Carrollian limit of quadratic gravity

Poula Tadros [1,2,*] and Ivan Kolář [1,†]

[1] *Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, Prague 180 00, Czech Republic*

[2] *Department of Applied Physics, Aalto University School of Science, FI-00076 Aalto, Finland*

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We study the Carrollian limit of the (general) quadratic gravity in four dimensions. We find that in order for the Carrollian theory to be a modification of the Carrollian limit of general relativity, the parameters in the action must depend on the speed of light in a specific way. By focusing on the leading and the next-to-leading orders in the Carrollian expansion, we show that there are four such nonequivalent Carrollian theories. Imposing conditions to remove tachyons (from the linearized theory), we end up with a classification of Carrollian theories according to the leading-order and next-to-leading-order actions. All modify the Carrollian limit of general relativity with quartic terms of the extrinsic curvature. To the leading order, we show that two theories are equivalent to general relativity, one to $R + R^2$ theory and one to the general quadratic gravity. To the next-to-leading order, two are equivalent to $R + R^2$ while the other two are equivalent to the general quadratic gravity. We study the two theories that are equivalent to $R + R^2$ to the leading order and write their magnetic limit actions.

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I. INTRODUCTION

The *quadratic gravity* can be derived as an effective field theory by truncating the expansion of the bosonic section of string theory with the first order being *general relativity (GR)* [1–5] or by imposing a maximal momentum to strings [6]. It has been studied even before the connection to string theory as a renormalizable theory of gravity [7–9]. It admits a wide class of black-hole and other spherically symmetric (exact) solutions [10–13]. Nevertheless, in general, it suffers from the presence of unphysical ghost and tachyonic degrees of freedom [8].

The *Carrollian limit* was first considered independently by Levy-Leblond [14] and Sen Gupta [15] as the ultralocal limit of the Poincaré group where the speed of light c approaches zero, $c \to 0$. However, at the time, due to the lack of physical application of this limit, it was only studied by mathematicians until 40 years later when the Carrollian limit was linked to many applications in physics. Now, Carrollian physics and Carrollian structures are studied in the context of representations of the Carroll group, i.e., Carroll particles [16–19], condensed matter physics [20–22], field theory [23–26], conformal field theory [27–30], fluid mechanics [31–36], cosmology [37,38], string theory [39–41], gravity [42–50] (it is regarded as the strong coupling limit of gravity theories [51]), black holes [19,52–56], null boundaries [28,57–59], and dynamics of particles near black-hole horizons [60–62].

The connection between the Carrollian limit and physics near black-hole horizons was shown in [52] utilizing the membrane paradigm [63–65] which is a paradigm showing that the physics of a black hole on a stretched horizon is dual to that of a relativistic fluid on a $(2+1)$-dimensional submanifold. Taking the Carrollian limit of both sides gives a duality between physics on the horizon and a Carrollian fluid. It was shown afterwards that there are two non-equivalent Carrollian limits of a relativistic theory called the *electric* and *magnetic limits*. The electric limit comes directly from the *leading order (LO)* in the *Carrollian expansion*, i.e., the expansion in c, while the magnetic limit is a certain truncation of the *next-to-leading order (NLO)* of this expansion.

In this paper we analyze the electric and the magnetic Carrollian limits of quadratic gravity, which is the first step toward the analysis of dynamics of particles near black-hole horizons. We study the electric limit of the general quadratic gravity theory and construct a classification of Carrollian theories from it, and the magnetic limit of the resulting ghost-free theories. Throughout the paper we use the units where Newton's constant G is set to $G = 1/(16\pi)$. The paper is organized as follows:

(i) In Sec. II, we review the pre-ultralocal (PUL) parametrization, which is suitable for the Carrollian expansion, and calculate the PUL versions of various tensors appearing in a general four-dimensional quadratic gravity action.

(ii) In Sec. III, we review the electric Carrollian limit of GR and show the ultralocality of the spacetime evolution.

*poulatadros9@gmail.com
†ivan.kolar@matfyz.cuni.cz

(iii) In Sec. IV, we perform the Carrollian expansion of quadratic gravity action. We show that the parameters α and β in the action must depend on c in a specific way; otherwise the resulting theory would be drastically different from the Carrollian limit of GR. Requiring the resulting theory to be a modification to the Carrollian limit of GR to LO or NLO gives four nonequivalent Carrollian theories.

(iv) In Sec. V, we study those limits one by one and derive conditions on α and β to remove tachyons (from the linearized theory) in each case to the LO and NLO.

(v) In Sec. VI, we study the magnetic limit of the ghost-free and tachyon-free theories.

(vi) The paper is concluded with a brief summary and discussions of our results in Sec. VII.

II. PRE-ULTRALOCAL PARAMETRIZATION

The *PUL parametrization* is a parametrization of the metric on a manifold using the decomposition of its tangent bundle in vertical and horizontal subbundles (see below). It is the most convenient parametrization of the spacetime for the analysis of Carrollian gravity since it is well adapted to the ultralocal structure of the Carrollian limit and it displays the speed of light c explicitly, which makes the calculations more obvious. In what follows, we briefly explain the mathematical background of the PUL parametrization. By following the calculations and notations from [46], we present the PUL version of the Riemannian tensor which will be used to calculate terms in quadratic gravity action in later sections.

Let $(M, \boldsymbol{g})$ be a $(d+1)$-dimensional Lorentzian manifold (with a mostly positive signature). Let us denote the tangent bundle of M by TM and define two subbundles of TM according to the signature of the metric: The first is called the *vertical bundle* Ver$M$ (or the timelike bundle), and it corresponds to the timelike direction; i.e., its fibers are endowed with a vector space isomorphic to the time coordinate. The second is referred to as the *horizontal bundle* Hor$M$ (or the spatial bundle), and it represents the remaining d spacelike directions. It is easy to prove that $TM = \mathrm{Ver}M \oplus \mathrm{Hor}M$. Furthermore, it generates a foliation of the manifold whose slices are the submanifolds of a constant time coordinate t. This foliation allows us to define orthogonal spatial and timelike sections as follows: Consider a covector T_μ and a vector V^μ from Ver$M$, where $\mu, \nu, \ldots = 1, 2, \ldots, d+1$ are tensor indices in TM. Next, we consider a symmetric tensor $\Pi_{\mu\nu}$ from Hor$M$, which is the induced metric (or the first fundamental form), and its inverse $\Pi^{\mu\nu}$.

By construction of the subbundles and the foliation we have

$$T_\mu V^\mu = -1, \quad -V^\mu T_\nu + \Pi^{\rho\mu}\Pi_{\rho\nu} = \delta^\nu_\mu, \quad T_\mu \Pi^{\mu\nu} = \Pi_{\mu\nu} V^\nu = 0, \tag{2.1}$$

The PUL parametrization of the metric $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}, \qquad g^{\mu\nu} = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}. \tag{2.2}$$

The metric, its inverse, and the spatial tensors can be written in terms of vielbeins as

$$\begin{aligned} g_{\mu\nu} &= \eta_{AB} E^A_\mu E^B_\nu, \qquad & g^{\mu\nu} &= \eta^{AB} \Theta^\mu_A \Theta^\nu_B, \\ \Pi_{\mu\nu} &= \eta_{ab} E^a_\mu E^b_\nu, \qquad & \Pi^{\mu\nu} &= \eta^{ab} \Theta^\mu_a \Theta^\nu_b, \end{aligned} \tag{2.3}$$

where E^A_μ and Θ^μ_A are the vielbeins. Indices $A, B, \ldots$ are vielbein labels running from 1 to $d+1$ (the dimension of TM) while $a, b, \ldots$ are vielbein labels running from 1 to d (the dimension of the Hor$M$). Comparing the PUL parametrization with the vielbein definition we get $E^A_\mu = (cT_\mu, E^a_\mu)$ and $\Theta^\mu_A = (-c^{-1} V^\mu, \Theta^\mu_a)$.

