Superradiant instability spectrum of the hydrodynamic vortex model

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We study analytically the superradiant instability properties of the hydrodynamic vortex model, an asymptotically flat acoustic geometry which, like the spinning Kerr black-hole spacetime, possesses an effective ergoregion. In particular, we derive a compact analytical formula for the complex resonant frequencies that characterize the long-wavelength dynamics of sound modes in this physically interesting acoustic spacetime.

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I. INTRODUCTION

The canonical Kerr black-hole solution of the vacuum Einstein field equations [1] is known to possess an *ergoregion* [2], a spacetime region which extends from the horizon $r_+ = M + \sqrt{M^2 - a^2}$ to $r_{\rm ergo} = M + \sqrt{M^2 - a^2\cos^2\theta}$, in which all physical observers are inevitably dragged by the rotation of the black hole [Here $\{M, a\}$ are respectively the mass and angular-momentum per unit mass of the Kerr black hole and θ is the polar angle of the stationary axisymmetric spacetime].

As pointed out by Zel'dovich [3] and by Press and Teukolsky [4,5], a corotating bosonic wave field that propagates in the black-hole ergosphere can be superradiantly amplified (that is, can extract rotational energy from the central spinning black hole) if its proper frequency is bounded from above by the relation $\omega < m\Omega_{\rm H}$, where m is the azimuthal harmonic index of the bosonic field mode and $\Omega_{\rm H}$ is the angular velocity of the black-hole horizon [2].

It is interesting to emphasize the fact that, although bosonic fields can be superradiantly amplified in the ergoregion, the asymptotically flat Kerr black-hole spacetime is known to be stable under perturbations of massless bosonic fields [5,6]. This important physical property of the composed Kerr-black-hole-massless-bosonic-field system is closely related to the well known absorption properties (ingoing boundary conditions) that characterize the classical black-hole horizon. In particular, the central black hole swallows (and also scatters away to infinity) the potentially dangerous amplified bosonic fields before they have the chance to develop exponentially growing instabilities inside the ergosphere.

It is worth mentioning that, in order to trigger superradiant instabilities in the spacetime of a spinning Kerr black hole, some additional confinement mechanism (which can be provided, for example, by a reflecting mirror which is placed around the central black hole [4,7] or, as in the case of amplified massive bosonic fields, by the mutual gravitational attraction between the central black hole and the fields [8]) is required in order to prevent the superradiantly amplified bosonic fields from radiating their energies to infinity [7,8].

Intriguingly, Friedman [9] has pointed out the physically important fact that, as opposed to the Kerr black-hole spacetime, spinning horizonless spacetimes (and, in general, rotating physical systems that have no absorptive boundaries) that possess ergoregions in which bosonic fields can be superradiantly amplified may generally be unstable to corotating bosonic perturbation fields.

In order to demonstrate this interesting physical phenomenon in the analogous setup of fluid dynamics [10,11], the physical properties of the *hydrodynamic vortex* model have recently been studied numerically in the physically important work [12]. This composed physical system describes a two-dimensional purely circulating flow of a vorticity free ideal fluid that can be described by a nontrivial effective spacetime metric [see Eq. (3) below]. In particular, this rotating acoustic spacetime has no absorptive horizons but, like the familiar spinning Kerr spacetime, it is characterized by the presence of an effective acoustic ergoregion whose radial boundary is defined by the circle at which the tangential velocity of the fluid equals the speed of propagating sound waves in the fluid [13] [see Eq. (4) below].

The highly interesting *numerical* results presented in [12] for the physical properties of the hydrodynamic vortex model have established the fact that, in accord with the prediction of [9], this horizonless physical system may develop exponentially growing instabilities due to the superradiant amplification phenomenon of linearized sound waves in the ergoregion of the effective spinning acoustic spacetime.

The main goal of the present paper is to explore, using *analytical* techniques, the superradiant instability spectrum that characterizes the physically interesting hydrodynamic vortex model. In particular, as we shall explicitly prove below, the complex resonant frequencies that characterize the dynamics of linearized sound waves in the spinning acoustic spacetime can be determined analytically in the dimensionless regime $C\omega \ll 1$ of small field frequencies, where C is the radius of the effective acoustic ergoregion.

