Nonequatorial scalar clouds supported by maximally spinning Kerr black holes

Shahar Hod
The Ruppin Academic Center, Emeq Hefer 40250, Israel
and The Hadassah Academic College, Jerusalem 91010, Israel

(Received 9 October 2023; accepted 20 November 2023; published 13 December 2023)

The physical and mathematical properties of nonequatorial ($m \neq l$) scalar clouds that are supported by extremal Kerr black holes are studied analytically in the dimensionless large-mass $M\mu \gg 1$ regime (here M is the mass of the central supporting black hole and $\{\mu, l, m\}$ are respectively the proper mass, the spheroidal harmonic index, and the azimuthal harmonic index of the linearized scalar field). In particular, we determine the discrete resonant spectrum $m/l = m/l(M\mu)$ that characterizes the composed black hole nonequatorial-scalar-field cloudy configurations. Interestingly, we reveal the existence of a critical dimensionless angular ratio, $(m/l)_{\max} = (1 + \frac{1}{4\sqrt{5}})^{-1} < 1$, above which maximally spinning Kerr black holes cannot support spatially regular bound-state configurations of the stationary massive scalar fields.

DOI: 10.1103/PhysRevD.108.124028

I. INTRODUCTION

Classical black-hole spacetimes posses engulfing horizons that act as one-way absorbing membranes for external matter fields. As first discussed by Wheeler [1,2], the presence of these absorbing boundaries suggests that asymptotically flat black-hole spacetimes with spatially regular horizons cannot support external static matter configurations.

Wheeler's influential no-hair conjecture [1,2], which suggests that static bound-state matter fields cannot be supported by black holes, has attracted much attention from physicists and mathematicians. In particular, early investigations of the composed Einstein-scalar field equations have yielded mathematically elegant no-hair theorems [3–6] that, in accord with the spirit of the no-hair conjecture, ruled out the existence of static spatially regular bound-state scalar configurations around central black holes.

However, later investigations (see [7–18] and references therein) of the composed Einstein-matter field equations have revealed the intriguing fact that, in some nontrivial field theories, curved black-hole spacetimes with spatially regular horizons are not necessarily bald; they can support static bound-state external matter configurations.

In addition, using analytical techniques it has been explicitly proved [19,20] that the physically intriguing phenomenon of superradiant scattering of scalar (and, in general, bosonic) fields in spinning black-hole spacetimes [21,22] allows central spinning black holes with spatially regular horizons to support stationary (rather than static) linearized massive scalar fields whose proper frequencies are in resonance [19,20,23–26],

$$\omega_{\text{field}} = m\Omega_{\text{H}}, \tag{1}$$

with the angular velocity that characterizes the horizon of the central supporting black hole, where $m \geq 1$ is the azimuthal harmonic index of the stationary scalar field. The supported stationary bound-state scalar configurations, which are characterized by the compact black-hole-field resonance relation (1), have received the nickname scalar 'clouds' in the linearized regime [19,20,23,24].

Interestingly, using direct numerical computations it was later demonstrated explicitly in the highly important works [23,24] that the nonlinearly coupled Einstein-scalar field equations are characterized by the existence of genuine nonvacuum (hairy) black-hole solutions in which self-gravitating (nonlinear) matter configurations that respect the black-hole-field resonance condition (1) are supported by central-spinning black holes with spatially regular horizons.

The proper frequencies of the stationary (marginally stable) bound-state scalar configurations are bounded by the inequalities $\mu/\sqrt{2} < \omega_{\text{field}} < \mu$ [27,28] which, taking cognizance of the resonance relation (1), imply that composed extremal-Kerr-black-hole-stationary-massive-scalar-field cloudy configurations are characterized by the mathematically compact dimensionless inequalities [19,29,30]

$$\frac{m}{2} < M\mu < \frac{m}{\sqrt{2}}. \tag{2}$$

Intriguingly, using numerical computations it has recently been revealed [31] that maximally spinning (extremal) Kerr black holes *cannot* support spatially regular equatorial [32] scalar clouds with the property

$$l = m \geq 3, \quad (3)$$

where l is the spheroidal harmonic index of the supported scalar field.