Following [46], we assume that all fields are analytic in c^2 and expand them as follows:

$$\begin{aligned} V^\mu &= v^\mu + c^2 M^\mu + O(c^4), \qquad & T_\mu &= \tau_\mu + c^2 N_\mu + O(c^4), \qquad & \Theta^\mu_a &= \theta^\mu_a + c^2 \pi^\mu_a + O(c^4), \\ E^a_\mu &= e^a_\mu + c^2 F^a_\mu + O(c^4), \qquad & \Pi^{\mu\nu} &= h^{\mu\nu} + c^2 \Phi^{\mu\nu} + O(c^4), \qquad & \Pi_{\mu\nu} &= h_{\mu\nu} + c^2 \Phi_{\mu\nu} + O(c^4), \end{aligned} \tag{2.4}$$

where $v^\mu, M^\mu, \tau_\mu, N_\nu, \theta^\mu_a, \pi^\mu_a, e^a_\mu, F^a_\mu, h^{\mu\nu}, \Phi^{\mu\nu}$ are fields used to define geometries in the Carrollian limit. These fields are not all independent but they are related by two constraints. Thus, we can write τ_μ and θ^μ_a in terms of the other fields. Including more orders in c^2 leads to defining more fields that interpolate between the Carrollian theory (LO in the expansion) and the full theory on the manifold. Expanding the first equation in (2.1), we get

$$\tau_\mu v^\mu + c^2(\tau_\mu M^\mu + N_\nu v^\mu) + c^4 N_\mu M^\mu = -1. \tag{2.5}$$

Comparing the LO and NLO terms we arrive at

$$\tau_\mu v^\mu = -1, \qquad \tau_\mu M^\mu + N_\nu v^\mu = 0. \tag{2.6}$$

Similarly, if we expand the second equation in (2.1), we obtain

$$-\tau_\nu v^\mu + h^{\mu\rho} h_{\rho\nu} + c^2(h^{\mu\rho}\Phi_{\rho\nu} + \Phi^{\mu\rho} h_{\rho\nu} - M^\mu \tau_\nu - v^\mu N_\mu) + c^4 \Phi^{\mu\rho}\Phi_{\rho\nu} = \delta^\mu_\nu, \tag{2.7}$$

which by comparison of LO and NLO terms gives

$$-\tau_\nu v^\mu + h^{\mu\rho}h_{\rho\nu} = \delta^\mu_\nu, \quad h^{\mu\rho}\Phi_{\rho\nu} + \Phi^\mu h_{\rho\nu} - M^\mu\tau_\nu - v^\mu N_\mu = 0. \tag{2.8}$$

Now, by expanding (2.2) we also get

$$h_{\mu\nu} + c^2\Phi_{\mu\nu} = \delta_{ab}e^a_\mu e^b_\nu + c^2\delta_{ab}(F^a_\mu e^b_\nu + e^a_\mu F^b_\nu) + c^4\delta_{ab}F^a_\mu F^b_\nu, \tag{2.9}$$

and after comparing the LO and NLO terms, we arrive at

$$h_{\mu\nu} = \delta_{ab}e^a_\mu e^b_\nu, \qquad \Phi_{\mu\nu} = \delta_{ab}(F^a_\mu e^b_\nu + e^a_\mu F^b_\nu). \tag{2.10}$$

Similarly,

$$h^{\mu\nu} = \delta^{ab}\theta^\mu_a\theta^\nu_b, \qquad \Phi^{\mu\nu} = \delta^{ab}(\theta^\mu_a\pi^\nu_b + \pi^\mu_a\theta^\nu_b). \tag{2.11}$$

We remark that the induced metric $\boldsymbol{h}$ and the set of all $\boldsymbol{v} \in \mathcal{V}$ give rise to the Carrollian spacetime $(\mathcal{C}, \mathcal{V}, \boldsymbol{h})$.

To derive a compatible connection with the PUL parametrization [46,59], we notice that V^μ and $\Pi_{\mu\nu}$ are invariant under Carroll boosts. Thus, they must be covariantly constant. Although this cannot determine a connection uniquely, it was argued in Appendix B of [46] that the most convenient choice is

$$C^\rho_{\mu\nu} = -V^\rho\partial_{(\mu}T_{\nu)} - V^\rho T_{(\mu}\pounds_V T_{\nu)} + \frac{1}{2}\Pi^{\rho\lambda}[\partial_\mu\Pi_{\nu\lambda} + \partial_\nu\Pi_{\lambda\mu} - \partial_\lambda\Pi_{\mu\nu}] - \Pi^{\rho\lambda}T_\nu\mathcal{K}_{\mu\lambda}, \tag{2.12}$$

where $\mathcal{K}_{\mu\lambda} = -\frac{1}{2}\pounds_V\Pi_{\mu\lambda}$ is the extrinsic curvature (or the second fundamental form). The connection $C^\rho_{\mu\nu}$ has a nonzero torsion given by

$$T^\rho_{\mu\nu} = 2\Pi^{\rho\lambda}T_{[\mu}\mathcal{K}_{\nu]\lambda}, \tag{2.13}$$

which, to the LO, reads

$$T^\rho_{\mu\nu} = 2h^{\rho\lambda}\tau_{[\mu}K_{\nu]\lambda}. \tag{2.14}$$

To proceed parametrizing the Riemann tensor of the Levi-Civita connection, we write its Christoffel symbols $\Gamma^\rho_{\mu\nu}$ in terms of the PUL fields using (2.2) and (2.3). The result is

$$\Gamma^\rho_{\mu\nu} = \frac{1}{c^2}\left[-\frac{1}{2}V^\rho V^\lambda\partial_\mu\Pi_{\nu\lambda} - \frac{1}{2}V^\rho V^\lambda\partial_\nu\Pi_{\lambda\mu} + \frac{1}{2}V^\rho V^\lambda\partial_\lambda\Pi_{\mu\nu}\right] + \frac{1}{2}[\Pi^{\rho\lambda}\partial_\mu\Pi_{\nu\lambda} + \Pi^{\rho\lambda}\partial_\nu\Pi_{\lambda\mu} - \Pi^{\rho\lambda}\partial_\lambda\Pi_{\mu\nu} + V^\rho V^\lambda\partial_\mu(T_\nu T_\lambda) + V^\rho V^\lambda\partial_\nu(T_\mu T_\lambda) - V^\rho V^\lambda\partial_\lambda(T_\nu T_\mu)] + c^2[\Pi^{\rho\lambda}\partial_\mu(T_\nu T_\lambda) - \Pi^{\rho\lambda}\partial_\nu(T_\mu T_\lambda) + \Pi^{\rho\lambda}\partial_\lambda(T_\nu T_\mu)]. \tag{2.15}$$

With the help of the coordinate expression of the Lie derivative we can rewrite $\Gamma^\rho_{\mu\nu}$ as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{c^2}[-V^\rho\mathcal{K}_{\mu\nu}] + [C^\rho_{\mu\nu} + \Pi^{\rho\lambda}T_\nu\mathcal{K}_{\mu\lambda}] + c^2[-T_{(\mu}\Pi^{\rho\lambda}B_{\nu)\lambda}], \tag{2.16}$$

where $B_{\mu\nu} = \partial_\mu T_\nu - \partial_\nu T_\mu$ is the exterior derivative of the covector T_μ, which is the same as Eq. (2.21) in [46]. Finally, we are equipped to parametrize the Riemann tensor of $\Gamma^\rho_{\mu\nu}$,

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\lambda} - \partial_\nu\Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda}. \tag{2.17}$$