II. DESCRIPTION OF THE SYSTEM

We shall analyze the dynamics of linearized sound waves in a vorticity free barotropic ideal fluid. The background (unperturbed) fluid velocity of a two-dimensional purely circulating flow is characterized by the relations [12]

$$v_r = v_z = 0; \qquad v_\phi = v_\phi(r), \tag{1}$$

where $\{r, \phi\}$ are the radial and azimuthal coordinates in the plane of flow, and z denotes the third spatial coordinate which is perpendicular to the plane (the xy plane) of flow. A locally irrotational (vorticity free) fluid flow is characterized by the simple functional relation [12]

$$v_\phi = \frac{C}{r} \tag{2}$$

for the tangential component of the velocity field, where the proportionality constant C characterizes the strength of circulation in the fluid flow. The angular momentum conservation law yields the relation [12] $\rho v_\phi r = \text{constant}$, which implies that the background density ρ of the fluid is a constant [see Eq. (2)]. The assumption of a barotropic fluid system then implies that the speed c of linearized sound waves in the fluid and the background pressure P of the fluid are also constants.

The circulating flow of the fluid in the hydrodynamic vortex model produces an acoustic spacetime whose effective two-dimensional geometry is described by the line-element [12–15]

$$ds^2 = -c^2\left(1 - \frac{C^2}{c^2 r^2}\right)dt^2 + dr^2 - 2C dt d\phi + r^2 d\phi^2 + dz^2. \tag{3}$$

Interestingly, this rotating acoustic geometry is characterized by the presence of an effective ergoregion whose radius [12–15]

$$r_{\text{ergo}} = \frac{|C|}{c} \tag{4}$$

is determined by the circle at which the tangential velocity of the fluid [see Eq. (2)] equals the speed c of propagating sound waves in the fluid. We shall henceforth use natural units in which $c = 1$ [16].

As shown in [11,12,17], the dynamics of linearized perturbation fields (sound modes) in the effective acoustic spacetime (3) is mathematically governed by the familiar Klein-Gordon wave equation [18]

$$\nabla^\nu \nabla_\nu \Psi = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu \Psi\right) = 0, \tag{5}$$

where g is the determinant of the effective line element (3). Substituting the field decomposition [19]

$$\Psi(t, r, \phi, z) = \frac{1}{\sqrt{r}}\sum_{m=-\infty}^{\infty} \psi_m(r;\omega)e^{im\phi}e^{-i\omega t} \tag{6}$$

into the Klein-Gordon wave equation (5) and using the line element (3) of the effective two-dimensional acoustic spacetime, one finds that the radial acoustic eigenfunctions $\psi_m(r;\omega)$ are determined by the ordinary differential equation [20,21]

$$\left[\frac{d^2}{dr^2} + \left(\omega - \frac{Cm}{r^2}\right)^2 - \frac{m^2 - \frac{1}{4}}{r^2}\right]\psi_m(r;\omega) = 0. \tag{7}$$

III. BOUNDARY CONDITIONS

Taking cognizance of the relation (2) for the tangential velocity field of the background fluid, one immediately realizes that the hydrodynamic description breaks down on the symmetry axis $r = 0$ of the spacetime. In order to describe a physically realistic system, it has been suggested in [12] to place an infinitely long supporting cylinder of finite proper radius R_0 at the center of the dynamical fluid system. In particular, as discussed in [12,17], the physically motivated boundary condition for the effective scalar eigenfunction Ψ at the surface of the central supporting cylinder is given by the functional relation [12,17]

$$\frac{d\Psi/dr}{\Psi}(r = R_0) = -\frac{i\rho\omega}{Z_\omega}, \tag{8}$$

where Z_ω is the frequency-dependent impedance of the cylinder [22], a physical parameter that quantifies the interaction of the propagating sound wave with the material of the scattering cylinder [12,17,22].

In addition, an asymptotically flat acoustic geometry is characterized by the physical boundary condition of purely outgoing waves at asymptotic infinity [see Eq. (6)]:

$$\Psi(r \to \infty) \sim \frac{e^{i\omega r}}{\sqrt{r}}. \tag{9}$$

That is, we consider purely outgoing waves at large distances from the central cylinder.