In particular, it has been shown [19,31] that extremal Kerr black holes can support stationary massive scalar clouds with the marginally allowed property $M\mu \to m/2$ [see (2)] in the restricted dimensionless regime

$$\bar{m} \leq \sqrt{\frac{2(l+1)}{3l}} \underset{l\gg1}{\to} \sqrt{\frac{2}{3}} \quad \text{for} \quad M\mu \to \frac{m}{2}, \quad (4)$$

where

$$\bar{m} \equiv \frac{m}{l}. \quad (5)$$

The main goal of the present paper is to study, using analytical techniques, the physical and mathematical properties of nonequatorial (with $m < l$) scalar clouds that are supported in maximally spinning Kerr black-hole spacetimes. In particular, below we shall determine the maximally allowed value of the dimensionless angular ratio $\bar{m}$ which is consistent with the existence of spatially regular stationary bound-state scalar clouds around maximally-spinning Kerr black holes.

Interestingly, we shall explicitly prove below that, in the large-mass $M\mu \gg 1$ regime with $M\mu > m/2$, the critical (maximally allowed) value of the dimensionless angular ratio $\bar{m}$ is a monotonically increasing function of the dimensionless field mass $M\mu$ with an asymptotic value $\bar{m}_{\text{max}}(M\mu \to m/\sqrt{2})$ which is *larger* than $\sqrt{2/3}$ [see Eq. (4)]. The analytically derived asymptotic value $\bar{m}_{\text{max}}(M\mu \to m/\sqrt{2})$ is proved to be smaller than 1 [see Eq. (40)], which rules out the existence of composed extremal-Kerr-black-hole-stationary-massive-scalar-field cloudy configurations with the equatorial property $l = m \gg 1$.

II. DESCRIPTION OF THE SYSTEM

We analyze the physical and mathematical properties of a physical system which is composed of a spatially regular stationary massive scalar field which is linearly coupled to a maximally rotating (extremal) Kerr black hole. The curved spacetime of an extremal Kerr black hole of mass M is described, using the Boyer-Lindquist spacetime coordinates (t, r, θ, ϕ), by the line element [33,34]

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - M\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[Mdt - (r^2+M^2)d\phi\right]^2, \quad (6)$$

where the spatially-dependent metric functions in (6) are given by the functional expressions $\Delta \equiv (r-M)^2$ and $\rho^2 \equiv r^2 + M^2\cos^2\theta$.

Extremal Kerr black-hole spacetimes are characterized by the degenerate functional relations

$$r_- = r_+ = \frac{J}{M} = M, \quad (7)$$

where $\{M, J, r_\pm\}$ are respectively the mass, angular momentum, and horizon radii of the black hole. The angular velocity that characterizes the black-hole degenerate horizon is given by the compact dimensionless relation [33,34]

$$M\Omega_{\text{H}} = \frac{1}{2}. \quad (8)$$

The Klein-Gordon equation

$$(\nabla^\nu\nabla_\nu - \mu^2)\Psi = 0 \quad (9)$$

determines the dynamics of the linearized massive scalar field Ψ in the black-hole spacetime. Taking cognizance of the curved line element (6) and using the field decomposition [35]

$$\Psi(t,r,\theta,\phi) = \int\sum_{l,m} e^{im\phi}S_{lm}(\theta;\epsilon)R_{lm}(r;\epsilon,\omega)e^{-i\omega t}d\omega \quad (10)$$

with the dimensionless physical quantity

$$\epsilon \equiv M\sqrt{\mu^2 - \omega^2}, \quad (11)$$

one obtains the coupled angular and radial scalar equations [36–41]

$$\frac{1}{\sin\theta}\frac{d}{\theta}\left(\sin\theta\frac{dS_{lm}}{d\theta}\right) + \left(K_{lm} + \epsilon^2\sin^2\theta - \frac{m^2}{\sin^2\theta}\right)S_{lm} = 0 \quad (12)$$

and

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR_{lm}}{dr}\right) + \Big[\left[(r^2+M^2)\omega - mM\right]^2 + \Delta\left[2mM\omega - \mu^2(r^2+M^2) - K_{lm}\right]\Big]R_{lm} = 0. \quad (13)$$

The discrete set of angular eigenvalues $\{K_{lm}(\epsilon)\}$, which effectively act as separation constants in Eqs. (12) and (13), can be determined from the angular differential equation (12) with the physically motivated requirement of regularity of the corresponding angular eigenfunctions $S_{lm}(\theta;\epsilon)$ at the poles $\theta = 0$ and $\theta = \pi$ [40].

