space" $L = L_2([0,1), dx)$ and defines $A_\pm(n) \coloneqq A_\pm(e_n) = \int dx e_n(x)^* A_\pm(x)$, etc., from which one finds $A_\pm(n)^* = A_\pm(-n)$,

$$\{A_\pm(n_1), A_\pm(n_2)\} = \pm 2(i n_2)\delta_{n_1+n_2,0}, \tag{2.17}$$

or in terms of commutators

$$[A_\pm(n_1), A_\pm(n_2)] = \pm 2 n_1 \delta_{n_1+n_2,0}. \tag{2.18}$$

This allows one to interpret $A_+(n)$ as an annihilation operator and $A_+(n)^*$ as a creation operator for $n > 0$, $A_-(n)$ as an annihilation operator and $A_-(n)^*$ as a creation operator for $n < 0$, while $A_+(0) = A_-(0) = (P_+)_0$ (zero mode). Similar remarks hold for $B_\pm$ and $C_\pm$ where $B_+(0) = B_-(0) = (P_-)_0$ and $C_+(0) = C_-(0) = (\Pi)_0$. This split with respect to the sign of n makes the discussion somewhat cumbersome as it requires one to introduce six different Fock spaces and a separate discussion of the zero mode sector.

Let us therefore introduce the quantities

$$A \coloneqq \frac{1}{\sqrt{2}}[\omega^{1/2} Q X_+ - i\omega^{-1/2} Q P_+],$$
$$B \coloneqq \frac{1}{\sqrt{2}}[\omega^{1/2} Q X_- - i\omega^{-1/2} Q P_-],$$
$$C \coloneqq \frac{1}{\sqrt{2}}[\omega^{1/2} Q \Phi - i\omega^{-1/2} Q \Pi], \tag{2.19}$$

where

$$\omega^2(.) = -(.)'' \eqqcolon -\Delta \tag{2.20}$$

is minus the Laplacian on S^1 and ω its positive square root. The quantities (2.19) are the standard annihilation operators of three massless Klein-Gordon fields where we have been careful to remove the zero mode on which the Laplacian is not invertible (if there would be a mass term, we would have $\omega^2 = m^2 - \Delta$ and in this case a separate discussion of the zero mode is not necessary).

For the zero modes we set

$$A_0 \coloneqq \frac{1}{\sqrt{2}}[\omega_0^{1/2} Q_\perp X_+ - i\omega_0^{-1/2} Q_\perp P_+],$$
$$B_0 \coloneqq \frac{1}{\sqrt{2}}[\omega_0^{1/2} Q_\perp X_- - i\omega_0^{-1/2} Q_\perp P_-],$$
$$C_0 \coloneqq \frac{1}{\sqrt{2}}[\omega_0^{1/2} Q_\perp \Phi - i\omega_0^{-1/2} Q_\perp \Pi], \tag{2.21}$$

where $\omega_0 > 0$ is an arbitrary parameter of dimension of inverse length. It is therefore natural to set it equal to $1/R$ but we will keep it unfixed for the moment.

For any operator valued distribution O and any smearing function f we set

$$\langle f, O\rangle \coloneqq \int dx f^*(x) O(x) \eqqcolon O(f^*). \tag{2.22}$$

Then, by promoting the Poisson brackets to commutators

$$[\langle f, A_0\rangle, \langle g, A_0\rangle^*] = \langle f, Q_\perp g\rangle, \quad [\langle f, A\rangle, \langle g, A\rangle^*] = \langle f, Q g\rangle, \tag{2.23}$$

and similar for the B and C sectors, all other commutators vanish. Here $*$ is the respective complex conjugate of (2.19) and (2.20) extended to an involution on linear combinations of products.

The relation among these annihilators is as follows:

$$A_\pm = P_+ \pm X'_+ = Q_\perp P_+ + Q(P_+ \pm X'_+)$$
$$= i\sqrt{\frac{\omega_0}{2}}[A_0 - A_0^*] + i\sqrt{\frac{\omega}{2}}[A - A^*] \pm \sqrt{\frac{1}{2\omega}}[A + A^*]'$$
$$= i\sqrt{\frac{\omega_0}{2}}[A_0 - A_0^*] + i\sqrt{2\omega}\{[Q_\pm A] - [Q_\pm A]^*\}, \quad (2.24)$$

where

$$Q_\pm = \frac{1}{2}\left[1_L \mp i\frac{\partial}{\omega}\right]Q \quad (2.25)$$

projects onto the positive/negative Fourier modes: $Q_\pm e_n = e_n$ if $n > / < 0$ and zero otherwise. Note that $Q_\pm$ is an orthogonal (i.e., self-adjoint) projection on the 1-particle Hilbert space L which commutes with Q, ∂, ω which can be seen by using the common eigenbasis e_n. As $[Q_\pm A]^* = Q_\mp A^*$ (Fock space, not L space adjoint), it follows that

$$i\sqrt{2\omega}A = Q_+ A_+ + Q_- A_-, \quad (2.26)$$

which demonstrates that the Fock space defined by declaring A as annihilation operators is the same as the tensor product of Fock spaces defined by declaring $Q_+ A_+, Q_- A_-$ as annihilators, which is exactly relation (2.18). Similar statements hold for the B and C sectors. It is thus equivalent but more economic to work with A rather than $A_\pm$, and we consider the Fock space $\mathcal{H}$ with Fock vacuum Ω annihilated by A_0 and A.

We compute the commutators corresponding to (2.13). We introduce the building blocks

$$E_0 \coloneqq \sqrt{\omega_0}[A_0 - A_0^*], \qquad E_\pm \coloneqq \sqrt{\omega} Q_\pm A \quad (2.27)$$

so that

$$A_\pm = i\left(\frac{1}{\sqrt{2}}E_0 + \sqrt{2}[E_\pm - E_\pm^*]\right). \quad (2.28)$$

Since we need $A_\pm^2$, there is an ordering ambiguity with respect to the term $(E_\pm - E_\pm^\dagger)^2$. We pick normal ordering with respect to the annihilators A and leave a possible normal ordering constant proportional to the algebraic unit 1 open for the moment; that is, we set

$$A_\pm^2(f) = -\left[\frac{1}{2}E_0^2(Q_\perp f) + 2E_0(1)(E_\pm(f) - E_\pm(f)^*)(f) + 2{:}(E_\pm - E_\pm^*)^2{:}\ (f)\right] \eqqcolon T_\pm^0(f) + T_\pm^1(f) + T_\pm^2(f), \quad (2.29)$$

where $:(.):$ denotes normal ordering. We have used in (2.29) that f is real valued. As $[E_0, E_\pm] = 0$, we find with $s, s' = \pm$,

$$[A_s^2(f), A_{s'}^2(g)] = [T_s^1(f), T_{s'}^1(g)] + [T_s^1(f), T_{s'}^2(g)] - [T_{s'}^1(g), T_s^2(f)] + [T_s^2(f), T_{s'}^2(g)]. \quad (2.30)$$

We have with

$$E_s(f) = \langle \omega^{1/2} Q_s f, A\rangle \quad (2.31)$$

that

$$\begin{aligned}
[T_s^1(f), T_{s'}^1(g)] &= 4E_0(1)^2[E_s(f) - E_s(f)^*, E_{s'}(g) - E_{s'}(g)^*] \\
&= -4E_0(1)^2\{[E_s(f), E_{s'}(g)^*] - [E_{s'}(g), E_s(f)^*]\} \\
&= -4E_0(1)^2\{\langle \omega^{1/2}Q_s f, \omega^{1/2}Q_{s'}g\rangle - \langle \omega^{1/2}Q_{s'}g, \omega^{1/2}Q_s f\rangle\} \\
&= -4\delta_{ss'}E_0(1)^2\{\langle f, \omega; Q_s g\rangle - \langle g, \omega Q_s g\rangle\} \\
&= -2\delta_{ss'}E_0(1)^2(-is)\{\langle f, g'\rangle - \langle g, f'\rangle\} \\
&= 4is\delta_{ss'}T_s^0([f, g]),
\end{aligned} \quad (2.32)$$

where we used that $2\omega Q_s = 1 - is\partial$. Next

$$\begin{aligned}
[T_s^1(f), T_{s'}^2(g)] &= 4E_0(1)[E_s(f) - E_s(f)^*, [E_{s'}]^2(g) + [E_{s'}^*]^2(g) - 2[E_{s'}^* E_{s'}](g)] \\
&= 4E_0(1)\{[E_s(f), [E_{s'}^*]^2(g) - 2[E_{s'}^* E_{s'}](g)] - [E_s(f)^*, [E_{s'}]^2(g) - 2[E_{s'}^* E_{s'}](g)]\} \\
&= 8E_0(1)\int dxdyf(x)g(y)[K_{ss'}(x,y)\{E_{s'}^*(y) - E_{s'}(y)\} - K_{s's}(y,x)\{E_{s'}^*(y) - E_{s'}(y)\}]
\end{aligned} \quad (2.33)$$

with the kernel

$$K_{ss'}(x,y) = [E_s(x), E^*_{s'}(y)] \Rightarrow K_{ss'}(x,y)^* 1_{\mathcal{H}} = [K_{ss'}(x,y)1_{\mathcal{H}}]^* = K_{s's}(y,x). \tag{2.34}$$

Explicitly,

$$\langle f, K_{ss'} \cdot g\rangle = [\langle f, E_s\rangle, \langle g, E_{s'}\rangle^*] = \delta ss' \langle f, \omega Q_s g\rangle =: \delta_{ss'}\langle f, K_s \cdot g\rangle. \tag{2.35}$$

Abbreviating $E^g_{s'}(y) = g(y)E_{s'}(y)$ we obtain

$$\begin{aligned}[T^1_s(f), T^2_{s'}(g)] &= -8E_0(1)\{\langle f, K_{ss'} \cdot \{E^g_{s'} - [E^g_{s'}]^*\}\rangle - \langle f, K^*_{ss'} \cdot \{E^g_{s'} - [E^g_{s'}]^*\}\rangle\} \\ &= -8E_0(1)\langle [K_{ss'} - K^*_{ss'}] \cdot f, \{E^g_{s'} - [E^g_{s'}]^*\}\rangle \\ &= -8E_0(1)\delta_{ss'}\{\langle f, \omega(Q_s - Q_{-s})\{E^g_s - [E^g_s]^*\}\rangle\} \\ &= 8E_0(1)\delta_{ss'}(is)\{\langle f, \{E^g_s - [E^g_s]^*\}'\rangle\},\end{aligned} \tag{2.36}$$

whence

$$\begin{aligned}[T^1_s(f), T^2_{s'}(g)] - [T^1_{s'}(g), T^2_s(f)] &= 8E_0(1)\delta_{ss'}(is)\{\langle f, \{E^g_s - [E^g_s]^*\}'\rangle - \langle g, \{E^f_s - [E^f_{s'}]^*\}'\rangle\} \\ &= -8E_0(1)\delta_{ss'}(-is)[E_s - (E_s)^*](fg' - f'g) \\ &= 4is\delta_{ss'}T^1_s([f,g]).\end{aligned} \tag{2.37}$$

Finally,

$$\begin{aligned}[T^2_s(f), T^2_{s'}(g)] = 4\int dx \int dy\, f(x)g(y)\{&[E_s(x)^2, [E_{s'}(y)^\dagger]^2 - 2E^*_{s'}(y)E_{s'}(y)] + [E_s(x)^*]^2, [E_{s'}(y)]^2 - 2E^*_{s'}(y)E_{s'}(y)] \\ &- 2[E_s(x)^*E_s(x), [E_{s'}(y)^*]^2 + [E_{s'}(y)]^2 - 2E^*_{s'}(y)E_{s'}(y)]\} \\ = 4\int dx \int dy\, f(x)g(y)\{&K_{ss'}(x,y)[2E_s(x)E_{s'}(y)^* + 2E_{s'}(y)^*E_s(x) - 4E_s(x)E_{s'}(y)] \\ &- K_{s's}(y,x)[2E_s(x)^*E_{s'}(y)2E_{s'}(y)E_s(x)^* - 4E_s(x)^\dagger E_{s'}(y)^*] \\ &- 2(K_{ss'}(x,y)E_s(x)^\dagger[2E_{s'}(y)^* - 2E_{s'}(y)] - K_{s's}(x,y)[2E_{s'}(y) - 2E_{s'}(y)^*]E_s(x))\} \\ = 4\int dx \int dy f(x)g(y)& \\ \times\{&K_{ss'}(x,y)[2K_{ss'}(x,y) + 4[E_{s'}(y)^* - E_{s'}(y)]E_s(x)] - K_{s's}(y,x)[2K_{s's}(y,x) - 4E_s(x)^*[E_{s'}(y)^* - E_{s'}(y)]] \\ &- 4(K_{ss'}(x,y)E_s(x)^*[E_{s'}(y)^* - E_{s'}(y)] - K_{s's}(y,x)[E_{s'}(y) - E_{s'}(y)^*]E_s(x))\}.\end{aligned} \tag{2.38}$$

Since $K_{ss'} = \delta_{ss'}K_s$ we can simplify (5.31) using $F_s(x) := E_s(x) - E_s(x)^\dagger$ and $f_x = f(x), g_y = g(y)$,

$$\begin{aligned}[T^2_s(f), T^2_{s'}(g)] &= 8\,\delta_{ss'}\int dx \int dy\, f(x)g(y)\{K_s(x,y)[K_s(x,y) - 2F_s(y)E_s(x)] - K_s(y,x)[K_s(y,x) + 2E_s(x)^*F_s(y)] \\ &\quad + 2K_s(x,y)E_s(x)^*F_s(y) + 2K_s(y,x)F_s(y)E_s(x)\} \\ &= 8\delta_{ss'}\int dx \int dy\, K_s(x,y)\{f_x g_y[K_s(x,y) - 2{:}E_s(x)F_s(y){:} \\ &\quad + 2{:}E_s(x)^*F_s(y){:}] - f_y g_x[K_s(x,y) + 2{:}E_s(y)^*F_s(x){:} - 2{:}E_s(y)F_s(x){:}]\} \\ &= 8\delta_{ss'}\int dx \int dy\, K_s(x,y)\{f_x g_y[K_s(x,y) - 2{:}F_s(x)F_s(y)]{:}] - f_y g_x[K_s(x,y) - 2{:}F_s(y)F_s(x){:}]\} \\ &= 8\,\delta_{ss'}\int dx \int dy\, K_s(x,y)\{f_x g_y - f_y g_x\}\{K_s(x,y) - 2{:}F_s(x)F_s(y)]{:}\},\end{aligned} \tag{2.39}$$

where we used that within the normal ordering symbol operator valued distributions commute. Using $F_s^f(x) = f(x)F_s(x)$ the second term in (2.39) can be written

$$\begin{aligned}
&-16\,\delta_{ss'}:\;[F_s^f(\omega Q_s \cdot F_s^g) - F_s^g(\omega Q_s \cdot F_s^f)]: \\
&= -8\,\delta_{ss'}(-is):[F_s^f((F_s^g)') - F_s^g((F_s^f)')]: \\
&= -8\,\delta_{ss'}(-is):[F_s^2]:(fg' - f'g) \\
&= 4\,i\,s\,\delta_{ss'}T_s^2([f,g]),
\end{aligned} \tag{2.40}$$

where we used that the operator ω is symmetric.

The first term in (2.39) can be evaluated as follows: Let $\mathbb{Z}_s = \{n \in \mathbb{Z}; sn \geq 0\}$ and then

$$\begin{aligned}
&(K_s \cdot f)(x) = 2\pi \sum_{n\in\mathbb{Z}_s} |n| e_n(x)\langle e_n, f\rangle \\
&\Rightarrow K_s(x,y) = 2\pi \sum_{n\in\mathbb{Z}_s} |n| e_n(x) e_n(y)^*.
\end{aligned} \tag{2.41}$$

Thus,

$$\begin{aligned}
\int dx \int dy\, f_x g_y K_s(x,y)^2 &= (2\pi)^2 \sum_{n_1,n_2\in\mathbb{Z}_s} |n_1 n_2| \langle f, e_{n_1+n_2}\rangle\langle e_{n_1+n_2}, g\rangle \\
&= (2\pi)^2 \sum_{n_1,n_2\in\mathbb{Z}_+} n_1 n_2 \langle f, e_{s(n_1+n_2)}\rangle\langle e_{s(n_1+n_2)}, g\rangle \\
&= (2\pi)^2 \sum_{n\in\mathbb{Z}_+} \langle f, e_{sn}\rangle\langle e_s, g\rangle \left[\sum_{n_1=0}^{n} n_1(n-n_2)\right] \\
&= (2\pi)^2 \sum_{n\in\mathbb{Z}_+} \langle f, e_{sn}\rangle\langle e_{sn}, g\rangle \left[n\frac{1}{2}n(n+1) - \frac{1}{6}n(n+1)(2n+1)\right] \\
&= (2\pi)^2 \frac{1}{6} \sum_{n\in\mathbb{Z}_+} \langle f, e_{sn}\rangle\langle e_{sn}, g\rangle [n^3 - n] \\
&= (2\pi)^2 \frac{s}{6} \sum_{n\in\mathbb{Z}_s} \langle f, e_{sn}\rangle\langle e_{sn}, g\rangle [(sn)^3 - (sn)] \\
&= (2\pi)^2 \frac{s}{6} \sum_{n\in\mathbb{Z}} \langle Q_s f, e_n\rangle\langle e_n, g\rangle [n^3 - n] \\
&= (2\pi)^2 \frac{s}{6} \sum_{n\in\mathbb{Z}} \langle Q_s f, e_n\rangle \left\langle \left[\left(\frac{-i\partial}{2\pi}\right)^3 - \frac{-i\partial}{2\pi}\right] e_n, g\right\rangle \\
&= (2\pi)^2 \frac{s}{6} \sum_{n\in\mathbb{Z}} \langle Q_s f, e_n\rangle \left\langle e_n, \left[\left(\frac{-i\partial}{2\pi}\right)^3 - \frac{-i\partial}{2\pi}\right] g\right\rangle \\
&= (2\pi)^2 \frac{s}{6} \left\langle Q_s f, \left[\left(\frac{-i\partial}{2\pi}\right)^3 - \frac{-i\partial}{2\pi}\right] g\right\rangle \\
&= (2\pi)^2 \frac{is}{6} \left\langle Q_s f, \left[\frac{1}{(2\pi)^3} g''' + \frac{1}{2\pi} g'\right]\right\rangle.
\end{aligned} \tag{2.42}$$

Thus, the first term in (2.39) can be written

$$\begin{aligned}
&8is\delta_{ss'}\frac{1}{6}(2\pi)^2\left\{\left\langle Q_s f, \left[\frac{1}{(2\pi)^3} g''' - \frac{1}{2\pi} g'\right]\right\rangle - \left\langle Q_s g, \left[\frac{1}{(2\pi)^3} g''' - \frac{1}{2\pi} g'\right]\right\rangle\right\} \\
&= 4is\delta_{ss'}\frac{1}{6}\left[(2\pi)^2\left\{\left\langle f, \left[\frac{1}{(2\pi)^3} g''' + \frac{1}{2\pi} g'\right]\right\rangle - \left\langle g, \left[\frac{1}{(2\pi)^3} f''' + \frac{1}{2\pi} f'\right]\right\rangle\right\}\right] \\
&=: 4is\delta_{ss'} c S(f,g),
\end{aligned} \tag{2.43}$$

where the term proportional $i\,s\,\partial/\omega$ in Q_s has dropped out as $\partial^2/\omega = -\omega$ and $\partial^4/\omega = \omega^3$ are symmetric operators on L. The term (2.43) displays the anomaly of the classical hypersurface deformation algebra or equivalently its central extension with central charge $c = \frac{1}{6}$ which is called the Virasoro algebra with that central charge.