Inserting (2.15), we obtain

$$R^\rho{}_{\lambda\mu\nu} = \frac{1}{c^2}[2V^\rho\nabla_{[\nu}\mathcal{K}_{\mu]\lambda} + 2V^\rho\mathcal{K}_{\lambda\sigma}C^\sigma_{[\nu\mu]} + 2V^\rho T_\lambda\mathcal{K}^\alpha_{[\nu}\mathcal{K}_{\mu]\alpha} + 2\mathcal{K}_{\lambda[\nu}\mathcal{K}^\rho_{\mu]}] + [\overset{c}{R}{}^\rho{}_{\lambda\mu\nu} + 2\nabla_{[\mu}(\mathcal{K}^\rho_{\nu]}T_\lambda) + 2C^\sigma_{[\mu\nu]}T_\lambda\mathcal{K}^\rho_\sigma + V^\rho\mathcal{K}_{\mu\sigma}T_{(\nu}B_{\lambda)}{}^\sigma - V^\rho\mathcal{K}_{\nu\sigma}T_{(\mu}B_{\lambda)}{}^\sigma + T_{(\mu}B_{\sigma)}{}^\rho V^\sigma\mathcal{K}_{\nu\lambda} - T_{(\nu}B_{\sigma)}{}^\rho V^\sigma\mathcal{K}_{\mu\lambda}] + c^2[\nabla_\nu(T_{(\mu}B_{\lambda)}{}^\rho) - \nabla_\mu(T_{(\nu}B_{\lambda)}{}^\rho) + 2C^\sigma_{[\nu\mu]}T_{(\sigma}B_{\lambda)}{}^\rho) - T_{(\mu}B_{\sigma)}{}^\rho T_\lambda\mathcal{K}^\sigma_\nu + T_{(\nu}B_{\sigma)}{}^\rho T_\lambda\mathcal{K}_\mu{}^\sigma] + c^4[T_{(\mu}B_{\sigma)}{}^\rho T_{(\nu}B_{\lambda)}{}^\sigma - T_{(\nu}B_{\sigma)}{}^\rho T_{(\mu}B_{\lambda)}{}^\sigma], \tag{2.18}$$

where $\overset{c}{R}{}^\rho{}_{\lambda\mu\nu} = \partial_\mu C^\rho_{\nu\lambda} - \partial_\nu C^\rho_{\mu\lambda} + C^\rho_{\mu\sigma}C^\sigma_{\nu\lambda} - C^\rho_{\nu\sigma}C^\sigma_{\mu\lambda}$ and indices lowering/raising for $\mathcal{K}_{\mu\nu}$ and $B_{\mu\nu}$ was done by the induced metric and its inverse.

III. CARROLLIAN EXPANSION OF GR

Having derived the PUL parametrization of the Riemann tensor in (2.18), we can now review the Carrollian expansion of the GR following [46]. Recall that the Einstein-Hilbert action in four dimensions ($d = 3$) is

$$S = c^3 \int R\sqrt{-g}d^4x. \tag{3.1}$$

Let us first calculate the PUL parametrization of the Ricci scalar R. By contracting ρ and μ in (2.18), we can write the Ricci tensor in the form

$$\begin{aligned} R_{\lambda\nu} = {}& \frac{1}{c^2}[-\nabla_\mu(V^\mu\mathcal{K}_{\nu\lambda}) - 2V^\mu C^\sigma_{[\mu\lambda]}\mathcal{K}_{\nu\sigma} + \mathcal{K}_{\nu\lambda}\mathcal{K} - \mathcal{K}_{\mu\lambda}\mathcal{K}^\mu_\nu] + \Big[\overset{c}{R}_{\lambda\nu} + \nabla_\mu(T_\lambda\mathcal{K}^\mu_\nu) - \nabla_\nu(T_\lambda\mathcal{K}) + 2C^\mu_{[\nu\beta]}T_\lambda\mathcal{K}_\mu{}^\beta \\ & + \mathcal{K}^\alpha_{(\nu}B_{\lambda)\alpha} - \frac{1}{2}V^\mu\mathcal{K}^\alpha_\nu T_\lambda B_{\mu\alpha} - \frac{1}{2}T_\nu V^\sigma B_{\sigma\alpha}\mathcal{K}^\alpha_\lambda\Big] + c^2[-\nabla_\mu(T_{(\nu}B_{\lambda)}{}^\mu) + 2C^\sigma_{[\nu\mu]}T_{(\sigma}B_{\lambda)}{}^\mu + T_{(\nu}B_{\sigma)}{}^\mu T_\lambda\mathcal{K}_\mu{}^\sigma] \\ & + c^4\Big[-\frac{1}{4}T_\nu T_\lambda B^{\mu\nu}B_{\mu\nu}\Big], \end{aligned} \tag{3.2}$$

where ∇_μ is the covariant derivative corresponding to the connection $C^\rho_{\mu\nu}$. Here, we also introduced the trace of the extrinsic curvature, $\mathcal{K} = \Pi^{\mu\nu}\mathcal{K}_{\mu\nu}$, and the Ricci tensor of the connection $C^\rho_{\mu\nu}$,

$$\overset{c}{R}_{\lambda\nu} = \partial_\mu C^\mu_{\nu\lambda} - \partial_\nu C^\mu_{\mu\lambda} + C^\mu_{\mu\sigma}C^\sigma_{\nu\lambda} - C^\mu_{\nu\sigma}C^\sigma_{\mu\lambda}. \tag{3.3}$$

The PUL parametrization of the Ricci scalar is obtained by contraction with the inverse metric and employing $\Pi^{\lambda\nu}\nabla_\mu(V^\mu\mathcal{K}_{\nu\lambda}) = \nabla_\nu(V^\nu\mathcal{K})$. The result is

$$\begin{aligned} R = {}& \frac{1}{c^2}[\mathcal{K}^2 - \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} - 2\nabla_\nu(V^\nu\mathcal{K})] + [-\overset{c}{R} + \Pi^{\lambda\nu}\nabla_\mu(T_\lambda\mathcal{K}_\nu{}^\mu) - \Pi^{\lambda\nu}\nabla_\nu(T_\lambda\mathcal{K}) + V^\lambda V^\nu\nabla_\mu(T_{(\nu}B_{\lambda)}{}^\mu) \\ & - V^\lambda V^\nu\nabla_\nu(T_{(\mu}B_{\lambda)}{}^\mu)] + c^2\Big[-\Pi^{\lambda\nu}\nabla_\mu(T_{(\nu}B_{\lambda)}{}^\mu) + \Pi^{\lambda\nu}\nabla_\nu(T_{(\mu}B_{\lambda)}{}^\mu) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\Big], \end{aligned} \tag{3.4}$$

where $\overset{c}{R} = \Pi^{\mu\nu}\overset{c}{R}_{\mu\nu}$. We used $V^\mu\overset{c}{R}_{\mu\nu} = 0$ in the calculations.

Using the relation $\nabla_\rho\Pi^{\mu\nu} = -V^{(\mu}\Pi^{\nu)\sigma}B_{\sigma\lambda}[\delta^\lambda_\rho - V^\lambda T_\rho]$, we can find that $\Pi^{\lambda\nu}\nabla_\mu(T_\lambda\mathcal{K}^\mu_\nu) = 0$, $\Pi^{\lambda\nu}\nabla_\nu(T_\lambda\mathcal{K}) = 0$, $V^\lambda V^\nu\nabla_\mu(T_{(\nu}B_{\lambda)}{}^\mu) = -V^\lambda\nabla_\mu(B^\mu_\lambda)$, and $-\Pi^{\lambda\nu}\nabla_\mu(T_{(\nu}B_{\lambda)}{}^\mu) = \frac{1}{2}B^{\mu\nu}B_{\mu\nu}$. Employing these identities, the Ricci scalar simplifies to

$$\begin{aligned} R = {}& \frac{1}{c^2}[\mathcal{K}^2 - \mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} - 2\nabla_\nu(V^\nu\mathcal{K})] \\ & + [-\overset{c}{R} - \nabla_\mu(V^\nu B^\mu_\nu)] + c^2\Big[\frac{1}{4}B^{\mu\nu}B_{\mu\nu}\Big]. \end{aligned} \tag{3.5}$$

Furthermore, we can separate the total derivative terms as they correspond to the boundary terms in actions of physical theories. Finally, the PUL parametrization of the Ricci scalar can be written in the form

$$\begin{aligned} R = {}& \frac{1}{c^2}[\mathcal{K}^2 - \mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu}] + [-\overset{c}{R}] + c^2\Big[\frac{1}{4}B^{\mu\nu}B_{\mu\nu}\Big] \\ & + \text{boundary terms}, \end{aligned} \tag{3.6}$$

where we collected all the boundary terms from all orders. Note that the boundary terms will be used in the calculation of quadratic curvature terms. (They are not important in this section since we compute the LO of GR.)