In the next section we shall use the large-mass (or, equivalently, the large-m) angular eigenvalues $\{K_{lm}(M\mu)\}$ [see Eq. (32)] [42] in order to analyze, using the differential equation (13), the spatial behavior of the radial eigenfunctions $\{R_{lm}(r;M\mu)\}$ that characterize the stationary bound-state scalar clouds.

III. RESONANT SPECTRUM OF THE COMPOSED EXTREMAL-KERR-BLACK-HOLE NONEQUATORIAL-MASSIVE-SCALAR-FIELD CLOUDY CONFIGURATIONS

In the present section we shall explore the physical and mathematical properties of the nonequatorial ($m < l$) scalar clouds (stationary bound-state linearized massive scalar field configurations) that are supported in the maximally-spinning Kerr black-hole spacetime (6). The proper frequencies of these spatially regular scalar configurations are characterized by the dimensionless functional relation (the black-hole-field resonance condition) [see Eqs. (1) and (8)]

$$M\omega = \frac{m}{2}. \tag{14}$$

Interestingly, as explicitly proved in [19], the radial equation (13) of the stationary massive scalar field in the maximally-spinning Kerr black-hole spacetime (6) can be solved analytically to yield the functional expression [the notation $(\beta \to -\beta)$ means "replace β by $-\beta$ in the preceding term"] [40,43]

$$R(x) = C_1 \times x^{-\frac{1}{2}+\beta} e^{-\epsilon x} M\left(\frac{1}{2}+\beta-\kappa, 1+2\beta, 2\epsilon x\right) + C_2 \times (\beta \to -\beta) \tag{15}$$

for the radial scalar eigenfunction, where $\{C_1, C_2\}$ are normalization constants, $M(a, b, z)$ is the confluent hypergeometric function [40],

$$\beta^2 \equiv K + \frac{1}{4} - 2m^2 + 2(M\mu)^2, \tag{16}$$

$$\kappa \equiv \frac{\alpha}{\epsilon} - \epsilon \quad \text{with} \quad \alpha \equiv (M\omega_c)^2 = \frac{m^2}{4}, \tag{17}$$

and

$$x \equiv \frac{r-M}{M} \tag{18}$$

is a dimensionless radial coordinate.

We consider supported field configurations that are spatially bounded (finite) at the black-hole horizon,

$$R(r = r_H) < \infty, \tag{19}$$

and decay asymptotically [19,20,23,24],

$$R(r \to \infty) \sim \frac{1}{r} e^{-\sqrt{\mu^2-\omega^2}\, r} \to 0, \tag{20}$$

at spatial infinity. The boundary condition (20) implies that the m-dependent proper frequency (14) of the stationary bound-state massive scalar field configurations should be characterized by the compact inequality

$$\omega^2 < \mu^2, \tag{21}$$

which implies the relation [see Eq. (11)] [44]

$$0 < \epsilon^2 \in \mathbb{R}. \tag{22}$$

As we shall now prove explicitly, the radial scalar function (15), supplemented by the physically motivated boundary conditions (19) and (20), determine the discrete resonance spectrum $\bar{m} = \bar{m}(M\mu; n)$ [45] that characterizes the composed extremal-Kerr-black-hole-nonequatorial-massive-scalar-field cloudy configurations.

From Eq. (15) one finds that, in the near-horizon ($x \ll 1$) region, the spatial functional behavior of the radial scalar eigenfunction is given by [19,40]

$$R(x \to 0) \to C_1 \times x^{-\frac{1}{2}+\beta} + C_2 \times x^{-\frac{1}{2}-\beta}, \tag{23}$$

which, together with Eq. (19), imply that well-behaved (spatially bounded) scalar configurations are characterized by the relations [19]

$$C_2 = 0 \tag{24}$$

and [46,47]

$$\Re\beta \geq \frac{1}{2}. \tag{25}$$

The asymptotic functional behavior of the scalar eigenfunction (15) at spatial infinity ($x \to \infty$) is given by [see Eq. (24)] [19,40]

$$R(x \to \infty) \to C_1 \times (2\epsilon)^{\kappa-\frac{1}{2}-\beta} \frac{\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\beta+\kappa)} \times x^{-1+\kappa}(-1)^{-\frac{1}{2}-\beta+\kappa} e^{-\epsilon x} + C_1 \times (2\epsilon)^{-\kappa-\frac{1}{2}-\beta} \times \frac{\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\beta-\kappa)} x^{-1-\kappa} e^{\epsilon x}. \tag{26}$$