Altogether

$$[A_s^2(f), A_{s'}^2(g)] = 4is\delta_{ss'}\{A_s^2([f,g]) + cS(f,g)\} \tag{2.44}$$

and similar for the B and C sector so that

$$[D_s(f), D_{s'}(g)] = i\delta_{ss'}[D_s([f,g]) + 3cS(f,g)]. \tag{2.45}$$

The Lie algebraic 2-cycle $S(f,g) = -S(g,f)$ is a 2-cocycle

$$S([f,g],h) + S([g,h],f) + S([h,f],g) = 0 \tag{2.46}$$

by construction but no 2-coboundary; i.e., there is no linear functional F on the space of test functions f such that $S(f,g) = F([f,g])$. Thus, the $D_s(f)$ cannot be modified by adding $3cF(f)\cdot 1$ to obtain a proper Lie algebra.

It should be noted that the result (2.46) is purely algebraic, and it just follows from $*$-algebraic relations and the chosen (normal) ordering. It is not necessary to assume a Fock representation; we just used the $*$-algebra generated by A_0 and A and their algebraic adjoints. In order that our intended Fock representation is defined by A_0, A, etc. (thereby replacing algebraic adjoint $*$ by Hilbert space adjoint $\dagger$), we must therefore check whether the constraints (and thus their adjoints as they are manifestly symmetric) are densely defined. Since $D_{s'}$ is a linear combination of the A_s^2, B_s^2, C_s^2, it will be sufficient to show that $A_s^2(f)$ is densely defined. Since $A_s^2(f)$ is a linear combination of the $T_s^j(f), j = 0,\ 1,\ 2$ [see (2.29)], it will be sufficient to consider those. Consider first the action of $A_s^2(f)$ on the Fock vacuum

$$\begin{aligned}
||T_s^0(F)\Omega||^2 &= ||\frac{\omega_0}{2}(Q_\perp F)A_0^\dagger\Omega||^2 = \left[\frac{\omega_0}{2}(Q_\perp F)\right]^2\\
||T_s^1(F)\Omega||^2 &= ||2\sqrt{\omega_0}A_0^\dagger E_s^\dagger(F)\Omega||^2 = 4\omega_0\langle F, \omega Q_s F\rangle\\
||T_s^2(F)\Omega||^2 &= 4||:(E_\pm - E_\pm^\dagger)^2:(f)\Omega||^2 = 4||(E_\pm^\dagger)^2(F)\Omega||^2\\
&= 8\int dxdyF(x)F(y)K_s(x,y)^2\\
&= (2\pi)^2\frac{is}{6}\left\langle Q_s f, \left[\frac{1}{(2\pi)^3}F''' + \frac{1}{2\pi}F'\right]\right\rangle\\
&= (2\pi)^2\frac{(is)^2}{12}\left\langle \omega^{-1}F', \left[\frac{1}{(2\pi)^3}F''' + \frac{1}{2\pi}f'\right]\right\rangle\\
&= (2\pi)^2\frac{1}{12}\left[\frac{1}{(2\pi)^3}\langle f, \omega^{-1}F''''\rangle + \frac{1}{2\pi}\langle f, \omega^{-1}F''\rangle\right]\\
&= (2\pi)^2\frac{1}{12}\left[\frac{1}{(2\pi)^3}\langle F, \omega^3 F\rangle - \frac{1}{2\pi}\langle F, \omega F\rangle\right],
\end{aligned} \tag{2.47}$$

where we used that for smooth, real valued, periodic functions F

$$\langle F, F'''\rangle = -\frac{1}{2}\langle(((F')^2)'\rangle = 0, \qquad \langle F, F'\rangle = -\frac{1}{2}\langle(F^2)'\rangle = 0. \tag{2.48}$$

Note that $[\omega/(2\pi)]^3 - [\omega/(2\pi)]$ has a spectrum in $\mathbb{N}$.

To show that the hypersurface deformation generators are indeed densely defined and symmetric in the chosen Fock representation we should check that they map Fock states into normalizable states. It is convenient not to work with Fock states directly but rather with the states

$$w[f]\Omega, w[f] = \exp(i\langle f, \Phi\rangle) \tag{2.49}$$

for the C sector and similar for the A and B sector. By choosing $f = \sum_{k=n}^\infty s_n b_n$ for some real valued orthonormal basis (ONB) of L one can generate all Fock states from the corresponding Weyl element $w[f]$ by taking suitable derivatives of (2.49) at $s_n = 0, n\in\mathbb{N}$. This shows that the $w[f]\Omega$ with f real valued span a dense subset. A short standard calculation reveals

$$E_s(x)w[f]\Omega = ig_s(x)w[f]\Omega; \qquad g_s(x) \coloneqq [Q_s f](x). \tag{2.50}$$

We establish the finiteness of the constraint operators on the Fock states only for the most difficult piece $T_s^2(F)$; the other pieces are left to the reader. We have

$$-T_s^2(F)w[f]\Omega = \int dxF(x)([E_s(x) + ig_s(x)^*]^\dagger)^2 w[f]\Omega. \tag{2.51}$$

Thus using the creation/annihilation algebra as in (2.47) and (2.50) a straightforward calculation reveals

$$\begin{aligned}
||T_s^2(F)w[f]\omega||^2 = ||w[f]\omega||^2 &\int dxF(x)\int dyF(y)\\
&\times [2K_s(x,y)^2 + 4K_s(x,y)g(x)g(y)\\
&+ g(x)^2 g(y)^2],
\end{aligned} \tag{2.52}$$

where $g = [Q_s + Q_{-s}]f = Qf$.

We now discuss the finiteness of (2.52). To be sure, if f is smooth, then finiteness is immediate. Therefore, with respect to the smooth and quasilocal waveletlike functions introduced in [21] for the purpose of renormalization, the following complications do not arise. However, the particular set of functions that were used for renormalization in [2–4] are only piecewise smooth (in fact, constant) and display finitely many discontinuities. We therefore consider these functions in what follows in order to pinpoint which convergence issues arise, why passing to smoother coarse-graining functions to define the renormalization flow is more convenient, and how one can still work with only piecewise smooth coarse-graining functions using zeta function regularization. Readers not interested in these issues can safely skip the rest of the following paragraph.

1. Zeta function regularization

The first term in (2.52) is, of course, the vacuum contribution (2.47) and thus independent of f. We already showed that it is finite in (2.47) for smooth F. The third term can be estimated by $\|F\|_\infty^2 \|f\|_\infty^4$ where $\|.\|_\infty$ denotes the supremum norm. Thus, it is finite even if f is a discontinuous but bounded function on [0, 1). The second term is given by (up to the factor of 4)

$$\langle F(Qf), [2\omega Q_s][F(Qf)]\rangle = \langle F(Qf), Q[\omega - is\partial]Q[F(Qf)]\rangle. \tag{2.53}$$

If f is at least C^1, then the piece $-is\partial$ vanishes by a similar calculation as in (2.70). If f has discontinuities but is periodic and together with F is real valued as is the case here, then this piece still vanishes if we define for a step function with $0 \le a < b < 1, x \in [0, 1)$

$$\chi'_{[a,b)}(x) = \delta(x,a) - \delta(x,b), \quad \chi_{[a,b)}(x) = \begin{cases} 1 & a < x < b \\ \frac{1}{2} & x = a \vee x = b \\ 0 & x < a \vee b < x \end{cases}. \tag{2.54}$$

The boundary values of the step function are uniquely selected by requiring

$$\langle \chi_{[a,b)}, \chi'_{[cd)}\rangle + \langle \chi'_{[a,b)}, \chi_{[cd)}\rangle = 0 \tag{2.55}$$

for all possible (namely 13) orderings of a, b, c, d. These values also ensure that the sum of step functions for a partition of [0, 1) equals unity at every point. Thus, even in the case of discontinuities (2.53) simplifies to

$$\begin{aligned} \langle F(Qf), Q\omega Q[F(Qf)]\rangle &= \langle [F(Qf)]', Q\omega^{-1}Q[F(Qf)]'\rangle \\ &= \langle F'(Qf), Q\omega^{-1}Q[F'(Qf)]\rangle \\ &\quad + 2\langle (Qf)', F; Q\omega^{-1}Q[F(Qf)]\rangle \\ &\quad + \langle F(Qf)', Q\omega^{-1}Q[F(Qf)']\rangle. \end{aligned} \tag{2.56}$$

We have explicitly, using the spectral theorem,

$$\begin{aligned} 2\pi G &:= 2\pi Q\omega^{-1}Q[F'(Qf)] \\ &= \sum_{n=1}^{\infty} n^{-1}[e_n\langle e_n, F(Qf)\rangle + e_{-n}\langle e_{-n}, F(Qf)\rangle] \end{aligned} \tag{2.57}$$

pointwise in [0, 1); thus, the modulus squared of (2.57) can be estimated from above by the Cauchy-Schwartz inequality and using $|e_n| = 1$ pointwise

$$\left[\sum_{n=1}^{\infty} n^{-2}\right]\left[\sum_{n\neq 0} |\langle e_n, F(Qf)\rangle| \le c\|F(Qf)\|_L^2\right] \le c(\|F\|_\infty \|(Qf)\|_\infty)^2, \tag{2.58}$$

where $c > 0$ is a constant. Thus since for bounded $L_2([0,1), dx)$ functions we have $\|.\|_L \le \|.\|_\infty$,

$$\begin{aligned} \|Q\omega^{-1}Q[F'(Qf)]\|_L &\le \|Q\omega^{-1}Q[F'(Qf)]\|_\infty \\ &\le c\|F\|_\infty \|(Qf)\|_\infty, \end{aligned} \tag{2.59}$$

which shows that the first term in (2.56) is finite due to $\|F'f\| < \infty$ and the Cauchy sequence (CS) inequality. The second term is also finite if Qf has finitely many discontinuities because the contributions of these discontinuities to the integral involving $(Qf)'$ amounts to a finite linear combination of evaluations of FG at those points and both functions have a finite supremum norm. The only potentially troublesome term is the last one which involves products of δ distributions. We evaluate it explicitly for the case encountered in the next sections, namely

$$Qf = \sum_{m=0}^{M} f(m)\chi_m(x), \qquad \chi_m(x) = \chi_{[x_m, x_{m+1})}(x) \tag{2.60}$$

with real valued $f(m)$ and characteristic functions χ_m of an interval where $M < \infty$ and $1 \equiv 0 = x_0 < x_1 < \cdots < x_{M-1} < 1$ is a partition of [0, 1). We find

$$\begin{aligned} \langle F(Qf)', Q\omega^{-1}[F(Qf)']\rangle &= \sum_{m_1,m_2} f(m_1)f(m_2)\{(F(Q\omega^{-1}Q[\chi'_{m_2}F])(x_{m_1}) - (F(Q\omega^{-1}Q[\chi'_{m_2}F])(x_{m_2})\} \\ &= -M^{-1}\sum_{m_1,m_2} [\partial_M f](m_1)f(m_2)\{F(Q\omega^{-1}Q[\chi'_{m_2}F]\}(x_{m_1}) \\ &= -M^{-1}\sum_{m_1,m_2} [\partial_M f](m_1)f(m_2)F(x_{m_1})\sum_{n=1}^{\infty}\frac{1}{n}[e_n(x_{m_1})\langle e_n, \chi'_{m_2}F\rangle + e_{-n}(x_{m_1})\langle e_{-n}, \chi'_{m_2}F\rangle] \end{aligned}$$

$$= -M^{-1}\sum_{m_1,m_2}[\partial_M f](m_1)f(m_2)F(x_{m_1})\sum_{n=1}^{\infty}\frac{1}{n}\{e_n(x_{m_1})[(e_{-n}F)(x_{m_2})-(e_{-n}F)(x_{m_2+1})]+\text{c.c.}\}$$

$$= M^{-2}\sum_{m_1,m_2}[\partial_M f](m_1)[\partial_M f](m_2)F(x_{m_1})F(x_{m_2})\sum_{n=1}^{\infty}\frac{1}{n}\{e_n(x_{m_1}-x_{m_2})+e_{-n}(x_{m_1}-x_{m_2})\} \tag{2.61}$$

with $(\partial_M f)(m) = M[f(m+1)-f(m)]$. It is the sum over $n\in\mathbb{N}$ in (2.61) that is problematic. We isolate and manipulate it as follows:

$$\sum_{n=1}^{\infty}\frac{1}{n}\{e_n(x_{m_1}-x_{m_2})+e_{-n}(x_{m_1}-x_{m_2})\} = 2\sum_{n=1}^{\infty}\frac{1}{n}\cos(k_M(m_1-m_2)n)$$
$$= 2\sum_{l=1}^{M-1}\frac{1}{l}\cos(k_M(m_1-m_2)l) + 2\sum_{l=0}^{M-1}\cos(k_M(m_1-m_2)l)\sum_{n=1}^{\infty}\frac{1}{l+nM}, \tag{2.62}$$

where we considered an equidistant partition, set $k_M = 2\pi/M$, and exploited periodicity modulo M. Then for any $0\le l\le M-1$ we consider

$$\sum_{n=1}^{\infty}\frac{1}{l+nM} = \sum_{n=1}^{\infty}\left[\frac{1}{l+nM}-\frac{1}{nM}\right]+\frac{1}{M}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}. \tag{2.63}$$

The first infinite sum in (2.63) converges absolutely for each l. The limit of the second sum marginally diverges to the simple pole (with residue unity) value of the Riemann zeta function. Consider

$$\gamma(N,\epsilon,\delta) \coloneqq \frac{1}{2}\sum_{n=1}^{N}\left[\frac{1}{n^{1+\delta+\epsilon}}+\frac{1}{n^{1+\delta-\epsilon}}\right]. \tag{2.64}$$

If we take the limits $\epsilon\to 0+, \delta\to 0+, N\to\infty$ in exactly this order, then we return to (2.63). As usual, regularization of infinities consists in interchanging limits that would be allowed if the sums involved would converge absolutely. We take the limits in the order $N\to\infty, \delta\to 0, \epsilon\to 0$. After $N\to\infty$ we obtain for $\delta>\epsilon>0$ the finite result

$$\gamma(\epsilon,\delta) \coloneqq \lim_{N\to\infty}\gamma(N,\epsilon,\delta) = \frac{1}{2}[\zeta(1+\delta+\epsilon)+\zeta(1+\delta-\epsilon)], \tag{2.65}$$

where ζ is the Riemann zeta function. It has an analytic extension to the whole complex plane except for its simple pole $z=1$. With this analytic extension being understood in (2.64) we can now take $\delta\to 0$,

$$\gamma(\epsilon) \coloneqq \lim_{\delta\to 0}\gamma(\epsilon,\delta) = \frac{1}{2}[\zeta(1+\epsilon)+\zeta(1-\epsilon)]. \tag{2.66}$$

Finally, we take $\epsilon\to 0$, which results in the principal value of the zeta function at unity

$$\gamma \coloneqq \lim_{\epsilon\to 0}\gamma(\epsilon) = [\text{pv}\zeta](1) = \lim_{\epsilon\to}\frac{1}{2}[\zeta(1+\epsilon)+\zeta(1-\epsilon)], \tag{2.67}$$

which turns out to be finite and equal to the *Euler-Mascheroni* constant [25]

$$\gamma = \lim_{N\to\infty}\left[-\ln(N)+\sum_{n=1}^{N}\frac{1}{n}\right], \tag{2.68}$$

which is numerically 0.58 in the second decimal precision.

This kind of regularization is, of course, standard in conformal field theory [26]. It would not be necessary if the functions f were smooth. In the smooth case exactly the same infinite sum of $1/n$ would occur but the difference would be that it is multiplied by n-dependent coefficients that either have compact support in n or lead to stronger decay rendering the sum absolutely convergent. Thus, in the smooth case the result of the calculation would be dominated by the respective and corresponding first term in (2.62) and (2.63). Note also that the proposed regularization can be considered as the regularization

$$\omega^{-1}\to\frac{1}{2}[\omega^{1+\delta+\epsilon}+\omega^{1+\delta-\epsilon}] \tag{2.69}$$

with $\delta>\epsilon>0$ and then taking the limits in the order described. This regularization is the price to pay when working with bounded discontinuous functions f but it extracts exactly the dominating terms that would arise if f was smooth. The motivation for using nonsmooth step functions is that they result in coarse-graining maps for purposes of renormalization with almost perfect properties as we will see in the next section. In [21] we introduce smooth coarse-graining maps which come very close to those step functions, for which the above regularization is not necessary and for which the finite result obtained here after regularization is exact. As these step functions are finite position resolution approximants of smooth continuum functions, our manipulation is physically justified. This can also be seen as follows: The absolute value of both terms in (2.63) can be bounded from above (after the above regularization) by c/M where $c = \gamma + \sum_{N=1}^{\infty} N^{-2}$ (observing that $n \le M-1$). If $f(m) = M\langle\chi_m^M, f\rangle$ for smooth f as we assume in the next section, with χ_m^M the characteristic function of the interval $[m/M, (m+1)/M)$, then the first term in (2.61) converges to the smooth continuum value $\langle Ff', Q\omega^{-1}QFf'\rangle$ as $M\to\infty$ while the two remaining terms can be bounded by $c\|Ff'\|^2/M$, which converges to zero. Accordingly our zeta function regularization (only necessary for nonsmooth finite resolution approximants) ensures that the continuum limit (taking the finite resolution regulator $M\to\infty$) agrees with the direct continuum result.

