Thermodynamics of charged black hole with scalar hair

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It is shown that energy, entropy and the first law of electrically charged black holes with scalar hair can be consistently described in a general Hamiltonian approach to black hole thermodynamics. In particular, we prove that the associated "hairy" Smarr formula is modified with respect to its standard form.

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I. INTRODUCTION

The concept of black hole entropy introduced in the 1970s [1] had a long-term influence on our understanding of the gravitational dynamics. During several decades, this influence has been limited to Riemannian black holes, obtained as solutions of Einstein's general relativity (GR) or higher derivative gravity [2,3]. However, in the early 1960s, there appeared a modern, gauge-field-theoretic theory of gravity, known as the Poincaré gauge theory (PG) [4–8]. In PG, spacetime is characterized by a Riemann-Cartan geometry in which both the torsion and the curvature influence the gravitational dynamics. As a consequence, PG offers new possibilities for exploring the interplay between dynamics, geometry and black hole entropy.

Through the years, many well-known black hole solutions of GR have been successfully generalized to the PG solutions with torsion [6], but a consistent analysis of their thermodynamic properties has long been missing. However, since recently, there exists a rather general Hamiltonian approach to black hole entropy [9] which extends the concept of entropy from Riemannian to Riemann-Cartan spacetimes. The approach offers an efficient description of entropy and the first law of black hole thermodynamics not only in PG, but also in its Riemannian (vanishing torsion) or teleparallel (vanishing curvature) subsectors [10,11].

The physics of black holes in the 1960s supported the idea that "a black hole has no hair," see Ruffini and Wheeler [12]. According to this no-hair conjecture, black holes cannot have any other charge except for mass, angular momentum, and electric/magnetic charge [13]. Since the 1990s, attempts to clarify the range of validity of this conjecture led to discovering a plethora of new, "hairy" black holes as counterexamples to the no-hair conjecture. Convincing results of our Hamiltonian approach in interpreting entropy as the canonical charge on horizon motivated us to examine its further extension to *hairy black holes*.

Martinez *et al.* [14] reported an exact four-dimensional black hole solution of GR with scalar hair, which is asymptotically locally AdS. Relying on our Hamiltonian approach, we found a simple and consistent description of its energy and entropy [15]. In the present paper, we focus our attention on the subsequent work of Martinez and Troncoso (MT) [16] representing a natural generalization of [14], with Maxwell field as an additional part of the matter sector.

The paper is organized as follows. In Sec. II, we present a PG-inspired tetrad formulation of the MT black hole as a Riemannian solution of GR. In Sec. III, we introduce general boundary terms at infinity and horizon, use them to calculate energy and entropy as the corresponding canonical charges, and verify the first law. In Sec. IV, we analyze boundary terms for the MT black hole considered as a solution of teleparallel gravity. Section V is devoted to concluding remarks, and the Appendix contains some technical details.

Our conventions are the same as in Ref. [15]. Latin indices $(i, j, \ldots)$ are the local Lorentz indices, greek indices $(\mu, \nu, \ldots)$ are the coordinate indices, and both run over 0, 1, 2, 3; the orthonormal coframe (tetrad) is $\vartheta^i = \vartheta^i{}_\mu dx^\mu$ (1-form), $\vartheta := \det(\vartheta^i{}_\mu)$, the dual basis (frame) is $e_i = e_j{}^\mu \partial_\mu$, and $\omega^{ij} = \omega^{ij}{}_\mu dx^\mu$ is the metric compatible connection (1-form); the metric components in the local Lorentz basis are $\eta_{ij} = (1, -1, -1, -1)$, and the totally antisymmetric symbol ε_{ijmn} is normalized by $\varepsilon_{0123} = 1$; the Hodge dual of a form α is denoted by ${}^\star\alpha$, and the wedge product of forms is implicitly understood.