Taking cognizance of the boundary condition (20), which characterizes the spatially regular bound-state massive scalar field configurations, one deduces that the coefficient $\Gamma(1+2\beta)/\Gamma(\frac{1}{2}+\beta-\kappa)$ of the blowing exponent in (26) must be zero. This observation yields the remarkably compact resonance equation [19,48]

$$\frac{1}{2}+\beta-\kappa=-n \quad \text{with} \quad n=0,1,2,\ldots \tag{27}$$

for the composed extremal-Kerr-black-hole-nonequatorial-stationary-massive-scalar-field cloudy configurations.

For later purposes we note that Eqs. (25) and (27) yield the characteristic inequality

$$\kappa \geq 1, \tag{28}$$

or equivalently [see Eq. (17)]

$$\epsilon \leq \frac{1}{2}\left(\sqrt{m^2+1}-1\right). \tag{29}$$

Intriguingly, the physically important numerical results presented in [31] indicate that, in the regime (3), the critical dimensionless angular ratio $\bar{m}_{\max}=\bar{m}_{\max}(M\mu)$ above which extremal Kerr black holes cannot support physically acceptable scalar clouds (stationary bound-state matter configurations with spatially regular scalar invariants) is a monotonically increasing function of the composed black-hole-scalar-field dimensionless mass parameter $M\mu$ (or, equivalently, of the spheroidal harmonic index l). Thus, in order to determine the critical (maximally allowed) value of the dimensionless angular ratio $\bar{m}$ which is consistent with the existence of spatially regular scalar clouds in maximally-spinning Kerr black-hole spacetimes, we shall analyze the properties of the composed extremal-Kerr-black-hole-stationary-bound-state-massive-scalar-field cloudy configurations in the eikonal large-mass regime

$$M\mu \gg 1 \quad \Longleftrightarrow \quad l \gg 1 \tag{30}$$

with the nonequatorial scalar property

$$\bar{m}<1. \tag{31}$$

Interestingly, and most importantly for our analysis, the angular scalar eigenvalues $K_{lm}(M\mu)$ can be determined analytically in the double asymptotic regime $\{m, M\mu\} \gg 1$ with $l-m \ll \sqrt{m^2+\epsilon^2}$ using a standard WKB analysis [49–51]. In particular, using a uniform asymptotic analysis of the angular differential equation (12) one finds the remarkably compact functional expression [42]

$$K_{lm}(\epsilon)=m^2-\epsilon^2+2l(1-\bar{m})\sqrt{m^2+\epsilon^2}+O(m). \tag{32}$$

From Eqs. (16) and (32) one obtains the eikonal large-mass expression

$$\beta=\sqrt{\epsilon^2-\frac{m^2}{2}+2m\sqrt{m^2+\epsilon^2}\cdot\left(\frac{1}{\bar{m}}-1\right)}\cdot[1+O(m^{-1})]. \tag{33}$$

Substituting Eqs. (17) and (33) into Eq. (27), one finds the relation

$$\frac{1}{2} \leq \sqrt{\epsilon^2-\frac{m^2}{2}+2m\sqrt{m^2+\epsilon^2}\cdot\left(\frac{1}{\bar{m}}-1\right)} = \frac{m^2}{4\epsilon}-\epsilon-n+O(1), \tag{34}$$

which yields the discrete resonance spectrum

$$\bar{m}(m,\epsilon;n)=\left[1+\frac{\frac{m^2}{2}-\epsilon^2+\left(\frac{m^2}{4\epsilon}-\epsilon-n\right)^2}{2m\sqrt{m^2+\epsilon^2}}\right]^{-1} \tag{35}$$

for the composed extremal-Kerr-black-hole-stationary-bound-state-massive-scalar-field cloudy configurations.

