Stress-energy tensor correlations across regular black holes horizons

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Hawking radiation can be regarded as a spontaneous and continuous creation of virtual particle-antiparticle pairs outside the event horizon of a black hole where strong tidal forces prevent the annihilation; the particle escapes to infinity contributing to the Hawking flux, while its corresponding antiparticle partner enters the event horizon and ultimately reaches the singularity. The aim of this paper is to investigate the energy density correlations between the Hawking particles and their partners across the event horizon of two models of nonsingular black holes by calculating the two-point correlation function of the density operator of a massless scalar field. This analysis is motivated by the fact that in acoustic black holes particle-partner correlations are signaled by the presence of a peak in the equal-time density-density correlator. Performing the calculation in a Schwarzschild black hole it was shown in Balbinot and Fabbri, [Quantum correlations across the horizon in acoustic and gravitational black holes, Phys. Rev. D **105**, 045010 (2022).] that the peak does not appear, mainly because of the singularity. It is then interesting to consider what happens when the singularity is not present. In the Hayward and Simpson-Visser nonsingular black holes we show that the density-density correlator remains finite when the partner particle approaches the hypersurface that replaces the singularity, opening the possibility that partner-particle correlations can propagate towards other regions of spacetime instead of being lost in a singularity.

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I. INTRODUCTION

One of the most important results of modern theoretical physics is the fact that black holes are not completely black, as predicted by general relativity, but they emit thermal radiation. This process was demonstrated by Hawking [1] in the context of quantum field theory in curved spacetime and it is caused by the dynamical gravitational field that acts during the collapse of a massive astrophysical body leading to the formation of a black hole. Even though the region inside the event horizon is causally disconnected from the rest of the spacetime, analog models of gravity have shown that the equal-time density-density correlator can be used to study the Hawking process when one point is taken outside the event horizon and the other inside. These models are based on the pioneering work of Unruh [2], who established that there exists an analogous phenomenon to Hawking radiation in condensed matter systems, where sound waves play the role of light and sound horizons the role of event horizons. This opened the possibility to study the Hawking radiation process in completely different physical systems. As an example, the behavior of a massless scalar field on a curved background is explained by the same equation that characterizes the propagation of sound waves in Eulerian fluids, which are described in terms of an acoustic metric that is a function of the background flow. In particular, the curvature of the acoustic geometry is induced by the inhomogeneities in the fluid flow, while flat Minkowski spacetime is recovered in the case of a homogeneous system. It is also possible to simulate a black hole using what is called an acoustic black hole, which is obtained whenever a subsonic flow is turned supersonic; sound waves in the supersonic region are dragged away by the flow and cannot propagate back toward the acoustic horizon separating the supersonic and subsonic regions [3]. However, differently from the case of astrophysical black holes, both the external and internal regions are accessible to experiments. In the case of acoustic black holes constructed from Bose-Einstein condensates, it has been predicted that the correlations between the Hawking particles and their

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partners will form a stationary peak in the equal-time density-density correlator, which appears at late times after the formation of the sonic horizon, with one point taken inside it and the other outside [3,4]. This striking feature has indeed been experimentally observed [5,6] and it is the most stringent evidence of Hawking-like (phonons in this case) radiation in an analog black hole. Inspired by these results, R. Balbinot and A. Fabbri [7] calculated the equal-time density-density correlator of a massless scalar field on a Schwarzschild black hole background, finding a result which is in disagreement with the acoustic black hole case; the expected peak signaling the particle-partner correlations does not appear. As we will review in detail below, the reason lies in the fact that when a Hawking particle emerges out of vacuum fluctuations in a region outside the event horizon, called quantum atmosphere, the corresponding partner has already entered the singularity. Hence the peak does not have sufficient time to form. On the opposite, in the acoustic case, the singularity does not exist and one can tune the experiment setup in such a way that the peak can always form.

The aim of this paper is to study the correlations in regular spacetimes by calculating the density-density correlator in the Hayward [8] and Simpson-Visser [9] nonsingular black holes. This choice is motivated by the fact that the latter represents a minimal modification to the Schwarzschild black hole that makes it regular, while the former possesses two horizons, of which the inner one is a Cauchy horizon, and therefore it has a causal structure that is similar to the usual charged and rotating solutions of general relativity. The result is that unlike in the Schwarzschild case, where the equal-time density-density correlator vanishes when the partner particle enters the singularity, in the regular black holes analyzed here the correlator remains finite when the partner particle approaches the hypersurfaces that replace the singularity.

The paper is organized as follows. In Sec. II we briefly recall the fundamental concepts of Hawking radiation. In Sec. III we review the results concerning a Schwarzschild black hole. In Sec. IV we generalize the procedure to discuss possible correlations in the Hayward and Simpson-Visser nonsingular black holes. Section V is devoted to the conclusions. All the mathematical details can be found in Appendixes A–E.

II. HAWKING RADIATION

We now briefly recall some fundamental aspects of Hawking radiation that will be useful in the rest of the paper. Let us consider a massless scalar field propagating in a black hole spacetime, which possesses asymptotically stationary regions in the past ("in"), corresponding to past null infinity, and in the future ("out"), given by future null infinity. Stationarity implies the existence of a timelike Killing vector field with respect to which one can uniquely specify positive frequency mode solutions to the field equation and the corresponding vacuum states, defined as the absence of particles according to all inertial observers in the asymptotic region of interest.[1] Let $|\mathrm{in}\rangle$ be the vacuum state in the "in" region and $|\mathrm{out}\rangle$ the one in the "out" region. One usually works in the Heisenberg picture, so that by assuming that the quantum state of the field in the "in" region is $|\mathrm{in}\rangle$, it will remain in that state during its subsequent evolution. However, as it was first shown by Hawking [1], inertial observers in the "out" region will detect a thermal distribution of particles at the temperature $T_H = \frac{k}{2\pi}$, where k is the surface gravity of the event horizon. One then concludes that particles have been created by the external time-dependent gravitational field acting between the two asymptotic stationary regions. In fact, by considering the field propagation in a Schwarzschild background, it is possible to write the $|\mathrm{in}\rangle$ vacuum state in the "out" region formally as [10]

$$|\mathrm{in}\rangle \propto e^{-\sum_{\omega}\frac{\pi\omega}{k}a_{\omega}^{bh\dagger}a_{\omega}^{out\dagger}}|\mathrm{out}\rangle, \tag{2.1}$$

where $a_{\omega}^{out\dagger}$ and $a_{\omega}^{bh\dagger}$ are the creation operators for respectively the outgoing modes reaching future null infinity and the trapped modes entering the horizon. The mathematical expressions of the states on which these operators act depend on the choice of the orthonormal set of modes that are exact solutions of the Klein-Gordon equation governing the excitation (usually a massless scalar field) [10,11]. In this formalism, the $|\mathrm{in}\rangle$ vacuum state represents a flux of entangled particles, one escaping to infinity and the other crossing the black hole horizon. This allows an intuitive picture of the Hawking process; the presence of a trapped region acts as an energy reservoir for the continuous and spontaneous creation of particle-antiparticle pairs outside of the event horizon, where strong tidal forces prevent their mutual annihilation. The particle, having positive Killing energy, escapes to infinity contributing to the Hawking flux, while its corresponding antiparticle partner enters the event horizon and ultimately reaches the singularity, depleting the trapped region due to its negative Killing energy.

III. QUANTUM CORRELATIONS IN THE SCHWARZSCHILD BLACK HOLE

In this section we review the calculation of the energy density correlations between Hawking quanta across the event horizon of a Schwarzschild black hole [7].

[1] Note that, in principle, it is not possible to define a complete set of positive frequency mode solutions at future null infinity ($\mathcal{I}^+$) since the latter is not a proper Cauchy surface. To form a complete Cauchy surface it is necessary to consider the union of $\mathcal{I}^+$ with the event horizon so that among the outgoing modes we must distinguish between the ones that are able to reach $\mathcal{I}^+$ and the ones that are trapped inside the horizon. However, an explicit expression of the latter, which would be difficult to obtain because there is no natural choice of time at the horizon, is not needed to evaluate the particle production at $\mathcal{I}^+$ [10].

A. Modeling gravitational collapse

Hawking radiation is produced by the time-dependent gravitational field during the collapse that leads to the formation of a black hole. In the spirit of the "no hair" theorem [12], the final result should be insensitive to the details of the collapse and thus one can work with the simplest solution to Einstein's equation describing the formation of a black hole through gravitational collapse, namely the Vaidya metric. This is obtained by expanding the mass parameter in the Schwarzschild metric from a constant to a function of the ingoing Eddington-Finkelstein coordinate v:

$$ds^2 = -\left(1 - \frac{2M(v)}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2. \quad (3.1)$$

Since the Ricci tensor has only one nonvanishing component given by

$$R_{vv} = \frac{2}{r^2}\frac{dM(v)}{dv}. \quad (3.2)$$

and the Ricci scalar is zero, the only nonvanishing component of the stress-energy tensor is

$$T_{vv} = \frac{L(v)}{4\pi r^2}, \quad (3.3)$$

where $L(v) = \frac{dM(v)}{dv}$. The physical interpretation is that the Vaidya solution describes a purely ingoing flux of massless radiation characterized by the function $L(v)$. If such influx is turned on at some advanced time v_i and turned off at v_f, then the spacetime geometry can be divided into three regions:

(i) A Minkowski vacuum region $v < v_i$;
(ii) An intermediate collapse region $v_i < v < v_f$;
(iii) The final Schwarzschild configuration $v > v_f$.

To discuss the Hawking radiation one cares about the "in" and "out" stationary regions, so we can ideally narrow the collapse region down to a single null surface. Therefore, we consider an ingoing shock wave located at some $v = v_0$ of the form $L(v) = M\delta(v - v_0)$, that is $M(v) = M\theta(v - v_0)$. The resulting spacetime is then obtained by patching portions of Minkowski and Schwarzschild spacetimes along $v = v_0$. For $v < v_0$ the metric is Minkowskian and can be written in double null form:

$$ds^2 = -du_{\rm in}dv + r^2(u_{\rm in}, v)d\Omega^2, \quad (3.4)$$

where $u_{\rm in} = t - r = v - 2r$. For $v > v_0$ the metric is the Schwarzschild one describing a black hole of mass M and horizon located at $r = 2M$. In double null form it reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dudv + r^2(u, v)d\Omega^2, \quad (3.5)$$

where

$$u = t - r^* = v - 2r^*, \quad (3.6)$$

and $r^* = \int (1 - \frac{2M}{r})^{-1} dr$ is the tortoise coordinate.