II. THE MT BLACK HOLE IN THE TETRAD FORMALISM

A. Dynamics

In our study of the MT black hole entropy, we use the general PG formalism [5–8], where the tetrad field ϑ^i and

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the metric compatible spin connection ω^{ij} are *a priori independent* dynamical variables, interpreted as the gauge potentials associated to the local Poincaré symmetry. The corresponding field strengths, the torsion $T^i := d\vartheta^i + \omega^i{}_k\vartheta^k$ and the curvature $R^{ij} = d\omega^{ij} + \omega^i{}_k\omega^{kj}$, characterize the Riemann-Cartan geometry of spacetime.

Consider a system of the gravitational field coupled to matter consisting of the scalar and electromagnetic field, described by the Lagrangian

$$L = L_G + L_\phi + L_{\rm em}, \tag{2.1}$$

where

$$L_G := -a_0{}^\star R \equiv -a_0{}^\star(\vartheta_i\vartheta_j)R^{ij}, \quad L_\phi := \frac{1}{2}d\phi^\star d\phi + {}^\star V(\phi),$$
$$L_{\rm em} := -\frac{1}{16\pi}F^\star F. \tag{2.2a}$$

Here, $a_0 = 1/16\pi G$, the gravitational part is *linear in curvature*,[1] the potential $V(\phi)$ describes a nonlinear self-interaction of the scalar field,

$$V(\phi) = \frac{k}{2\ell^2}\cosh^4\left(\frac{\phi}{\sqrt{k}}\right), \tag{2.2b}$$

where $k = 12a_0$ and $F = dA$ is the electromagnetic field strength.

Since the gravitational Lagrangian is linear in curvature and matter fields do not depend on the spin connection, the variation of the complete Lagrangian (2.1) with respect to ω^{ij} yields the condition of vanishing torsion, whereupon ω^{ij} becomes a Riemannian connection [11,15]. Thus, we end up with a Riemannian spacetime, the subcase of PG with $T^i = 0$, where the tetrad field ϑ^i is the only independent gravitational variable. A complementary description of the MT black hole, based on the teleparallel geometry with $R^{ij} = 0$, is given in Sec. IV.

After introducing the matter covariant momenta

$$H_\phi := \frac{\partial L_\phi}{\partial d\phi} = {}^\star d\phi, \quad H_{\rm em} := \frac{L_{\rm em}}{\partial F} = -\frac{1}{4\pi}{}^\star F, \tag{2.3}$$

the variation of the Lagrangian (2.1) with respect to ϕ, A and ϑ^i yields the field equations

$$\mathcal{E}_\phi := -dH_\phi + \partial_\phi{}^\star V = 0, \tag{2.4a}$$

$$\mathcal{E}_{\rm em} := -dH_{\rm em} = 0, \tag{2.4b}$$

$$\mathcal{E}_i := E_i + \tau_i + \mathcal{T}_i = 0, \tag{2.4c}$$

[1]In the framework of PG, the Lagrangian L_G defines Einstein–Cartan theory of gravity. For matter consisting of scalar and Yang-Mills fields, the resulting theory coincides with GR [6].

where

$$E_i := \frac{\partial L_G}{\partial\vartheta^i}, \qquad \tau_i =: \frac{\partial L_\phi}{\partial\vartheta^i}, \qquad \mathcal{T}_i := \frac{\partial L_{\rm em}}{\partial\vartheta^i}, \tag{2.5}$$

are the gravitational and matter energy-momentum currents (3-forms), respectively. It is not difficult to verify the validity of the matter field equations (2.4a) and (2.4b). In the Appendix, we prove the validity of the gravitational field equation (2.4c) for $k = 12a_0$, first by showing that it coincides with Einstein's equation, and then by performing an explicit calculation.