IV. THE CRITICAL VALUE OF THE DIMENSIONLESS ANGULAR RATIO $\bar{m}$

In the present section we shall determine the critical (maximally allowed) value of the dimensionless angular ratio $\bar{m}$ above which the extremal Kerr black holes cannot support spatially regular stationary scalar clouds. To this end, we first point out that, for given values of the dimensionless physical parameters $\{m,\epsilon\}$, the expression on the rhs of (35) is maximized for [see Eqs. (29) and (34)]

$$n^*=\frac{m^2}{4\epsilon}-\epsilon+O(1), \tag{36}$$

which in the eikonal large-mass (large-m) regime yields

$$\bar{m}(m,\epsilon;n=n^*)=\left[1+\frac{\frac{1}{2}-(\epsilon/m)^2}{2\sqrt{1+(\epsilon/m)^2}}\right]^{-1}. \tag{37}$$

In addition, we note that the expression (37) is a monotonically increasing function of the dimensionless ratio ϵ/m. Thus, it is maximized by the maximally allowed ratio [see Eq. (29) in the eikonal $m \gg 1$ regime] [52,53]

$$\left(\frac{\epsilon}{m}\right)_{\max}=\frac{1}{2}\cdot[1+O(m^{-1})], \tag{38}$$

which yields the critical (maximally allowed) angular ratio

$$\bar{m}_{\max}=\bar{m}(\epsilon/m=1/2;n=n^*)=\frac{4\sqrt{5}}{1+4\sqrt{5}} \tag{39}$$

in the dimensionless large mass $M\mu \gg 1$ regime.

V. SUMMARY

Recent analytical [19,20] and numerical [23,24] studies of the Einstein-scalar field equations have revealed the intriguing fact that asymptotically flat spinning black holes can support spatially regular bound-state (linearized as well as self-gravitating) matter configurations which are made of minimally coupled scalar fields that are characterized by the black-hole-field resonance relation (1).

In a very interesting paper [31] (see also [19]) a remarkable observation was made according to which maximally-spinning Kerr black holes *cannot* support spatially regular equatorial field configurations with the angular property $l = m \geq 3$, where $\{l, m\}$ are respectively the spheroidal and azimuthal harmonic indices of the field.

Motivated by this intriguing observation, in the present compact paper we have studied, using analytical techniques, the physical and mathematical properties of non-equatorial (with $m < l$) stationary bound-state scalar clouds that are supported by maximally spinning Kerr black-hole spacetimes. In particular, solving the Klein-Gordon differential equation (9) of the composed extremal-Kerr-black-hole-stationary-massive-scalar-field system in the dimensionless large-mass $M\mu \gg 1$ regime, we have determined the critical (maximally allowed) value [see Eqs. (5) and (39)] [54]

$$\left(\frac{m}{l}\right)_{\max} = \frac{4\sqrt{5}}{1+4\sqrt{5}} < 1 \quad \text{for} \quad M\mu \gg 1 \qquad (40)$$

of the dimensionless angular ratio.

It is worth stressing the fact that the physical significance of the analytically derived critical angular ratio (40) in the Einstein-scalar field theory stems from the fact that, in the dimensionless large mass $M\mu \gg 1$ regime, it determines the regime of existence of composed extremal-Kerr-black-hole-stationary-bound-state-massive-scalar-field cloudy configurations.

ACKNOWLEDGMENTS

This research is supported by the Carmel Science Foundation. I would like to thank Yael Oren, Arbel M. Ongo, Ayelet B. Lata, and Alona B. Tea for helpful discussions.