To guarantee continuity of the global metric at $v = v_0$, one needs to impose the condition

$$r(u_{\rm in}, v_0) = r(u, v_0), \quad (3.7)$$

which leads to the following relation between the retarded null coordinates inside and outside v_0:

$$u = u_{\rm in} - 4M \ln\left|\frac{u_{\rm in}}{4M}\right|. \quad (3.8)$$

Exploiting the arbitrariness of v_0, we have set $v_0 = 4M$ to simplify calculations. Inverting (3.8) one can formally extend the coordinate $u_{\rm in}$ in the exterior region in terms of the Lambert function [13]

$$u_{\rm in} = -4MW(\pm e^{-\frac{u}{4M}}). \quad (3.9)$$

The positive sign holds in the exterior region and the minus sign in the interior one.

B. Density-density correlator

Let us now consider a massless scalar field propagating in the Vaidya spacetime. We assume the quantum state of the field to be the Minkowski vacuum $|{\rm in}\rangle$ at past null infinity. Then, we neglect the backscattering of the modes induced by the curvature of spacetime, impose reflecting boundary conditions at the origin $r = 0$ in the Minkowski region, and require regularity of the modes there. This corresponds to work on the effective $(1 + 1)$ metric [10,14,15]

$$ds^2_{(2)} = -\left(1 - \frac{2M(v)}{r}\right)dv^2 + 2dvdr. \quad (3.10)$$

As shown in Appendix B, the density-density correlator of the scalar field is given by the action of a differential operator on the Wightman function, which is defined as

$$G^+(x, x') = \langle {\rm in}|\phi(x)\phi(x')|{\rm in}\rangle. \quad (3.11)$$

$G^+(x, x')$ can be easily computed by expanding the field on the complete set of positive frequency mode solutions to the field equation at past null infinity [14], namely

$$G^+(x, x') = -\frac{1}{4\pi}\ln\frac{(u_{\rm in} - u'_{\rm in})(v - v')}{(u_{\rm in} - v')(v - u'_{\rm in})}. \quad (3.12)$$

As stressed in [7], to fully characterize the correlation functions, one needs to specify the observer's state and we choose the free-falling one on a radial path with velocity vector u^α. Due to the conformal flatness of the metric (3.10), the energy density of the field ϕ is equivalent to its pressure density measured by this class of observers. Thus (for more details see Appendix A)

$$\rho = T_{\alpha\beta} u^\alpha u^\beta, \tag{3.13}$$

where $T_{\alpha\beta}$ is the stress-energy tensor of a massless scalar field,

$$T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \frac{g_{\alpha\beta}}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{3.14}$$

To obtain a simple expression for the energy density it is convenient to map the metric (3.10) into the "out" region of the Painlevé-Gullstrand coordinates [16]

$$ds^2_{(2)} = -f dT^2 - 2V dT dr + dr^2, \tag{3.15}$$

where $f = 1 - \frac{2M}{r}$, $V = -\sqrt{\frac{2M}{r}}$ and T is the Painlevé time given by

$$T = v + \int \left(\frac{\sqrt{1-f} - 1}{f} \right) dr. \tag{3.16}$$

As also shown in Appendix A, the energy density measured by radial free-falling observers in Painlevé-Gullstrand coordinates is

$$\rho = T_{rr}. \tag{3.17}$$

Writing this expression in double null coordinates allows us to obtain the following result for the density-density correlator (see Appendix B for details):

$$\begin{aligned} G(x, x') &= \langle \mathrm{in}|\rho(x)\rho(x')|\mathrm{in}\rangle \\ &= \langle \mathrm{in}| \frac{T_{uu}(x)T_{u'u'}(x')}{(1+V(x))^2(1+V(x'))^2} + \frac{T_{uu}(x)T_{v'v'}(x')}{(1+V(x))^2(1-V(x'))^2} \\ &\quad + \frac{T_{vv}(x)T_{u'u'}(x')}{(1-V(x))^2(1+V(x'))^2} + \frac{T_{vv}(x)T_{v'v'}(x')}{(1-V(x))^2(1-V(x'))^2} |\mathrm{in}\rangle. \end{aligned} \tag{3.18}$$

As pointed out in [10], (see also [17–19]), the expectation value of a physical observable in the $|\mathrm{in}\rangle$ vacuum state which describes gravitational collapse (and which is reproduced at late time by the Unruh state in an eternal Schwarzschild black hole) can be written as the sum of two contributions, one describing the particle creation induced by the formation of a black hole and the other being the vacuum polarization that is constant in time and does not contribute to the Hawking radiation. The vacuum polarization terms come in fact from the expectation value of the stress-energy tensor computed in the Boulware state, which is chosen to correspond to the usual Minkowski vacuum at future null infinity. However, this vacuum state is pathological at the horizon, in the sense that the expectation values of physically relevant quantities diverge as the horizon is approached. Being empty at infinity, the Boulware state corresponds to the absence from the vacuum of blackbody radiation at the black hole temperature and so its physical realization would be the state describing vacuum polarization outside a massive spherical body of radius slightly larger than its Schwarzschild radius. Since we are interested in studying energy-density correlations between Hawking quanta we therefore have to neglect vacuum polarization effects that do not describe black holes evaporation. As shown in [7,20], this accounts for retaining only the term coming from the u sector of the correlator, since in a two-dimensional spacetime all the other terms receive contributions only from the vacuum polarization. Therefore, the key ingredient for the description of density correlations is the T_{uu} part, which, as reported in Appendix B, is obtained from

$$\begin{aligned} &\frac{1}{(1+V(r))}\frac{1}{(1+V(r'))}\partial_u\partial_{u'}\langle \mathrm{in}|\phi(x)\phi(x')|\mathrm{in}\rangle_{|T=T'} \\ &= \frac{1}{16\pi}\frac{1}{1-\sqrt{\frac{2M}{r}}}\frac{1}{1-\sqrt{\frac{2M}{r'}}}\frac{1}{(u_{\mathrm{in}}-4M)(u'_{\mathrm{in}}-4M)}\frac{1}{\cosh^2 \ln\sqrt{-\frac{u_{\mathrm{in}}}{u'_{\mathrm{in}}}}}\Bigg|_{T=T'}. \end{aligned} \tag{3.19}$$

At late retarded time ($u \to +\infty$, $u_{\rm in} \to 0$) the relation between the retarded null coordinates inside and outside the horizon reported in Eq. (3.9) can be approximated as

$$u_{\rm in} \simeq \pm 4Me^{-\frac{u}{4M}}, \qquad (3.20)$$

where $u_{\rm in} < 0$ for $r > 2M$ and $u_{\rm in} > 0$ for $r < 2M$. In this limit one obtains

$$\frac{1}{(1+V(r))}\frac{1}{(1+V(r'))}\partial_u\partial_{u'}\langle {\rm in}|\phi(x)\phi(x')|{\rm in}\rangle_{|T=T'} \simeq \frac{1}{16\pi}\frac{1}{1-\sqrt{\frac{2M}{r}}}\frac{1}{1-\sqrt{\frac{2M}{r'}}}\frac{1}{16M^2}\frac{1}{\cosh^2(\frac{u-u'}{8M})}\Bigg|_{T=T'}. \qquad (3.21)$$

This function has a maximum for $u = u'$, i.e. along the trajectory of the Hawking particle and its partner, as in the case of analog black holes. However, one has still to verify whether this maximum condition is fulfilled or not. Considering the relation between Eddington-Finkelstein coordinates and Painlevé time (3.16), the condition $u = u'$ at equal times can be written as

$$r + 2\sqrt{2Mr} + 4M\ln\left(\sqrt{\frac{r}{2M}} - 1\right) = r' + 2\sqrt{2Mr'} + 4M\ln\left(1 - \sqrt{\frac{r'}{2M}}\right). \qquad (3.22)$$

In analogy to what happens for acoustic black holes, one would expect that at late times and for r sufficiently far away from the horizon, the equal-time density-density correlator should show a peak along (3.22), see Refs. [3,4,7].

The key ingredient to investigate this possibility is the location of the region of spacetime where Hawking quanta emerge out of vacuum fluctuations.

Calculating the radius of a radiating body using the Stephan-Boltzmann law, Giddings [21] was able to show that the Hawking particles originate from a region outside the event horizon of the black hole, called "quantum atmosphere", located at a distance $o(1/k)$, k being the horizon surface gravity. This result was later corroborated by the work of other authors [22,23], using a detailed analysis of the renormalized vacuum expectation value of the stress-energy tensor of a massless scalar field in the Schwarzschild spacetime. The same conclusion was obtained also for acoustic black holes [4,7].

If we now plot the two functions entering the left- and right-hand sides of Eq. (3.22) (Fig. 1) we see that the right-hand side is always smaller or equal than zero since $0 < r' < 2M$. As a consequence, to find a solution, also the left-hand side has to be smaller or equal than zero and this happens for $r \lesssim 2.6M$. Therefore, Eq. (3.22) has real solutions only if $r \lesssim 2.6M$, for which the corresponding r' is located between zero and $2M$. We then conclude that when the Hawking particle emerges from the quantum atmosphere out of vacuum fluctuations at a distance of the order of $4M$ from the horizon, the corresponding partner has already been swallowed by the singularity and the correlations are lost. On the other hand, for solutions with a nonvanishing $r' < 2M$, the correlator $G(x, x')$ is dominated by the coincidence limit and the peak, signaling the particle-partner correlations, does not appear.