Hence, the (electric) Carrollian limit of the GR action is

$$S = c^2 \int [K^2 - K^{\mu\nu}K_{\mu\nu}]ed^4x, \tag{3.7}$$

where $K_{\mu\nu} = -\frac{1}{2}\pounds_v h_{\mu\nu}$ and $e = \det(\tau_\mu, e^a_\mu)$.

Varying this action we get the constraints

$$K^2 - K^{\mu\nu}K_{\mu\nu} = 0, \qquad h^{\nu\alpha}\nabla_\alpha[K_{\mu\nu} - Kh_{\mu\nu}] = 0, \tag{3.8}$$

and the evolution equation

$$\pounds_v K_{\mu\nu} = -2K^\alpha_\mu K_{\nu\alpha} + KK_{\mu\nu}. \tag{3.9}$$

IV. CARROLLIAN EXPANSION OF QUADRATIC GRAVITY

Quadratic gravity is a theory where quadratic curvature terms are added to the action, which makes it renormalizable [7,8]. It also emerges from string theory by imposing a cutoff for the maximum possible momenta [6]. The action for the theory is given by

$$S = c^3 \int [R - \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2] \sqrt{-g} d^4 x. \tag{4.1}$$

In Sec. III we computed the PUL parametrization of R. Now, we will do the same also for the two other terms in the action, $R^{\mu\nu}R_{\mu\nu}$ and R^2. Using (3.2), we can find the PUL parametrization of $R^{\mu\nu}R_{\mu\nu}$,

$$R^{\mu\nu}R_{\mu\nu} = \frac{1}{c^4} \overset{(-4)}{R^{\mu\nu}R}_{\mu\nu} + \frac{1}{c^2} \overset{(-2)}{R^{\mu\nu}R}_{\mu\nu} + \overset{(0)}{R^{\mu\nu}R}_{\mu\nu} + c^4 \overset{(4)}{R^{\mu\nu}R}_{\mu\nu}, \tag{4.2}$$

where

$$\begin{aligned}
\overset{(-4)}{R^{\mu\nu}R}_{\mu\nu} &= \Pi^{\nu\alpha}\Pi^{\lambda\beta}\nabla_\mu(V^\mu \mathcal{K}_{\alpha\beta})\nabla_\rho(V^\rho \mathcal{K}_{\nu\lambda}) - 2\mathcal{K}^{\alpha\beta}\mathcal{K}\nabla_\mu(V^\mu \mathcal{K}_{\alpha\beta}) + \mathcal{K}^{\lambda\nu}\mathcal{K}_{\lambda\nu}\mathcal{K}^2 - V^\mu V^\nu \nabla_\mu \mathcal{K}\nabla_\nu \mathcal{K} \\
&\quad + 2\mathcal{K}_{\alpha\beta}\mathcal{K}^{\alpha\beta}V^\nu\nabla_\nu\mathcal{K} - (\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu})^2, \\
\overset{(-2)}{R^{\mu\nu}R}_{\mu\nu} &= -2\overset{c}{R}{}^{\lambda\nu}\nabla_\mu(V^\mu\mathcal{K}_{\lambda\nu}) - \nabla_\mu(V^\mu\mathcal{K}_{\alpha\beta})\mathcal{K}^{\rho\beta}B^\alpha{}_\rho - \frac{1}{2}\mathcal{K}^{\rho\alpha}B^\beta{}_\rho V^\mu\nabla_\mu\mathcal{K}_{\alpha\beta} + 2\overset{c}{R}_{\lambda\nu}\mathcal{K}^{\lambda\nu}\mathcal{K} + \mathcal{K}^{\lambda\nu}\mathcal{K}\mathcal{K}^\alpha_\lambda B_{\nu\alpha} \\
&\quad - \Pi^{\nu\alpha}\nabla_\mu\mathcal{K}^\mu_\alpha\nabla_\rho\mathcal{K}^\rho_\nu + 2\Pi^{\nu\alpha}\nabla_\mu\mathcal{K}^\mu_\alpha\nabla_\nu\mathcal{K} + V^\lambda\nabla_\mu\mathcal{K}^\mu_\alpha\mathcal{K}^{\rho\alpha}B_{\lambda\rho} - \Pi^{\nu\alpha}\nabla_\nu\mathcal{K}\nabla_\alpha\mathcal{K} - V^\lambda\nabla_\alpha\mathcal{K}\mathcal{K}^{\alpha\epsilon}B_{\lambda\epsilon} \\
&\quad - 2V^\nu V^\alpha\nabla_\mu(B_\alpha{}^\mu)\nabla_\nu\mathcal{K} - V^\alpha\mathcal{K}^{\sigma\lambda}B_{\sigma\alpha}V^\rho\mathcal{K}^\beta_\lambda B_{\rho\beta} + 2V^\lambda\nabla_\mu(B_\lambda{}^\mu)\mathcal{K}^{\alpha\beta}\mathcal{K}_{\alpha\beta}, \\
\overset{(0)}{R^{\mu\nu}R}_{\mu\nu} &= \frac{1}{2}\mathcal{K}^{\alpha\beta}\mathcal{K}_{\alpha\beta}B^{\mu\nu}B_{\mu\nu} - \Pi^{\nu\alpha}\nabla_\alpha\mathcal{K}\nabla_\rho(B_\nu{}^\rho) - \frac{1}{4}V^\sigma\mathcal{K}^{\nu\rho}B_{\sigma\rho}\nabla_\mu(B_\nu{}^\mu) - \frac{1}{2}V^\lambda V^\alpha\nabla_\mu(B_\alpha{}^\mu)\nabla_\nu(B_\lambda{}^\nu) \\
&\quad + \frac{3}{2}\overset{c}{R}{}^{\alpha\lambda}\mathcal{K}^\beta_\lambda B_{\alpha\beta} + \overset{c}{R}{}^{\mu\nu}\overset{c}{R}_{\mu\nu} + \frac{1}{4}\mathcal{K}_{\beta\lambda}\mathcal{K}^{\rho\lambda}B^{\nu\beta}B_{\nu\rho} + \frac{1}{4}\mathcal{K}^{\beta\lambda}\mathcal{K}^{\alpha\rho}B_{\alpha\beta}B_{\lambda\rho}, \overset{(4)}{R^{\mu\nu}R}_{\mu\nu} = \frac{1}{16}(B^{\alpha\beta}B_{\alpha\beta})^2.
\end{aligned} \tag{4.3}$$

By expanding this expression to the LO, we arrive at

$$\begin{aligned}
R^{\mu\nu}R_{\mu\nu} = \frac{1}{c^4}[&h^{\nu\alpha}h^{\lambda\beta}\nabla_\mu(v^\mu K_{\alpha\beta})\nabla_\rho(v^\rho K_{\nu\lambda}) - 2K^{\alpha\beta}K\nabla_\mu(v^\mu K_{\alpha\beta}) + K^{\lambda\nu}K_{\lambda\nu}K^2 \\
&- v^\mu v^\nu\nabla_\mu K\nabla_\nu K + 2K_{\alpha\beta}K^{\alpha\beta}v^\nu\nabla_\nu K - (K^{\mu\nu}K_{\mu\nu})^2].
\end{aligned} \tag{4.4}$$