B. Geometry

The metric of the MT spacetime is static and spherically symmetric,

$$ds^2 = f^2dt^2 - \frac{dr^2}{g^2} - r^2(d\rho^2 + \sinh^2\rho\, d\varphi^2), \tag{2.6a}$$

where

$$f^2 = N^2C^{-1}, \quad g^2 = N^2C^2, \quad N^2 := \frac{r^2}{\ell^2} - 1 + G\frac{q^2}{\ell^2},$$
$$C := 1 + G\frac{q^2}{r^2}. \tag{2.6b}$$

The only nontrivial parameter of the solution is q; it is proportional to the electric charge (Sec. III C). The horizon radius is defined as the positive root of $N^2 = 0$,

$$r_+ = \sqrt{\ell^2 - Gq^2}. \tag{2.7}$$

The tetrad field associated to the MT metric (2.6) is chosen in the diagonal form

$$\vartheta^0 := f dt, \qquad \vartheta^1 := \frac{dr}{g}, \qquad \vartheta^2 = r d\rho,$$
$$\vartheta^3 := r \sinh\rho d\varphi. \tag{2.8}$$

The horizon area is given by

$$\mathcal{A}_H = \int_{S_H}\vartheta^2\vartheta^3 = r_+^2\sigma, \tag{2.9}$$

where σ is the horizon area normalized to $r_+ = 1$. The black hole temperature, which is determined by surface gravity, does not depend on the electric charge parameter q,

$$\kappa := g\partial_r f|_{r_+} = \frac{1}{\ell} \Rightarrow T := \frac{1}{2\pi\ell}, \tag{2.10}$$

see also [17]. The Riemannian spin connection reads (with $c = 2, 3$)

$$\omega^{01} = -\frac{g}{f} f' \vartheta^0, \qquad \omega^{1c} = \frac{g}{r} \vartheta^c, \qquad \omega^{23} = \frac{\cosh\theta}{r \sinh\theta} \vartheta^3. \tag{2.11}$$

One can show that the tetrad (2.8) combined with the matter fields

$$\phi := \sqrt{k}\,\mathrm{atanh}\left(\sqrt{\frac{Gq^2}{r^2 + Gq^2}}\right), \qquad A := -\frac{q}{\sqrt{r^2 + Gq^2}} \frac{\vartheta^0}{f}, \tag{2.12}$$

solves the field equation (2.4) provided

$$k = 12a_0 = \frac{3}{4\pi G}. \tag{2.13}$$

For technical details, see the Appendix.

The nonvanishing value of V at $q = 0$ plays the role of an effective cosmological constant, $V(0) = 6a_0/\ell^2 =: -2\Lambda_{\text{eff}}$, associated to an *asymptotically AdS background*. Indeed, for $q = 0$, the metric functions take the form $f^2 = g^2 = r^2/\ell^2 - 1$. As a consequence, the curvature scalar R is not maximally symmetric but contains additional $O(r^{-1})$ terms, which prevents the MT black hole to be a solution of the quadratic curvature gravity; for details, see Ref. [15].

III. BOUNDARY TERMS

A systematic Hamiltonian approach to conserved charges was originally proposed by Regge and Teitelboim [18]. A covariant version of that approach, introduced later by Nester and collaborators [19], turned out to be an important step towards a deeper understanding of the nature of conserved charges. Relying on these developments, we proposed an extension of these ideas to the thermodynamics of black holes [9]. Technically, the canonical gauge generator G is regularized by adding a suitable boundary term Γ, such that the improved generator $\tilde{G} := G + \Gamma$ becomes a differentiable functional on the phase space. The form of Γ depends on the adopted boundary conditions. In gauge theories, G is a linear combination of constraints, so that it vanishes on shell, and the boundary term Γ defines the corresponding conserved charge.