Interestingly, the Schrödinger-like ordinary differential equation (7), supplemented by the physically motivated boundary conditions (8) and (9), determine the complex resonant frequencies $\{\omega(C,R_0,m)\}$ which characterize the dynamics of linearized sound waves in the effective acoustic spacetime (3). It is worth emphasizing that resonant field frequencies with $\Im\omega > 0$ [see Eq. (6)] are associated with superradiantly *unstable* modes that grow exponentially in time. As we shall explicitly show in the next section, the complex resonant frequencies of the hydrodynamic vortex model can be studied analytically in the dimensionless regime $C\omega \ll 1$ of small field frequencies.

IV. THE RESONANCE EQUATION AND ITS REGIME OF VALIDITY

In the present section we shall analyze the Schrödinger-like differential equation (7) which determines the radial behavior of the acoustic eigenfunctions $\psi_m(r)$. As we shall explicitly prove below, the characteristic radial equation (7) can be solved *analytically* in the two asymptotic radial regions $r \ll m/\omega$ and $r \gg C$. We shall then show that, for small resonant frequencies in the regime

$$C\omega \ll 1, \tag{10}$$

one can use a functional matching procedure in the overlapping region $C \ll r \ll m/\omega$ in order to determine analytically the complex resonance spectrum $\{\omega(C,R_0,m;n)\}$ that characterizes the dynamics of linearized sound waves in the hydrodynamic vortex model.

We shall first solve the Schrödinger-like differential equation (7) in the radial region

$$r \ll m/\omega, \tag{11}$$

in which case one may approximate (7) by

$$\left[\frac{d^2}{dr^2} + \left(\frac{Cm}{r^2}\right)^2 - \frac{m^2-\frac{1}{4}}{r^2}\right]\psi_m = 0. \tag{12}$$

Here we have used the strong inequality $\omega^2 \ll m^2/r^2$ [see Eq. (11)]. In addition, we have used the strong inequality $Cm\omega/r^2 \ll m^2/r^2$ which stems from the small-frequency assumption $C\omega \ll 1 \le m$ [see Eq. (10)].

The general mathematical solution of (12) can be expressed in terms of the Bessel functions of the first and second kinds (see Eq. 9.1.53 of [23]):

$$\psi_m(r) = A_1 r^{\frac{1}{2}} \cdot J_m\left(\frac{Cm}{r}\right) + A_2 r^{\frac{1}{2}} \cdot Y_m\left(\frac{Cm}{r}\right), \tag{13}$$

where the normalization constants $\{A_1, A_2\}$ are determined by the physical boundary condition (8) of the wave field at the surface $r = R_0$ of the central supporting cylinder. In particular, substituting (13) into (8), one finds the dimensionless ratio [see Eq. (6)]

$$\frac{A_2}{A_1} = -\frac{J'_m(\frac{Cm}{R_0}) - \frac{i\rho\omega R_0^2}{Z_\omega Cm}\cdot J_m(\frac{Cm}{R_0})}{Y'_m(\frac{Cm}{R_0}) - \frac{i\rho\omega R_0^2}{Z_\omega Cm}\cdot Y_m(\frac{Cm}{R_0})}, \tag{14}$$

where a prime $'$ denotes a derivative of the Bessel function with respect to its argument Cm/r. Using Eq. (9.1.27c) of [23], one can express the dimensionless ratio (14) in the form

$$\frac{A_2}{A_1} = -\frac{\frac{R_0}{C}(1+\frac{i\rho\omega R_0}{Z_\omega m})\cdot J_m(\frac{Cm}{R_0}) - J_{m-1}(\frac{Cm}{R_0})}{\frac{R_0}{C}(1+\frac{i\rho\omega R_0}{Z_\omega m})\cdot Y_m(\frac{Cm}{R_0}) - Y_{m-1}(\frac{Cm}{R_0})}. \tag{15}$$

Using the small argument,

$$\frac{Cm}{r} \ll 1, \tag{16}$$

asymptotic behaviors of the Bessel functions (see Eqs. (9.1.7) and (9.1.9) of [23]), one finds from (13) the expression

$$\psi_m(r) = A_1(m!)^{-1}\left(\frac{Cm}{2}\right)^m r^{-m+\frac{1}{2}} - A_2\pi^{-1}(m-1)!\left(\frac{Cm}{2}\right)^{-m} r^{m+\frac{1}{2}} \tag{17}$$

for the radial acoustic eigenfunction that characterizes the linearized perturbation modes of the hydrodynamic vortex model in the intermediate radial region [see Eqs. (11) and (16)]

$$Cm \ll r \ll m/\omega. \tag{18}$$

We shall next solve the Schrödinger-like radial differential equation (7) in the region

$$r \gg C, \tag{19}$$

in which case one may approximate (7) by

$$\left(\frac{d^2}{dr^2} + \omega^2 - \frac{m^2-\frac{1}{4}}{r^2}\right)\psi_m = 0. \tag{20}$$