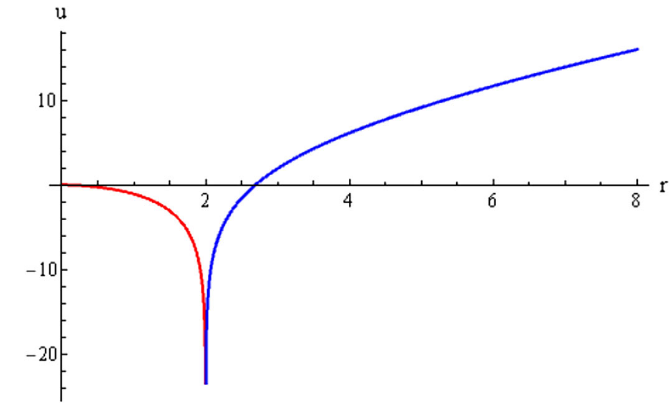

FIG. 1. Plot of the left-hand side of Eq. (3.22) (u') as a function of $r' < 2M$ in red and of the right-hand side (u) for $r > 2M$ in blue.

This result is in striking disagreement with what happens in acoustic black holes and it is due to the presence of the quantum atmosphere and of the central singularity.

For later comparison with other black hole type curved spacetime backgrounds, as well as to better understand the behavior of the correlator in regions of physical interest, it is useful to study the correlator in the limits $r' \to 0$, $r \to 2M$ and $r \to \infty$. Using the results of Appendix B, the T_{uu} correlator can be written as

$$G(x, x') = \frac{1}{16\pi^2}\frac{1}{(1+V(x))^2(1+V(x'))^2} \times \left(\frac{du_{\rm in}}{du}\right)^2\left(\frac{du'_{\rm in}}{du'}\right)^2\frac{1}{(u_{\rm in} - u'_{\rm in})^4}. \qquad (3.23)$$

We note that fixing the time variables, (3.23) takes the form of the product of a function of r, a function of r' and another of both r and r'. However, the latter affects the form of the correlator only in the coincidence limit or when the denominator diverges, which occurs when one of the two particles is at infinity. All the mathematical details regarding the calculation of the limits of the correlator are described in Appendix C. When $r' \to 0$ we can focus on the parts that depend only on r',

$$\lim_{r'\to 0}\frac{1}{(1 \pm V(r'))^2} = 0, \qquad (3.24)$$

and

$$\lim_{r'\to 0} \frac{du'_{\rm in}}{du'} = \infty. \tag{3.25}$$

Thus, one needs to consider

$$\lim_{r'\to 0} \frac{1}{(1+V(r'))^2} \left(\frac{du'_{\rm in}}{du'}\right)^2 = 0. \tag{3.26}$$

Therefore, when the partner particle approaches the singularity, the density-density correlator vanishes.

When $r \to 2M$ one has

$$\lim_{r\to 2M} \frac{1}{(1+V(r))^2} = \infty, \tag{3.27}$$

$$\lim_{r\to 2M} \frac{du_{\rm in}}{du} = 0. \tag{3.28}$$

In this case we have to study the following limit:

$$\lim_{r\to 2M} \frac{1}{(1+V(r))^2} \left(\frac{du_{\rm in}}{du}\right)^2 = 4e^{-\frac{T}{2M}}. \tag{3.29}$$

In Eq. (3.29), T is the Painlevé time it takes to go from the quantum atmosphere ($r_{qa} \sim \frac{1}{k} = 4M$) to the horizon ($r = 2M$):

$$T = \int_{4M}^{2M} \frac{1}{V} dr = \frac{4}{3}(2\sqrt{2}-1)M. \tag{3.30}$$

The same result is obtained for $r' \to 2M$. Therefore, the correlator remains finite when one of the Hawking quanta approaches the horizon.

When $r \to \infty$, the part of the correlator that depends on both r and r' dominates because $u_{\rm in} \to u \to \infty$. This implies that the correlator goes to zero when the Hawking particle reaches spatial infinity.

IV. QUANTUM CORRELATIONS IN NONSINGULAR BLACK HOLES

As discussed in the previous section, energy density correlations between Hawking particles and their partners across the event horizon of a Schwarzschild black hole are lost because of the presence of the quantum atmosphere and of the singularity. It would then be interesting to study what happens if the singularity is not present.

A. Nonsingular black holes

One of the fundamental results of classical general relativity is the Penrose singularity theorem [24], which proves that, under general assumptions, the gravitational collapse of a sufficiently massive astrophysical body will lead to the formation of trapped surfaces and thus of a singularity. However, the appearance of singularities is usually accompanied by an unlimited increase of spacetime curvature, and in these conditions, Einstein's equation is not expected to work anymore because the quantum corrections become of the same order of the main classical terms. Therefore, singularities in general relativity are usually regarded as a problem of this classical theory, which may be solved by its quantization. In the absence of a completely satisfactory theory of quantum gravity, one can still provide qualitative arguments for the existence of nonsingular black holes, violating at least one of the assumptions of the Penrose theorem, typically the strong energy condition [25,26]. One possibility is to study quantum effects of gravitation with the semiclassical approach in which matter fields are quantized in the usual way, while the spacetime geometry is treated classically. However, in the case of black hole spacetimes, close to the singularity the curvature reaches order unity in Planck units and so quantum fields begin to dominate on the geometry. Whether gravitation can still be treated classically at this level is far from being certain, but one can wonder if quantum polarization effects can provide a mechanism to slow down the infinite rise of curvature and to maintain it bounded to Planckian magnitude [27]. It is possible to outline in a qualitative way the behavior of the corrected curvature when quantum effects are taken into account assuming that the $\langle -T^t_t \rangle$ component of the stress-energy tensor is proportional to the curvature squared[2] with a coefficient (a^2), that depends on the number and on the nature of the quantum fields [28]. One then arrives at the following expression:

$$\frac{M(r)}{r^3} = \frac{1}{a^2 + \left(\frac{r}{r_Q}\right)}, \tag{4.1}$$

where $M(r)$ is the mass function and $r_Q = M^{1/3}$ is the radius at which quantum effects become important. For $r \gg r_Q$ one recovers the curvature of the Schwarzschild black hole, while for smaller radii it depends on the sign of a^2. If $a^2 < 0$ the curvature diverges at $r = |a^{2/3}| r_Q$, while if $a^2 > 0$ the curvature remains bounded and constant. Therefore, a spherically symmetric, uncharged black hole should be described by the Schwarzschild solution down to a critical radius r_Q where quantum effects produce a smooth transition towards a constant curvature, de Sitter core [29]. In this region, the strong energy condition does not hold and thus the singularity can be avoided.

It is interesting to notice that the same conclusion was obtained also in [30], where it was supposed that, because of quantum effects, the spacetime curvature should always be subject to an upper bound of Planckian magnitude. At the limiting curvature, the density is so high that all the particles lose their identity and matter undergoes a

[2]Here by curvature squared we intend the Kretschmann scalar, which for the Schwarzschild black hole is $K \sim \frac{M^2}{r^6}$.

transition into a vacuumlike state described by the stress-energy tensor $T_{\mu\nu} = -\rho g_{\mu\nu}$, where ρ is a positive, constant energy density. In this way, the strong energy condition is violated and gravity acts in such a way that the trajectories of freely falling test particles moving along causal geodesics behave as if they were repulsed from the center. In the following, we consider a quantized scalar field that propagates on the nonsingular geometries of the Simpson-Visser and Hayward black holes. As a first approximation, we neglect the backreaction of the field on the background geometry. As a consequence, the stress-energy tensor associated to the scalar field cannot contribute to energy condition violations.

1. Hayward nonsingular black hole

We now present an explicit example of a nonsingular black hole obtained by applying the argument presented above. As previously discussed, assuming that the $\langle -T^t_{\ t}\rangle$ component of the stress-energy tensor is proportional to the curvature squared because of quantum polarization effects, the mass function can be written as [see Eq. (4.1)]

$$M(r) = \frac{Mr^3}{r^3 + 2ML^2}, \tag{4.2}$$

where $2L^2 = a^2 > 0$. It is then possible to consider the following spherically symmetric metric:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{4.3}$$

where

$$f(r) = 1 - \frac{2M(r)}{r} = 1 - \frac{2Mr^2}{r^3 + 2L^2M}. \tag{4.4}$$

This is called the Hayward metric [8].

Asymptotically, for $r \gg M$, $f(r)$ behaves as

$$f(r) \sim 1 - \frac{2M}{r}, \tag{4.5}$$

reproducing a Schwarzschild spacetime with mass M.

For $r \to 0$, we have

$$f(r) \sim 1 - \frac{r^2}{L^2}, \tag{4.6}$$

which gives a de Sitter spacetime with cosmological constant $\Lambda = \frac{3}{L^2}$. Hayward's model is the most popular of a class of black holes that have a de Sitter structure near the central singularity. In the literature, these are often called nonsingular black holes with a de Sitter core (for a recent review see Ref. [31]).

It is possible to check that the Hayward metric is not singular by computing the curvature invariants, which are regular everywhere, and also that the stress-energy tensor that generates such a spacetime violates the strong energy condition.

The Hayward spacetime contains a trapped region, the boundary of which is given by the solutions of the equation $f(r) = 0$. In particular, it is possible to define a critical mass $M^* = \frac{3\sqrt{3}L}{4}$ such that, if $M > M^*$, $f(r)$ has three real roots, if $M = M^*$, $f(r)$ has one multiple root, if $M < M^*$, $f(r)$ has one real root and a complex pair. However, in the $M > M^*$ case one of the three roots is negative and thus it cannot be accepted because r should be greater or equal then zero (and real). The same happens for the real root in the $M < M^*$ case. Since L gives the approximate length scale below which quantum effects of gravity become dominant, one might expect L to be of the order of the Planck length. Therefore, for black holes formed by gravitational collapse, M is much greater than the critical mass M^*,[3] and the roots of $f(r)$ can be approximated as [8]

$$r_1 \simeq 2M - \frac{L^2}{2M}, \tag{4.7}$$

$$r_2 \simeq L + \frac{L^2}{4M}. \tag{4.8}$$

It is important to note that the Hayward spacetime is not globally hyperbolic because the inner horizon at $r = r_2$ is a Cauchy horizon. This is a common characteristic of black holes with a de Sitter core.