The PUL parametrization of R^2 can be computed from (3.4),

$$\begin{aligned}
R^2 = &\frac{1}{c^4}[\mathcal{K}^4 - 2\mathcal{K}^2\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} - 4\mathcal{K}^2\nabla_\nu(V^\nu\mathcal{K}) + (\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu})^2 + 4\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu}\nabla_\nu(V^\nu\mathcal{K}) + 4\nabla_\mu(V^\mu\mathcal{K})\nabla_\nu(V^\nu\mathcal{K})] \\
&+ \frac{1}{c^2}[-\mathcal{K}^2\overset{c}{R} - \mathcal{K}^2\nabla_\mu(V^\lambda B^\mu_\lambda) + \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\overset{c}{R} + \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\nabla_\rho(V^\lambda B^\rho_\lambda) + 2\overset{c}{R}\nabla_\mu(V^\mu\mathcal{K}) + 2\nabla_\mu(V^\mu\mathcal{K})\nabla_\nu(V^\lambda B^\nu_\lambda)] \\
&+ \left[\frac{1}{2}\mathcal{K}B^{\mu\nu}B_{\mu\nu} + (\overset{c}{R})^2 - \frac{1}{2}\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu}B^{\sigma\rho}B_{\sigma\rho} - B^{\mu\nu}B_{\mu\nu}\nabla_\rho(V^\rho\mathcal{K}) + \nabla_\mu(V^\lambda B^\mu_\lambda)\nabla_\rho(V^\sigma B^\rho_\sigma)\right. \\
&\left.+ 2\overset{c}{R}\nabla_\mu(V^\lambda B^\mu_\lambda)\right] + c^2\left[-\frac{1}{4}\overset{c}{R}B^{\mu\nu}B_{\mu\nu} - B^{\mu\nu}B_{\mu\nu}\nabla_\rho(V^\sigma B^\rho_\sigma)\right] + c^4\left[\frac{1}{16}(B_{\mu\nu}B^{\mu\nu})^2\right],
\end{aligned} \tag{4.5}$$

and its Carrollian expansion to the LO is

$$R^2 = \frac{1}{c^4}[K^4 - 2K^2K^{\mu\nu}K_{\mu\nu} - 4K^2\nabla_\nu(v^\nu K) + (K^{\mu\nu}K_{\mu\nu})^2 + 4K^{\mu\nu}K_{\mu\nu}\nabla_\nu(v^\nu K) + 4\nabla_\mu(v^\mu K)\nabla_\nu(v^\nu K)]. \tag{4.6}$$

Substituting (3.6), (4.4), and (4.6) into the action (4.1) we get

$$\begin{aligned}
S = \int \{&c^2[K^2 - K^{\mu\nu}K_{\mu\nu}] - \alpha[h^{\nu\alpha}h^{\lambda\beta}\nabla_\mu(v^\mu K_{\alpha\beta})\nabla_\rho(v^\rho K_{\nu\lambda}) - 2K^{\alpha\beta}K\nabla_\mu(v^\mu K_{\alpha\beta}) + K^{\lambda\nu}K_{\lambda\nu}K^2 \\
&- v^\mu v^\nu\nabla_\mu K\nabla_\nu K + 2K_{\alpha\beta}K^{\alpha\beta}v^\nu\nabla_\nu K - (K^{\mu\nu}K_{\mu\nu})^2] + \beta[K^4 - 2K^2K^{\mu\nu}K_{\mu\nu} - 4K^2\nabla_\nu(v^\nu K) \\
&+ (K^{\mu\nu}K_{\mu\nu})^2 + 4K^{\mu\nu}K_{\mu\nu}\nabla_\rho(v^\rho K) + 4\nabla_\mu(v^\mu K)\nabla_\nu(v^\nu K)]\}ed^4x.
\end{aligned} \tag{4.7}$$

Interestingly, this formula can be rewritten purely by the extrinsic curvature $K_{\mu\nu}$ and its Lie derivatives along $\boldsymbol{v}$. In order to do that, we first write

$$\pounds_{\boldsymbol{v}} K_{\mu\nu} = v^\sigma \nabla_\sigma K_{\mu\nu} + K_{\sigma\nu} \nabla_\mu v^\sigma + K_{\sigma\mu} \nabla_\nu v^\sigma - K_{\mu\nu} \nabla_\sigma v^\sigma + v^\sigma T^\rho_{\sigma\mu} K_{\rho\nu} + v^\sigma T^\rho_{\sigma\nu} K_{\rho\mu}, \tag{4.8}$$

where $T^\rho_{\mu\nu}$ is the torsion of the connection defined in Sec. II. Since the PUL-parametrization vector v^μ is covariantly constant by definition and $T^\rho_{\mu\nu}$ is given by (2.14), the relation reduces to

$$\pounds_{\boldsymbol{v}} K_{\mu\nu} = v^\sigma \nabla_\sigma K_{\mu\nu} - K^\sigma_{(\mu} K_{\nu)\sigma}. \tag{4.9}$$

Substituting in (4.7) and using the fact that $v^\sigma \nabla_\sigma$ acts on scalars simply as $\pounds_{\boldsymbol{v}}$, we get

$$\begin{aligned} S = \int \{ & c^2 [K^2 - K^{\mu\nu} K_{\mu\nu}] - \alpha [h^{\nu\alpha} h^{\lambda\beta} \pounds_{\boldsymbol{v}} K_{\alpha\beta} \pounds_{\boldsymbol{v}} K_{\nu\lambda} + 2 \pounds_{\boldsymbol{v}} K_{\nu\lambda} K^{\sigma(\nu} K^{\lambda)}_\sigma + K^\sigma_{(\alpha} K_{\beta)\sigma} K^{\rho(\alpha} K^{\beta)}_\rho \\ & - (K_{\mu\nu} K^{\mu\nu})^2 - 2 K^{\alpha\beta} K \pounds_{\boldsymbol{v}} K_{\alpha\beta} - 2 K^{\alpha\beta} K K^\sigma_{(\alpha} K_{\beta)\sigma} + K^2 K^{\mu\nu} K_{\mu\nu} - (\pounds_{\boldsymbol{v}} K)^2 + 2 K_{\mu\nu} K^{\mu\nu} \pounds_{\boldsymbol{v}} K] \\ & + \beta [K^4 - 2K^2 K_{\mu\nu} K^{\mu\nu} - 4K^2 \pounds_{\boldsymbol{v}} K + (K_{\mu\nu} K^{\mu\nu})^2 + 4 K_{\mu\nu} K^{\mu\nu} \pounds_{\boldsymbol{v}} K + (\pounds_{\boldsymbol{v}} K)^2] \} e d^4 x. \end{aligned} \tag{4.10}$$

Note that only the first two terms have the factor c^2. Thus, assuming α and β being independent of c, the Carrollian limit of the theory would exclude the first two terms coming from the Carrollian limit of the Ricci scalar. This means that the resulting theory would not couple to R, and it will be drastically different from the Carrollian limit of GR [cf. (3.7)]. Hence, α and β should depend on c. In this case, we get an infinite number of nonequivalent Carrollian theories, but only four of them modify GR to LO or NLO. Notice that this limit is, as expected, ultralocal since there are no space derivatives in the Lagrangian, and therefore, there would not be space derivatives in the field equations. This means that the evolution of a point cannot be affected by neighboring points no matter how close they are.

Similar calculations were done in [66] by rescaling specific terms in the action. However, our approach gives more freedom to rescale terms differently and gives more nonequivalent theories. Other papers considered specific solutions for $f(R)$ gravity [67–69]. A general classification of theories for the most general quadratic gravity theory will be provided in the next section.

V. THEORIES FROM THE CARROLLIAN LIMIT OF QUADRATIC GRAVITY

In this section, we study Carrollian theories resulting from the Carrollian limit of quadratic gravity. Different (nonequivalent) theories arise from assuming different dependencies of α and β on the speed of light c in (4.10).

TABLE I. This table summarizes some possible Carrollian theories arising from quadratic gravity that couple to R at most in the NLO. We list the theories with factors of c with non-negative powers since negative c dependencies are clearly not modifications of the Carrollian limit of GR. For example, although (0, 0) cannot be a modification to the Carrollian limit of GR, we can say that R terms are a NLO modification of this theory. There are other geometries which are modifications to the listed geometries such as (0, 4) which can be regarded as a next-to-next-to-leading-order modification of (0, 2) while GR itself is the NLO. We can extend the list indefinitely adding more geometries modifying GR to higher orders but here we focus on the LO and NLO.