Here we have used the strong inequality $C^2m^2/r^4 \ll m^2/r^2$ [see Eq. (19)]. In addition, we have used the strong inequality $Cm\omega/r^2 \ll m^2/r^2$ which stems from the small-frequency assumption $C\omega \ll 1 \le m$ [see Eq. (10)].

The general mathematical solution of (20) can be expressed in terms of the Bessel functions of the first and second kinds (see Eq. (9.1.49) of [23]):

$$\psi_m(r) = B_1 r^{\frac{1}{2}} \cdot J_m(\omega r) + B_2 r^{\frac{1}{2}} \cdot Y_m(\omega r), \tag{21}$$

where $\{B_1, B_2\}$ are normalization constants which, as we shall explicitly show below, can be determined by a functional matching procedure. Using the small argument,

$$\omega r \ll 1, \tag{22}$$

asymptotic behaviors of the Bessel functions (see Eqs. (9.1.7) and (9.1.9) of [23]), one finds from (21) the expression

$$\psi_m(r) = B_1 (m!)^{-1} \left(\frac{\omega}{2}\right)^m r^{m+\frac{1}{2}} - B_2 \pi^{-1} (m-1)! \left(\frac{\omega}{2}\right)^{-m} r^{-m+\frac{1}{2}} \tag{23}$$

for the radial acoustic eigenfunction that characterizes the linearized perturbation modes of the hydrodynamic vortex model in the intermediate radial region [see Eqs. (19) and (22)]

$$C \ll r \ll 1/\omega. \tag{24}$$

Interestingly, from Eqs. (18) and (24) one learns that, for small resonant frequencies, there is an overlap radial region which is determined by the strong inequalities

$$Cm \ll r_o \ll 1/\omega, \tag{25}$$

in which the expressions (17) and (23) for the radial acoustic eigenfunction $\psi_m(r)$ of the hydrodynamic vortex model are both valid. Note, in particular, that the two expressions (17) and (23) for the eigenfunction $\psi_m(r)$ share the same radial functional behavior. One can therefore determine the coefficients $\{B_1, B_2\}$ of the characteristic radial eigenfunction (21) by matching the expressions (17) and (23) in their overlap radial region (25). This functional matching procedure yields the expressions

$$B_1 = -A_2 \pi^{-1} (m-1)! m! \left(\frac{Cm\omega}{4}\right)^{-m} \tag{26}$$

and

$$B_2 = A_1 A_2 B_1^{-1} \tag{27}$$

for the normalization constants of the radial acoustic eigenfunction (21).

We are now in a position to derive the characteristic resonance equation which determines the complex resonant frequencies of the hydrodynamic vortex model. Using Eqs. (9.2.1) and (9.2.2) of [23], one finds that the radial eigenfunction (21) is characterized by the large-r asymptotic behavior

$$\psi(r \to \infty) = B_1 \sqrt{2/\pi\omega} \cdot \cos(\omega r - m\pi/2 - \pi/4) + B_2 \sqrt{2/\pi\omega} \cdot \sin(\omega r - m\pi/2 - \pi/4). \tag{28}$$

Taking cognizance of the boundary condition (9), which characterizes the asymptotic spatial behavior of the radial eigenfunctions of the hydrodynamic vortex model, one deduces from (28) the simple relation

$$B_2 = iB_1. \tag{29}$$

Substituting (29) into (27), one finds the relation

$$iB_1^2 = A_1 A_2, \tag{30}$$

which yields the compact resonance equation [see Eq. (26)]

$$\left(\frac{Cm\omega}{4}\right)^{2m} = i\left[\frac{(m-1)!m!}{\pi}\right]^2 \cdot \frac{A_2}{A_1} \tag{31}$$

for the complex resonant frequencies that characterize the dynamics of linearized perturbation modes in the hydrodynamic vortex model. It is worth emphasizing again that the analytically derived resonance condition (31) is valid in the low frequency regime [see Eq. (25)]

$$Cm\omega \ll 1, \tag{32}$$

which corresponds to the small dimensionless ratio

$$\frac{A_2}{A_1} \ll 1. \tag{33}$$

Since each inequality in (25) roughly corresponds to an order-of-magnitude difference between two physical quantities [that is, $Cm/r_o \lesssim 10^{-1}$ and $r_o/(1/\omega) \lesssim 10^{-1}$], the analytically derived resonance condition (31) for the characteristic resonant frequencies of the hydrodynamic vortex model is expected to be valid in the dimensionless low frequency regime $Cm\omega \lesssim 10^{-2}$.