Thermodynamic variables of the MT black hole are defined by the gravitational and matter contributions to the boundary integral $\Gamma := \Gamma_\infty - \Gamma_H$, determined by the following variational equations:

$$\delta\Gamma_\infty = \oint_{S_\infty} \delta B(\xi), \qquad \delta\Gamma_H = \oint_{S_H} \delta B(\xi), \tag{3.1a}$$

$$\begin{aligned} \delta B(\xi) := {} & (\xi\rfloor\vartheta^i)\delta H_i + \delta\vartheta^i(\xi\rfloor H_i) + \frac{1}{2}(\xi\rfloor\omega^{ij})\delta H_{ij} \\ & + \frac{1}{2}\delta\omega^{ij}(\xi\rfloor\delta H_{ij}) - \delta\phi(\xi\rfloor H_\phi) \\ & + (\xi\rfloor A)\delta H_{\text{em}} + \delta A(\xi\rfloor H_{\text{em}}). \end{aligned} \tag{3.1b}$$

Here, ξ is the Killing vector for local translations (for static black holes, $\xi = \partial_t$), and (S_∞, S_H) are components of the boundary of the spacial section Σ of spacetime, located at infinity and horizon, respectively. When the boundary integrals $(\Gamma_\infty, \Gamma_H)$ exist as finite solutions of the variational equation (3.1), they define *energy and entropy* as the canonical charges at infinity and horizon, respectively.[2] The upper line in (3.1b) describes the gravitational, and the lower one the matter (scalar and Maxwell) contributions to δB. The variation δ is assumed to satisfy the following general requirements:

(r1) On S_∞, δ acts only on the parameters of the solution, excluding thereby background contributions and consequently, eliminating the need for subtraction terms.

(r2) On S_H, surface gravity is constant, $\delta\kappa = 0$.[3]

The derivation of the formula (3.1) is based on the regularity of the corresponding canonical gauge generator [18], which is ensured by the condition

$$\delta\Gamma \equiv \delta\Gamma_\infty - \delta\Gamma_H = 0. \tag{3.2}$$

This formula relates the variations of energy, entropy and Maxwell charge in a way that represents the *first law of black hole thermodynamics*.

A. The gravitational contribution

In Riemannian spacetime, where $H_i = 0$, the nontrivial covariant momenta are $H_{ij} = -2a_0{}^\star(\vartheta_i\vartheta_j)$, and the gravitational contribution to the boundary term reads

[2] The existence and finiteness of $(\Gamma_\infty, \Gamma_H)$ is not *a priori* guaranteed, it strongly depends on the adopted boundary conditions. Based on a set of naturally chosen boundary conditions, our analysis of a number of typical PG black holes (including the Kerr–Newman–AdS spacetime with torsion) shows that:

– the variational equation (3.1) are δ-integrable, and
– the resulting boundary integrals $(\Gamma_\infty, \Gamma_H)$ are finite; see Refs. [9,10,15].

These results strongly support physical relevance of the canonical formalism defined by Eq. (3.1).

[3] For stationary black holes, there exists a Killing vector field ξ that is normal to the event horizon; it can be used to define κ and show that $\delta\kappa = 0$ over the horizon, see Wald [20]. The analysis does not depend on the existence or nonexistence of torsion, hence, κ is constant also in the framework of PG.

$$\begin{aligned}\delta B_G &= \omega^{01}{}_t \delta H_{01} + \delta\omega^{12} H_{12t} + \delta\omega^{13} H_{13t} \\ &= 2a_0 f' g \delta(\vartheta^2\vartheta^3) - 2a_0 \delta\left(\frac{g}{r}\vartheta^2\right) f\vartheta^3 \\ &\quad + 2a_0\delta\left(\frac{g}{r}\vartheta^3\right) f\vartheta^2 \\ &= 2a_0 f' g\delta(r^2\sigma) - 4a_0(\delta g) f r\sigma, \end{aligned} \tag{3.3}$$

where we use $\omega^{ij}{}_t \equiv \xi\rfloor\omega^{ij}$ and $H_{ijt} \equiv \xi\rfloor H_{ij}$.

Energy is obtained by calculated δB_G at infinity. Since the factor $r^2|_\infty$ does not depend on the black hole parameter q, the first term vanishes, so that the integration over S_∞ yields

$$(\delta\Gamma_G)_\infty = -12a_0 \frac{q\delta q}{\ell^2} r\sigma + O_1. \tag{3.4a}$$

The O_1 term is ignorable and, as we shall see, the potentially divergent term will be exactly canceled by the corresponding scalar field contribution.