Carrollian theories from quadratic gravity		
Theory	Action contributing to the LO	Type of modification to the Carrollian limit of GR
(0, 0)	$S = c^3 \int [-\alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2] \sqrt{-g} d^4 x$	*Not a modification of GR*
(0, 2)	$S = c^3 \int -\alpha R^{\mu\nu} R_{\mu\nu} \sqrt{-g} d^4 x$	*Not a modification of GR*
(2, 0)	$S = c^3 \int \beta R^2 \sqrt{-g} d^4 x$	*Not a modification of GR*
(2, 2)	$S = c^3 \int [R - \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2] \sqrt{-g} d^4 x$	*Modifies GR to the LO*
(2, 4)	$S = c^3 \int [R - \alpha R^{\mu\nu} R_{\mu\nu}] \sqrt{-g} d^4 x$	*Modifies GR to the LO with* $R^{\mu\nu} R_{\mu\nu}$ *terms and the NLO by* R^2 *terms*
(4, 2)	$S = c^3 \int [R + \beta R^2] \sqrt{-g} d^4 x$	*Modifies GR to the LO with* R^2 *terms and the NLO by* $R^{\mu\nu} R_{\mu\nu}$ *terms*
(4, 4)	$S = c^3 \int R \sqrt{-g} d^4 x$	*Modifies GR in the NLO*

Thus, we classify them as such and denote them by (n, m), where $\alpha = c^n\alpha'$ and $\beta = c^m\beta'$. The relevant theories are listed in Table I. As mentioned above, not all theories are modifications to GR. For example, the theories with negative powers of c in α or β but also (0,0), (0,2), and (2,0) are not physically interesting since they are drastically different from GR at LO. It is easy to see that dependencies with odd powers of c ultimately converge to one of the theories in Table I. Theories with higher-power dependencies on c cannot modify GR to the LO nor the NLO but to higher orders; however, since α and β dependencies on c are nonperturbative assumptions, having higher powers of c in the action without being an overall factor can lead to inconsistencies in the Galilean limit. Thus, in what follows, we focus only on the four interesting Carrollian theories (2,2), (2,4), (4,2), and (4,4).

A. (2, 2) Carrollian theory

Consider the case where α and β are quadratic in the speed of light, $\alpha = c^2\alpha'$, $\beta = c^2\beta'$, with α' and β' being constants independent of c. We will study the resulting action to the LO, i.e., the electric limit. From Table I, the action is

$$S = c^3 \int [R - \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2]\sqrt{-g} d^4x. \tag{5.1}$$

Writing $\alpha = c^2\alpha'$ and $\beta = c^2\beta'$, where α' and β' are c independent constants, we can write the action as

$$S = \int c^3[R - c^2\alpha' R^{\mu\nu} R_{\mu\nu} + c^2\beta' R^2]\sqrt{-g} d^4x, \tag{5.2}$$

which in the LO of the Carrollian expansion gives

$$\begin{aligned} S = c^2 \int \{ &[K^2 - K^{\mu\nu}K_{\mu\nu}] - \alpha'[h^{\nu\alpha}h^{\lambda\beta}\pounds_v K_{\alpha\beta}\pounds_v K_{\nu\lambda} + 2\pounds_v K_{\nu\lambda}K^{\sigma(\nu}K^{\lambda)}_{\sigma} + K^{\sigma}_{(\alpha}K_{\beta)\sigma}K^{\rho(\alpha}K^{\beta)}_{\rho} - 2K^{\alpha\beta}K\pounds_v K_{\alpha\beta} \\ &- 2K^{\alpha\beta}KK^{\sigma}_{(\alpha}K_{\beta)\sigma} + K^2K^{\mu\nu}K_{\mu\nu} - (\pounds_v K)^2 + 2K_{\mu\nu}K^{\mu\nu}\pounds_v K - (K_{\mu\nu}K^{\mu\nu})^2] + \beta'[K^4 - 2K^2K_{\mu\nu}K^{\mu\nu} \\ &- 4K^2\pounds_v K + (K_{\mu\nu}K^{\mu\nu})^2 + 4K_{\mu\nu}K^{\mu\nu}\pounds_v K + (\pounds_v K)^2]\} e d^4x. \end{aligned} \tag{5.3}$$

Since the Carrollian expansion and the weak-field regime are not conflicting, the conditions to find tachyons remain the same. In [7] it was found that the additional degrees of freedom have masses of[1]

$$m_0 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{-\alpha}}, \qquad m_2 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{\alpha - 3\beta}}. \tag{5.4}$$

The conditions to avoid tachyons are (at any order of the Carrollian expansion)

$$\alpha \le 0, \qquad \alpha - 3\beta \ge 0, \tag{5.5}$$

which translates to

$$\alpha' \le 0, \qquad \alpha' - 3\beta' \ge 0, \tag{5.6}$$

in the case of (2, 2) theory.

B. (2, 4) Carrollian theory

Let us now investigate the case where $\alpha = c^2\alpha'$ and $\beta = c^4\beta'$. The action is

$$S = c^3 \int [R - c^2\alpha' R^{\mu\nu} R_{\mu\nu} + c^4\beta' R^2]\sqrt{-g} d^4x. \tag{5.7}$$

To the LO in the Carrollian expansion, we get the action

$$\begin{aligned} S = c^2 \int \{ &[K^2 - K^{\mu\nu}K_{\mu\nu}] - \alpha'[h^{\nu\alpha}h^{\lambda\beta}\pounds_v K_{\alpha\beta}\pounds_v K_{\nu\lambda} + 2\pounds_v K_{\nu\lambda}K^{\sigma(\nu}K^{\lambda)}_{\sigma} + K^{\sigma}_{(\alpha}K_{\beta)\sigma}K^{\rho(\alpha}K^{\beta)}_{\rho} \\ &- 2K^{\alpha\beta}K\pounds_v K_{\alpha\beta} - 2K^{\alpha\beta}KK^{\sigma}_{(\alpha}K_{\beta)\sigma} + K^2K^{\mu\nu}K_{\mu\nu} - (\pounds_v K)^2 + 2K_{\mu\nu}K^{\mu\nu}\pounds_v K - (K_{\mu\nu}K^{\mu\nu})^2]\} e d^4x. \end{aligned} \tag{5.8}$$

Notice that this theory is the same as the Carrollian limit of $R - \alpha R_{\mu\nu}R^{\mu\nu}$. The conditions (5.5) to the LO reduce to $\alpha' = 0$. Thus, to the LO, the theory without tachyons is the same as the Carrollian limit of GR.

Assuming α' and β' to be of the same numerical order, the conditions to the LO and NLO, respectively, are

$$\alpha' = 0, \qquad \beta' \le 0. \tag{5.9}$$

Thus, the theory without tachyons to the NLO would be

$$S = c^3 \int [R_{\mathrm{NLO}} + c^4\beta'(R^2)_{\mathrm{LO}}]\sqrt{-g} d^4x, \tag{5.10}$$

where R_{NLO} is the Ricci scalar expanded to the NLO and $(R^2)_{\mathrm{LO}}$ is the LO of the Carrollian expansion of R^2.

[1] We remark that α and β in our convention have opposite signs to the convention used in [7].

C. (4, 2) Carrollian theory

Considering the dependencies are $\alpha = c^4\alpha'$ and $\beta = c^2\beta'$, the action is

$$S = c^3 \int [R - c^4\alpha' R^{\mu\nu}R_{\mu\nu} + c^2\beta' R^2]\sqrt{-g}d^4x. \quad (5.11)$$

The corresponding LO action reads

$$S = c^2 \int [(K^2 - K^{\mu\nu}K_{\mu\nu}) + \beta'[K^4 - 2K^2 K_{\mu\nu}K^{\mu\nu} - 4K^2 \pounds_v K + (K_{\mu\nu}K^{\mu\nu})^2 + 4K_{\mu\nu}K^{\mu\nu}\pounds_v K + (\pounds_v K)^2]]ed^4x. \quad (5.12)$$

In this case the conditions (5.5) then reduce to $\beta' \leq 0$. Expanding the conditions to the NLO we obtain

$$\beta' \leq 0, \qquad \alpha' = 0. \quad (5.13)$$

Hence, this theory is equivalent to the Carrollian limit of $R - \beta R^2$ theory to all orders with NLO action being the same as (5.10).