V. THE SUPERRADIANT INSTABILITY SPECTRUM OF THE HYDRODYNAMIC VORTEX MODEL

As emphasized above, in the present analytical study we focus on the low-frequency resonance spectrum which characterizes the dynamics of sound waves in the hydrodynamic vortex model. In particular, in the dimensionless regime

$$\omega R_0 \ll mZ_\omega/\rho \tag{34}$$

of small resonant frequencies, one can approximate the dimensionless ratio (15) by (here we have used Eq. (9.1.27c) of [23])

$$\frac{A_2}{A_1} = -\frac{J'_m(\frac{Cm}{R_0})}{Y'_m(\frac{Cm}{R_0})}. \tag{35}$$

It is worth noting that, in general, the frequency-dependent impedance diverges as an inverse power law of the frequency in the small frequency $\omega \to 0$ limit [22]. Thus, one finds that the ratio $\omega R_0/(mZ_\omega/\rho) \to 0$ approaches zero faster than ω^1 in the low frequency regime that we explore analytically here [22]. Note that the small-frequency relation (34) corresponds to the Neumann-type boundary condition $\frac{d\Psi}{dr}(r = R_0) = 0$ [12] for the linearized perturbation fields at the surface $r = R_0$ of the central cylinder [see Eq. (8)].

Substituting the dimensionless ratio (35) into the analytically derived resonance equation (31), one finds the simple resonance relation

$$C\omega(C,R_0,m) = \frac{4}{m}\left[\frac{(m-1)!m!}{\pi}\right]^{1/m}\cdot\left|\frac{J'_m(\frac{Cm}{R_0})}{Y'_m(\frac{Cm}{R_0})}\right|^{1/2m}\times e^{\pm i\pi/4m}, \tag{36}$$

which characterizes the dynamics of linearized wave fields in the effective acoustic spacetime (3). The $+/-$ signs in the analytically derived functional expression (36) refer respectively to negative/positive values of the dimensionless ratio $J'_m(Cm/R_0)/Y'_m(Cm/R_0)$. Here we have used the relation $\pm i = e^{i\pi(\pm\frac{1}{2}+2n)}$, where the integer n is the resonance parameter of the acoustic field mode.

Since the low frequency resonances of the hydrodynamic vortex model that we explore in the present paper correspond to the strong inequality (33), one deduces from (35) that our analytical study is valid for central supporting cylinders whose radii lie in the vicinity of the discrete critical radii [24]

$$R^*_0(C,m;k) = \frac{Cm}{j'_{m,k}}; \qquad k = 1,2,3,\ldots \tag{37}$$

for which $J'_m(Cm/R^*_0) = 0$ [and thus also $\omega[C, R^*_0(C,m;k), m] = 0$, see Eq. (36)], where $j'_{m,k}$ is the kth positive zero of the function $J'_m(x)$ [23,25]. As explicitly shown in [24], the discrete set (37) of critical cylinder radii support the marginally stable static resonances (with $\Re\omega = \Im\omega = 0$) of the hydrodynamic vortex model.

In particular, as shown numerically in [12] and analytically in [24], the hydrodynamic vortex model is characterized by the existence of a discrete set of critical cylinder radii, $\{R^*_0(C,m;k)\}_{k=1}^{k=\infty}$, that support spatially regular *static* ($\Re\omega = \Im\omega = 0$) acoustic field configurations. Interestingly, it has been shown [12,24] that, for given values of the fluid-field parameters $\{C, m\}$, these marginally stable field configurations mark the onset of the exponentially growing superradiant instabilities in the hydrodynamic vortex model.

VI. SUMMARY AND DISCUSSION

The superradiant instability properties of the hydrodynamic vortex model, an asymptotically flat horizonless acoustic geometry that possesses an ergoregion, were studied *analytically*. In particular, we have derived the compact analytical formula (36) for the parameter-dependent complex resonances of the composed fluid-cylinder system.