The Hayward nonsingular black hole can be obtained from gravitational collapse generalizing the Vaidya metric used in the previous section to model the formation of a Schwarzschild black hole. Also in this case, it is enough to promote the mass parameter M in the metric (4.3) from a constant to a function of the ingoing null coordinate v [8].

2. Simpson-Visser nonsingular black hole

Another popular model of regular spacetime is the one proposed by Simpson and Visser [9]. In contrast with the case studied above, this model does not have a de Sitter core [31]. This is an example of "black bounce" spacetime [32], which is a regular spacetime, where the area radius always remains nonzero, thereby leading to a "throat". In the case that we are going to analyze, the idea is to introduce a minimal modification to the Schwarzschild metric in order to make it regular. The line element can be written as

[3] This is the case for the calculation of the density-density correlator, that is when the geometry has already settled down after the collapse and the backreaction of Hawking evaporation on the metric can still be neglected.

$$ds^2 = -\left(1 - \frac{2M}{\sqrt{r^2+a^2}}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{\sqrt{r^2+a^2}}\right)} + (r^2+a^2)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{4.9}$$

Depending on the value of the parameter a, this metric represents either:

(i) The ordinary Schwarzschild spacetime ($a = 0$);
(ii) A traversable wormhole geometry (in the Morris-Thorne sense [33]) with a two-way timelike throat ($a > 2M$);
(iii) A one-way wormhole geometry with an extremal null throat ($a = 2M$);
(iv) A nonsingular black hole ($a < 2M$).

In particular, we will be interested in the latter case.

The horizons locations are obtained by solving the equation $f(r) = 0$, with $f(r) = 1 - \frac{2M}{\sqrt{r^2+a^2}}$. One has $r_\pm = \pm\sqrt{4M^2 - a^2}$, which are real, nondegenerate, solutions if and only if $a < 2M$, corresponding to the case under study. Therefore, when $a < 2M$ there will be symmetrically placed r coordinate values, $r_+ > 0$ and $r_- < 0$, which correspond to a pair of horizons.

The hypersurface $r = 0$ is a spacelike, spherical hypersurface which marks the boundary between our Universe and a bounce into a separate copy of it. This implies that for negative values of the r coordinate we have "bounced" into another universe. It is possible to compute the curvature invariants to directly check the regularity of the spacetime. Then, studying the stress-energy tensor that gives rise to such a spacetime, one can show that the null energy condition is violated, and so are all the energy conditions.

A peculiar feature of the Simpson-Visser nonsingular black hole is the fact that it is not possible to describe its formation from gravitational collapse generalizing the Vaidya metric if one keeps the parameter a constant and nonvanishing because it implies the existence of a throat. In fact, it can be easily seen from the metric (4.9) that when $M = 0$ one has a wormhole instead of flat space. Therefore, starting with zero mass and then considering an increasing mass function one has a transition from a wormhole to a nonsingular black hole [34]. This implies that a more complete model of gravitational collapse is needed to describe the onset of a black bounce. However, this is not a problem for the calculation of the density-density correlator, which is the focus of this work.

B. Density-density correlator

To study the density-density correlator in nonsingular black hole spacetimes we follow the procedure adopted in Sec. III, where we performed the same calculation on a Schwarzschild background spacetime. In particular, we neglect backscattering of the modes induced by the curvature of spacetime and consider a $(1+1)$-dimensional theory describing the propagation of a massless scalar field in the $(1+1)$-dimensional section of the Vaidya-like spacetime described by the line element

$$ds^2_{(2)} = -f(r,v)dv^2 + 2dvdr. \tag{4.10}$$

In this way, the Wightman function is the same as (3.12) because any two-dimensional metric describing the "in" region is conformal to the Minkowski one, and therefore the field equation has the same mode solutions at past null infinity. Then, we consider the quantum state of the system to be the $|\mathrm{in}\rangle$ vacuum of the "in" region. Therefore, the correlator that we want to study is

$$G(x,x') = \langle \mathrm{in}|\rho(x)\rho(x')|\mathrm{in}\rangle, \tag{4.11}$$

evaluated in the black hole region, where one point (x) is taken outside the (outer) horizon and the other (x') inside. It is also assumed that the spacetime has already settled down after the collapse (or the wormhole to black hole transition in the Simpson-Visser case) and that the backreaction of Hawking evaporation on the metric can be neglected. In this way, one has to deal with static, spherically symmetric metrics for which it is possible to introduce Painlevé coordinates, describing free-falling observers, in the same way as in Schwarzschild, and write the metric in the form (3.15). Therefore, the calculation of the correlator proceeds exactly as in the Schwarzschild case and the fundamental element to study the presence of correlations between the Hawking quanta and their corresponding partners across the horizon, namely the T_{uu} correlator, can be written in the form (3.23). The differences with the Schwarzschild spacetime are in the expressions of the null coordinates in the "in" and "out" regions (u_{in} and u, respectively) and the relations between them. Therefore, the explicit expression of the correlator will be different for the Hayward, Simpson-Visser and Schwarzschild black holes. Nevertheless, it is possible to generalize the procedure that was used for the Schwarzschild black hole in the following way:

(i) Compute the tortoise coordinate;
(ii) Impose the continuity of the metric (4.10) describing the transition from the "in" region to the "out" one on the transition null shell at $v = v_0$;
(iii) Obtain the relation between the retarded null coordinates outside and inside v_0, check that the function $u(u_{\mathrm{in}})$ is invertible and compute $\frac{du}{du_{\mathrm{in}}}$. Then, $\frac{du_{\mathrm{in}}}{du} = (\frac{du}{du_{\mathrm{in}}})^{-1}$;
(iv) Invert the function $u(u_{\mathrm{in}})$; Unfortunately, for the Hayward and Simpson-Visser black holes this can be done only in particular cases, when the relation between u and u_{in} can be approximated with a simpler expression. In particular, for the Hayward black hole we will consider three limits: $r' \to r_2$, $r \to r_1$ and $r \to \infty$. For the Simpson-Visser black

hole the interesting regions are $r' \to 0$, $r \to r_+$ and $r \to \infty$.

(v) Write u in Painlevé coordinates;

(vi) Study the behavior of the density correlator in the limits mentioned above.

1. Density correlations in the Hayward spacetime

In the Hayward case, the calculation exploits the fact that the metric reduces to the Schwarzschild one asymptotically, while it reproduces the de Sitter metric close to the origin and to the Cauchy horizon in the $M \gg L$ limit that is considered. When the inner point approaches the Cauchy horizon we have (see Appendix D for details)

$$\lim_{r' \to r_2} \frac{1}{(1+V(r'))^2} = \infty. \tag{4.12}$$

$$\lim_{r' \to r_2} \frac{du'_{\rm in}}{du'} = 0. \tag{4.13}$$

Therefore, one has to consider

$$\lim_{r' \to r_2} \frac{1}{(1+V(r'))^2} \left(\frac{du'_{\rm in}}{du'}\right)^2 = 4r_2^2 e^{-\frac{4r_1}{r_2}+\frac{2T'}{r_2}}. \tag{4.14}$$

The fact that this expression is not vanishing on the inner horizon implies that the density correlator remains finite when one of the two points approaches this hypersurface.

It is then important to verify that when $L \to 0$, which implies $r_2 \to 0$, so that there is no Cauchy horizon but a singularity at $r = 0$, the density correlator vanishes according to (3.26).

Looking at Eq. (4.14) there are two possibilities:

(i) If $-4r_1 + 2T' > 0$, the correlator diverges when $L \to 0$;

(ii) If $-4r_1 + 2T' < 0$, the correlator goes to zero when $L \to 0$.

T' is the Painlevé time it takes for the partner particle to travel the distance between the quantum atmosphere and the Cauchy horizon:

$$T' = \int_{r_{QA}}^{r_2} \frac{1}{V} dr = -\int_{r_{QA}}^{r_2} \left(\sqrt{\frac{2Mr^2}{r^3 + 2L^2 M}}\right)^{-1} dr, \tag{4.15}$$

where r_{QA} is the radius of the quantum atmosphere, $r_{QA} \sim \frac{1}{k}$, with $k = \frac{3}{4M} - \frac{1}{r_1}$ being the surface gravity of the outer horizon of the Hayward black hole. For example, taking $M = 1$ and $L = 0.001$, the integral can be solved numerically, giving $T' \simeq -4.23 < 0$. However, as L decreases, T' increases because r_2 becomes smaller and thus it could be that $-4r_1 + 2T'$ becomes positive when $L \to 0$. Nevertheless, this does not happen because when $L \to 0$, T' becomes the Painlevé time that it takes for the partner particle to go from the quantum atmosphere to $r = 0$ in a Schwarzschild black hole, which is

$$\lim_{L \to 0} T' = \frac{8}{3}\sqrt{2}M. \tag{4.16}$$

Therefore,

$$\lim_{L \to 0}(-4r_1 + 2T') = \frac{-24 + 16\sqrt{2}}{3} M \simeq -1.37M < 0. \tag{4.17}$$

This implies that, in the $L \to 0$ limit one gets back the Schwarzschild result, as expected.

In the $M \gg L$ case, the Hayward black hole differs from the Schwarzschild one only close to $r = 0$. Outside the Cauchy horizon the effect of the L parameter is just an infinitesimal shift of the outer horizon, which does not have any appreciable consequence. Therefore, when $r \to r_1$ and $r \to \infty$ every element of the density-density correlator reduces to its Schwarzschild counterpart. We can then conclude that the correlator remains finite when one of the two points hits the outer horizon and it vanishes asymptotically.

2. Density correlations in the Simpson-Visser spacetime

We finally discuss what happens in a Simpson-Visser nonsingular black hole. In this case, the idea is to perform a Taylor expansion of the relevant quantities in the limits of interest (see Appendix E for details).