[1] R. Ruffini and J. A. Wheeler, Phys. Today **24**, No. 1, 30 (1971).
[2] B. Carter, in *Black Holes, Proceedings of 1972 Session of Ecole d'ete de Physique Theorique*, edited by C. De Witt and B. S. De Witt (Gordon and Breach, New York, 1973).
[3] J. E. Chase, Commun. Math. Phys. **19**, 276 (1970).
[4] J. D. Bekenstein, Phys. Rev. Lett. **28**, 452 (1972).
[5] C. Teitelboim, Lett. Nuovo Cimento **3**, 326 (1972).
[6] I. Pena and D. Sudarsky, Classical Quantum Gravity **14**, 3131 (1997).
[7] P. Bizoń, Phys. Rev. Lett. **64**, 2844 (1990); M. S. Volkov and D. V. Gal'tsov, Sov. J. Nucl. Phys. **51**, 1171 (1990); H. P. Kuenzle and A. K. M. Masood-ul-Alam, J. Math. Phys. (N.Y.) **31**, 928 (1990).
[8] G. Lavrelashvili and D. Maison, Nucl. Phys. **B410**, 407 (1993).
[9] P. Bizoń and T. Chamj, Phys. Lett. B **297**, 55 (1992); M. Heusler, S. Droz, and N. Straumann, Phys. Lett. B **268**, 371 (1991); **271**, 61 (1991); **258**, 21 (1992).
[10] B. R. Greene, S. D. Mathur, and C. O'Neill, Phys. Rev. D **47**, 2242 (1993); T. Torii, K. Maeda, and T. Tachizawa, Phys. Rev. D **51**, 1510 (1995).
[11] N. Straumann and Z. H. Zhou, Phys. Lett. B **243**, 33 (1990).
[12] P. Bizoń and R. M. Wald, Phys. Lett. B **267**, 173 (1991).
[13] N. E. Mavromatos and E. Winstanley, Phys. Rev. D **53**, 3190 (1996).
[14] M. S. Volkov and D. V. Galt'sov, Phys. Rep. **319**, 1 (1999).
[15] P. Bizoń and T. Chmaj, Phys. Rev. D **61**, 067501 (2000).
[16] G. V. Lavrelashvili and D. Maison, Phys. Lett. B **343**, 214 (1995).
[17] M. S. Volkov, O. Brodbeck, G. V. Lavrelashvili, and N. Straumann, Phys. Lett. B **349**, 438 (1995).
[18] P. Bizoń, Phys. Lett. B **259**, 53 (1991).
[19] S. Hod, Phys. Rev. D **86**, 104026 (2012).
[20] S. Hod, Eur. Phys. J. C **73**, 2378 (2013); Phys. Rev. D **90**, 024051 (2014); Phys. Lett. B **739**, 196 (2014); Classical Quantum Gravity **32**, 134002 (2015); Phys. Lett. B **751**, 177 (2015); **758**, 181 (2016); S. Hod and O. Hod, Phys. Rev. D **81**, 061502(R) (2010); S. Hod, J. High Energy Phys. 01 (2017) 030.
[21] Ya. B. Zel'dovich, Pis'ma Zh. Eksp. Teor. Fiz. **14**, 270 (1971) [JETP Lett. **14**, 180 (1971)]; Zh. Eksp. Teor. Fiz. **62**, 2076 (1972) [Sov. Phys. JETP **35**, 1085 (1972)]; A. V. Vilenkin, Phys. Lett. **78B**, 301 (1978).
[22] W. H. Press and S. A. Teukolsky, Nature (London) **238**, 211 (1972); Astrophys. J. **185**, 649 (1973).
[23] C. A. R. Herdeiro and E. Radu, Phys. Rev. Lett. **112**, 221101 (2014); C. L. Benone, L. C. B. Crispino, C. Herdeiro, and E. Radu, Phys. Rev. D **90**, 104024 (2014); C. A. R. Herdeiro and E. Radu, Phys. Rev. D **89**, 124018 (2014); Int. J. Mod. Phys. D **23**, 1442014 (2014).
[24] Y. Brihaye, C. Herdeiro, and E. Radu, Phys. Lett. B **739**, 1 (2014); J. C. Degollado and C. A. R. Herdeiro, Phys. Rev. D **90**, 065019 (2014); C. Herdeiro, E. Radu, and H. Rúnarsson, Phys. Lett. B **739**, 302 (2014); C. Herdeiro and E. Radu, Classical Quantum Gravity **32**, 144001 (2015);

C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D **24**, 1542014 (2015); **24**, 1544022 (2015); P. V. P. Cunha, C. A. R. Herdeiro, E. Radu, and H. F. Rúnarsson, Phys. Rev. Lett. **115**, 211102 (2015); B. Kleihaus, J. Kunz, and S. Yazadjiev, Phys. Lett. B **744**, 406 (2015); C. A. R. Herdeiro, E. Radu, and H. F. Rúnarsson, Phys. Rev. D **92**, 084059 (2015); C. Herdeiro, J. Kunz, E. Radu, and B. Subagyo, Phys. Lett. B **748**, 30 (2015); C. A. R. Herdeiro, E. Radu, and H. F. Rúnarsson, Classical Quantum Gravity **33**, 154001 (2016); Int. J. Mod. Phys. D **25**, 1641014 (2016); Y. Brihaye, C. Herdeiro, and E. Radu, Phys. Lett. B **760**, 279 (2016); Y. Ni, M. Zhou, A. C. Avendano, C. Bambi, C. A. R. Herdeiro, and E. Radu, J. Cosmol. Astropart. Phys. 07 (2016) 049; M. Wang, arXiv:1606.00811.