Our analytical matching procedure is valid in the dimensionless small-frequency regime (10) and it is therefore convenient to define the dimensionless physical quantity

$$\Delta R \equiv \frac{R_0 - R^*_0(C,m;k)}{R^*_0(C,m;k)} \quad \text{with} \quad \Delta R \ll 1, \tag{38}$$

in terms of which one can expand the small dimensionless ratio in Eq. (35) in the form

$$\frac{J'_m(\frac{Cm}{R_0})}{Y'_m(\frac{Cm}{R_0})} = -j'_{m,k}\cdot\frac{J''_m(j'_{m,k})}{Y'_m(j'_{m,k})}\cdot\Delta R\cdot[1+O(\Delta R)]. \tag{39}$$

Here we have used the Taylor expansions $J'_m(Cm/R_0) = J'_m(Cm/R^*_0) + J''_m(Cm/R^*_0)\cdot(-Cm/R^*_0)\cdot\Delta R + O[(\Delta R)^2]$ with $J'_m(Cm/R^*_0) \equiv 0$ and $Y'_m(Cm/R_0) = Y'_m(Cm/R^*_0)\cdot[1+O(\Delta R)]$. Using Eq. (9.1.31) of [23], one can write $\frac{J'_m(Cm/R_0)}{Y'_m(Cm/R_0)} = -j'_{m,k}\cdot\frac{J_{m-2}(j'_{m,k})-2J_m(j'_{m,k})+J_{m+2}(j'_{m,k})}{2[Y_{m-1}(j'_{m,k})-Y_{m+1}(j'_{m,k})]}\cdot\Delta R\cdot[1+O(\Delta R)]$ for the ratio (39), which is now expressed in terms of the Bessel functions themselves.

Substituting the ratio (39) into (36), one obtains the functional relation

$$C\omega(C,R_0,m) = \frac{4}{m}\left[\frac{(m-1)!m!}{\pi}\right]^{1/m}\cdot\left|j'_{m,k}\cdot\frac{J''_m(j'_{m,k})}{Y'_m(j'_{m,k})}\cdot\Delta R\right|^{1/2m}\times e^{\pm i\pi/4m} \tag{40}$$

for the low frequency resonances that characterize the dynamics of linearized sound modes in the effective acoustic spacetime, where the $+/-$ signs in (40) refer respectively to negative/positive values of the dimensionless quantity ΔR. It can be checked directly that the coefficient $-j'_{m,k}\frac{J''_m(j'_{m,k})}{Y'_m(j'_{m,k})}$ in (39) is a positive definite expression, which implies that the ratio $\frac{J'_m(Cm/R_0)}{Y'_m(Cm/R_0)}$ is positive/negative for positive/negative values of the dimensionless quantity ΔR [see Eq. (39)].

Intriguingly, from the analytically derived functional relation (40) one deduces the characteristic inequality

$$\Im\omega > 0 \quad \text{for } \Delta R < 0, \tag{41}$$

which describes exponentially growing supperradiant instability modes of the hydrodynamic vortex model [see Eq. (6)].

Finally, it is interesting to point out that the low frequency resonance expression (40) of the hydrodynamic vortex model can be further simplified in the following two asymptotic regimes:

(1) In the $m \gg 1$ limit of large harmonic indices, one finds

$$C\omega(m \gg 1, k) = \frac{4m}{e^2} \cdot |\Delta R|^{1/2m} \times e^{\pm i\pi/4m}. \tag{42}$$

Here we have used the relations $(m!)^{1/m} \to m/e$ and $|j'_{m,k} \cdot J''_m(j'_{m,k})/Y'_m(j'_{m,k})|^{1/2m} \to 1$ in the asymptotic $m \gg 1$ regime (see Eqs. (9.5.16) and (9.5.20) of [23]). It is important to note that, taking cognizance of (10), one finds that the asymptotic $m \gg 1$ formula (42) for the complex resonances of the hydrodynamic vortex model is valid in the $\Delta R \ll m^{-2m}$ regime.