The term of the density correlator that depends only on r' remains finite when $r' \to 0$. In fact one has

$$\lim_{r' \to 0} \frac{1}{(1 \pm V(r'))^2} \left(\frac{du'_{\rm in}}{du'}\right)^2 = \frac{1}{\left(1 \pm \sqrt{\frac{2M}{a}}\right)^2} \left[\frac{\sqrt{\frac{(2m-a)^2}{a^2} T'^2 + 4a^2} - 4M}{\sqrt{\frac{(2m-a)^2}{a^2} T'^2 + 4a^2}}\right]^2, \tag{4.18}$$

Therefore, the density correlator remains finite and nonvanishing when the inner point approaches $r = 0$ (unless the outer point reaches infinity, as it will be discussed below). In particular, T' is the Painlevé time it takes for the partner particle to go from the quantum atmosphere to $r = 0$, which is given by

$$T' = \int_{r_{QA}}^{0} \frac{1}{V} dr = -\int_{r_{QA}}^{0} \left(\sqrt{\frac{2M}{\sqrt{r^2 + a^2}}}\right)^{-1} dr. \tag{4.19}$$

$r_{QA} \sim \frac{1}{k}$ is the radius of the quantum atmosphere, with $k = \frac{1}{4M}\sqrt{1 - \frac{a^2}{4M^2}}$ being the surface gravity of the horizon of the

Simpson-Visser black hole located at r_+. The result of the integral (4.19) can be expanded for $M \gg a$,[4]

$$T' \simeq \frac{8}{3}\sqrt{2}M + \frac{\sqrt{\frac{\pi}{2}}\Gamma\left(\frac{1}{4}\right)}{6\sqrt{M}\Gamma\left(\frac{3}{4}\right)}a^{3/2} + \frac{3}{4\sqrt{2}M}a^2. \tag{4.20}$$

Note that, when $a = 0$ one gets back the corresponding Painlevé time in the Schwarzschild black hole (4.16). Then, one should verify that the result (4.18), with the " + " sign in the first factor, reduces to the Schwarzschild one when $a \to 0$. Performing an expansion for $M \gg a$ one obtains

$$\lim_{r' \to 0} \frac{1}{(1+V(r'))^2}\left(\frac{du'_{\rm in}}{du'}\right)^2 = \frac{a}{2M} + o(a^{3/2}) \underset{a\to 0}{\longrightarrow} 0. \tag{4.21}$$

Therefore, when $a \to 0$ the density correlator vanishes and we get back the corresponding Schwarzschild result (3.26).

When one of the two points approaches the event horizon at $r = r_+$ one has, for $M \gg a$,

$$\lim_{r \to r_+} \frac{1}{(1+V(r))^2}\left(\frac{du_{\rm in}}{du}\right)^2 \simeq 4e^{-\frac{T}{2M}} + \frac{e^{-\frac{3T}{4M}}}{8M^3}(-12M + 9Me^{\frac{T}{4M}} + 2Te^{\frac{T}{4M}})a^2 + o(a^3). \tag{4.22}$$

It can be noted that the zeroth-order term coincides with the Schwarzschild result (3.29), as expected. In fact, far from $r = 0$, the Simpson-Visser "regularization" has only the effect of shifting the horizon by an infinitesimal amount in the $M \gg a$ limit. In this spirit, when the outer point reaches infinity, the density correlator goes to zero. This can be easily checked because, in this limit, $u_{\rm in} \to u \to \infty$ and therefore (3.23) vanishes.

V. CONCLUSIONS

We have studied the two-point correlation function of the density operator of a massless scalar field propagating in two nonsingular black hole spacetimes to investigate the energy density correlations between the Hawking quanta escaping to infinity and their partners that enter the event horizon. The analysis focused, in particular, on the Hayward and the Simpson-Visser black holes and it was inspired by the work done for an analog black hole generated using a Bose-Einstein condensate. Here, a peak in the density-density correlator, which has also been measured experimentally, signals the presence of particle-partner correlations. In the Schwarzschild case this peak does not show as when a Hawking particle emerges out of the quantum atmosphere the corresponding partner has already been swallowed by the singularity and their mutual correlations are lost. We have therefore investigated the behavior of the density-density correlator when the singularity is not present considering the Hayward and Simpson-Visser nonsingular black holes. This has been done generalizing the procedure already used in the Schwarzschild case. However, for these regular spacetimes it is not possible to compute analytically (mainly because the functions involved are not invertible) the correlator for any couple of points outside and inside the event horizon and so we have not been able to show the presence of a peak directly. Nevertheless, we have demonstrated that the correlator remains finite on the hypersurface that replaces the singularity, in opposition to what happens with the Schwarzschild black hole. In addition, when the black hole mass is much larger than the Planck length, the spacetime structure of the Hayward and Simpson-Visser nonsingular black holes is similar to that of the Schwarzschild one apart from the region close to $r = 0$ and thus one can suppose that also the form of the correlator will be qualitatively similar. However, in this case the characteristic peak will appear due to the fact that the partner particle does not enter a singularity but it continues to travel in other regions of spacetime where the correlator is finite. Therefore, energy density correlations can propagate in other copies of our universe towards the hypersurfaces that replace the singularity in regular black hole spacetimes. To confirm these conclusions a numerical calculation of the correlator should be performed, which could be done by generalizing the methods used in [35,36]. We hope to report soon on this possibility.

APPENDIX A

In Sec. III we have considered the energy density operator of a massless scalar field measured by free-falling observers in Painlevé-Gullstrand coordinates, which are obtained introducing a new time coordinate, called the Painlevé time, defined as

$$T = t + \int \frac{\sqrt{1-f}}{f}dr, \tag{A1}$$

where $f = 1 - \frac{2M}{r}$ for a Schwarzschild black hole. Note that the Painlevé time coincides with the proper time of the inertial observers free-falling radially from initial zero velocity.

The two-dimensional Schwarzschild line element in these (T, r) coordinates is

$$ds^2 = -fdT^2 - 2VdTdr, \tag{A2}$$

where $V = -\sqrt{1-f}$. The four-velocity of the observers that are considered is

[4]As usual we work on a stationary geometry. Since a should be a Planckian cutoff at which quantum effects of gravity become dominant, it is reasonable to assume $M \gg a$.

$$u^\alpha = (1, V). \tag{A3}$$

($u^\alpha u_\alpha = -1$, as can be easily verified). Recalling Eq. (3.13), the energy density is given by

$$\rho = T_{\alpha\beta}u^\alpha u^\beta = T_{TT} + 2VT_{Tr} + V^2 T_{rr}. \tag{A4}$$

Due to conformal invariance the trace of the stress-energy tensor is zero, $T_{\alpha\beta} = 0$. Therefore, we have

$$T_{\alpha\beta}g^{\alpha\beta} = -T_{TT} - 2VT_{Tr} + fT_{rr} = 0. \tag{A5}$$

where $g^{\alpha\beta}$ is the inverse of the two-dimensional Schwarzschild metric in Painlevé-Gullstrand coordinates,

$$g^{\alpha\beta} = \begin{pmatrix} -1 & -V \\ -V & f \end{pmatrix}. \tag{A6}$$

Substituting this result in Eq. (A4) we obtain the following simple expression for the density operator,

$$\rho = (f + V^2)T_{rr} = T_{rr}. \tag{A7}$$

As discussed in Appendix B, to calculate the two-point correlation function one has to derive twice the Wightman function (3.12). Since the latter is written in null coordinates, it is better to express also the density operator in the same coordinate system. Recalling the expression of the energy-momentum tensor of a massless scalar field (3.14), the components in null coordinates are

$$T_{uu} = \partial_u\phi\partial_u\phi, \tag{A8}$$

$$T_{vv} = \partial_v\phi\partial_v\phi, \tag{A9}$$

$$T_{uv} = T_{vu} = 0. \tag{A10}$$

Applying the usual tensor transformation rules to T_{rr} we get

$$\begin{aligned} T_{rr} &= \frac{\partial x^\alpha}{\partial r}\frac{\partial x^\beta}{\partial r}T_{\alpha\beta} \\ &= \left(\frac{\partial u}{\partial r}\right)^2 T_{uu} + \left(\frac{\partial v}{\partial r}\right)^2 T_{vv} + 2\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}T_{uv}. \end{aligned} \tag{A11}$$

The relation between the Eddington-Finkelstein coordinates and the Painlevé-Gullstrand ones is

$$u = t - r^* = T - \int \frac{\sqrt{1-f}+1}{f}dr, \tag{A12}$$

$$v = t + r^* = T - \int \frac{\sqrt{1-f}-1}{f}dr. \tag{A13}$$

Deriving and substituting into Eq. (A11) one obtains the following expression for the density operator written in (u, v) coordinates:

$$\begin{aligned} \rho = T_{rr} &= \left(\frac{\sqrt{1-f}+1}{f}\right)^2 T_{uu} + \left(\frac{\sqrt{1-f}-1}{f}\right)^2 T_{vv} \\ &= \frac{T_{uu}}{(1+V)^2} + \frac{T_{vv}}{(1-V)^2}. \end{aligned} \tag{A14}$$

APPENDIX B

Given the expression (3.18) of the density-density correlator, we want to write it in null coordinates, which in turn can be expressed in Painlevé-Gullstrand coordinates, in order to study how it behaves as a function of the position of the two points outside and inside the event horizon.