With this understanding, the hypersurface generators are densely defined on the span of Fock states.

2. Comments on the space of solutions to the constraints

For completeness, we close this section with a few remarks on the actual solution of the quantum constraints which are mostly standard. These will not be of any relevance for the rest of the paper, and the reader not interested in these remarks can safely jump to the next section.

Not even the Fock vacuum is in the kernel of any of them, not to speak of the joint kernel. Indeed, there can be no joint zero eigenvector v of all the constraints except the zero vector due to the anomaly

$$0 = [D_s(f), D_s(g)]v = is3cS(f,g)v. \tag{2.70}$$

In solving the constraints, we thus look not for joint zero eigenvectors (zero is not in the joint point spectrum) but for generalized joint eigenvectors (distributions), i.e., linear functionals l on a dense and invariant [under the action of the $D_s(f)$] domain $\mathcal{D}$ such that

$$l[D_s(f)v] = 0 \quad \forall\ f, \quad s, \quad v\in\mathcal{D}. \tag{2.71}$$

Note that the finite linear span of Fock states is dense but not invariant. However, Eq. (2.71) also does not work for any such choice of invariant domain, because if $\mathcal{D}$ is invariant, then any such l also satisfies $l[[D_s(f), D_s(g)]v] = iscS(f,g)l[v] = 0$; i.e., l vanishes identically on $\mathcal{D}$. We thus resort, as it is common practice, to solving the equations $D_s(f) = 0$ not in the strong operator topology but in the weak operator topology. That is, we look for a proper subspace $\mathcal{D}\subset\mathcal{H}$ in the domain of the $D_s(f)$ such that the $D_s(f)\mathcal{D}\subset\bar{\mathcal{D}}_\perp$; i.e., the image of $\mathcal{D}$ under any $D_s(f)$ lies in the orthogonal complement of (the completion of) $\mathcal{D}$. That is, for any $v, v'\in\mathcal{D}$ we impose for all s, f,

$$\langle v, [D_s(f) - a\langle f\rangle 1_{\mathcal{H}}]v'\rangle = 0, \tag{2.72}$$

where a possible normal ordering constant a was introduced. In other words, with respect to the split $\mathcal{H} = \bar{\mathcal{D}}\oplus\bar{\mathcal{D}}_\perp$ all operators $D_s(f)$ contain no diagonal block corresponding to $\mathcal{D}$. A well-known choice of $\mathcal{D}$ consists in the solution to the system of equations

$$[D_s(e_n) - a\delta_n 1_{\mathcal{H}}]v = 0; \quad \forall\ s, \qquad n\ge 0. \tag{2.73}$$

Since $D_s(e_n)^\dagger = D_s(e_{-n})$ it follows that (2.73) implies (2.72) for all f. The system (2.73) does not suffer from the anomaly because for m, $n\ge 0$,

$$\begin{aligned}[D_s(e_m) - a\delta_n 1_{\mathcal{H}}, D_{s'}(e_n) - a\delta_n 1_{\mathcal{H}}] &= i\delta_{ss'}[D_s([e_m, e_n]) + 3cS(e_m, e_n)]\\ &= i\delta_{ss'}[2\pi i(m-n)D_s(e_{m+n}) - 3ic(m^3 - m - n^3 + n)\delta_{m+n,0}]\\ &= -2\pi\delta_{ss'}(m-n)D_s(m+n)\end{aligned} \tag{2.74}$$

as the second term only contributes for $m = n = 0$ if m, $n\ge 0$ but then the prefactor vanishes. Thus the right-hand side (rhs) of (2.75) is nonvanishing iff $m+n>0$ so that the system of conditions (2.73) is consistent. Of course, other choices of $\mathcal{D}$ are equally valid such as imposing (2.73) for $n\le 0$ only for both values of s or using (2.73) with $n\ge 0$ for $s=+$ and with $n\le 0$ for $s=-$.

Alternatively, to actually solve (2.73) we could use master constraint methods [27]; i.e., we set

$$M := \sum_{s,n\ge 0} m_n D_s(e_n)^\dagger D_s(e_n), \tag{2.75}$$

where $m_n > 0$ are coefficients that decay sufficiently fast in order that M be densely defined in the Fock space. Then any solution v of (2.73) solves $Mv = 0$, and conversely any solution of $Mv = 0$ solves $\langle v, Mv\rangle = 0$ and therefore (2.73). The task is now to solve for the ground states of the master constraint M. One will look for them in the form

$$v = v_{AB}\otimes v_c, \qquad v_{AB}\in\mathcal{H}_A\otimes\mathcal{H}_B, \qquad v_C\in\mathcal{H}_C, \tag{2.76}$$

where v_C is any Fock state and v_{AB} is to be determined in dependence on v_C. In this way, the physical Hilbert space is isomorphic to $\mathcal{H}_C$.

This is, of course, expected as the PFT should be equivalent to the massless Klein Gordon field on the cylinder. Indeed, the natural gauge fixing conditions $T = t, X = x$ reproduce this theory which one immediately arrives at using the corresponding reduced phase space quantization. The actual solution of PFT is beyond the scope of the present work in which we are just interested in studying how the system behaves under renormalization.

III. HAMILTONIAN RENORMALIZATION OF HAMILTONIAN SYSTEMS

This section is to recall the essential elements from [4,11] to which the reader is referred for more information.

We introduce some coordinate $x \in [0,1)$ and equidistant lattices Λ_M on [0, 1) with M points $x_m = \frac{m}{M}, m \in \mathbb{Z}_M := \{0,1,2,\ldots,M-1\}$. Among the numbers $M \in \mathbb{N}$ we introduce the relation $M < M'$ iff $\frac{M'}{M} \in \mathbb{N}$ which means that Λ_M is a sublattice of $\Lambda_{M'}$. It is not difficult to see that this defines a partial order and that $\mathbb{N}$ is directed with respect to it.

The space of complex valued sequences $\{f_M(m)\}_{m\in\mathbb{Z}_M}$ is denoted by L_M and given a Hilbert space structure by

$$\langle f_M, f'_M\rangle_{L_M} := \frac{1}{M}\sum_{m\in\mathbb{Z}_M} f_M(m)^* f'_M(m). \tag{3.1}$$

Let $\chi_{[a,b)}$ be the characteristic function of the left closed, right open interval $[a,b) \subset [0,1)$ where for any $S \subset \mathbb{R}$ we have $\chi_S(x) = 1$ if $x \in S$ and $\chi_s(x) = 0$ else. Then for $x \in [0,1)$

$$\chi^M_m(x) := \chi_{[\frac{m}{M},\frac{m+1}{M})}(x). \tag{3.2}$$

Consider the embedding [recall $L = L_2([0,1),dx)$]

$$I_M\colon\ L_M \to L;\quad (I_M f_M)(x) := \sum_{m\in\mathbb{Z}_M} f_M(m)\chi^M_m(x), \tag{3.3}$$

which is, in fact, an isometry

$$\langle I_M f_M, I_M f'_M\rangle_L = \langle f_M, f'_M\rangle_{L_M} \tag{3.4}$$

and thus allows the interpretation of (3.1) as the Riemann sum approximation of $\langle f, f'\rangle_L$ with $f_M(m) := f(m/M)$ and $f'_M(m) = f'(m/M)$.

For $M < M'$ we construct the embeddings

$$I_{MM'}\colon\ L_M \to L_{M'};\qquad I_{MM'} := I^\dagger_{M'} I_M. \tag{3.5}$$

The operator $I^\dagger_M$ can be worked out explicitly

$$[I^\dagger_M f](m) = M\langle \chi^M_m, f\rangle_L. \tag{3.6}$$

It is also an isometry

$$\langle I_{MM'} f_M, I_{MM'} f'_M\rangle_{L_{M'}} = \langle f_M, f'_M\rangle_{L_M}, \tag{3.7}$$

and these embeddings automatically obey the consistency conditions for all $M < M' < M''$,

$$I_{M'M''}\circ I_{MM'} = I_{MM''}. \tag{3.8}$$

This follows from the identity

$$I_{M'} I^\dagger_{M'} I_M = I_M, \tag{3.9}$$

which in turn is due to the property of the χ^M_m to define partitions of [0, 1) which are nested for $M < M'$, that is,

$$\chi^M_m = \sum_{l=0}^{k-1} \chi^{M'}_{km+l},\qquad k = \frac{M'}{M}. \tag{3.10}$$

We can also work out $I_{MM'}$ explicitly ($k := M'/M$)

$$\begin{aligned}(I_{MM'} f_M)(m') &= M'\langle \chi^M_{m'}, I_M f_M\rangle_L\\ &= M' \sum_{m\in\mathbb{Z}_M} f_M(m)\langle \chi^M_{m'}, \chi^M_m\rangle_L\\ &= M' \sum_{m\in\mathbb{Z}_M} f_M(m)\left(\frac{1}{M'}\sum_{l=0}^{k-1}\delta_{m',mk+l}\right)\\ &= f_M\left(\left[M\frac{m'}{M'}\right]\right),\end{aligned} \tag{3.11}$$

where [.] denotes the floor function (Gauss bracket). We remark at this point that the objects I_M and $I_{MM'}$ are never changed by the renormalization flow to be defined below.

Consider a scalar field ϕ on [0, 1) with conjugate momentum π. Note that geometrically π is a scalar density of weight one on [0, 1) as one can see from the Poisson bracket

$$\{\pi(x), \phi(y)\} = \delta(x,y). \tag{3.12}$$

We consider real density one valued test functions f and real density zero valued test functions F on [0, 1). Then the real numbers

$$\phi(f) := \langle f, \phi\rangle,\qquad \pi(F) := \langle F, \pi\rangle \tag{3.13}$$

are invariant under diffeomorphisms of [0, 1), and we have

$$\{\pi(F), \phi(f)\} = F(f). \tag{3.14}$$

One can construct the abstract *-algebra (even C^*-algebra) $\mathfrak{A}$ generated by the Weyl elements

$$w(f,F) = \exp(i[\phi(f) + \pi(F)]) \tag{3.15}$$

and the corresponding Weyl relations that follow from the reality of (3.13) and (3.14).

Representations of $\mathfrak{A}$ can be constructed from a state (positive, normalized, linear functional) $\hat{\omega}$ on it via the Gel'fand-Naimark-Segal construction [19]. This delivers a Hilbert space $\mathcal{H}$, a representation ρ of $\mathfrak{A}$ by bounded operators on $\mathcal{H}$, and a vector $\Omega \in \mathcal{A}$ cyclic for $\rho(\mathfrak{A})$. If $\mathcal{H}$ is separable, we always find an Abelian sub-*-algebra $\mathfrak{B}$ of $\mathfrak{A}$ for which Ω is still cyclic. For instance, we can pick an ONB $e_I, I \in \mathbb{Z}$ with $b_0 \coloneqq \Omega$ of $\mathcal{H}$ and consider the Abelian group of unitary operators $U_I, I \in \mathbb{Z}, U_I^\dagger = U_{-I}$ such that $U_I e_J = e_{I+J}$. Then we find $a_I \in \mathfrak{A}$ such that $\rho(a_I) = U_I$ and $\mathfrak{B}$ is generated by those a_I. See [11] for more details and more general cases. Of course, the a_I may in general be very complicated (in general infinite) linear combinations of the Weyl elements (3.15). Still it follows that $\mathcal{H}$ can be thought of as $L_2(\Delta(\mathfrak{B}), d\nu)$ where $\Delta(\mathfrak{B})$ is the Gel'fand spectrum (space of "characters," i.e., homomorphisms $\chi: \mathfrak{B} \to \mathbb{C}$ equipped with the Gel'fand topology) of $\mathfrak{B}$ and ν a probability measure thereon. More precisely, there is a unitary map $U: \mathcal{H} \to L_2(\Delta(\mathfrak{B}), d\nu)$ with $[U\rho(b)\Omega](\chi) \coloneqq \hat{b}(\chi) \coloneqq \chi(b)$ which is essentially the Gel'fand isomorphism.

We will assume that $\mathfrak{B}$ can be generated by the $w(f) \coloneqq w(f, F = 0)$ so that we can identify the space of characters with the space of fields ϕ and ν as a probability measure on that space. Indeed, this is the case in Fock representations $\hat{\omega} = \langle \cdots \rangle_{\mathcal{H}}$ in which $w(f)\Omega$ is essentially $\exp(i\langle [2\omega]^{-1/2} f, A\rangle_L^\dagger)\Omega$ up to a phase where $A = (\sqrt{\omega}\phi - i\sqrt{\omega}^{-1}\pi)/\sqrt{2}$ is the annihilator. Thus arbitrary linear combinations of Fock states $\langle f_1, A\rangle^\dagger \cdots \langle f_n, A\rangle^\dagger \Omega$ can be obtained by taking derivatives at $s_1 = \cdots = s_n = 0$ of $w(\sum_k s_k [2\omega]^{1/2} f_k)\Omega$ establishing that the span of the $w(f)\Omega$ is dense. Then ν is the Gaussian measure with covariance $1/(2\omega)$ [3]

$$\nu(w(f)) \coloneqq \langle \Omega, w(f)\Omega\rangle_{\mathcal{H}} = \exp\left(-\frac{1}{4}\langle f, \omega^{-1} f\rangle_L\right). \tag{3.16}$$

Given the injections $I_M; L_M \to L$ we may restrict ϕ to the subspace $I_M L_M$; i.e., we define a scalar field ϕ_M on the lattice Λ_M by

$$\phi_M(f_M) \coloneqq \phi(I_M f_M) \Rightarrow \phi_M = I_M^\dagger \phi, \tag{3.17}$$

which provides a natural "discretization." Here $\phi(f) \coloneqq \langle f, \phi\rangle$ for real valued f. As $(I_M^\dagger \phi)(m) = M\langle \chi_m^M, \phi\rangle$ approaches $\phi(x)$ in the limit $M \to \infty$ for $m = xM$ we see that the density zero valued ϕ is smeared against the density one valued discretized δ distribution $M\chi^M$ which is diffeomorphism covariant. We may likewise define a discretized momentum $\pi_M = M^{-1} I_M^\dagger \pi = \langle \chi_\cdot^M, \pi\rangle$, which smears the density one valued π against the density zero valued χ^M which is also covariant. Together this ensures that ϕ_M and π_M are conjugate on Λ_M,

$$\begin{aligned}\{\pi_M(m), \phi_M(m')\} &= M\{\pi(\chi_m^M), \phi(\chi_{m'}^M)\} \\ &= M\langle \chi_m^M, \chi_{m'}^M\rangle_L = \delta_{m,m'}.\end{aligned} \tag{3.18}$$

Although this is geometrically more natural, we will instead use

$$\pi_M(m) \coloneqq [I_M^\dagger \pi](m), \quad \{\pi_M(m), \phi_M(m')\} = M\delta_{m,m'} \tag{3.19}$$

so that ϕ_M and π_M are conjugate not in the sense of a Kronecker δ but rather a discrete δ distribution.

Given a function $H[\phi, \pi]$ on the continuum phase space coordinated by the variables ϕ and π we may try to define a discretized function

$$H_M[\phi_M, \pi_M] \coloneqq H[I_M\phi_M, I_M\pi_M], \tag{3.20}$$

where the approximation $I_M I_M^\dagger \to 1_L$ as $M \to \infty$ was used. This indeed works as long as H depends on π and ϕ only algebraically. However, when derivatives are involved, the simple prescription (3.20) may cause trouble because the functions χ^M are not differentiable. This can be improved by passing to alternative, smoother coarse-graining maps I_M [21] which lead to coarse-graining maps $I_{MM'}$ satisfying the consistency conditions (3.8) which are essential for the renormalization scheme. For the examples discussed in [21] it turns out that the *natural discretization* $\partial_M \coloneqq I_M^\dagger \partial I_M$ is a well-defined and antisymmetric discrete derivative operator on L_M.

To keep the presentation simple and to see into which problems one may run using step functions, we take the usual point of view that the prescription (3.20) is as good as any other as long as $H_M[\phi_M, \pi_M]$ converges to $H[\phi, \pi]$ in the continuum limit $M \to \infty$. Noting that $\phi_M(m) = M\langle \chi_m^M, \phi\rangle$ approaches $\phi(x)$ as $m, M \to \infty$ if we keep $x = m/M$ fixed, we may therefore discretize, e.g., $\phi'(x)$ by

$$I_M(\partial_M \phi_M), \tag{3.21}$$

where

$$[\partial_M f_M](m) \coloneqq \frac{M}{2}[f_M(m+1) - f_M(m-1)] \tag{3.22}$$

is the antisymmetric, next neighbor, first order lattice derivative. There are an infinite number of prescriptions such as (3.22) which have the correct continuum limit in the sense mentioned above, and therefore using any such prescription introduces a *discretization ambiguity* into the functions $H_M[\phi_M, \pi_M]$. This ambiguity is drastically reduced if one uses the natural discretization using smoother functions χ^M with all the desired properties as indicated above.

Given a continuum measure ν we may construct a family of measures ν_M by

$$\nu_M(w_M[f_M]) := \nu(w[I_M f_M]), \quad w_M[f_M] = \exp(i\langle f_M, \phi_M\rangle_{L_M}), \tag{3.23}$$

which are automatically *cylindrically consistent*, i.e., for all $M < M'$,

$$\nu_{M'}(w_{M'}[I_{MM'} f_M]) = \nu_M(w_M[f_M]), \tag{3.24}$$

i.e., integrating the excess degrees of freedom in artificially writing the function w_M of ϕ_M as the function $w_{M'}$ of $\phi_{M'}$, which, however, depends on $\phi_{M'}$ only in terms of the *blocked* variables $I^\dagger_{MM'}\phi_{M'}$ does not change the result. Conversely, under relatively mild technical assumptions [2], a cylindrically consistent family of measures ν_M on quantum configuration spaces K_M can be extended to a measure ν on a space K called the *projective limit* of the K_M. In that sense, a cylindrically consistent family is as good as the continuum definition but the practical advantage of the family is that the ν_M are easier to compute.