D. (4, 4) Carrollian theory

If we consider $\alpha = c^4\alpha'$ and $\beta = c^4\beta'$, then the action reads

$$S = c^3 \int [R - c^4\alpha' R^{\mu\nu}R_{\mu\nu} + c^4\beta' R^2]\sqrt{-g}d^4x. \quad (5.14)$$

For this theory, the LO action is the same as GR. At the NLO and higher orders it will receive corrections from both R^2 and $R_{\mu\nu}R^{\mu\nu}$ terms. The conditions (5.5) are the same as in the (2, 2) case.

VI. THE MAGNETIC LIMIT

In this section we study the magnetic limit of the theories (2, 4) and (4, 2) because these two theories are free from tachyons and ghosts if $\beta' \leq 0$. The magnetic limit is obtained by truncating the NLO action such that the resulting action is invariant under Carroll symmetries. In the case of quadratic gravity we have to truncate it the same way as GR; i.e., we have to put all the NLO fields to zero. It is well known that the NLO captures all the dynamics of the Carrollian limit [46]. Thus, the field equations from the magnetic limit leads to corrections to the dynamics in GR and even more solutions that are nonexistent in GR.

A. The magnetic limit of (2, 4)

Imposing the truncation

$$M^\mu = N_\mu = \Phi_{\mu\nu} = \Phi^{\mu\nu} = 0, \quad (6.1)$$

we get the LO and NLO of the terms of (5.10) to be

$$R_{\text{LO}} = K^2 - K_{\mu\nu}K^{\mu\nu}, \quad R_{\text{NLO}} = -\overset{c}{R},$$
$$(R^2)_{\text{LO}} = (K^2 - K_{\mu\nu}K^{\mu\nu})(K^2 - K_{\mu\nu}K^{\mu\nu} + 4\pounds_v K) - 4(\pounds_v K)^2. \quad (6.2)$$

As shown in the previous section, the LO of this theory is identical to the LO of GR; i.e., the constraints and the evolution equation are the same as (3.8) and (3.9). In the NLO, the LO constraints and evolution equations must hold, so they serve as constraints to the NLO field equations. Thus, taking the trace of (3.9) we get

$$h^{\mu\nu}\pounds_v K_{\mu\nu} = -2K_{\mu\nu}K^{\mu\nu} + K^2, \quad (6.3)$$

and then noting that

$$\pounds_v K = h^{\mu\nu}\pounds_v K_{\mu\nu} + 2K_{\mu\nu}K^{\mu\nu}, \quad (6.4)$$

we get

$$\pounds_v K = K^2. \quad (6.5)$$

Thus, the constraints on the magnetic action Lagrangian are

$$K^2 - K^{\mu\nu}K_{\mu\nu} = 0, \quad h^{\nu\alpha}\nabla_\alpha[K_{\mu\nu} - Kh_{\mu\nu}] = 0, \quad \pounds_v K = K^2. \quad (6.6)$$

Notice that this is not a general equation. It is valid only in (2, 4) and the theories where LO is identical to GR.

Using the above relations, we can write the action for the magnetic limit of (2, 4) as

$$S = -\int d^4xe[-\overset{c}{R} + \beta'[(K^2 - K_{\mu\nu}K^{\mu\nu})(K^2 - K_{\mu\nu}K^{\mu\nu} + 4\pounds_v K) - 4(\pounds_v K)^2] + \lambda_1(K^2 - K_{\mu\nu}K^{\mu\nu}) + \beta'\lambda_2(\pounds_v K - K^2)], \quad (6.7)$$

where λ_1 and λ_2 are Lagrange multipliers. As expected, the theory (2, 4) modifies the magnetic limit of GR with quartic terms in the extrinsic curvature and imposes an additional constraint. It would be interesting to see how these terms modify the dynamics of different solutions of the field equations, especially black holes. We expect that this theory has more solutions than the Carrollian limit of GR, namely those corresponding to the Schwarzchild-Bach black holes [12,13]. If this is the case, then one should examine if some terms can be considered as a flux that is analogous to the magnetic field in [61]. Here, however, the flux would come from the theory instead of being turned on by hand.

B. The magnetic limit of (4, 2)

This case is more complicated than (2, 4) since the LO is more involved than that of GR. We first study the constraints and the evolution equations for the LO, and then move on to the NLO. To the LO, the action is

$$S = \int d^4xe[(K^2 - K_{\mu\nu}K^{\mu\nu})(1 + \beta'[K^2 - K^{\mu\nu}K_{\mu\nu} + 4£_vK]) - \beta'(£_vK)^2]. \tag{6.8}$$

Varying with respect to v^μ and $h^{\mu\nu}$, we get the constraints

$$(K^2 - K_{\mu\nu}K^{\mu\nu})(1 + \beta'[K^2 - K^{\mu\nu}K_{\mu\nu} + 4£_vK]) - \beta'(£_vK)^2 = 0, \tag{6.9a}$$

$$h^{\mu\rho}\nabla_\mu(K_{\rho\nu} - Kh_{\rho\nu} + 2\beta'[K_{\rho\nu}(-3(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + 4£_vK) - Kh_{\rho\nu}(K^2 - K_{\alpha\beta}K^{\alpha\beta} + 2£_vK)]) = 0. \tag{6.9b}$$

Varying with respect to $h^{\mu\nu}$ and using the constraints, the evolution equation is

$$\begin{aligned}&2(KK_{\mu\nu} - K^\sigma_\mu K_{\nu\sigma})(1 + \beta'(2(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + 4£_vK)) + 2(2\beta'(K^2 - K_{\alpha\beta}K^{\alpha\beta}) - \beta'£_vK)(£_vK_{\mu\nu} - 4K^\sigma_\mu K_{\sigma\nu})\\&+ £_v[(Kh_{\mu\nu} - K_{\mu\nu})(1 + \beta'(2(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + 4£_vK))] - 8\beta'£_v[K_{\mu\nu}(2(K^2 - K_{\alpha\beta}K^{\alpha\beta}) - £_vK)]\\&+ 2\beta'£_v£_v[2(K^2 - K_{\alpha\beta}K^{\alpha\beta}) - £_vK] = 0.\end{aligned} \tag{6.10}$$

As expected, setting $\beta' = 0$, the equation reduces to the evolution equation of GR. The corrections to GR due to the R^2 term are quartic in the extrinsic curvature.

After truncation the NLO action reads

$$S = c^3 \int e[\overset{c}{R} + \beta'(-K^2 + K_{\mu\nu}K^{\mu\nu} + 2£_vK)(\overset{c}{R} + \nabla_\mu(v^\lambda b^\mu_\lambda))]d^4x. \tag{6.11}$$

However, the equations for the LO must also, so we have to add (6.9) to the Lagrangian as a constraint,

$$\begin{aligned}S = c^3 \int e[&\overset{c}{R} + \beta'(-K^2 + K_{\mu\nu}K^{\mu\nu} + 2£_vK)(\overset{c}{R} + \nabla_\mu(v^\lambda b^\mu_\lambda)) + \lambda((K^2 - K_{\mu\nu}K^{\mu\nu})(1 + \beta'[K^2 - K^{\mu\nu}K_{\mu\nu} + 4£_vK])\\&- \beta'(£_vK)^2)]d^4x,\end{aligned} \tag{6.12}$$

where λ is a Lagrange multiplier. Notice that the field equations for this action must include (6.10).