As remarked in Sec. III B, the fundamental object to study possible energy-density correlations between a Hawking particle and its partner across the event horizon of a black hole in a two-dimensional spacetime is the T_{uu} correlator. Applying the Wick's theorem, it can be written as a differential operator applied to the Wightman function:

$$\langle \mathrm{in}|T_{uu}(x)T_{u'u'}(x')|\mathrm{in}\rangle = (\partial_u\partial_{u'}\langle \mathrm{in}|\phi(x)\phi(x')|\mathrm{in}\rangle)^2. \tag{B1}$$

Then, we have to calculate

$$\begin{aligned} \partial_u\partial_{u'}\langle \mathrm{in}|\phi(x)\phi(x')|\mathrm{in}\rangle &= -\frac{1}{4\pi}\partial_u\partial_{u'}\ln\frac{(u_{\mathrm{in}}-u'_{\mathrm{in}})(v-v')}{(u_{\mathrm{in}}-v')(v-u'_{\mathrm{in}})} \\ &= -\frac{1}{4\pi}\frac{du_{\mathrm{in}}}{du}\frac{du'_{\mathrm{in}}}{du'}\frac{1}{(u_{\mathrm{in}}-u'_{\mathrm{in}})^2}, \end{aligned} \tag{B2}$$

where we have used the expression (3.12) for the Wightman function. Let us now recall Eq. (3.8), which gives u as a function of u_{in}. Since x is taken outside the horizon, $u_{\mathrm{in}} < 0$ and we have

$$u = u_{\mathrm{in}} - 4M\ln\left(-\frac{u_{\mathrm{in}}}{4M}\right). \tag{B3}$$

Since this function is invertible when $u_{\mathrm{in}} < 0$ one has $\frac{du_{\mathrm{in}}}{du} = (\frac{du}{du_{\mathrm{in}}})^{-1}$. Taking the derivative of (B3) with respect to u_{in} we get

$$\frac{du}{du_{\mathrm{in}}} = \frac{u_{\mathrm{in}} - 4M}{u_{\mathrm{in}}}, \tag{B4}$$

from which

$$\frac{du_{\mathrm{in}}}{du} = \frac{u_{\mathrm{in}}}{u_{\mathrm{in}} - 4M}. \tag{B5}$$

Repeating the same calculation for the "primed" sector (remember that x' is inside the horizon and thus $u'_{\rm in} > 0$) one obtains

$$\frac{du'_{\rm in}}{du'} = \frac{u'_{\rm in}}{u'_{\rm in} - 4M}. \tag{B6}$$

Now we can insert the results (B5) and (B6) into Eq. (B2):

$$\partial_u \partial_{u'} \langle {\rm in}|\phi(x)\phi(x')|{\rm in}\rangle = -\frac{1}{4\pi}\frac{u_{\rm in}u'_{\rm in}}{(u_{\rm in}-u'_{\rm in})^2}\frac{1}{(u_{\rm in}-4M)(u'_{\rm in}-4M)}. \tag{B7}$$

It is convenient to use the following identity

$$-\frac{u_{\rm in}u'_{\rm in}}{(u_{\rm in}-u'_{\rm in})^2} = \frac{1}{4\cosh^2 \ln\sqrt{-\frac{u_{\rm in}}{u'_{\rm in}}}}. \tag{B8}$$

Therefore, the T_{uu} correlator (B1) finally reads

$$\langle {\rm in}|T_{uu}(x)T_{u'u'}(x')|{\rm in}\rangle = \frac{1}{256\pi^2}\frac{1}{\cosh^4 \ln\sqrt{-\frac{u_{\rm in}}{u'_{\rm in}}}}\frac{1}{(u_{\rm in}-4M)^2(u'_{\rm in}-4M)^2}. \tag{B9}$$

For completeness, we also report the calculation of the other three terms in the correlator (3.18). In particular, repeating the same procedure for the v-sector we obtain

$$\langle {\rm in}|T_{vv}(x)T_{v'v'}(x')|{\rm in}\rangle = (\partial_v\partial_{v'}\langle {\rm in}|\phi(x)\phi(x')|{\rm in}\rangle)^2. \tag{B10}$$

Using again the Wightman function (3.12) we have

$$\begin{aligned}\partial_v\partial_{v'}\langle {\rm in}|\phi(x)\phi(x')|{\rm in}\rangle &= -\frac{1}{4\pi}\partial_u\partial_{u'}\ln\frac{(u_{\rm in}-u'_{\rm in})(v-v')}{(u_{\rm in}-v')(v-u'_{\rm in})}\\ &= -\frac{1}{4\pi}\frac{1}{(v-v')^2}.\end{aligned} \tag{B11}$$

Thus, the T_{vv} correlator reads

$$\langle {\rm in}|T_{vv}(x)T_{v'v'}(x')|{\rm in}\rangle = \frac{1}{16\pi^2}\frac{1}{(v-v')^4}. \tag{B12}$$

The "mixed" correlators can be computed in a similar way,

$$\langle {\rm in}|T_{uu}(x)T_{v'v'}(x')|{\rm in}\rangle = 2(\partial_u\partial_{v'}\langle {\rm in}|\phi(x)\phi(x')|{\rm in}\rangle)^2, \tag{B13}$$

$$\langle {\rm in}|T_{vv}(x)T_{u'u'}(x')|{\rm in}\rangle = 2(\partial_v\partial_{u'}\langle {\rm in}|\phi(x)\phi(x')|{\rm in}\rangle)^2, \tag{B14}$$

with

$$\begin{aligned}\partial_u\partial_{v'}\langle {\rm in}|\phi(x)\phi(x')|{\rm in}\rangle &= \frac{1}{4\pi}\frac{du_{\rm in}}{du}\frac{1}{(u_{\rm in}-v')^2}\\ &= \frac{1}{4\pi}\frac{u_{\rm in}}{u_{\rm in}-4M}\frac{1}{(u_{\rm in}-v')^2}\end{aligned} \tag{B15}$$

and

$$\begin{aligned}\partial_v\partial_{u'}\langle {\rm in}|\phi(x)\phi(x')|{\rm in}\rangle &= \frac{1}{4\pi}\frac{du'_{\rm in}}{du'}\frac{1}{(v-u'_{\rm in})^2}\\ &= \frac{1}{4\pi}\frac{u'_{\rm in}}{u'_{\rm in}-4M}\frac{1}{(u'_{\rm in}-v)^2}.\end{aligned} \tag{B16}$$

APPENDIX C

In this appendix, we report the calculation of the limits of the correlator in a Schwarzschild black hole.

Let's start considering the limit $r' \to 0$. We have

$$\lim_{r'\to 0} V(r') = \lim_{r'\to 0} -\sqrt{\frac{2M}{r'}} = \infty, \tag{C1}$$

and so

$$\lim_{r'\to 0}\frac{1}{(1\pm V(r'))^2} = 0. \tag{C2}$$

On the other hand, at $v = v_0 = 4M$,

$$u = 4M - 2r^*. \tag{C3}$$

Thus, using the expression (3.9) for $u_{\rm in}$,

$$\lim_{r'\to 0} u'_{\rm in} = -4MW(-e^{-1-\frac{|r^*|}{2M}}) = 4M, \tag{C4}$$

where we have used $\lim_{r'\to 0} r^* = 0$ and $W(-e^{-1}) = -1$, considering the principal branch of the Lambert function [13]. This implies that

$$\lim_{r'\to 0}\frac{du'_{\rm in}}{du'} = \lim_{r'\to 0}\frac{u'_{\rm in}}{u'_{\rm in}-4M} = \infty. \tag{C5}$$

Then, as discussed in Sec. III, the object that determines the behavior of the correlator in this limit is

$$\lim_{r'\to 0}\frac{1}{(1+V(r'))^2}\left(\frac{du'_{\rm in}}{du'}\right)^2. \tag{C6}$$

This can be recast in the following way:

$$\lim_{r'\to 0}\frac{N}{D} = \frac{0}{0}, \tag{C7}$$

where

$$N = r'(u'_{\mathrm{in}})^2, \tag{C8}$$

$$D = ((\sqrt{r'} - \sqrt{2M})^2)(u'_{\mathrm{in}} - 4M)^2. \tag{C9}$$

The idea is then to apply l'Hopital's theorem,

$$\lim_{r'\to 0} \frac{N}{D} = \lim_{r'\to 0} \frac{dN/dr'}{dD/dr'}. \tag{C10}$$

Writing u'_{in} as a function of u' and then u' in Painlevé coordinates it is possible to compute the derivatives and then take the limit. The result is

$$\lim_{r'\to 0} \frac{1}{(1+V(r'))^2} \left(\frac{du'_{\mathrm{in}}}{du'}\right)^2 = 0. \tag{C11}$$

In the $r \to 2M$ limit one has

$$\lim_{r\to 2M} V(r) = \lim_{r\to 2M} -\sqrt{\frac{2M}{r}} = -1. \tag{C12}$$

Thus,

$$\lim_{r\to 2M} \frac{1}{(1+V(r))^2} = \infty. \tag{C13}$$

On the other hand,

$$\lim_{r\to 2M} u_{\mathrm{in}} = 0, \tag{C14}$$

since $\lim_{r\to 2M} u = \infty$. So,

$$\lim_{r\to 2M} \frac{du_{\mathrm{in}}}{du} = \lim_{r\to 2M} \frac{u_{\mathrm{in}}}{u_{\mathrm{in}} - 4M} = 0. \tag{C15}$$

In this case one has to consider

$$\lim_{r\to 2M} \frac{1}{(1+V(r))^2} \left(\frac{du_{\mathrm{in}}}{du}\right)^2. \tag{C16}$$

Looking at the expression (3.20) for u_{in} at late times, the derivative term can be written as

$$\frac{du_{\mathrm{in}}}{du} = \frac{e^{-\frac{u}{4M}}}{e^{-\frac{u}{4M}} + 1}. \tag{C17}$$

From Eq. (3.16) one obtains

$$\begin{aligned} u &= T - \int \frac{\sqrt{1-f}+1}{f} dr \\ &= T - r - 2\sqrt{2Mr} - 4M \ln\left|\sqrt{\frac{r}{2M}} - 1\right|, \end{aligned} \tag{C18}$$

which can be expanded for $r \to 2M$,

$$u \simeq T - 4M \ln\left(\frac{r}{2M} - 1\right). \tag{C19}$$

Therefore, in this limit, one has

$$e^{-\frac{u}{4M}} \simeq e^{-\frac{T}{4M}} \left(\frac{r}{2M} - 1\right). \tag{C20}$$

The limit (C16) can then be written as

$$\lim_{r\to 2M} \frac{N}{D} = \frac{0}{0}, \tag{C21}$$

where in this case

$$N = r\left[e^{-\frac{T}{4M}} \left(\frac{r}{2M} - 1\right)\right]^2, \tag{C22}$$

$$D = (\sqrt{r} - \sqrt{2M})^2 \left[e^{-\frac{T}{4M}} \left(\frac{r}{2M} - 1\right) + 1\right]^2. \tag{C23}$$

Making use again of l'Hopital's theorem, the final result is

$$\lim_{r\to 2M} \frac{1}{(1+V(r))^2} \left(\frac{du_{\mathrm{in}}}{du}\right)^2 = 4e^{-\frac{T}{2M}}. \tag{C24}$$

APPENDIX D

We apply the procedure discussed in Sec. IV to study the two-point correlation function of the density operator of a massless scalar field on a Hayward nonsingular black hole background. In particular, we consider one point x outside the outer horizon and the other x' between the two horizons.