Consider the Hilbert spaces $\mathcal{H}_M = L_2(K_M, d\nu_M)$ and the embeddings

$$J_M : \mathcal{H}_M \to \mathcal{H} = L_2(\Phi, d\nu); \quad w_M[f_M]\Omega_M \mapsto w[I_M f_M]\Omega, \tag{3.25}$$

which by construction are isometries. Here $\nu_M(.) = \langle \Omega_M, .\Omega_M\rangle_{\mathcal{H}_M}$ and $\nu(.) = \langle \Omega, .\Omega\rangle_{\mathcal{H}}$. It is also not difficult to see that the J_M inherit from the I_M the consistency properties

$$J_{M'M''} J_{MM'} = J_{MM''} \quad \forall \;\; M < M' < M'', \tag{3.26}$$

where $J_{MM'} = J^\dagger_{M'} J_M, M < M'$. It follows that $\mathcal{H}$ is the inductive limit of the $\mathcal{H}_M$ [19]. Given a symmetric quadratic form H on $\mathcal{H}$ with dense domain $\mathcal{D}$ spanned by the $w[f]\Omega$ we may construct the symmetric quadratic forms $H_M := J^\dagger_M H J_M$ which are automatically consistently defined: For any $M < M'$ we have

$$J^\dagger_{MM'} H_{M'} J_{MM'} = H_M. \tag{3.27}$$

Moreover, given $J_M\psi_M, J_{M'}\psi'_{M'} \in \mathcal{H}$ with $\psi_M \in \mathcal{D}_M$, $\psi'_{M'} \in \mathcal{D}_{M'}$ in the dense set of the span of vectors $w_M[f_M]\Omega_M$, etc., we find $M'' > M, M'$ and can compute

$$\langle J_M\psi_M, H J_{M'}\psi_{M'}\rangle_{\mathcal{H}} = \langle J_{MM''}\psi_M, H_{M''} J_{M'M''}\psi'_{M'}\rangle_{\mathcal{H}_{M''}}; \tag{3.28}$$

i.e., for all practical purposes the family of quadratic forms H_M is as good as H but easier to compute. Note that H is *not* the inductive limit of the H_M [19] for two reasons: First, while H_M are actually operators and not only quadratic forms (as the systems labeled by M only depend on finitely many degrees of freedom), the object H is in general not. Second, for H to be the inductive limit of the H_M we require the much stronger *intertwiner property* $J_M H_M = H J_M$ which implies $H_M = J^\dagger_M H J_M$ but not vice versa. We remark at this point the objects J_M and $J_{MM'}$ will change during the flow to be defined below as they map vacua to vacua which do flow.

The problem that one encounters in quantizing a classical Hamiltonian system with canonical variables ϕ, π and Hamiltonian H is this: Provide a representation ρ of the $*$-algebra generated by the $\phi(f), \pi(F)$ [or the C^*-algebra generated by the $w(f, F)$] that supports "the" Hamiltonian H as a self-adjoint operator. We have used inverted commas as this task is ill-defined as it stands: The classical function H typically is ill-defined when naively substituting the classical ϕ, π by their corresponding operator valued distributions. The strategy of constructive QFT is to come up with quantizations of the simpler, well-defined (since finite dimensional—if both UV regulator M and IR regulator R are present) discretized Hamiltonian systems defined by ϕ_M, π_M, H_M and then restrict the discretization ambiguities inherent in these systems by inverting the logic: the automatic consistency *properties* of discretizations descending from continuum quantum theories (sometimes called "blocking from the continuum") are imposed as consistency *conditions* which are twofold when the renormalization flow reaches a fixed point and thus qualifies as a continuum theory.

That is, we start from a family of triples $(\mathcal{H}^{(0)}_M, \Omega^{(0)}_M, H^{(0)}_M)$ obtained by some prescription and then define a sequence ("renormalization flow") of such triples $(\mathcal{H}^{(0)}_M, \Omega^{(0)}_M, H^{(0)}_M)$ by the following rules:

(1) The maps for $M < M'$,

$$J^{(n)}_{MM'} w_M[f_M]\Omega^{(n+1)}_M := w_{M'}[I_{MM'} f_M]\Omega^{(n)}_{M'}, \tag{3.29}$$

are imposed to be isometries; that is, the corresponding measures are defined by

$$\nu^{(n+1)}_M(w_M[f_M]) := \nu^{(n)}_{M'}(w_{M'}[I_{MM'} f_M]). \tag{3.30}$$

(2) Using these we set

$$H^{(n+1)}_M := J^\dagger_{MM'} H^{(n)}_{M'} J_{MM'}. \tag{3.31}$$

The idea is then to look for fixed points $J_{MM'}, \Omega_M, \mathcal{H}_M, \nu_M, H_M$ of this flow for which then all consistency conditions are satisfied by construction and which therefore defines a continuum theory. The hope is then that at fixed points all but finitely many (so-called relevant parameters) of the free parameters that coordinate the discretization ambiguities also assume fixed values, thus rendering the theory predictive.

In practice one cannot use (3.29)–(3.31) for all $M < M'$ since for $M < M'_1, M'_2, M'_1 \neq M'_2$; e.g., the definitions (3.30) and (3.31) generically do not agree when using $M' = M'_1$ or $M' = M'_2$, respectively. Thus, one usually picks a fixed $M'(M)$ satisfying $M'(M) > M$, a popular choice being $M'(M) = 2M$. Then, relying on the intuition of *universality*, the fixed point is hoped for not to depend on the choice $M'(M)$, so that at the fixed point the consistency conditions indeed hold for all $M < M'$.

An automatic feature of this renormalization scheme is that for all M the fixed point vacuum Ω_M is a ground state of the fixed point Hamiltonian H_M if this is true for the initial data $\Omega_M^{(0)}$ and $H_M^{(0)}$: This follows inductively from

$$H_M^{(n+1)}\Omega_M^{(n+1)} = [J^{(n)}_{MM'(M)}]^\dagger H^{(n)}_{M'(M)} J^{(n)}_{MM'(M)}\Omega_M^{(n+1)} = [J^{(n)}_{MM'(M)}]^\dagger H^{(n)}_{M'(M)}\Omega^{(n)}_{M'(M)} = 0. \tag{3.32}$$

This condition is necessary in order to make the renormalization scheme compatible with Wilsonian renormalization of the Euclidean (path integral) formulation from which the present scheme was derived via OS reconstruction [4,11].

IV. HAMILTONIAN RENORMALIZATION OF CONSTRAINED SYSTEMS

As mentioned, the scheme reviewed in the previous section was motivated using the Euclidean formulation of a QFT which needs as a minimal input a self-adjoint Hamiltonian H on a Hilbert space $\mathcal{H}$ bounded from below with vacuum Ω. From these one can attempt to construct the associated Gibbs measure μ on the space of field histories, and when this exists, it satisfies a minimal set of Euclidean axioms (in particular, reflection positivity) ensuring that $(H, \mathcal{H}, \Omega)$ can be recovered from μ.

When we consider constrained Hamiltonian systems, in particular when there is no Hamiltonian but just a set of Hamiltonian constraints, we are strictly speaking leaving that framework. One can return to it by using the reduced phase space formulation in which one gauge fixes the Hamiltonian constraints thereby ending up with a true Hamiltonian again that just acts on the gauge invariant (or true) degrees of freedom [28], and this is the strategy followed so far [11]. However, in this paper we want to explore a different route:

The observation is that the two renormalization steps (3.30) and (3.31) actually do not rely on H being bounded from below or that Ω is the vacuum of H. Thus we propose to "abuse" (3.30) and (3.31) and use them also for constrained Hamiltonian systems. In other words, we keep (3.30) as it is and apply (3.31) to each constraint operator separately.

This proposal raises the following immediate questions and concerns:

(1) The classical continuum constraints are of the form $H(F) = \int dx F(x) H(x)$ where F is a smearing function and $H(x)$ is the Hamiltonian constraint density. Thus the essential difference between a true Hamiltonian system and a constrained Hamiltonian system (apart from the fact that true Hamiltonian densities are typically bounded from below at least classically) is that for the true Hamiltonian the only allowed smearing function is $F = 1$ while for the constrained case the space of the smearing function is infinite dimensional. The question is now how F should be treated when we discretize $H(F)$. There are two extreme and equally natural points of view:
   (a) The first is that for each F the function $H(F)$ is simply an independent object and should be treated just as a true Hamiltonian. That is, the function F remains as it is; it is not discretized.
   (b) The second is that F should be treated on equal footing with the phase space variables ϕ, π and thus should be discretized, perhaps by the same map $I_M^\dagger$, perhaps by another. This, of course, introduces yet more discretization ambiguities into the quantization and also requires one to invent a flow equation on the space of discretized smearing functions F_M when stating (3.31).

Note that the second point of view is often taken for granted in lattice inspired approaches to constrained systems [29]. One may think that the first point of view in fact provides a natural choice of discretization of F as follows:

Suppose that we actually have the continuum theory, i.e., the Hilbert space $\mathcal{H}$ and the constraints $H(F)$, at our disposal. Then the idea is to define a map $E_M\colon L \to L_M$ via the identity

$$H_M(E_M F) \coloneqq \sum_{m \in \mathbb{Z}_M} (E_M F)(m) H_M(m) \coloneqq J_M^\dagger H(F) J_M, \tag{4.1}$$

which assumes that the rhs can actually be written in this local form. This is unfortunately already not the case even for the PFT considered here. The reason for this to happen is that H when written in terms of polynomials of annihilation and creation operators involves nonlocal integral kernels. While these do get discretized by means of J_M this leads to an effective E_M which maps $L \to L_M^N$ where $N \geq 2$ is the polynomial degree. We will demonstrate this explicitly below for PFT.

This establishes that viewpoints i. and ii. are drastically different; i.e., a map $E_M\colon L \to L_M$ generically cannot be induced via (4.1). Instead, according to viewpoint ii. we consider as an extra structure maps $\tilde{I}_M L_M \to L$ and $\tilde{I}_{MM'}\colon L_M \to L_{M'}$ and define

$$H_M(F_M) \coloneqq \sum_{m \in \mathbb{Z}_M} F_M(m) H_M(m) \coloneqq J_M^\dagger H(\tilde{I}_M F_M) J_M. \tag{4.2}$$

This is consistently defined

$$H_M(F_M) = J^\dagger_{MM'} H_{M'}(\tilde{I}_{MM'} F_M) J_{MM'} \tag{4.3}$$

due to $J_{M'} J_{MM'} = J_M$ and provided that $\tilde{I}_{M'} \tilde{I}_{MM'} = \tilde{I}_M$. We may reduce the ambiguity and actually consider $\tilde{I}_M = I_M, \tilde{I}_{MM'} = I_{MM'}$; however, this choice is inconvenient for the following reason: While we can certainly compute the commutator $[H_M(F_M), H_M(G_M)]$ directly, which is well-defined, one would like to see the deviation from the continuum computation by using the identity

$$\begin{aligned}
&[H_M(F_M), H_M(G_M)] \\
&= J^\dagger_M\{[H(I_M F_M), H(I_M G_M)] \\
&\quad - H(I_M F_M)(1_\mathcal{H} - P_M) H(I_M G_M) \\
&\quad + H(I_M G_M)(1_\mathcal{H} - P_M) H(I_M F_M)\} J^\dagger_M,
\end{aligned} \tag{4.4}$$

where we defined $P_M = J_M J^\dagger_M$, which is a projection in $\mathcal{H}$ due to the isometry of J_M. The first term gives the cylindrical projection of the continuum algebra, which in our case is the Virasoro algebra. The second and third terms should vanish as $M \to \infty$ because J_M becomes the identity in $\mathcal{H}$. Therefore, Eq. (4.4) appears to be an appropriate way to monitor how the cylindrically projected theories approach the correct continuum. The catch is that we know that in PFT the commutator $[H(I_M F_M), H(I_M G_)]$ depends on the first and third derivatives of the $I_M F_M, I_M G_M$ which are, however, not even continuous. Accordingly, if we want to use (4.4) we should instead use $\tilde{I}_M$ and $\tilde{I}_{MM'}$ which are at least C^3 and which share all the properties of $\tilde{I}_M$ and $\tilde{I}_{MM'}$. Thus such maps constructed from wavelets [30] suggest themselves, and we will give more details below.

To summarize this part of the discussion, for the purpose of this paper we take viewpoint i. and leave F and G undiscretized, and then with $H_M(F) = J^\dagger_M H(F) J_M$ the computation

$$\begin{aligned}
[H_M(F), H_M(G)] = J^\dagger_M\{&[H(F), H(G)] \\
&- H(F)(1_\mathcal{H} - P_M) H(G) \\
&+ H(G)(1_\mathcal{H} - P_M) H(F)\} J^\dagger_M
\end{aligned} \tag{4.5}$$

is unproblematic. To avoid confusion note that (4.5) is supposed to yield the Virasoro algebra, as $M \to \infty$, including the central term; i.e., the anomaly as compared to the classical computation (Witt algebra) should be present. We thus want to check that the Virasoro algebra is recovered without anomaly, not the Witt algebra.

(2) As noted in the previous section, due to the central term in the Virasoro algebra, there cannot be a joint vacuum Ω for all the constraints $H(F)$. This is even more the case for the $H_M(F)$ at finite resolution because they typically do not close as it is plain to see from (4.5); hence, the states Ω_M that arise at the fixed point cannot be joint vacua for the $H_M(F)$.

This is no obstacle for the renormalization scheme when applied separately to the $H(F)$ because the $H_M(F)$ are operators (and not only quadratic forms) of systems with finitely many degrees of freedom, and thus one does not expect the usual problems in finding a domain that is typical for QFT (infinitely many degrees of freedom) especially if $H(F)$, even when normal ordered, contains terms that are monomials made solely from creation operators. Thus, we expect to find dense domains $D_M(F)$ for $H_M(F)$ and by construction $J_{MM'} D_M(F) \subset D_{M'}(F)$. However, a problem may occur when we compute commutators such as (4.5) because the domains $D_M(F)$ may depend on F and it may be the case that $H_M(F) D_M(F) \not\subset D_{M(F')}$ [31]. At least it is true that at finite M the domains are invariant $H_M(F) D_M(F) \subset D_M(F)$ because they are just finite linear combinations of monomials [and not infinite linear combinations as in the case of $H(F)$] of creation and annihilation operators. Thus a minimal requirement for (4.5) to be meaningful is that the $H_M(F)$ have a dense, invariant domain D_M independent of F and then by construction $J_{MM'} D_M \subset D_{M'}$.

Since the span D of the $J_M D_M$ is dense in the inductive limit $\mathcal{H}$ on which by construction is a form domain of $H(F)$, this then also makes the fixed point $H(F)$ densely defined as a quadratic form. However, this does not ensure that the commutators of the $H(F)$ are well-defined because matrix elements of the formal expression $H(F)H(F')$, which can be formally computed by invoking resolutions of the identity in terms of an ONB made from vectors in D, may diverge, which is a potential danger even if $H(F)$ can be promoted to an operator especially if D is not invariant for $H(F)$. It is here where a joint cyclic vacuum would be very convenient to build a common dense operator domain upon. In the absence of it, the construction of such a domain may be very difficult, if it exists at all. In PFT we know that this problem does not occur, despite the nonexistence of such a joint vacuum, as a common dense (but not invariant) operator domain is given explicitly by the span of the chosen Fock states. However, it may be in more complicated theories, especially if the domains depend on F, which in unfortunate cases can have nondense intersections [31].

(3) Note that our renormalization scheme constructs a single Hilbert space $\mathcal{H}$ (or measure ν) but an infinite number of quadratic forms $H(F)$ if a simultaneous fixed point of the respective flow equations exists at all. While the flow equations for ν and $H(F)$ are tightly coupled, the flow equations for the various $H(F)$ are treated as independent for each choice

of F. Now it could happen that these latter equations have several different fixed points for each choice of F that are reached depending on the choice of initial discretization $H_M^{(0)}(F)$. Then the corresponding fixed point family $H_M(F)$ may depend rather discontinuously on F and thus would probably not coincide with the result of *blocking from the continuum* $H_M(F) := J_M H(F) J_M$. We will see that this does not happen because the flow acts directly on the constraint function which does not depend on F during the flow if the initial discretization does not.

In the next section we examine whether these issues arise in the Hamiltonian renormalization of PFT.

V. HAMILTONIAN RENORMALIZATION OF PFT

Since the constraint operators are of the form

$$D_+ = [A_+^2 - A_-^2] \otimes 1_B \otimes 1_C + 1_A \otimes 1_B \otimes C_+^2,$$
$$D_+ = 1_A \otimes [B_+^2 - B_-^2] \otimes 1_C - 1_A \otimes 1_B \otimes C_-^2, \tag{5.1}$$

it will be sufficient to consider one of the sectors A, B, C only, say C. Our first task is to pick initial discretizations of the $C_{\pm,M}^{(0)}$ and corresponding Hilbert space measures $\nu_M^{(0)}$ on $K_M = \mathbb{R}^M$. As suggested by the considerations of Sec. II we build $C_{\pm,M}^{(0)}$ out of $C_{0,M}^{(0)}$ and $C_M^{(0)}$. We define in parallel to the continuum [see (2.24) and (2.25)]

$$\begin{aligned}
\Phi_M &:= I_M^\dagger \Phi, \\
\Pi_M &:= I_M^\dagger \Pi, \\
QM_\perp f_M &:= \langle 1, f_M \rangle_{L_M} 1, \\
Q^M &:= 1_{L_M} - Q_\perp^M, \\
C_{0,M}^{(0)} &:= \frac{1}{\sqrt{2}}\left[\sqrt{\omega_0} Q_\perp^M \Phi_M - i \frac{1}{\sqrt{\omega_0}} Q_\perp^M M \Pi_M\right], \\
C_M^{(0)} &:= \frac{1}{\sqrt{2}}\left[\sqrt{\omega_M^{(0)}} Q^M \Phi_M - i \frac{1}{\sqrt{\omega_M^{(0)}}} Q^M \Pi_M\right], \\
C_{s,M}^{(0)} &:= i\sqrt{\omega_0/2}[C_{0,M}^{(0)} - (C_{0,M}^{(0)})^\dagger] + i\sqrt{2\omega_M^{(0)}}[Q_s^{M(0)} C_M^{(0)} - (Q_s^{M(0)} C_M^{(0)})^\dagger)], \\
(\omega_M^{(0)})^2 &:= -(\partial_M^{(0)})^2, \\
(\partial_M^{(0)} f_M)(m) &:= (2M)^{-1}[f_M(m+1) - f_M(m-1)], \\
Q_s^{M(0)} &:= \frac{1}{2}\left[1_{L_M} - is\frac{\partial_M^{(0)}}{\omega_M^{(0)}}\right] Q^M, \\
D_{s,M}^{(0)} &:= :[C_{s,M}^{(0)}]^2: \ .
\end{aligned} \tag{5.2}$$

Here the adjoint operation and normal ordering is with respect to the Fock Hilbert space structure $\mathcal{H}_M^{(0)}$ defined by the annihilation operators $C_{0,M}^{(0)}$ and $C_M^{(0)}$ with Fock vacuum $\Omega_M^{(0)}$. Note that $Q_\perp^M, Q^M, i\partial_M, \omega_M$ are self-adjoint on L_M and that $Q_\perp, Q^M, Q_s^M$ are orthogonal projections in L_M with $Q_\perp^M Q^M = Q_+^M Q_-^M = 0$ and $1_{L_M} = Q_\perp^M + Q^M$, $Q^M = Q_+^M + Q_-^M$. Note that $Q_\perp^M$ is the projection on the constant function 1 that equals unity.