Now, we study a special case of the above equations where we treat $£_vK$ as an independent variable. Varying the action with respect to v^σ, we get the equations

$$(K^2 - K_{\mu\nu}K^{\mu\nu})(1 + \beta'[K^2 - K^{\mu\nu}K_{\mu\nu} + 4£_vK]) - \beta'(£_vK)^2 = 0,$$
$$h^{\rho\sigma}\nabla_\sigma(Kh_{\rho\mu} - K_{\rho\mu}) = 0. \tag{6.13}$$

Varying the action with respect to $£_vK$ and assuming $£_vK \neq 0$, we get

$$£_vK = 2(K^2 - K_{\mu\nu}K^{\mu\nu}). \tag{6.14}$$

From (6.13) and (6.14), we get the equations

$$£_vK = \frac{-2}{5\beta'}, \qquad K^2 - K_{\mu\nu}K^{\mu\nu} = \frac{-1}{5\beta'}. \tag{6.15}$$

Varying the action with respect to $h^{\mu\nu}$ and using (6.15) we get

$$£_vK_{\mu\nu} = -2K^\sigma_\mu K_{\sigma\nu} + Kh_{\mu\nu}. \tag{6.16}$$

Collecting the independent field equations we get the system

$$£_vK = \frac{-2}{5\beta'}, \qquad K^2 - K_{\mu\nu}K^{\mu\nu} = \frac{-1}{5\beta'},$$
$$-2K^\sigma_\mu K_{\sigma\nu} + Kh_{\mu\nu} = £_vK_{\mu\nu}, \qquad h^{\rho\sigma}\nabla_\sigma(Kh_{\rho\mu} - K_{\rho\mu}) = 0. \tag{6.17}$$

It turns out that this system solves (6.9) and (6.10). Thus, the solutions to the system (6.17) are also solutions to the full (4, 2) equations at LO. Notice that this system solves the full theory but the converse is not true. This means that a solution for (6.17) is a solution for the full theory but its set

TABLE II. After imposing the conditions to remove tachyons, the set of resulting theories consists either of the full Stelle gravity to various orders or variations of $R+R^2$ theories. It is worth mentioning that, as said before, theories with odd powers of c will be equivalent to one of the theories above, and higher powers of c may be problematic in the Galilean limit. Note that the LO actions possess Carrollian symmetries by construction so they are Carrollian theories, but the NLO actions do not. The NLO of the Carrollian expansion does not preserve Carrollian symmetry in general; however, certain truncation recovers the symmetries resulting in the magnetic Carrollian limit of the theory.

Carrollian theories from quadratic gravity after removing tachyons			
Theory	Action contributing to the LO	Action contributing to the NLO	Conditions
(2, 2)	$S=c^3\int[R_{\rm LO}-c^2\alpha'(R^{\mu\nu}R_{\mu\nu})_{\rm LO}+c^2\beta'(R^2)_{\rm LO}]\sqrt{-g}d^4x$	$S=c^3\int[R_{\rm NLO}-c^2\alpha'(R^{\mu\nu}R_{\mu\nu})_{\rm NLO}+c^2\beta'(R^2)_{\rm NLO}]\sqrt{-g}d^4x$	$\alpha'\leq 0,\ \alpha'-3\beta'\geq 0$
(2, 4)	$S=c^3\int[R_{\rm LO}]\sqrt{-g}d^4x$	$S=c^3\int[R_{\rm NLO}+c^4\beta'(R^2)_{\rm LO}]\sqrt{-g}d^4x$	$\alpha'=0,\ \beta'\leq 0$
(4, 2)	$S=c^3\int[R_{\rm LO}+c^2\beta'(R^2)_{\rm LO}]\sqrt{-g}d^4x$	$S=c^3\int[R_{\rm NLO}+c^4\beta'(R^2)_{\rm NLO}]\sqrt{-g}d^4x$	$\alpha'=0,\ \beta'\leq 0$
(4, 4)	$S=c^3\int[R_{\rm LO}]\sqrt{-g}d^4x$	$S=c^3\int[R_{\rm NLO}-c^4\alpha'(R^{\mu\nu}R_{\mu\nu})_{\rm NLO}+c^4\beta'(R^2)_{\rm NLO}]\sqrt{-g}d^4x$	$\alpha'\leq 0,\ \alpha'-3\beta'\geq 0$

of solutions is only a subset of that of the full theory. It is also worth mentioning that this system cannot reproduce GR without a cosmological constant; i.e., it is not valid for $\beta'=0$.

Notice that (6.17) is similar to Eq. (4.18) in [46], which describe GR with a cosmological constant, except for when the evolution equation is the same as GR without a cosmological constant. Modifications to the gravitational sector to reproduce a cosmological constant (without adding a cosmological constant term in the Lagrangian) were studied in $f(R)$ gravity [70]. Thus, we can interpret the effect of the R^2 term to be an effective cosmological constant with the value $-1/(10\beta')$. We will leave the solutions of this system of equations to future works. Now we use them as constraints to write the action for the magnetic limit.

For the special case where $£_v K$ is considered independent, the NLO action reads

$$S=c^3\int e\left[\overset{c}{R}+\beta'(-K^2+K_{\mu\nu}K^{\mu\nu}+2£_v K)(\overset{c}{R}+\nabla_\mu(v^\lambda b^\mu_\lambda))+\lambda_1\left(£_v K+\frac{2}{5\beta'}\right)+\lambda_2\left(K^2-K_{\mu\nu}K^{\mu\nu}+\frac{1}{5\beta'}\right)\right]d^4x, \quad (6.18)$$

where λ_1 and λ_2 are Lagrange multipliers and $b_{\mu\nu}=\partial_\mu\tau_\nu-\partial_\nu\tau_\mu$.

It is clear that the action contains a cosmological term. This is a direct result of the emergence of an effective cosmological constant in the LO equations. Like the magnetic limit action of (2, 4), this action modifies the magnetic limit of GR but with a nonzero cosmological constant. Applying this to the general magnetic limit action (6.12) we conclude that it includes a cosmological term in addition to terms that can be interpreted as flux. Notice that magnetic limits are no longer ultralocal due to the presence of spatial derivatives of the metric in the form of the Ricci scalar and terms containing the covariant derivative of $b_{\mu\nu}$. This allows some dynamics that were absent in the electric limit.

VII. CONCLUSIONS

In the present paper, we studied the electric and magnetic Carrollian limits of quadratic gravity. We calculated the PUL parametrization of terms with quadratic curvature in the action. After the Carrollian expansion, we saw that such terms are of the order of c^{-4} while the Ricci scalar term is only of the order of c^{-2}. From that, we concluded that the Carrollian limit of quadratic gravity requires α and β to depend on c in a particular way so that the resulting theory is a modification of GR. We classified different limits according to the dependencies of α and β on c. For example, the three of them (0, 0) (no dependence on c), (0, 2), and (2, 0) are not GR modifications because to the LO only the terms of order c^{-4} survive, i.e., only the quadratic terms in curvature but not the Ricci scalar. The only four theories that are modifications of GR (to the LO and NLO) are summarized in Table II together with the corresponding modifications.

Focusing on the ghost-free theories, namely (2, 4) and (4, 2), we see that (2, 4) is the same as GR to the LO, so the electric limit and the constraints to the magnetic limit are the same as those of GR. However, to the NLO the theory has extra terms which can be interpreted as an additional flux. In the case of (4, 2) the LO and the NLO are equivalent to that of $R+\beta' R^2$ theory. The constraints and the evolution equations are in general much more complicated. However, there is a special case where the LO equations reduce to GR with a cosmological constant, and this means that the full theory gives rise to an emergent cosmological constant in addition to the extra terms which, as in the (2, 4) case, can be interpreted as an additional flux.

More work has to be done to study the field equations for these theories to the LO and NLO. It would be interesting to compare each case with GR to understand what modifications can arise from different quartic terms of the extrinsic curvature. Another direction for future research is to calculate the Galilean limit of quadratic gravity. Since the dependence of α and β on c is not a perturbative assumption, the higher powers of c in the action may be problematic in the Galilean limit. In the current classification the most attractive options for future study are (2, 4) and (4, 2) since, after imposing the tachyon removing conditions, we get the Carrollian limit of $R + \beta R^2$, a renormalizable theory with no ghosts or tachyons (only if β is positive) which is deduced directly from the string theory. We plan to study black-hole solutions for these theories. Since $R + R^2$ theories have more black-hole solutions than GR, a direction for future work is to study black-hole solutions for their actions. These should coincide with the Carrollian limit of Schwarzchild-Bach solutions. It is interesting to analyze the dynamics of Carrollian particles on horizons of various black-hole solutions and compare the dynamics with that of [61] as well as study the modifications arising from the quartic terms.

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