Let's start by giving the explicit expression of the tortoise coordinate,

$$r^* = \int \frac{1}{f(r)} dr, \tag{D1}$$

where $f(r)$ is that of Eq. (4.4). Performing the integral, one obtains

$$\begin{aligned} r^* &= 2M \sum_{i:f(r_i)=0} \frac{r_i \ln|r - r_i|}{3r_i - 4M} + r + C \\ &\simeq 2M\left[\frac{r_1}{3r_1 - 4M} \ln\left|\frac{r}{r_1} - 1\right| + \frac{r_2}{3r_2 - 4M} \ln\left|\frac{r}{r_2} - 1\right|\right] + r. \end{aligned} \tag{D2}$$

In the last step, we have assumed $M \gg L$ in order to approximate the roots of $f(r)$ as in Eqs. (4.7) and (4.8). We consider $r' > r_2$ (which means that the inner point is taken outside of the Cauchy horizon) and so the argument of the second logarithm is always positive.

It is easy to verify that for $L \to 0$, $r_1 \to 2M$ and $r_2 \to 0$. So, $r^* \to 2M \ln|\frac{r}{2M} - 1|$, which is the tortoise coordinate of a Schwarzschild black hole.

Next, we have to impose the continuity of the metric (4.10) describing the collapse on the null shell at $v = v_0$, which results in the condition $r(u_{\rm in}, v_0) = r(u, v_0)$. We choose $v_0 = 2r_1$ so that the Schwarzschild results are obtained in the limit $L \to 0$. The relation between u and $u_{\rm in}$ is

$$u = u_{\rm in} - 4M\left[\frac{r_1}{3r_1 - 4M}\ln\left|-\frac{u_{\rm in}}{2r_1}\right| + \frac{r_2}{3r_2 - 4M}\ln\left[\frac{2(r_1 - r_2) - u_{\rm in}}{2r_2}\right]\right]. \tag{D3}$$

To compute the density correlator (3.23) one needs an expression for $\frac{du_{\rm in}}{du}$. This is easily obtained because the function $u(u_{\rm in})$ is invertible in the subdomains $u_{\rm in} < 0$ and $u_{\rm in} > 0$ and thus $\frac{du_{\rm in}}{du} = (\frac{du}{du_{\rm in}})^{-1}$. Then, one needs only to differentiate (D3) in the regions inside and outside the outer horizon, which is the surface at which $u_{\rm in}$ changes sign.

For the point outside the horizon $(r > r_1)$, $u_{\rm in} < 0$. Performing the derivative of (D3) and inverting the result, one obtains

$$\frac{du_{\rm in}}{du} = \left[1 - \frac{4Mr_1}{3r_1 - 4M}\frac{1}{u_{\rm in}} + \frac{4Mr_2}{3r_2 - 4M}\frac{1}{2(r_1 - r_2) - u_{\rm in}}\right]^{-1}. \tag{D4}$$

For the point inside the outer horizon $(r_2 < r' < r_1)$, $u'_{\rm in} > 0$, but the derivative is the same of the point outside.

Note that when $L \to 0$, $\frac{du_{\rm in}}{du} \to \frac{u_{\rm in}}{u_{\rm in} - 4M}$ which, once again, is the Schwarzschild result. To study the dependence of the density correlator on the position of the two points one needs, first of all, to write $u_{\rm in}$ as a function of u by inverting (D3). Unfortunately, this function cannot be inverted analytically and thus we consider only the limits discussed in Sec. IV.

Taking into account the fact that the Hayward metric describes a particular case of a nonsingular black hole with a de Sitter core, one can approximate $f(r)$ as

$$f(r) \sim \begin{cases} 1 - \left(\frac{r}{r_2}\right)^2 & \text{if } r < r_Q \\ 1 - \frac{r}{r_1} & \text{if } r > r_Q, \end{cases} \tag{D5}$$

where we are still assuming $M \gg L$, so that $r_1 \simeq 2M$ and $r_2 \simeq L$. Therefore, when $r \to r_2$, the Hayward metric can be approximated by the de Sitter one, with

$$f(r) \sim 1 - \left(\frac{r}{r_2}\right)^2. \tag{D6}$$

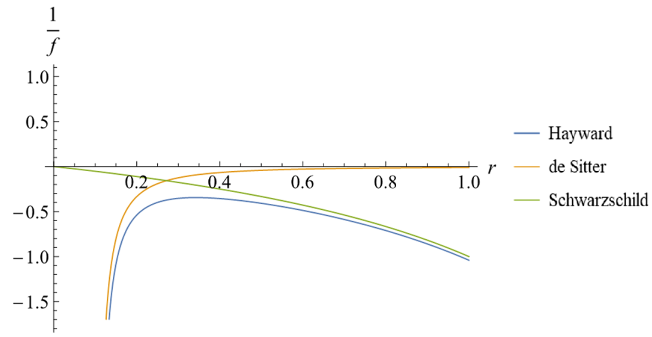

FIG. 2. Plot of $1/f$ for the Hayward (blue), de Sitter (orange), and Schwarzschild (green) spacetimes. In this plot $M = 1$, $L = 0.1$, $r_2 = 0.1025$.

Then, in this limit the tortoise coordinate can be approximated as

$$r^* \simeq \int \frac{1}{1 - (\frac{r}{r_2})^2} dr = -\frac{r_2}{2}\ln|r_2 - r| + C. \tag{D7}$$

Since $r > r_2$ this can be written as

$$r^* \simeq -\frac{r_2}{2}\ln\left(\frac{r}{r_2} - 1\right). \tag{D8}$$

To make more evident the fact that this approximation is reasonable one can plot the integrand $(1/f)$ for Hayward, together with the same function for de Sitter and Schwarzschild (Fig. 2). It is evident that the de Sitter function well reproduces the Hayward one in the $r \to r_2$ limit. One then has

$$u = 2r_1 + r_2 \ln\left[\frac{2(r_1 - r_2) - u_{\rm in}}{2r_2}\right]. \tag{D9}$$

Inverting this function one obtains

$$u_{\rm in} = 2(r_1 - r_2) - 2r_2 e^{-\frac{2r_1}{r_2} + \frac{u}{r_2}}. \tag{D10}$$

It is also convenient to consider $\frac{du_{\rm in}}{du}$ in this limit,

$$\frac{du_{\rm in}}{du} = -\frac{2(r_1 - r_2) - u_{\rm in}}{r_2}. \tag{D11}$$

When the outer point approaches the outer horizon $r \to r_1$, $f(r)$ can be approximated as

$$f(r) \sim 1 - \frac{r_1}{r}, \tag{D12}$$

which is the same of the Schwarzschild black hole, simply with the horizon location slightly shifted from $2M$ to r_1. Therefore, in this limit, one has

$$u_{\rm in} = -2r_1 W(e^{-\frac{u}{2r_1}}). \tag{D13}$$

Asymptotically, when $r \to \infty$, $f(r) \to 1$ and thus there is no distinction between r^* and r. Consequently, $u_{\rm in} = u$.

Finally, in order to analyze the behavior of the density-density correlator as a function of the positions of the two points we need to write u in Painlevé coordinates. This is done by approximating $f(r)$ as before in the interesting limits.

In particular, when $r \to r_2$, one can use the de Sitter core approximation. Thus, in this limit

$$\int \frac{\sqrt{1-f}+1}{f} dr \simeq \int \frac{\frac{r}{r_2}+1}{1-\left(\frac{r}{r_2}\right)^2} = -r_2 \ln(r-r_2). \tag{D14}$$

Therefore, when $r \to r_2$ one has

$$u = T + r_2 \ln(r - r_2). \tag{D15}$$

Instead, when $r \to r_1$ it is possible to use the Schwarzschild approximation for $f(r)$ and so u is written exactly as in the Schwarzschild case, simply with $2M$ replaced by r_1.

Finally, when $r \to \infty$, $f(r) \to 1$ and thus the Painlevé time and the ordinary Schwarzschild time coincide, as well as the tortoise coordinate and the usual radial coordinate. So, in this limit,

$$u = T - r. \tag{D16}$$

We now have all the elements to discuss the behavior of the density correlator (3.23) in the $r' \to r_2$ limit for the Hayward black hole,

$$\lim_{r' \to r_2} V(r') = \lim_{r' \to r_2} \left(-\frac{r'}{r_2}\right) = -1, \tag{D17}$$

and thus

$$\lim_{r' \to r_2} \frac{1}{(1+V(r'))^2} = \infty. \tag{D18}$$

$\frac{du'_{\rm in}}{du'}$ can be approximated as in Eq. (D11), with $u_{\rm in}$ given by (D10) and u by (D15). So,

$$\lim_{r' \to r_2} u' = -\infty, \tag{D19}$$

which implies

$$\lim_{r' \to r_2} u'_{\rm in} = 2(r_1 - r_2), \tag{D20}$$

and

$$\lim_{r' \to r_2} \frac{du'_{\rm in}}{du'} = 0. \tag{D21}$$

Then,

$$\begin{aligned}
\lim_{r' \to r_2} \frac{1}{(1+V(r'))^2} \left(\frac{du'_{\rm in}}{du'}\right)^2 &= \lim_{r' \to r_2} \frac{1}{(1-\frac{r'}{r_2})^2} \left[-\frac{2(r_1-r_2) - 2(r_1-r_2) - 2r_2 e^{-\frac{2r_1}{r_2}} e^{\frac{T'}{r_2} + \ln(r'-r_2)}}{r_2}\right]^2 \\
&= \frac{r_2^2}{(r'-r_2)^2} \left[\frac{2r_2 e^{-\frac{2r_1}{r_2}} e^{\frac{T'}{r_2}} (r'-r_2)}{r_2}\right]^2 \\
&= 4r_2^2 e^{-\frac{4r_1}{r_2} + \frac{2T'}{r_2}}.
\end{aligned} \tag{D22}$$

APPENDIX E

According to the general procedure discussed in Sec. IV, we start giving an explicit expression for the tortoise coordinate

$$\begin{aligned}
r^* &= \int \frac{1}{f(r)} dr = \int \left(1 - \frac{2M}{\sqrt{r^2+a^2}}\right)^{-1} dr \\
&= r - \frac{8M^2}{\sqrt{4M^2-a^2}} \operatorname{artanh}\left[\frac{2M + r - \sqrt{r^2+a^2}}{\sqrt{4M^2-a^2}}\right] + 2M \operatorname{artanh}\left[\frac{r}{\sqrt{r^2+a^2}}\right] + C,
\end{aligned} \tag{E1}$$

where C is an integration constant.