An immediate observation is that

$$Q_\perp^M \Phi_M = \langle 1, \Phi_M \rangle_{L_M} = \frac{1}{M}\sum_m 1(m)\Phi_M(m) = \frac{1}{M}\sum_m (I_M^\dagger \Phi)(m) = \sum_m \langle \chi_m^M, \Phi \rangle_L = \langle 1, \Phi \rangle = Q_\perp \Phi \tag{5.3}$$

and similarly for $Q_\perp^M \Pi_M = Q_\perp \Pi$ so that in fact

$$C_{0,M}^{(0)} = C_0 \tag{5.4}$$

is actually the same as in the continuum in the initial discretization. We will see that this property is preserved by the renormalization flow so that the zero modes remain unrenormalized.

We proceed to the flow equation for the Fock measure. We have

$$\begin{aligned}\langle f_M, \Phi_M\rangle_{L_M} &= \langle Q_\perp^M f_M, \Phi_M\rangle_{L_M} + \langle Q^M f_M, \Phi_M\rangle_{L_M} \\ &= \langle [2\omega_0]^{-1/2} Q_\perp^M f_M, C_{0,M}^{(0)}\rangle_{L_M} + \langle [2\omega_0]^{-1/2} Q_\perp^M f_M, C_{0,M}^{(0)\dagger}\rangle_{L_M} \\ &\quad + \langle [2\omega_M^{(0)}]^{-1/2} Q^M f_M, C_M^{(0)}\rangle_{L_M} + \langle [2\omega_M^{(0)}]^{-1/2} Q^M f_M, C_M^{(0)\dagger}\rangle_{L_M}. \end{aligned} \tag{5.5}$$

Thus, the initial measure family has generating functional of moments

$$\begin{aligned}\nu_M^{(0)}(w_M[f_M]) &= \langle \Omega_M^{(0)}, \exp(i\langle f_M, \phi_M\rangle)\Omega_M^{(0)}\rangle_{\mathcal{H}^{(0)}} \\ &= \exp\left(-\frac{1}{4}[\langle Q_\perp^M f_M, \omega_0^{-1} Q_\perp^M f_M\rangle_{L_M} + Q^M f_M, [\omega_M^{(0)}]^{-1} Q^M f_M\rangle_{L_M}]\right). \end{aligned} \tag{5.6}$$

It is a family of Gaussian measures with covariances (kernels on L_M)

$$K_M^{(0)} = \frac{1}{2}[Q_\perp^M \omega_0^{-1} Q_\perp^M + Q^M [\omega_M^{(0)}]^{-1} Q^M]. \tag{5.7}$$

This is exactly as for the $1+1$ Klein-Gordon field treated in the first reference of [6] except that there we assumed a nonvanishing mass p so that the projections $Q_\perp^M$ and Q^M are not necessary and the initial covariance is just $[2\omega_M^{(0)}(p)]^{-1}$ with $[\omega_M^{(0)}(p)]^2 = [\omega_M^{(0)}]^2 + p^2$.

To study the flow of (5.7) we can borrow the results of [6] as follows: In [6] we used the spectral theorem to write

$$[2\omega_M(p)]^{-1} = \int_{\mathbb{R}} \frac{dk}{2\pi}[k^2 + (\omega_M^{(0)}(p))^2]^{-1} \tag{5.8}$$

by the residue theorem where due to $p \neq 0$ there is no real pole of the holomorphic integrand. Here, instead of integrating over the real line, we consider the path

$$c_\rho\colon \mathbb{R} \to \mathbb{C}; c_\rho(k) = \begin{cases} k & |k| > \rho \\ -\rho e^{i\frac{\pi}{2}(\frac{k}{\rho}+1)} & |k| = \rho \end{cases}, \tag{5.9}$$

where $\rho > 0$ is arbitrarily small thus avoiding the real pole $k = 0$. Then

$$Q^M [2\omega_M^{(0)}]^{-1} Q^M = \lim_{\rho\to 0+} \int_{c_\rho} \frac{dk}{2\pi}[k^2 + (\omega_M^{(0)})^2]^{-1}. \tag{5.10}$$

By the flow equation

$$\nu_M^{(n+1)}(w_M(f_M)) \coloneqq \nu_{M'(M)}^{(n)}(w_{M'(M)}(I_{MM'(M)} f_M)) \tag{5.11}$$

the measure family stays always inside the Gaussian class and (5.12) translates into a flow of covariances

$$K^{(n+1)} = I_{MM'(M)}^\dagger K_{M'(M)}^{(n)} I_{MM'(M)}, \tag{5.12}$$

where $M'(M) > M$ is the fixed higher resolution that enters the concrete implementation of the blocking equations. As in [6] we will choose $M'(M) = 2M$ for simplicity.

We note that

$$\begin{aligned} Q_\perp^{M'} I_{MM'} f_M &= \langle 1, I_{MM'} f_M\rangle_{L_{M'}} = \frac{1}{M'} \sum_{m' \in \mathbb{Z}_{M'}} f_M\left(\left[M\frac{m'}{M'}\right]\right) \\ &= \frac{1}{M'} \sum_{m\in\mathbb{Z}_M} f_M(m)\left[\sum_{l=0}^{M'/M-1} 1\right] = \frac{1}{M}\sum_{m\in\mathbb{Z}_M} f_M(m) = \langle 1, f_M\rangle_{L_M} \\ &= Q_\perp^M f_M = I_{MM'} Q_\perp^M f_M, \end{aligned} \tag{5.13}$$

where in the last step we used that $I_{MM'}c = c$ if c is a constant. Thus,

$$Q_\perp^{M'} I_{MM'} = I_{MM'} Q_\perp^M; \tag{5.14}$$

i.e., the family of projections $Q_\perp^M$ is equivariant with respect to the coarse-graining maps $I_{MM'}$. Similarly

$$Q^{M'} I_{MM'} = (1_{L_{M'}} - Q_\perp^{M'}) I_{MM'} = I_{MM'} - I_{MM'} Q_\perp^M = I_{MM'}(1_{L_M} - Q_\perp^M) = I_{MM'} Q^M. \tag{5.15}$$

It follows from (5.7) and (5.12) that the covariance always takes the form

$$K_M^{(n)} = \frac{1}{2}[Q_\perp^M [\omega_{0,M}^{(n)}]^{-1} Q_\perp^M + Q^M [\omega_M^{(n)}]^{-1} Q^M]; \tag{5.16}$$

in particular, the projections Q^M and $Q_\perp^M$ are not changed under the flow. Moreover, we have separated the flow

$$[\omega_{0,M}^{(n+1)}]^{-1} = I_{MM'(M)}^\dagger [\omega_{0,M'(M)}^{(n)}]^{-1} I_{MM'(M)},$$
$$[\omega_M^{(n+1)}]^{-1} = I_{MM'(M)}^\dagger [\omega_{M'(M)}^{(n)}]^{-1} I_{MM'(M)}, \tag{5.17}$$

The obvious fixed point of the first equation in (5.17) is

$$[\omega_{0,M}^{(n)}]^{-1} = \omega_0^{-1} Q_\perp^M; \tag{5.18}$$

i.e., the zero modes remain unrenormalized as promised. As for the second equation, we can in view of (5.10) immediately copy the results of [6]: Instead of the parameter $q^2 := k^2 + p^2$ used there we just use $q^2 = k^2$. All other relations remain *literally identical*. As the flow equations in [6] depend analytically on q^2, we infer that the fixed point covariance ω_M is the same as in [6] except that $p = 0$ and that it appears sandwiched between Q^M,

$$K_M = \frac{1}{2}[Q_\perp^M \omega_0^{-1} Q_\perp^M + Q^M \omega_M^{-1} Q^M], \tag{5.19}$$

and moreover K_M agrees with the covariance obtained by blocking from the continuum.

Next we turn to the smeared constraints. Here we enter new territory as compared to [6], first due to the presence of the projections $Q_s^{M(0)}$ and second because the constraints do not annihilate the Fock vacuum. We focus just on the part of $D_s(f)$ quadratic in the nonzero mode fields as this term by itself also satisfies the Virasoro algebra [see Sec. II where this term was denoted by $T_s^2(f)$], and it is also this term alone that leads to the anomaly. The other terms denoted $T_s^0(f)$ and $T_s^1(f)$ can be treated by similar methods. We start with the continuum expression and write it in terms of integral kernels

$$D_s(F) = \int dx F(x) \int dy \int dz [\kappa_s^1(x; y, z) C(y)^\dagger C(z) + \kappa_s^2(x; y, z) C(y) C(z) + \kappa_s^2(x; y, z)^* C(y)^\dagger C(z)^\dagger], \tag{5.20}$$

where $\kappa_s^1(x; y, z)^* = \kappa_s^1(x; z, y)$ and $\kappa_s^2(x; y, z) = \kappa_s^2(x; z, y)$. We block from the continuum and compute $[D_s(f)]_M := J_M^\dagger D_s(f) J_M$,

$$\langle w_M[f_M]\Omega_M, [D_s(F)]_M w_M[g_M]\Omega_M\rangle_{\mathcal{H}_M} = \langle w[I_M f_M]\Omega, D_s(F) w[I_M g_M]\Omega\rangle_{\mathcal{H}}. \tag{5.21}$$

We have for any f, g

$$\langle w[f]\Omega, D_s(F) w[g]\Omega\rangle_{\mathcal{H}} = \int dx F(x) \int dy \int dz [\kappa_s^1(x; y, z)\langle C(y) w[f]\Omega, C(z) w[g]\Omega\rangle + \kappa_s^2(x; y, z)\langle w[f]\Omega, C(y) C(z) w[g]\Omega\rangle + \kappa_s^2(x; y, z)^*\langle C(y) C(z) w[f]\Omega, w[g]\Omega\rangle \tag{5.22}$$

and

$$C(y) w[f]\Omega = w[f] w[f]^{-1} C(y) w[f]\Omega = w[f](C(x) - i[\phi(f), C(y)])\Omega = i[C(x), \phi(f)] w[f]\Omega = [(2\omega)^{-1/2} Q f](y) w[f]\Omega$$
$$C(y) C(z) w[f]\Omega = [(2\omega)^{-1/2} Q f](z) C(y) w[f]\Omega = [(2\omega)^{-1/2} Q f](z)[(2\omega)^{-1/2} Q f](y) w[f]\Omega. \tag{5.23}$$

Abbreviating $\sigma = (2\omega)^{-1/2} Q$ we thus find

$$\langle w[f]\Omega, D_s(F) w[g]\Omega\rangle = \langle w[f]\Omega, w[g]\Omega\rangle \int dx F(x) \int dy \int dz (\sigma f)(y)(\sigma g)(z)[\kappa_s^1(x;y,z) + \kappa_s^2(x;y,z) + \kappa_s^2(x;y,z)^*]. \tag{5.24}$$

Applied to $f = I_M f_M$ and $g = I_M g_M$ we obtain due to $J_M^\dagger J_M = 1_{\mathcal{H}_M}$

$$\begin{aligned}&\langle w_M[f_M]\Omega_M, [D_s(F)]_M w_M[g_M]\Omega_M\rangle\\&= \langle w_M[f_M]\Omega_M, w_M[g_M]\Omega_M\rangle \int dxF(x)\int dy\int dz(\sigma I_M f_M)(y)(\sigma I_M g_M)(z)[\kappa_s^1(x;y,z)+\kappa_s^2(x;y,z)+\kappa_s^2(x;y,z)^*]\\&= \langle w_M[f_M]\Omega_M, w_M[g_M]\Omega_M\rangle \sum_{m_1,m_2\in\mathbb{Z}_M} f_M(m_1)g_M(m_2)\int dxF(x)\int dy\int dz\sigma_M(y,m_1)\sigma_M(z,m_2)\\&\quad\times[\kappa_s^1(x;y,z)+\kappa_s^2(x;y,z)+\kappa_s^2(x;y,z)^*]\\&=: \langle w_M[f_M]\Omega_M, w_M[g_M]\Omega_M\rangle\\&\quad\times\sum_{m_1,m_2\in\mathbb{Z}_M} f_M(m_1)g_M(m_2)\int dxF(x)[\kappa_{s,M}^1(x;m_1,m_2)+\kappa_{s,M}^2(x;m_1,m_2)+\kappa_{s,M}^2(x;m_1,m_2)^*]\end{aligned}\tag{5.25}$$

with $\sigma_M(x,m) \coloneqq (\sigma\chi_m^M)(x)$. Now in terms of

$$C_M = \frac{1}{\sqrt{2}}[\sqrt{\omega_M}Q^M\Phi_M - i\sqrt{\omega_M}^{-1}Q_M\Pi_M],\tag{5.26}$$

where ω_M^{-1} is the fixed point covariance that we obtained from the flow of the measures and which annihilates Ω_M. We find with the abbreviation $\hat\sigma_M = [2\omega_M]^{-1/2}Q^M$ and the Ansatz

$$\begin{aligned}[D_s(F)]_M = \sum_{\hat m_1,\hat m_2\in\mathbb{Z}_M}\int dxF(x)[&\hat\kappa_{s,M}^1(x;\hat m_1,\hat m_2)C_M(\hat m_1)^\dagger C_M(\hat m_2)\\&+\hat\kappa_{s,M}^2(x;\hat m_1,\hat m_2)C_M(\hat m_1)C_M(\hat m_2)+\hat\kappa_{s,M}^2(x;\hat m_1,\hat m_2)^*C_M(\hat m_1)^\dagger C_M(\hat m_2)^\dagger]\end{aligned}\tag{5.27}$$

with

$$\hat\kappa_{s,M}^1(x;\hat m_1,\hat m_2)^* = \hat\kappa_{s,M}^1(x;\hat m_2,\hat m_1), \hat\kappa_{s,M}^2(x;\hat m_1,\hat m_2) = \hat\kappa_{s,M}^2(x;\hat m_2,\hat m_1)\tag{5.28}$$

by exactly the same calculation

$$\begin{aligned}&\langle w_M[f_M]\Omega_M, [D_s(F)]_M w_M[g_M]\Omega_M\rangle_{\mathcal{H}_M}\\&= \langle w_M[f_M]\Omega_M, w_M[g_M]\Omega_M\rangle\times\sum_{m_1,m_2\in\mathbb{Z}_M} f_M(m_1)g_M(m_2)\int dxF(x)\\&\quad\times\sum_{\hat m_1,\hat m_2}\hat\sigma_M(\hat m_1,m_1)\hat\sigma_M(\hat m_2,m_2)[\hat\kappa_{s,M}^1(x;\hat m_1,\hat m_2)+\hat\kappa_{s,M}^2(x;\hat m_1,\hat m_2)+\hat\kappa_{s,M}^2(x;\hat m_1,\hat m_2)^*]\\&=: \langle w_M[f_M]\Omega_M, w_M[g_M]\Omega_M\rangle\times\sum_{m_1,m_2\in\mathbb{Z}_M} f_M(m_1)g_M(m_2)\int dxF(x)\\&\quad\times[\hat\kappa_{s,M}^{1\prime}(x;m_1,m_2)+\hat\kappa_{s,M}^{2\prime}(x;m_1,m_2)+\kappa_{s,M}^2(x;m_1,m_2)^*].\end{aligned}\tag{5.29}$$

Comparing (5.25) and (5.29) we obtain an exact match iff for $j = 1, 2$

$$\begin{aligned}\hat\kappa_{s,M}^{j\prime}(x;m_1,m_2) = \kappa_{s,M}^j(x;m_1,m_2) \Leftrightarrow &\int dy\int dz\kappa_s^j(x;y,z)\sigma_M(y,m_1)\sigma_M(z,m_2)\\&= \sum_{\hat m_1,\hat m_2}\hat\kappa_{s,M}^j(x;\hat m_1,\hat m_2)\hat\sigma_M(\hat m_1,m_1)\hat\sigma_M(\hat m_2,m_2),\end{aligned}\tag{5.30}$$

which determines the discrete kernels $\hat\kappa_{s,M}^j(x;\hat m_1,\hat m_2)$ in terms of the continuum kernels $\kappa_s^j(x,;y,z)$.