(2) In the $k \gg m$ limit of small cylinder radii, one finds using Eq. (9.5.13) of [23]

$$j'_{m,k} = k\pi[1 + O(m/k)] \quad \text{for } k \gg m \tag{43}$$

[this yields $R^*_0(C, m; k \gg m) = \frac{Cm}{k\pi} \ll C$, see Eq. (37)], which yields [see Eq. (40)]

$$C\omega(m, k \gg m) = \frac{4}{m}[(m-1)!m!]^{1/m} \cdot \left|\frac{k}{\pi} \cdot \Delta R\right|^{1/2m} \times e^{\pm i\pi/4m}. \tag{44}$$

Here we have used the relation $J''_m(j'_{m,k})/Y'_m(j'_{m,k}) \to -1$ for $k \gg m$ (see Eqs. (9.2.1) and (9.2.2) of [23]). It is important to note that, taking cognizance of (10), one finds that the asymptotic $k \gg m$ formula (44) for the complex resonances of the hydrodynamic vortex model is valid in the $\Delta R \ll k^{-1}$ regime.

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[1] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).

[2] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).

[3] Ya. B. Zel'dovich, Pis'ma Zh. Eksp. Teor. Fiz. **14**, 270 (1971); [JETP Lett. **14**, 180 (1971)]; Zh. Eksp. Teor. Fiz. **62**, 2076 (1972); [Sov. Phys. JETP **35**, 1085 (1972)]; A. V. Vilenkin, Phys. Lett. B **78**, 301 (1978).

[4] W. H. Press and S. A. Teukolsky, Nature (London) **238**, 211 (1972).

[5] W. H. Press and S. A. Teukolsky, Astrophys. J. **185**, 649 (1973).

[6] B. F. Whiting, J. Math. Phys. (N.Y.) **30**, 1301 (1989).

[7] Carlos A. R. Herdeiro, J. C. Degollado, and H. F. Rúnarsson, Phys. Rev. D **88**, 063003 (2013); S. Hod, Phys. Rev. D **88**, 064055 (2013); **88**, 124007 (2013); **90**, 027501 (2014); Phys. Lett. B **736**, 398 (2014); Eur. Phys. J. C **74**, 3137 (2014); J. C. Degollado and C. A. R. Herdeiro, Phys. Rev. D **89**, 063005 (2014); S. R. Dolan, S. Ponglertsakul, and E. Winstanley, Phys. Rev. D **92**, 124047 (2015); R. Li and J. Zhao, Eur. Phys. J. C **74**, 3051 (2014); Phys. Lett. B **740**, 317 (2015); R. Li, J. Zhao, X. Wu, and Y. Zhang, Eur. Phys. J. C **75**, 142 (2015); R. Li, Y. Tian, H. Zhang, and J. Zhao, Phys. Lett. B **750**, 520 (2015); R. Li, J. K. Zhao, and Y. M Zhang, Commun. Theor. Phys. **63**, 569 (2015); N. S. Gual, J. C. Degollado, P. J. Montero, J. A. Font, and C. Herdeiro, Phys. Rev. Lett. **116**, 141101 (2016); S. Ponglertsakul, E. Winstanley, and S. R. Dolan, Phys. Rev. D **94**, 024031 (2016); S. Hod, Phys. Lett. B **755**, 177 (2016); Y. Huang, D. J. Liu, and X. Z. Li, arXiv:1606.00100.

[8] T. Damour, N. Deruelle, and R. Ruffini, Lett. Nuovo Cimento **15**, 257 (1976); T. M. Zouros and D. M. Eardley, Ann. Phys. (N.Y.) **118**, 139 (1979); S. Detweiler, Phys. Rev. D **22**, 2323 (1980); H. Furuhashi and Y. Nambu, Prog. Theor. Phys. **112**, 983 (2004); S. R. Dolan, Phys. Rev. D **76**, 084001 (2007); S. Hod and O. Hod, Phys. Rev. D **81**, 061502(R) (2010); H. R. Beyer, J. Math. Phys. (N.Y.) **52**, 102502 (2011); S. Hod, Phys. Rev. D **84**, 044046 (2011); Y. S. Myung, Phys. Rev. D **84**, 024048 (2011); S. Hod, Phys. Lett. B **708**, 320 (2012); **713**, 505 (2012); J. P. Lee, J. High Energy Phys. 01 (2012) 091; Mod. Phys. Lett. A **27**, 1250038 (2012); S. Hod, Phys. Rev. D **86**, 104026 (2012); Phys. Lett. B **718**, 1489 (2013); S. R. Dolan, Phys. Rev. D **87**, 124026 (2013); H. Witek, V. Cardoso, A. Ishibashi, and U. Sperhake, Phys. Rev. D **87**, 043513 (2013); J. C. Degollado and C. A. R. Herdeiro, Gen. Relativ. Gravit. **45**, 2483 (2013); R. Li, Eur. Phys. J. C **73**, 2274 (2013); S. J. Zhang, B. Wang, and E. Abdalla, arXiv:1306.0932; H. Witek, arXiv:1307.1145; Y. S. Myung, Phys. Rev. D **88**, 104017 (2013); R. Li, Phys. Rev. D **88**, 127901 (2013); S. Hod, Phys. Lett. B **739**, 196 (2014); B. Arderucio, arXiv:1404.3421; M. O. P. Sampaio,