Imposing the continuity of the metric (4.10) one obtains again the condition $r(u_{\rm in}, v_0) = r(u, v_0)$, which, written explicitly, gives

$$\frac{v_0-u_{\rm in}}{2}-\frac{8M^2}{\sqrt{4M^2-a^2}}\mathrm{artanh}\left[\frac{2M+\frac{v_0-u_{\rm in}}{2}-\sqrt{\left(\frac{v_0-u_{\rm in}}{2}\right)^2+a^2}}{\sqrt{4M^2-a^2}}\right]+2M\mathrm{artanh}\left[\frac{\frac{v_0-u_{\rm in}}{2}}{\sqrt{\left(\frac{v_0-u_{\rm in}}{2}\right)^2+a^2}}\right]+C=\frac{v_0-u}{2}. \tag{E2}$$

Since v_0 is arbitrary, in this case it is convenient to choose $v_0=0$ to simplify the calculation. However, this is different from what we have done for the Schwarzschild and the Hayward black holes, where we have chosen v_0 to be twice the radius of the outer horizon. This affects the relation between the inner and outer retarded null coordinates, and therefore it will not be enough to take the $a\to 0$ limit to check the consistency with the Schwarzschild case in all the expressions that involve $u_{\rm in}$ and u. In order to do that, we will have to write everything in Painlevé coordinates and only at that stage verify the agreement with the Schwarzschild results for $a\to 0$.

We continue by writing u as a function of $u_{\rm in}$ as

$$u=u_{\rm in}+\frac{16M^2}{\sqrt{4M^2-a^2}}\mathrm{artanh}\left[\frac{2M-\frac{u_{\rm in}}{2}-\sqrt{\frac{u_{\rm in}^2}{4}+a^2}}{\sqrt{4M^2-a^2}}\right]-4M\mathrm{artanh}\left[\frac{-\frac{u_{\rm in}}{2}}{\sqrt{\frac{u_{\rm in}^2}{4}+a^2}}\right]+C. \tag{E3}$$

The next step to study the density correlator (3.23) is to compute $\frac{du_{\rm in}}{du}$. This can be easily done since the function $u(u_{\rm in})$ is invertible and therefore $\frac{du_{\rm in}}{du}=(\frac{du}{du_{\rm in}})^{-1}$. Then, one only needs to differentiate (E3),

$$\frac{du_{\rm in}}{du}=\left(\frac{du}{du_{\rm in}}\right)^{-1}=\frac{\sqrt{u_{\rm in}^2+4a^2}-4M}{\sqrt{u_{\rm in}^2+4a^2}}. \tag{E4}$$

Remember that the aim is to discuss how the density correlator changes as we move the two points inside and outside the horizon. Therefore, we need to write $u_{\rm in}$ as a function of the radial coordinate. The strategy is always the same; first invert (E3) and then write u in Painlevé coordinates. Unfortunately, this can be done only in the limits discussed in Sec. IV, where the relation between u and $u_{\rm in}$ can be properly approximated.

Let us start by considering $r'\to 0$.

In this limit, the tortoise coordinate can be expanded as

$$r^*=C-\frac{8M^2}{\sqrt{4M^2-a^2}}\mathrm{artanh}\left[\frac{2M-a}{\sqrt{4M^2-a^2}}\right]-\frac{a}{2M-a}r+o(r^3). \tag{E5}$$

When $a\to 0$, this should give back the Schwarzschild result, for which

$$\lim_{r\to 0} r^*=0. \tag{E6}$$

However,

$$\lim_{a\to 0}\left[-\frac{8M^2}{\sqrt{4M^2-a^2}}\mathrm{artanh}\left[\frac{2M-a}{\sqrt{4M^2-a^2}}\right]\right]=\infty. \tag{E7}$$

So, the idea is to choose the integration constant C to cancel the zeroth-order term. In this way,

$$r^*=-\frac{a}{2M-a}r+o(r^3). \tag{E8}$$

Then, in this limit, the relation between u and $u_{\rm in}$ becomes

$$u=-\frac{a}{2M-a}u_{\rm in}, \tag{E9}$$

which gives

$$u_{\rm in}=-\frac{2M-a}{a}u. \tag{E10}$$

When $r\to r_+$ one can expand $1/f(r)$ as

$$\frac{1}{f(r)}\simeq\frac{4M^2}{\sqrt{4M^2-a^2}}\frac{1}{r-\sqrt{4M^2-a^2}}. \tag{E11}$$

Integrating the leading order one obtains

$$r^*\simeq\frac{4M^2}{\sqrt{4M^2-a^2}}\ln\left|\frac{r}{\sqrt{4M^2-a^2}}-1\right|. \tag{E12}$$

In this limit, the relation between u and $u_{\rm in}$ becomes

$$\frac{4M^2}{\sqrt{4M^2-a^2}}\ln\left(-\frac{u_{\rm in}}{2\sqrt{4M^2-a^2}}-1\right)=-\frac{u}{2}, \qquad \text{(E13)}$$

from which

$$u=-\frac{8M^2}{\sqrt{4M^2-a^2}}\ln\left(-\frac{u_{\rm in}}{2\sqrt{4M^2-a^2}}-1\right). \qquad \text{(E14)}$$

Inverting this expression one obtains

$$u_{\rm in}=-2\sqrt{4M^2-a^2}-2\sqrt{4M^2-a^2}e^{-\frac{4M^2-a^2}{8M^2}u}. \qquad \text{(E15)}$$

Finally, when $r\to\infty$, $f(r)\to 1$ and thus $r^*=r$, which implies $u_{\rm in}=u$.

One has then to write u in Painlevé coordinates, which can be done in the limits mentioned above.

When $r\to 0$ one has

$$u=T-\int\frac{\sqrt{1-f}+1}{f}dr\simeq T+\frac{\sqrt{a}(\sqrt{a}+\sqrt{2M})}{2M-a}r. \qquad \text{(E16)}$$

Expanding for $r\to r_+$ one obtains

$$\begin{aligned} u&=T-\int\frac{\sqrt{1-f}+1}{f}dr\\ &\simeq T-\frac{8M^2}{\sqrt{4M^2-a^2}}\ln\left|\frac{r}{\sqrt{4M^2-a^2}}-1\right|. \end{aligned} \qquad \text{(E17)}$$

When $r\to\infty$ the Painlevé time and the Schwarzschild time coincide, as well as the tortoise coordinate and the radial coordinate, and so

$$u=T-r. \qquad \text{(E18)}$$

We now have all the elements to study the density correlator in the relevant limits for the Simpson-Visser nonsingular black hole.

Let us start considering the limit for $r'\to 0$. Since

$$\lim_{r'\to 0}V(r')=-\sqrt{\frac{2M}{a}}, \qquad \text{(E19)}$$

the factors $\frac{1}{(1+V(r'))^2}$ and $\frac{1}{(1-V(r'))^2}$ remain finite in this limit.

Then, when r' is close to zero, u' can be approximated as in Eq. (E16), and using the result (E10) one obtains

$$u'_{\rm in}\simeq-\frac{2M-a}{a}T'-\frac{\sqrt{a}+\sqrt{2M}}{\sqrt{a}}r'. \qquad \text{(E20)}$$

This implies that the term of the density correlator that depends only on r' remains finite when $r'\to 0$. Using the Eqs. (E19) and (E20), together with (E4), one arrives at the result (4.18).

One can then study the limit for $r\to r_+$ of the density correlator,

$$\lim_{r\to r_+}V(r)=-1, \qquad \text{(E21)}$$

and thus

$$\lim_{r\to r_+}\frac{1}{(1+V(r))^2}=\infty. \qquad \text{(E22)}$$

Then, from Eq. (E17),

$$\lim_{r\to r_+}u=\infty. \qquad \text{(E23)}$$

Using Eq. (E15), one has

$$\lim_{r\to r_+}u_{\rm in}=-2\sqrt{4M^2-a^2} \qquad \text{(E24)}$$

and applying (E4),

$$\lim_{r\to r_+}\left(\frac{du_{\rm in}}{du}\right)=0. \qquad \text{(E25)}$$

Therefore, one has to compute the following limit:

$$\lim_{r\to r_+}\frac{1}{(1+V(r))^2}\left(\frac{du_{\rm in}}{du}\right)^2. \qquad \text{(E26)}$$

This can be written as

$$\lim_{r\to r_+}\frac{N}{D}=\frac{0}{0}, \qquad \text{(E27)}$$

where

$$N=\sqrt{r^2+a^2}\left\{\sqrt{\left[-2r_+-2r_+e^{-\frac{r_+}{8M^2}T}\left(\frac{r}{r_+}-1\right)\right]^2+4a^2}-4M\right\}^2, \qquad \text{(E28)}$$

$$D=[(r^2+a^2)^{1/4}-\sqrt{2M}]^2\left\{\sqrt{\left[-2r_+-2r_+e^{-\frac{r_+}{8M^2}T}\left(\frac{r}{r_+}-1\right)\right]^2+4a^2}\right\}^2. \qquad \text{(E29)}$$

Applying l'Hopital's theorem and expanding for $M\gg a$ one obtains the result (4.22).

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