The question is whether the flow $n\mapsto[D^{(n)}(F)]_M$ starting from (5.2) actually yields this fixed point. Before we answer this question we note that (5.27) is simply not of the form

$$\int dxF(x)\sum_m E_M(x;m)D_{s,M}(m),\tag{5.31}$$

which would yield a natural map (kernel) $E_M; L \mapsto L_M$ (see the discussion of item 1, viewpoint i. in Sec. IV). It is not even of the form

$$\int dx F(x) \sum_{m_1,m_2} E_M(x; m_1, m_2) D_{s,M}(m_1, m_2) \tag{5.32}$$

in terms of a bikernel $E_M\colon L \mapsto L_M \times L_M$ because there are three independent monomials of annihilation and creation operators involved, not only one. Thus, blocking from the continuum does not give rise to such a *natural* kernel or bikernel which would allow us to consider the discretized constraints as $[C(F)]_M$ as smeared with a discretized function or bifunction. However, one may introduce such an interpretation *by hand* by restricting F to be of the form $\tilde{I}_M F_M$ where $\hat{I}_M$ should be sufficiently differentiable and has all the properties of I_M (see again the discussion of item 1, viewpoint ii. in Sec. IV). Such $\tilde{I}_M$ will indeed be provided in [21].

To study the actual flow of the constraints we note that

$$\kappa_s^1(x; y, z) = \kappa_s(x, y)^* \kappa_s(x, z), \qquad \kappa_s^2(x; y, z) = \kappa_s(x, y)\kappa_s(x, z), \qquad \kappa_s(x, y) = [Q_s \sqrt{2\omega}](x, y), \tag{5.33}$$

while $\sigma = (2\omega)^{-1/2} Q$ and $\sigma_M = \sigma \circ I_M$ so that

$$\kappa_{s,M}^1(x; m_1, m_2) = \kappa_{s,M}(x, m_1)^* \kappa_{s,M}(x, m_2), \qquad \kappa_{s,M}^2(x; m_1, m_2) = \kappa_{s,M}(x, m_1)\kappa_s(x, m_2),$$
$$\kappa_{s,M}(x, m) = [\kappa_s \ \sigma_M](x, m) = [Q_s \circ I_M](x, m). \tag{5.34}$$

Accordingly we conclude that

$$\hat{\kappa}_{s,M}^1(x; m_1, m_2) = \hat{\kappa}_{s,M}(x, m_1)^* \hat{\kappa}_{s,M}(x, m_2),$$
$$\hat{\kappa}_{s,M}^2(x; m_1, m_2) = \hat{\kappa}_{s,M}(x, m_1)\hat{\kappa}_s(x, m_2), \qquad \hat{\kappa}_{s,M}(x, m) = [Q_s I_M \sqrt{2\omega_M}](x, m) \tag{5.35}$$

because with $\hat{\sigma}_M = (2\omega_M)^{-1/2} Q_M$ we have

$$[\hat{\kappa}_{s,M} \circ \hat{\sigma}_M](x, m) = [Q_s I_M Q_M](x, m) = [Q_s Q I_M](x, m) = [Q_s I_M](x, m) = \kappa_{s,M}(x, m). \tag{5.36}$$

To see whether these fixed point values of the kernels are reached from the initial discretization we write

$$\hat{\kappa}_{s,M}^{1(n)}(x; m_1, m_2) = \hat{\kappa}_{s,M}^{(n)}(x, m_1)^* \hat{\kappa}_{s,M}^{(n)}(x, m_2),$$
$$\hat{\kappa}_{s,M}^{2(n)}(x; m_1, m_2) = \hat{\kappa}_{s,M}^{(n)}(x, m_1)\hat{\kappa}_s^{(n)}(x, m_2), \qquad \hat{\kappa}_{s,M}^{(n)}(x, m) = \left[Q_s I_M \sqrt{2\omega_M^{(n)}}\right](x, m), \tag{5.37}$$

and by the literally identical calculation we obtain

$$\hat{\sigma}_M^{(n)}(x, m) = [2\omega_M^{(n)}]^{-1/2} Q_M \tag{5.38}$$

in terms of which the flow equation reads

$$\sum_{\hat{m}_1,\hat{m}_2} \kappa_{s,M}^{j(n+1)}(x, \hat{m}_1, \hat{m}_2)\hat{\sigma}_M^{(n+1)}(\hat{m}_1, m_1)\hat{\sigma}_M^{(n+1)}(\hat{m}_2, m_2)$$
$$= \sum_{\hat{m}'_1,\hat{m}'_2} \kappa_{s,M'}^{j(n)}(x, \hat{m}'_1, \hat{m}'_2)(\hat{\sigma}_{M'}^{(n)} \circ I_{MM'}(\hat{m}'_1, m_1))$$
$$\times (\hat{\sigma}_{M'}^{(n)} \circ I_{MM'}(\hat{m}'_2, m_2)), \tag{5.39}$$

which is equivalent to

$$\hat{\kappa}_{s,M}^{(n+1)} \circ \hat{\sigma}_M^{(n+1)} = \hat{\kappa}_{s,M'}^{(n)} \circ \hat{\sigma}_{M'}^{(n)} \circ I_{MM'} \tag{5.40}$$

or

$$\hat{\kappa}_{s,M}^{(n+1)} \circ Q_M = \hat{\kappa}_{s,M'}^{(n)} \circ [\omega_{M'}^{(n)}]^{-1/2} \circ I_{MM'}[\omega_M^{(n)}]^{1/2} \circ Q_M, \tag{5.41}$$

where the sequence $n \mapsto \omega_M^{(n)}$ was constructed explicitly from the measure flow and satisfies for $M'(M) = 2M$,

$$I_{MM'(M)}^\dagger [\omega_{M'(M)}^{(n)}]^{-1} I_{MM'(M)} = [\omega_M^{(n+1)}]^{-1}. \tag{5.42}$$

Starting with

$$\hat{\kappa}_{s,M}^{(0)} = I_M Q_{s,M}^{(0)} [\omega_M^{(0)}]^{1/2}, \tag{5.43}$$

one finds from (5.41) using the consistency of the maps $I_{M_2M_3} I_{M_1M_2} = I_{M_1M_3}$ for $M_1 < M_2 < M_3$

$$\hat{\kappa}^{(n)}_{s,M} = I_{2^n M} Q^{(0)}_{s,2^n M} I_{M,2^n M} [\omega^{(n)}_M]^{1/2}. \tag{5.44}$$

Taking the limit $n \to \infty$ we get due to limit values $I_\infty = 1_L$, $Q^{(0)}_{s,\infty} = Q_s$, $I_{M,\infty} = I_M$, $\omega^{(\infty)}_M$ formally

$$\hat{\kappa}^{(\infty)}_{s,M} = \hat{\kappa}_{s,M}. \tag{5.45}$$

However, it must be shown if and in what sense the sequence (5.44) actually runs into the limit (5.45) which coincides with that blocked from the continuum. This will be done in the next section.

VI. DISCRETE VIRASORO ALGEBRA

The current section is the most important one of the present paper as it answers the question whether the continuum algebra is visible at finite resolution, how large its finite resolution anomaly is, and in what sense that anomaly is simply a finite resolution artifact and converges to zero as we increase the resolution. We thus consider the finite resolution M constraint operators on $\mathcal{H}_M$,

$$D_{sM}(F) \coloneqq J^\dagger_M D_s(F) J_M, \tag{6.1}$$

and compute the finite resolution anomaly

$$\alpha_M(F,s;G,t) \coloneqq [D_{sM}(F), D_{tM}(G)] - J^\dagger_M [D_s(F), D_t(G)] J_M = -J^\dagger_M [D_s(F) P^\perp_M D_t(G) - D_t(G) P^\perp_M D_s(F)] J_M, \tag{6.2}$$

where

$$P^\perp_M = 1_\mathcal{H} - P_M, \qquad P_M = J_M J^\perp_M = P^2_M = P^\dagger_M \tag{6.3}$$

is an orthogonal projection thanks to the isometry $J^\dagger_M J_M = 1_{\mathcal{H}_M}$. The finite resolution anomaly vanishes only when the constraint operators preserve the subspaces $P_M \mathcal{H}$ of $\mathcal{H}$ which is generically not the case and certainly for PFT it is not.

Heuristically the anomaly vanishes as we increase the resolution $M \to \infty$ as we expect that $P^\perp_M \to 0$. The rest of this section is devoted to showing that this is the case rigorously in a suitable operator topology. In fact, showing that $\alpha_M(s,F;t,G)$ as $M \to \infty$ is a delicate issue and must be defined appropriately. This is because we change the Hilbert space $\mathcal{H}_M$ on which α_M is defined. Hence we cannot simply probe the anomaly, say with respect to the weak operator topology on $\mathcal{H}_M$, that is, fixing $\psi_M, \psi'_M \in \mathcal{H}_M$, considering the matrix elements

$$\langle \psi_M, \alpha_M(s,F;t,G) \psi'_M \rangle_{\mathcal{H}_M} \tag{6.4}$$

and taking $M \to \infty$ at fixed ψ_M, ψ'_M as these depend themselves on M. However, what we can do is to consider fixed $\psi, \psi' \in \mathcal{H}$ independent of M and probe the anomaly with $\psi_M \coloneqq J^\dagger_M \psi, \psi'_M \coloneqq J^\dagger_M \psi'$. Accordingly we study the large M behavior of

$$\langle J^\dagger_M \psi, \alpha_M(s,F;t,G) J^\dagger_M \psi' \rangle_{\mathcal{H}_M}. \tag{6.5}$$

It will be sufficient to study one of the two terms in (6.2), i.e., the matrix element

$$\begin{aligned}&\langle \psi, P_M D_s(F) P^\perp_M D_t(G) P_M \psi' \rangle_\mathcal{H} \\ &\quad = \langle D_s(F) P_M \psi, P^\perp_M D_t(G) P_M \psi' \rangle_\mathcal{H},\end{aligned} \tag{6.6}$$

where it used the symmetry of all operators involved.

There are several issues with (6.6) that require clarification: First of all, one would like to take ψ, ψ' from the dense domain $\mathcal{D}$ given by the span of the Weyl vectors $w[f]\Omega$; however, to be useful we need an explicit formula for $J^\dagger_M \psi, P_M \psi$ for $\psi \in \mathcal{D}$, which is not available from [4,6,7]. We derive this formula below. Next, as expected, the range of $J^\dagger_M \mathcal{D}$ is in $\mathcal{D}_M$, which is the span of the $w[I_M f_M]\Omega$ that is dense in $P_M \mathcal{H}$. However, as $I_M f_M$ is a step function, it is not clear that $D_s(F) w[I_M f_M]\Omega$ is well-defined, i.e., a normalizable element of $\mathcal{H}$. It is for this reason that we considered also the case of discontinuous functions f such as $I_M f_M$ as the domain of the constraint operators in Sec. II, and we showed that after suitable regularization we have indeed $D_s(F) w[I_M f_M]\Omega \in \mathcal{H}$. Finally, the image of $\mathcal{D}$ or $P_M \mathcal{D}$ is not invariant under the constraints so that evaluation of the matrix elements of $P^\perp_M$ between vectors in $\mathcal{D}$ is again not directly possible. In fact, in order to evaluate $P^\perp_M$ on say $D_s(F) P_M w[f]\Omega$ one would need to know how to write it as a linear combination of the $w[g]\Omega$, a task which has no obvious solution. One could think that one can avoid this complication and use the fact that $\mathcal{D}$ is dense in $\mathcal{H}$. Thus given ϵ we find $\tilde{\psi} \in \mathcal{D}$ which differs in norm from $D_s(F) P_M w[f]\Omega$ by at most ϵ. If that $\tilde{\psi}$ would only depend on s, F, ϵ one could indeed restrict consideration to the limit of the matrix elements $\langle \tilde{\psi}, P^\perp_M \tilde{\psi}' \rangle$ with $\tilde{\psi}, \tilde{\psi}' \in \mathcal{D}$ because $\|P^\perp_M\| = 1$ is bounded. Unfortunately, such $\tilde{\psi}$ does depend on M, and without explicitly knowing how it does so, it is not possible to estimate the limit of (6.6). The fact that also $\|P_M\| = 1$ does not help as P_M stands between $D_s(F)$ and ψ.

We are therefore forced to have a detailed look at (6.6). A simplification can be obtained by observing that

$$\begin{aligned}&|\langle D_s(F) P_M \psi, P^\perp_M D_t(G) P_M \psi' \rangle| \\ &\quad \leq \langle D_s(F) P_M \psi, P^\perp_M D_s(F) P_M \psi \rangle^{1/2} \\ &\qquad \times \langle D_t(G) P_M \psi', P^\perp_M D_t(G) P_M \psi' \rangle^{1/2}\end{aligned} \tag{6.7}$$

thanks to the CS inequality and the projector property $(P_M^\perp)^2 = (P_M^\perp)^\dagger = P_M^\perp$. Thus (6.6) converges to zero as $M \to \infty$ for all s, F, ψ, t, G, ψ' if and only if

$$\langle D_s(F) P_M \psi, P_M^\perp D_s(F) P_M \psi \rangle \tag{6.8}$$

converges to zero for all $s, F, \psi \in \mathcal{D}$: That convergence of (6.6) implies convergence of (6.8) followed by choosing $t = s, G = F, \psi' = \psi$. The next convergence of (6.8) for all $\psi \in \mathcal{D}$ implies in particular convergence of

$$\langle D_s(F) P_M w[f]\Omega, P_M^\perp D_s(F) P_M w[f]\Omega \rangle \tag{6.9}$$

for the choice $\psi = w[f]\Omega$ and conversely convergence of (6.9) implies convergence of (6.8) for finite linear combinations of the $w[f]\Omega$, that is, general $\psi \in \mathcal{D}$ again by the CS inequality.

Accordingly we will prove that (6.9) converges to zero. Our first task is to compute $P_M w[f]\Omega$. We begin by computing $J_M^\dagger w[f]\Omega$,

$$\begin{aligned}
&\langle w_M[g_M]\Omega_M, J_M^\dagger w[f]\Omega \rangle_{\mathcal{H}_M} \\
&= \langle J_M w_M[g_M]\Omega_M, w[f]\Omega \rangle_{\mathcal{H}} \\
&= \langle w[I_M g_M]\Omega, w[f]\Omega \rangle_{\mathcal{H}} \\
&= \langle \Omega, w[f - I_M g_M]\Omega \rangle_{\mathcal{H}} \\
&= \exp\left(-\frac{1}{2} C(f - I_M g_M, f - I_M g_M)\right),
\end{aligned} \tag{6.10}$$

where we have written out the continuum covariance

$$2C = Q^\perp \omega_0^{-1} Q^\perp + Q \omega^{-1} Q \tag{6.11}$$

as a symmetric bilinear form on $L \times L$. We can also consider it as an operator defined by

$$\langle f, Cg \rangle_L \coloneqq C(f, g). \tag{6.12}$$

We will make use of these two meanings of C as appropriate, and it is clear from the context which meaning is used, respectively. We also remind the reader of the covariance at resolution M,

$$2C_M = I_M^\dagger 2C I_M = Q_M^\perp \omega_0^{-1} Q_M^\perp + Q_M \omega_M^{-1} Q_M, \tag{6.13}$$

where equivariance $Q I_M = I_M Q_M$ was used. Note that both C and C_M considered as operators on L and L_M, respectively, have, in contrast to ω and ω_M an inverse, explicitly

$$\frac{1}{2} C_M^{-1} = I_M^\dagger 2C I_M = Q_M^\perp \omega_0 Q_M^\perp + Q_M \omega_M Q_M \tag{6.14}$$

and similar for C^{-1}.

We make the Ansatz

$$J_M^\dagger w[f]\Omega = \kappa_M(f) w_M[f_M(f)]\Omega_M \tag{6.15}$$

for numbers $\kappa_M(f)$ and vectors $f_M(f) \in L_M$ to be determined. Plugging (6.15) into (6.10) we find

$$\exp\left(-\frac{1}{2} C(f - I_M g_M, f - I_M g_M)\right) = \kappa_M(f) \exp\left(-\frac{1}{2} C_M(f_M(f) - g_M, f_M(f) - I_M g_M)\right), \tag{6.16}$$

which is uniquely solved by

$$f_M(f) = C_M^{-1} I_M^\dagger C f, \kappa_M(f) = \exp\left(-\frac{1}{2}[C(f,f) - C_M(f_M(f), f_M(f))]\right). \tag{6.17}$$

Note that $\kappa_M(f)$ can be simplified

$$\begin{aligned}
C_M(f_M(f), f_M(f)) &= C(I_M f_M(f), I_M f_M(f)) = \langle I_M f_M(f), C I_M f_M(f) \rangle L \\
&= \langle f_M(f), I_M^\dagger C I_M f_M(f) \rangle_{L_M} = \langle f_M(f), C_M f_M(f) \rangle_{L_M} \\
&= \langle f_M(f), I_M^\dagger C f \rangle_{L_M} = \langle f, [C(I_M C_M^{-1} I_M^\dagger) C] f \rangle L.
\end{aligned} \tag{6.18}$$

It follows

$$P_M w[f]\Omega = \kappa_M(f) w[f^M(f)]\Omega, \qquad f^M(f) = I_M f_M(f) = (I_M C_M^{-1} I_M^\dagger) C f. \tag{6.19}$$

It is instructive to verify the projection property $P_M^2 = P_M$ and the isometry property $J_M^\dagger J_M = 1_{\mathcal{H}_M}$, which relies on $\kappa_M(I_M f_M) = 1$ and $f^M(I_M f_M) = I_M f_M$ for any $f_M \in L_M$.

The next task is to compute $D_s(F)P_M w[f]\Omega$, which given (6.19) can be done, of course, using the explicit expression of $D_s(F)$ in terms of creation and annihilation operators. However, to be useful, we must write $D_s(F)P_M w[f]\Omega$ in the form of linear combinations of $w[h]\Omega$ again because in order to apply $P_M^\perp$ to it, whose action follows from (6.19), its action is only known in closed form on vectors in $\mathcal{D}$ and not on Fock states. The other option would be to expand $P_M w[f]\Omega$ into Fock states. While this is possible, it leads to very complex expressions. We therefore choose the former route which also has the advantage to maximally benefit from the identity $P_M^\perp P_M = 0$.