[25] G. García and M. Salgado, Phys. Rev. D **99**, 044036 (2019); **101**, 044040 (2020); **104**, 064054 (2021).

[26] We use natural units in which $G = c = \hbar = 1$.

[27] S. Hod, Phys. Lett. B **708**, 320 (2012).

[28] Note that the mass parameter μ of the scalar field stands for $\mu/\hbar$ and it therefore has the dimensions of $(\text{length})^{-1}$.

[29] Here M is the mass of the central supporting Kerr black hole.

[30] Note that maximally-spinning (extremal) Kerr black holes are characterized by the dimensionless relation $M\omega = m/2$ [see Eq. (8) below].

[31] G. García and M. Salgado, Phys. Rev. D **108**, 104012 (2023).

[32] We use here the term "equatorial clouds" to describe stationary supported scalar configurations which are characterized by the maximally allowed angular ratio $m/l = 1$.

[33] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).

[34] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).

[35] The angular eigenfunctions $S_{lm}(\theta;\epsilon)$, which are solutions of the scalar angular equation (12), are known as the spheroidal harmonic functions. In the simple $\epsilon \to 0$ limit they become the familiar spherical harmonic functions.

[36] S. A. Teukolsky, Phys. Rev. Lett. **29**, 1114 (1972); Astrophys. J. **185**, 635 (1973).

[37] T. Hartman, W. Song, and A. Strominger, J. High Energy Phys. 03 (2010) 118.

[38] A. Ronveaux, *Heun's Differential Equations* (Oxford University Press, Oxford, UK, 1995); C. Flammer, *Spheroidal Wave Functions* (Stanford University Press, Stanford, 1957).

[39] P. P. Fiziev, arXiv:0902.1277; R. S. Borissov and P. P. Fiziev, arXiv:0903.3617; P. P. Fiziev, Phys. Rev. D **80**, 124001 (2009); Classical Quantum Gravity **27**, 135001 (2010).

[40] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1970).

[41] S. Hod, Phys. Rev. Lett. **100**, 121101 (2008); Phys. Lett. B **717**, 462 (2012); Phys. Rev. D **87**, 064017 (2013);

[42] S. Hod, Phys. Lett. B **746**, 365 (2015).

[43] For brevity, we shall henceforth omit the discrete harmonic indices $\{l, m\}$ that characterize the angular behavior of the supported stationary scalar fields.

[44] The relation $\epsilon > 0$ can be assumed without loss of generality.

[45] Here $n = 0, 1, 2, \ldots$ is the resonance parameter of the composed maximally-spinning-black-hole-stationary-bound-state-scalar-field cloudy configurations [see Eq. (27) below].

[46] It is important to note that it was proved in the physically interesting paper [31] that stationary matter configurations that respect the boundary condition (19) at the black-hole horizon are characterized by spatially regular scalar invariants. We shall therefore consider linearized bound-state scalar clouds that are characterized by the physically motivated relation $\beta \geq 1/2$ [see Eqs. (23) and (24)].

[47] We shall assume $\Re\beta > 0$ without loss of generality.

[48] Here we have used Equation (6.1.7) of [40].

[49] L. D. Landau and E. M. Liftshitz, *Quantum Mechanics*, 3rd ed. (Pergamon, New York, 1977), Chap. VII.

[50] J. Heading, *An Introduction to Phase Integral Methods* (Wiley, New York, 1962).

[51] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978), Chap. 10.

[52] Note that the maximally allowed ratio $(\epsilon/m)_{\max} = 1/2 \cdot [1 + O(m^{-1})]$ corresponds to the maximally allowed dimensionless field mass $(M\mu)_{\max} \to m/\sqrt{2}$ [see Eqs. (2), (11), and (14)].

[53] Note that the relation (38) implies $n^* = 0$ in Eq. (36).

[54] Note that $m/l \in \mathbb{Q}$.