C. Herdeiro, and M. Wang, Phys. Rev. D **90**, 064004 (2014); S. Hod, Phys. Rev. D **91**, 044047 (2015); H. M. Siahaan, Int. J. Mod. Phys. D **24**, 1550102 (2015); S. Hod, Phys. Lett. B **749**, 167 (2015); J. W. Gerow and A. Ritz, Phys. Rev. D **93**, 044043 (2016); S. Hod, Phys. Lett. B **758**, 181 (2016); Y. Huang and D. J. Liu, Phys. Rev. D **94**, 064030 (2016).

[9] J. L. Friedman, Commun. Math. Phys. **63**, 243 (1978).

[10] It is worth emphasizing that, as originally shown by Unruh [11], the wave equation that governs the dynamics of linearized sound waves in physical fluid systems [see Eq. (7) below] is mathematically analogous to the familiar Klein-Gordon wave equation that governs the dynamics of linearized massless scalar fields in curved spacetimes.

[11] W. G. Unruh, Phys. Rev. Lett. **46**, 1351 (1981).

[12] L. A. Oliveira, V. Cardoso, and L. C. B. Crispino, Phys. Rev. D **89**, 124008 (2014).

[13] T. R. Slatyer and C. M. Savage, Classical Quantum Gravity **22**, 3833 (2005).

[14] F. Federici, C. Cherubini, S. Succi, and M. P. Tosi, Phys. Rev. A **73**, 033604 (2006).

[15] S. R. Dolan, L. A. Oliveira, and L. C. B. Crispino, Phys. Rev. D **82**, 084037 (2010); S. R. Dolan, L. A. Oliveira, and L. C. B. Crispino, Phys. Rev. D **85**, 044031 (2012).

[16] Note that, in these units, the proportionality constant C [see Eq. (2)] has the dimensions of length.

[17] V. Cardoso, A. Coutant, M. Richartz, and S. Weinfurtner, Phys. Rev. Lett. **117**, 271101 (2016).

[18] As shown in [11,12,17], the gradient of the effective scalar field Ψ characterizes the linearized perturbation modes in the background flow of the fluid.

[19] Note that in (6) the effective scalar field Ψ is expanded in terms of linearized acoustic wave modes with cylindrical symmetry.

[20] Note that the azimuthal periodicity of the angular eigenfunction $e^{im\phi}$ implies that the azimuthal harmonic index $|m|$, which characterizes the linearized acoustic field mode, is an integer.

[21] It is worth pointing out that the ordinary differential equation (7), which determines the spatial behavior of the radial acoustic eigenfunctions, is invariant under the symmetry transformation $m \to -m$ with $C \to -C$. One can therefore assume, without loss of generality, that $m > 0$ and $C > 0$.

[22] M. Lax and H. Feshbach, J. Acoust. Soc. Am. **20**, 108 (1948); S. Aso and R. Kinoshita, J. Text. Mach. Soc. Jpn. **9**, 40 (1964); M. Delany and E. Bazley, Appl. Acoust. **3**, 105 (1970); H. S. Seddeq, Aust. J. Basic Appl. Sci. **3**, 4610 (2009); S. W. Rienstra and A. Hirschberg, http://www.win.tue.nl/~sjoerdr/papers/boek.pdf.

[23] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1970).

[24] S. Hod, Phys. Rev. D **90**, 027501 (2014).

[25] K. T. Tang, *Mathematical Methods for Engineers and Scientists3: Fourier Analysis, Partial Differential Equations and Variational Models* (Springer, New York, 2006).