We note that (we pick the C sector for definiteness and focus only on the corresponding contribution to the constraints)

$$w[h]\Omega = \exp(i\langle h,\Phi\rangle)\Omega = \exp(i\langle C^{-1/2}h, A_C^\dagger + A_C\rangle_L)\Omega$$
$$= \exp\left(-\frac{1}{2}\langle h, Ch\rangle \exp(i\langle C^{1/2}h, A_C^\dagger + \rangle_L\right)\Omega, \qquad (6.20)$$

using well-known Fock space techniques (Baker Campbell Hausdorff formula). Here we have denoted the annihilation operator of the C sector by A_C in order not to confuse it with the covariance C. Thus, we find the functional derivatives

$$\frac{\delta}{\delta h(y)} w[C^{-1}h + g]\Omega = [-g(y) - (C^{-1}h)(y) + i(C^{-1/2}A_C^\dagger)(y)]w[C^{-1}h + g]\Omega$$
$$\frac{\delta^2}{[\delta h(y)][\delta h(z)]} w[C^{-1}h + g]\Omega = \{-C^{-1}(y,z) + [-g(y) - (C^{-1}h)(y) + i(C^{-1/2}A_C^\dagger)(y)]$$
$$\times [-g(z) - (C^{-1}h)(z) + i(C^{-1/2}A_C^\dagger)(z)]w[C^{-1}h + g]\Omega, \qquad (6.21)$$

i.e., at $h = 0$

$$\left(\frac{\delta}{\delta h(y)} w[C^{-1}h + g]\Omega\right)_{h=0} = [-g(y) + i(C^{-1/2}A_C^\dagger)(y)]w[g]\Omega$$
$$\left(\frac{\delta^2}{[\delta h(y)][\delta h(z)]} w[C^{-1}h + g]\Omega\right)_{h=0} = \{-C^{-1}(y,z) + [-g(y) + i(C^{-1/2}A_C^\dagger)(y)][-g(z) + i(C^{-1/2}A_C^\dagger)(z)]\}w[g]\Omega. \qquad (6.22)$$

Here we used that all expressions just depend on creation operators that mutually commute.

Recall the constraint operator

$$-D_s(F) = \int dx F(x) \int dy \int dz \{Q_s(x,y)Q_s(x,z)(C^{-1/2}A_C)(y)(C^{-1/2}A_C)(z)$$
$$+ Q_s^*(x,y)Q_s^*(x,z)(C^{-1/2}A_C^\dagger)(y)(C^{-1/2}A_C^\dagger)(z) - 2Q_s^*(x,y)Q_s(x,z)(C^{-1/2}A_C^\dagger)(y)(C^{-1/2}A_C)(z)\}, \qquad (6.23)$$

where $Q_s(x,y)$ is the integral kernel of the projection Q_s. We have explicitly

$$[C^{-1/2}A_C](y)w[g]\Omega = w[g](w[-g][C^{-1/2}A_C](y)w[g])\Omega = w[g]([C^{-1/2}A_C](y) - i[\phi[g],(C^{-1/2}A_C)(y)])\Omega$$
$$= -iw[g][A_C^*[C^{1/2}g],(C^{-1/2}A_C)(y)])\Omega = -ig(y)w[g]\Omega, \qquad (6.24)$$

whence

$$-D_s(F)w[g]\Omega = \int dx F(x) \int dy \int dz \{-Q_s(x,y)Q_s(x,z)g(y)g(z)$$
$$+ Q_s^*(x,y)Q_s^*(x,z)(C^{-1/2}A_C^\dagger)(y)(C^{-1/2}A_C^\dagger)(z) - 2iQ_s^*(x,y)Q_s(x,z)(C^{-1/2}A_C^\dagger)(y)g(z)\}w[g]\Omega$$
$$= \int dx \int dy \int dz \{\{(-Q_s(x,y)Q_s(x,z) + Q_s^*(x,y)Q_s^*(x,z) - 2Q_s(y,z)^* Q(x,z))g(y)g(z) - C^{-1}(x,y)\}$$
$$+ Q_s^*(x,y)Q_s^*(x,z)\{(C^{-1/2}A_C^\dagger)(y)(C^{-1/2}A_C^\dagger)(z) + 2ig(y)(C^{-1/2}A_C^\dagger)(z) + C^{-1/2}(y,z) - g(y)g(z)\}$$
$$- 2\{(Q_s^*(x,y)Q_s(x,z) + Q_s^*(x,y)Q_s^*(x,z))[i(C^{-1/2}A_C^\dagger)(y) - g(y)]g(z)\}\}w[g]\Omega \qquad (6.25)$$

with $Q = Q_s + Q_s^*$. We evaluate (6.25) for $g = f^M(f)$, multiply it from the left with $\kappa_M(f)P_M^\perp$, and use (6.22) to obtain the identity

$$P_M^\perp D_s[F]P_M w[f]\Omega = \left\{P_M^\perp \int dx F(x)\int dy \int dz \left\{Q_s(x,y)^* Q_s(x,z)^* \frac{\delta^2}{[\delta h(y)][\delta h(z)]} - 2Q_s(x,y)^* Q(x,z)[f^M(f)](z)\frac{\delta}{\delta h(y)}\right\}\right.$$
$$\left.\times \kappa_M(f) w[C^{-1}h + f_M(f)]\Omega\right\}_{h=0}, \tag{6.26}$$

where the terms in (6.25) that do not involve creation operators could be dropped because at $h=0$ we get $P_M^\perp \kappa_M(f)w[C^{-1}h + f_M(f)]\Omega = P_M^\perp P_M w[f]\Omega = 0$.

Formula (6.26) is the desired expression because $P_M^\perp$ can be pulled past the functional derivatives where it hits $w[C^{-1}h + f^M(f)]\Omega$ and can be evaluated. Let $h' = C^{-1}h, g = f^M(f)$. Then due to the projector property $f^M(g) = g$ and $\kappa_M(g) = 1$ whence

$$\begin{aligned}
&P_M^\perp w[h'+g]\Omega = w[h'+g]\Omega - \kappa_M(h'+g)w[f^M(h'+g)]\Omega,\\
&f^M(h'+g) = f^M(h') + g,\\
&\kappa_M(h'+g) = \kappa_M(h')\kappa_M(g)\exp(\langle h', C(1 - I_M C_M^{-1} I_M^\dagger)g\rangle_L) = \kappa_M(h');
\end{aligned} \tag{6.27}$$

therefore,

$$\kappa_M(f)P_M^\perp w[h'+g]\Omega = \kappa_M(f)(w[h'+g] - \kappa_M(h')w[f^M(h')+g])\Omega = [w[h'] - \kappa_M(h')]w[f^M(h')]P_M w[f]\Omega. \tag{6.28}$$

We can now evaluate (6.8):

$$\begin{aligned}
\|P_M^\perp D_s(F)P_M w[f]\Omega\|^2 = &\left\{\int dx F(x)\int dy\int dz\int dx' F(x')\int dy'\int dz'\right.\\
&\times\left\{Q_s(x,y)Q_s(x,z)\frac{\delta^2}{[\delta h(y)][\delta h(z)]} - 2Q_s(x,y)Q(x,z)g(z)\frac{\delta}{\delta h(y)}\right\}\\
&\times\left\{Q_s^*(x',y')Q_s^*(x',z')\frac{\delta^2}{[\delta\hat h(y')][\delta\hat h(z')]} - 2Q_s^*(x',y')Q(x',z')g(z')\frac{\delta}{\delta\hat h(y')}\right\}\\
&\left.\times\langle[w[h'] - \kappa_M(h')]w[f^M(h')]P_M w[f]\Omega, [w[\hat h'] - \kappa_M(\hat h')]w[f^M(\hat h')]P_M w[f]\Omega\rangle_{\mathcal H}\right\}_{h=\hat h=0}
\end{aligned} \tag{6.29}$$

with $g = f^M(f), h' = C^{-1}h$, and $\hat h' = C^{-1}\hat h$. We have

$$\begin{aligned}
&\langle[w[h'] - \kappa_M(h')]w[f^M(h')]P_M w[f]\Omega, [w[\hat h'] - \kappa_M(\hat h')]w[f^M(\hat h')]P_M w[f]\Omega\rangle\\
&= \kappa_M(f)^2\langle[w[h'] - \kappa_M(h')]w[f^M(h')]w[g]\Omega, [w[\hat h'] - \kappa_M(\hat h')]w[f^M(\hat h')]w[g]\Omega\rangle\\
&= \kappa_M(f)^2\langle[w[h'] - \kappa_M(h')]w[f^M(h')]\Omega, [w[\hat h'] - \kappa_M(\hat h')]w[f^M(\hat h')]\Omega\rangle\\
&= \kappa_M(f)^2\langle P_M^\perp w[h']\Omega, P_M^\perp w[\hat h']\Omega\rangle\\
&= \kappa_M(f)^2\langle w[h']\Omega, P_M^\perp w[\hat h']\Omega\rangle\\
&= \kappa_M(f)^2\langle\Omega, [w[\hat h' - h'] - \kappa_M(\hat h')w[f^M(\hat h') - h']\Omega\rangle\\
&= \kappa_M(f)^2\left[\exp\left(-\frac{1}{2}\langle\hat h' - h', C(\widehat{h'} - h')\rangle\right) - \kappa_M(\hat h')\exp\left(-\frac{1}{2}\langle f^M(\hat h') - h', C(f^M(\widehat{h'}) - h')\rangle\right)\right].
\end{aligned} \tag{6.30}$$

Before evaluating the functional derivatives we can simplify (6.30):

$$\begin{aligned}
\langle h', Cf^M(\hat{h}')\rangle &= \langle f^M(h'), C\hat{h}'\rangle \\
\langle f^M(\hat{h}'), Cf^M(\hat{h}')\rangle &= \langle \hat{h}', Cf^M(\hat{h}')\rangle \kappa_M(\hat{h}') \exp\left(-\frac{1}{2}\langle f^M(\hat{h}') - h', C(f^M(\widehat{h'}) - h')\rangle\right) \\
&= \exp\left(-\frac{1}{2}[\langle \hat{h}', C[\hat{h}' - f^M(\hat{h}')]\rangle + \langle f^M(\hat{h}') - h', C[f^M(\hat{h}') - h']\rangle]\right) \\
&= \exp\left(-\frac{1}{2}[\langle[\hat{h}' - h']C[\hat{h}' - h']\rangle + 2\langle h', C(\hat{h}' - f^M(\hat{h}'))\rangle]\right). \qquad (6.31)
\end{aligned}$$

Accordingly, Eq. (6.30) can be rewritten as (reintroducing $h = Ch', \hat{h} = C\hat{h}'$)

$$\kappa_M(f)^2 \exp\left(-\frac{1}{2}\langle -h, C^{-1}[\hat{h} - h]\rangle\right)[1 - \exp(-\langle h, [C^{-1} - I_M C_M^{-1} I_M^\dagger]\hat{h}\rangle)]. \qquad (6.32)$$

It will be convenient to define the symmetric kernels $K = C^{-1}, \Delta K = C^{-1} - I_M C_M^{-1} I_M^\dagger$. In carrying out the double, triple, and fourfold functional derivatives of (6.32) at $h = \hat{h} = 0$ we use arguments familiar from Wick's theorem in perturbative QFT: as (6.32) is a linear combination of two exponentials $E(H) = \exp(B(H,H)/2)$ of a quadratic polynomial B in $H = (h, \hat{h}')$, their derivatives are schematically

$$\begin{aligned}
&E' = (BH)E, E'' = [B + (BH)^2]E, E''' = [3B^2 H + (BH)^3]E, \\
&E'''' = [3B^2 + 3(BH)^2 B + 3(B^2 H)(BH) + (BH)^4]E, \qquad (6.33)
\end{aligned}$$

so that at $H = 0$ only second and fourth derivatives survive. To simplify the notation we set

$$E_1 := \exp\left(-\frac{1}{2}\langle \hat{h} - h, K[\hat{h} - h]\rangle\right), \qquad E_2 := \exp(-\langle h, [\Delta K]\hat{h}\rangle), \qquad E_{j,y} := \frac{\delta}{\delta h(y)}, \qquad E_{j,y'} := \frac{\delta}{\delta \hat{h}(y')}, \qquad (6.34)$$

with $j = 1, 2$ and similar for z, z'. Then

$$\begin{aligned}
(E_1 E_2)_{,yy'} &= E_{1,yy'} E_2 + E_1 E_{2,yy'} + E_{1,y} E_{2,y'} + E_{1,y'} E_{2,y}, \\
(E_1 E_2)_{,yy'zz'} &= [E_{1,yy'zz'} E_2 + E_{1,yy'} E_{2,zz'}] + [E_{1,zz'} E_{2,yy'} + E_1 E_{2,yy'zz'}] \\
&\quad + [E_{1,yz} E_{2,y'z'} = E_{1,yz'} E_{2,y'z}] + [E_{1,y'z} E_{2,yz'} + E_{1,y'z'} E_{2,yz}] + \cdots, \qquad (6.35)
\end{aligned}$$

where $\cdots$ denotes odd order derivatives which vanish at $H = 0$. We have at $H = 0$

$$\begin{aligned}
&E_{1,yz} = -K(y,z), \qquad E_{1,y'z'} = -K(y',z'), \qquad E_{1,yz'} = K(y,z'), \\
&E_{2,yz} = 0, \qquad E_{2,y'z'} = 0, \qquad E_{2,yz'} = [\Delta K](y,z'), \\
&E_{1,yy'zz'} = K(y,z)K(y',z') + K(y,y')K(z,z') + K(y,z')K(z,y'), \\
&E_{2,yy'zz'} = [\Delta K](y,y')[\Delta K](z,z') + [\Delta K](y,z')[\Delta K](z,y'). \qquad (6.36)
\end{aligned}$$

Collecting all terms we find at $H = 0$

$$[E_1(1 - E_2)]_{,yy'} = [\Delta K](y,y'), \qquad (6.37)$$

$$\begin{aligned}
[E_1(1 - E_2)]_{,yy'zz'} &= K(y,y')[\Delta K](z,z') + K(y,z')[\Delta K](z,y')K(z,y')[\Delta K](y,z') \\
&\quad - [\Delta K](y,y')[\Delta K](z,z') - [\Delta K](y,z')[\Delta K](z,y'), \qquad (6.38)
\end{aligned}$$

where importantly both terms proportional to $E_{1,yy'zz'}$ have canceled so that all functional derivatives contain at least one factor of ΔK, which we expect to imply the convergence to zero of (6.8), which now can be vastly simplified to

$$\kappa_M(f)^2 \int dx F(x) \int dx' F(x') \{3K_s(x,x')[\Delta K]_s(x,x') - 2([\Delta K]_s(x,x'))^2 + 4g'(x)g'(x')[\Delta K]_s(x,x')\}, \tag{6.39}$$

where using $Q_s^*(y,z) = Q_{-s}(y,z) = Q_s(z,y)$,

$$K_s(x,x') = \int dy \int dz Q_s(x,y) Q_s(x',z) K(y,z) = [Q_s K Q_s](x,x'), \tag{6.40}$$

and similar for $[\Delta K]_s(x,x')$. Here

$$g'(x) = [Q I_M C_M^{-1} I_M^\dagger C f](x) = (Q(K - [\Delta K])Cf)(x), \qquad \kappa_M(x) = \exp\left(-\frac{1}{2}\langle Cf, [\Delta K]Cf\rangle\right). \tag{6.41}$$

Since P_M is a projection, we have $\|P_M\| = 1$; thus,

$$\|P_M w[f]\Omega\| = \kappa_M(f)\|w[f^M(f)]\Omega\| = \kappa_M(f)\| \leq \|P_M\|\|w[f]\Omega\| = 1, \tag{6.42}$$

and it will be sufficient to show that the integral term in (6.39) converges. Also we focus on $s = +$, the case $s = -$ being completely analogous. Obviously then, the convergence or not of (6.8) rests on the properties of Δ_K and g'. We begin with the term

$$\int dx \int dx' F(x)F(x')K_+(x,x')[\Delta K]_+(x,x') = \int dx \int dx' F(x)F(x')K_+(x,x')[\Delta K]_-(x',x), \tag{6.43}$$

where in the second step we used that $[\Delta K](y,z) = [\Delta K](z,y)$ and $Q_s(y,z) = Q_s^*(z,y) = Q_{-s}(z,y)$. We expand into the Fourier basis

$$\begin{aligned} K_+(x,x') &= \sum_{n,n' \in \mathbb{Z}} e_n(x)\langle e_n, Q_+ K Q_+ e_{n'}\rangle e_{-n'}(x') = \sum_{n,n'>0} e_n(x)\langle e_n, K e_{n'}\rangle e_{-n'}(x'), \\ [\Delta K]_-(x',x) &= \sum_{n,n'<0} e_n(x')\langle e_n, [\Delta K] e_{n'}\rangle e_{-n'}(x), \\ F(x) &= \sum_{|n|<n_0} \hat{F}(n) e_n(x) = F^*(x), \qquad \hat{F}^*(n) = \hat{F}(-n), \end{aligned} \tag{6.44}$$

where we assume that F *has compact momentum support* $|n| < n_0$. Presumably what follows can also be shown under milder decay assumptions on the Fourier modes $\hat{F}(n)$ (e.g., rapid decrease in $n \in \mathbb{Z}$), but we will be satisfied if convergence can be proved for this class of smearing functions of the constraint. Then (6.43) turns into

$$\sum_{|n_1|,|n_2|<n_0} \hat{F}(n_1); \quad \hat{F}^*(n_2) \sum_{m,n>0} \langle e_m, K e_n\rangle \sum_{m',n'<0} [\Delta K]_{m',n'} \delta_{n_1+m-n'} \delta_{-n_2-n+m'}. \tag{6.45}$$

This implies the constraints on the range of m, n, m', n',

$$\begin{aligned} & n' = n_1 + m < 0, \quad m' = n_2 + n < 0, \quad m = n' - n_1 > 0, \qquad n = m' - n_2 > 0 \\ \Rightarrow\ & 0 < m < -n_1 < n_0, \quad 0 < n < -n_2 < n_0, \quad 0 > n' > n_1 > -n_0, \quad 0 > m' > n_2 > -n_0; \end{aligned} \tag{6.46}$$

thus, the compact momentum support propagates to the m, n, m', n' modes. For bounded values of m, n the modulus of the matrix element $|\langle e_m, K e_n\rangle|$ is uniformly bounded, and we are left to study the behavior of $\langle e_n, [\Delta K] e_{n'}\rangle$ at fixed values of $n, n' \neq 0$ (of equal sign). We have

$$\begin{aligned} \langle e_n, [\Delta K] e_{n'}\rangle &= \langle e_n, [C^{-1} - I_M C_M^{-1} I_M^\dagger] e_{n'}\rangle \\ &= 2[\omega(n)\delta_{n,n'} - \sum_{\tilde{n} \in \mathbb{Z}_M} \omega_M(\tilde{n})\langle e_n, I_M e_{\tilde{n}}^M\rangle \langle I_M e_{\tilde{n}}^M, e_{n'}\rangle, \end{aligned} \tag{6.47}$$

where in the second step we expanded into the spectral basis $e^M_{\tilde{n}} \in L_M$ of ω_M given by $e^M_{\tilde{n}}(m) = e_{\tilde{n}}(x^M_m), x^M_m = \frac{m}{M}$, $m \in \mathbb{Z}_M$. The eigenvalues $\omega_M(\tilde{n})$ follow from the definition $C_M = I^\dagger_M C I_M$, i.e.,

$$Q_M \omega_M^{-1} Q_M = I^\dagger_M Q \omega^{-1} Q I_M \tag{6.48}$$

from which

$$\begin{aligned} Q_M \omega_M^{-1} Q_M e^M_{\tilde{n}} &= \sum_{0 \neq n, n' \in \mathbb{Z}} I^\dagger_M e_n \langle e_n, \omega^{-1} e_{n'} \rangle \langle e_{n'}, I_M e^M_{\tilde{n}} \rangle \\ &= \sum_{\tilde{n}' \in \mathbb{Z}_M} e^M_{\tilde{n}'} \sum_{n \neq 0} \langle I_M e^M_{\tilde{n}'}, e_n \rangle \omega^{-1}(n) \langle e_n, I_M e^M_{\tilde{n}} \rangle. \end{aligned} \tag{6.49}$$

Here we need the Fourier modes of the characteristic functions χ^M_m of the interval $[x^M_m, x^M_{m+1})$,

$$(I^\dagger_M e_n)(m) = M \langle \chi^M_m, e_n \rangle = M e_n(x^M_m) \frac{e^{ik_M n} - 1}{2\pi i n}, \quad k_M = \frac{2\pi}{M}. \tag{6.50}$$

We note that (6.50) does not have compact momentum support and also does not decay rapidly. This has some bearing further below. It follows

$$\begin{aligned} \langle e^M_{\tilde{n}}, I^\dagger_M e_n \rangle &= \langle I_M e^M_{\tilde{n}}, e_n \rangle \\ &= \sum_{m \in \mathbb{Z}_M} [e^M_{\tilde{n}}(m)]^* e^M_n(m) \frac{e^{ik_M n} - 1}{2\pi i n} \\ &= \left[\sum_m e^M_{n - \tilde{n}}(m) \right] \frac{e^{ik_M n} - 1}{2\pi i n} \\ &= M \delta_{\tilde{n}, \hat{n}} \frac{e^{ik_M n} - 1}{2\pi i n}, \end{aligned} \tag{6.51}$$

where $\hat{n} \in \mathbb{Z}_M$ and $n = \hat{n} + lM, l \in \mathbb{Z}$ uniquely decomposes a general integer n into a multiple l of M and a remainder $\hat{n} \in \mathbb{Z}_M = \{0, 1, \ldots, M-1\}$. Accordingly

$$\begin{aligned} Q_M \omega_M^{-1} Q_M e^M_{\tilde{n}} &= \sum_{\tilde{n}'} e^M_{\tilde{n}'} \sum_{n \neq 0} \omega(n)^{-1} \delta_{\tilde{n}, \hat{n}} \delta_{\tilde{n}', \hat{n}} \\ &\quad \times \frac{2M^2 [1 - \cos(k_M \hat{n})]}{[2\pi n]^2} \\ &= [1 - \delta_{\tilde{n}, 0}] e^M_{\tilde{n}} \sum_{l \in \mathbb{Z}} \omega(\tilde{n} + lM)^{-1} \\ &\quad \times \frac{2M^2 [1 - \cos(k_M \tilde{n})]}{[2\pi(\tilde{n} + lM)]^2}, \end{aligned} \tag{6.52}$$

whence for $M > n > 0$,

$$\begin{aligned} \omega_M(n)^{-1} &= \sum_l \omega(n + lM)^{-1} \frac{2[1 - \cos(k_M n)]}{[k_M(n + lM)]^2} \\ &= \omega(n)^{-1} \frac{2[1 - \cos(k_M n)]}{[k_M n]^2} \\ &\quad + \sum_{l \neq 0} \omega(n + lM)^{-1} \frac{2[1 - \cos(k_M n)]}{[k_M(n + lM)]^2}. \end{aligned} \tag{6.53}$$

Since $\omega(n) = 2\pi|n|$, at fixed n the first term in (6.53) converges to $\omega^{-1}(n)$ as $M \to \infty$ while the modulus of the second is bounded by the series

$$\frac{4}{(2\pi)^3 M} \sum_{l=1}^\infty \left[\frac{1}{[l + \frac{n}{M}]^3} + \frac{1}{[l - \frac{n}{M}]^3} \right] < \frac{4}{(2\pi)^3 M} \sum_{l=1}^\infty \left[\frac{1}{l^3} + \frac{1}{[l - \frac{1}{2}]^3} \right] \tag{6.54}$$

for $n < M/2$ and thus converges to zero as M^{-1}. Accordingly $\omega_M(n) - \omega(n) = O(1/M)$ at fixed n. Then (6.47) becomes

$$\begin{aligned} \langle e_n, [\Delta K] e_{n'} \rangle &= 2 \left[\omega(n) \delta_{n,n'} - \sum_{\tilde{n} \in \mathbb{Z}_M} \omega_M(\tilde{n}) \sum_{m_1, m_2} \frac{e^{ik_M n'} - 1}{2\pi i n'} \left[\frac{e^{ik_M n} - 1}{2\pi i n} \right]^* e^M_{\tilde{n} - n}(m_1) e^M_{n' - \tilde{n}}(m_2) \right] \\ &= 2 \left[\omega(n) \delta_{n,n'} - \sum_{\tilde{n} \in \mathbb{Z}_M} \omega_M(\tilde{n}) \delta_{\tilde{n}, \hat{n}} \delta_{\tilde{n}, \hat{n}'} \frac{2[1 - \cos(k_M \tilde{n})]}{(2\pi)^2 n n'} \right], \end{aligned} \tag{6.55}$$

where $n = \hat{n} + lM, n' = \hat{n}' + l'M$, and $\hat{n}, \hat{n}' \in \mathbb{Z}_M$. Since $0 > n, n' > -n_0$, and eventually $M > n_0$, we have $l = l' = -1$ and $\hat{n} = M + n = \tilde{n}, \hat{n}' = M + n' = \tilde{n}$; therefore, $n = n'$ in the second term of (6.56) and $\tilde{n} = n + M$

$$\langle e_n, [\Delta K] e_{n'} \rangle = 2\delta_{n,n'} \left[\omega(n) - \omega_M(M + n) \frac{2[1 - \cos(k_M(n))]}{(k_M n)^2} \right]. \tag{6.56}$$

Note that for $-M < -n_0 < n < 0$ we have

$$\begin{aligned}\omega_M(M+n)^{-1} &= \sum_{l\in\mathbb{Z}}\omega(M+n+lM)^{-1}\frac{2[1-\cos(k_M(M+n))]}{[k_M(M+n+lM)]^2}\\ &= \sum_{l\in\mathbb{Z}}\omega(n+lM)^{-1}\frac{2[1-\cos(k_M n)]}{[k_M(n+lM)]^2}\to\omega(n)^{-1}=\omega(-n)^{-1}\end{aligned}\tag{6.57}$$

as $M\to\infty$. Thus, indeed, Eq. (6.43) converges to zero.

Next consider

$$\int dx\int dx' F(x)F(x')[\Delta K]_+(x,x')[\Delta K]_+(x,x') = \int dx\int dx' F(x)F(x')[\Delta K]_+(x,x')[\Delta K]_-(x',x).\tag{6.58}$$

By the same argument as above, if F has compact momentum support, then (6.58) is a quadratic polynomial in the $\langle e_n,[\Delta K]_{n'}\rangle$ with M independent coefficients where either $n_0>n,n'>0$ or $-n_0<n,n'<0$ and hence converges to zero.

Finally consider

$$\int dx F(x)\int dx' F(x')g'(x)g'(x')[\Delta K]_s(x,x'),\tag{6.59}$$

where

$$g'(x) = [QI_M C_M^{-1}I_M^\dagger Cf](x).\tag{6.60}$$

We note that $QI_M = I_M Q_M, [C_M,Q_M]=[C,Q]=0$ implies that $I_M^\dagger Q = q - mI_M^\dagger$ whence by the now familiar argument

$$g'(x) = [QI_M C_M^{-1}I_M^\dagger CQf](x)\tag{6.61}$$

so that

$$\begin{aligned}g' &= \sum_{n,n'\neq 0}\sum_{\tilde n}\omega_M(\tilde n)e_{n'}\langle e_{n'},I_M e^M_{\tilde n}\rangle\langle I_M e^M_{\tilde n},e_n\rangle\omega(n)^{-1}\hat f(n)\\ &= \sum_{l,l'\in\mathbb{Z}}\sum_{\tilde n}\omega_M(\tilde n)e_{\tilde n+l'M}\frac{2[1-\cos(k_M\tilde n)]}{k_M^2(\tilde n+lM)(\tilde n+l'M)}\omega(\tilde n+lM)^{-1}\hat f(\tilde n+lM).\end{aligned}\tag{6.62}$$

It follows that g' does not have compact momentum support n' even if f does. Therefore, $F(x)g'(x)$ also does not have compact momentum support even if F does. It is not even clear that (6.62) converges. This feature of f' is again due to the fact that the functions χ^M_m are discontinuous. If one would replace them by χ^{M,n_0}_m where $\hat\chi^{M,n_0}_m$ is the Fourier expansion of χ^M_m restricted to modes $|n|<n_0$, then $\chi^{M,n_0}_m\to\chi^M_m$ in the L norm, and if we define $I^{n_0}_M,[I^{n_0}_M]^\dagger$ like $I_M,I_M^\dagger$ with χ^M_m replaced by χ^{M,n_0}_m and first take the limit $M\to\infty$ in (6.59) and then $n_0\to\infty$, then (6.59) vanishes as $M\to\infty$. This regularization using the momentum cutoff n_0 is similar to the zeta function regularization of Sec. III and is justified by the following argument: while the χ^M_m have all the necessary features in order to define a renormalization flow, they are not the only choice. There are other, smoother choices [21] satisfying the same necessary requirements but those have a built-in compact momentum support of order M. In that case the sum over l, l' in (6.62) disappears and the compact momentum support of f propagates to that of g, and then, e.g., $g=Qf$ *even exactly* for sufficiently large M. Then also Fg' have compact momentum support, and the same argument as was made for (6.43) and (6.58) can be used to show that (6.59) converges to zero without any regulator. Since the choice of the χ^M_m is quite arbitrary subject to a minimal set of requirements and since one wants to probe functions f of compact momentum support using their $I_M I_M^\dagger f$ approximants, such a smooth choice of χ^M_m is simply more convenient. With respect to any choice we have convergence of $I_M I_M^\dagger\to 1_L$ in the L_2 sense, but the finite resolution approximants have additional smoothness or momentum compactness properties while others do not and those additional properties turn out to be important in the present convergence analysis. The strict proof that with the choice of I_M made in [21] expression (6.59) converges to zero is given in Sec. 5 of [21] and also provides the argument that was missing at the end of the previous section to establish convergence of the flow of constraints.

We conclude this section with the remark that the functions χ^M_m used in [21] are smooth with compact momentum support and that smooth smearing functions F, f of constraints and Weyl elements, respectively, are of rapid decrease in the momentum mode label n. Thus, with respect to those functions all estimates of this section pass through without any regularization and convergence is established.

VII. DISCRETIZED SMEARING FUNCTIONS OF THE CONSTRAINTS

As we have seen, the embeddings J_M do not induce a canonical map $E_M\colon\ L \to L_M$ such that (we drop the index s for the purpose of this section)

$$D_M(F) \coloneqq J_M^\dagger D(F) J_M \eqqcolon \tilde{D}_M(E_M F). \tag{7.1}$$

However, we may use the map $E_M \coloneqq I_M^\dagger$ to define the family of discretized smearing functions $F_M \coloneqq I_M^\dagger F$,

$$\tilde{D}_M(F_M) \coloneqq J_M^\dagger D(I_M F_M) J_M = D_M(p_M F), \tag{7.2}$$

where

$$p_M = I_M I_M^\dagger\colon\ L \to L \tag{7.3}$$

is a projection due to isometry $I_M^\dagger I_M = 1_{L_M}$. This defines a consistent family of quadratic forms in the sense that for any $M < M'$,

$$J_{MM'}^\dagger \tilde{D}_{M'}(I_{MM'} F_M) J_{MM'} = \tilde{D}_M(F_M) \tag{7.4}$$

with $I_{MM'} = I_{M'}^\dagger I_M$ thanks to $I_{M'} I_{MM'} = I_M$ and $J_{M'} J_{MM'} = J_M$. We can therefore compute

$$[\tilde{D}_M(F_M), \tilde{D}_M(G_M)] = J_M^\dagger([D(p_M F), D(p_M G)] - D(p_M F)(1 - P_M) D(p_M G) + D(p_M G)(1 - P_M) D(p_M F)) J_M \tag{7.5}$$

and modulo the central term we have in our case

$$[D(p_M F), D(p_M G)] = D([p_M F, p_M G]). \tag{7.6}$$

The new smearing function in (7.6) is given by

$$[p_M F, p_M G] \coloneqq [p_M F]'[p_M G] - [p_M F][p_M G]' = p_M([p_M F, p_M G]) + (1 - p_M)([p_M F, p_M G]). \tag{7.7}$$

Thus (7.5) becomes

$$[\tilde{D}_M(F_M), \tilde{D}_M(G_M)] = \tilde{D}_M(\kappa_M(F_M, G_M)) \tag{7.8}$$

modulo the central term and the corrections involving $1_{\mathcal{H}} - P_M$ and $1_L - p_M$. Here the discretized structure functions are defined by

$$\kappa_M(F_M, G_M) \coloneqq I_M^\dagger \kappa(I_M F_M, I_M G_M), \qquad \kappa(F, G) = [F, G], \tag{7.9}$$

which are well-defined if the functions χ_m^M defining I_M are sufficiently differentiable. We have already seen in the previous section that the correction involving $1_{\mathcal{H}} - P_M$ converges to zero if F has compact momentum support. That is no longer the case for F replaced by $p_M F$ if the functions χ_m^M are step functions, but it is the case when those functions themselves have compact momentum support as those in [21]. The functions χ_M^m in general span a closed, finite dimensional subspace $V_M \subset L$ and their derivatives $[\chi_m^M]'$ may or may not lie in V_M (for the case [21] they actually do). However, the products $\chi_{m'}^M [\chi_m^M]'$ are no longer in V_M so that the term proportional to $1_L - p_M$ does not vanish automatically. If, however, F, G have compact momentum support, then the projections $p_M F$ coincide with F for sufficiently large M because V_M roughly involves all Fourier modes up to order $|n| \leq M$ and thus also $[F, G]$ eventually lies in V_M and the correction involving $1 - p_M$ eventually vanishes.

If F, G do not have compact momentum support but are smooth, then their Fourier transforms are of rapid decrease in the mode label n. In this case the terms involving $1_L - p_M$ are not exactly zero for sufficiently large M but do converge to zero rapidly. Thus we see that with respect to the coarse-graining maps of [21] the correction terms of type $1_{\mathcal{H}} - P_M, 1_L - p_M$ of the discrete Virasoro algebra converge to zero in the weak operator topology of $\mathcal{H}$ and that in particular the central term of the Virasoro algebra is correctly reproduced.

VIII. CONCLUSION AND OUTLOOK

In the present work we have investigated the question whether Hamiltonian renormalization in the sense of [4,6,7,11], while derived in the context of ordinary Hamiltonian systems, can be "abused" to study also generally covariant Hamiltonian systems with an infinite number of Hamiltonian constraints rather than a single Hamiltonian. We have chosen parametrized field theory on the $1 + 1$ cylinder to test related questions where the exact quantum theory is known.

We have explicitly demonstrated that indeed the general framework of [11] can be applied, although the system does not exhibit a common vacuum vector Ω for all constraint operators due to the central term in the Virasoro algebra. The renormalization flow indeed finds the correct fixed point theory. This enabled us to study the constraint algebra at finite resolution. That finite resolution algebra *generically does not close* (including the central term). However, it does not close for a simple mathematical reason: The constraints at finite resolution are forced to map states in the Hilbert space of given finite resolution to themselves. However, to achieve closure, matrix elements with states at higher resolution are needed. These are restored as we increase the resolution and explains why the failure of closure is parametrized by the projection $1_{\mathcal{H}} - P_M$ where P_M projects on the given finite resolution subspace. In that sense the failure to close *does not represent an anomaly but just a finite size artifact*. In QFTs that are not exactly solvable one can distinguish between true anomalies and these artifacts by studying whether their size decreases as we increase the resolution.

In addition, we could address the question if and in what sense smearing functions of constraint operators can or should also be discretized when probing them at finite resolution. Namely, while it is not necessary or even natural to do so, one can use the coarse-graining map that was employed for reasons of renormalization also for those smearing functions. This leads to an additional finite size artifact in the finite resolution constraint algebra parametrized by $1_L - p_M$ where now p_M projects on smearing functions (rather than Hilbert space states) of finite resolution. This is because the commutator of constraints is smeared by a bilinear expression in two smearing functions and typically derivatives thereof of finite order. Those aggregates generically leave the subspace $p_M L$. However, again these corrections converge to zero as we increase the resolution for coarse-graining maps with sufficient smoothness.

In the convergence proofs that we supplied it was important that the functions that define the coarse-graining maps of the renormalization flow display sufficient smoothness as otherwise the estimates that were needed do not hold: the Fourier transform of a merely piecewise smooth function is not of rapid decrease and displays the Gibbs phenomenon at the discontinuities [32]; i.e., the partial Fourier transform of the function at finite resolution has points within the resolution size away from the discontinuity that differ from the function by a size *independent of* the resolution.

We will use the lessons learned for more complicated and physically more interesting constrained QFT such as PFT in higher dimensions and the $U(1)^3$ model for quantum gravity [33] which present the next logical step in the degree of complexity as in these models the constraint algebra (hypersurface deformation algebra) no longer closes with structure constants but only structure functions.

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