Entropy is determined by calculating $\delta\Gamma_G$ at horizon:

$$\begin{aligned}(\delta\Gamma_G)_H &= 2a_0(gf')_H \delta r_+^2\sigma = 2a_0\kappa\delta r_+^2\sigma = T\delta S, \\ S &:= \frac{r_+^2\sigma}{4G}. \end{aligned} \tag{3.4b}$$

B. The scalar field contribution

Now, consider the first term in the lower line of (3.1b):

$$\delta B_\phi = -\delta\phi(\xi\rfloor{}^\star d\phi) = -\delta\phi\phi' g f\vartheta^2\vartheta^3. \tag{3.5}$$

The calculation of energy yields

$$(\delta\Gamma_\phi)_\infty = -\delta_q\phi\phi' g f r^2\sigma = k\frac{q\delta q}{\ell^2} r\sigma + O_1. \tag{3.6}$$

Hence, potentially divergent contributions to energy stemming from gravity and the scalar matter cancel each other. To be more precise, starting with the integrals of δB_G and δB_ϕ over the boundary surface S_r located at finite r, and continuing with the limit $r\to\infty$, one obtains

$$\begin{aligned}\lim_{r\to\infty}[(\delta\Gamma_G)_r + (\delta\Gamma_\phi)_r] &= \lim_{r\to\infty}\left[(-12a_0 + k)\frac{q\delta q}{\ell^2} r\sigma + O_1\right] \\ &= 0. \end{aligned} \tag{3.7}$$

The MT black hole is a solution of the field equations only for $k = 12a_0$, see Eq. (2.13). As a consequence, the coefficient $(-12a_0 + k)$ vanishes, which implies $E = 0$. The result follows from the specific asymptotic behavior of dynamical variables, induced by the presence of the scalar field.

On the other hand, Eq. (3.5) with $g(r_+) = 0$ yields a vanishing contribution to entropy,

$$(\delta\Gamma_\phi)_H = 0. \tag{3.8}$$

C. The Maxwell contribution

Using the electromagnetic potential A defined in Eq. (2.12), one can calculate the corresponding covariant momentum

$$H_{\rm em} = -\frac{1}{4\pi}{}^\star F = \frac{1}{4\pi}\frac{q}{r^2}\vartheta^2\vartheta^3, \tag{3.9}$$

and obtain the asymptotic electric charge:

$$Q = \int_{S_\infty} H_{\rm em} = \frac{q\sigma}{4\pi}. \tag{3.10}$$

Now, using the last two boundary terms in (3.1b), one finds that the electromagnetic contribution to energy vanishes,

$$(\delta\Gamma_{\rm em})_\infty = \int_{S_\infty} A_t \delta H_{\rm em} = 0, \tag{3.11}$$

where we used $A_t = O_1$. Combining this result with (3.7), one concludes that the complete energy of the MT black hole vanishes.

The electromagnetic contribution at horizon takes the standard form,

$$(\delta\Gamma_{\rm em})_H = \int_{S_H} A_t\delta H_{\rm em} = \frac{q}{\ell}\frac{\delta(q\sigma)}{4\pi} = \Phi\delta Q, \tag{3.12}$$

where Φ is the electromagnetic potential

$$\Phi := A_t\Big|_{r_+}^{\infty} = \frac{q}{\ell}. \tag{3.13}$$

D. The first law

The form of the boundary terms at infinity implies that energy (mass) of the MT black hole vanishes:

$$\delta E = \delta\Gamma_\infty = 0. \tag{3.14a}$$

Similarly, the sum of the boundary terms at horizon also vanishes:

$$\delta\Gamma_H = T\delta S + \Phi\delta Q = 0. \tag{3.14b}$$

Hence, the first law (3.2) takes the form

$$\delta\Gamma_\infty = \delta\Gamma_H \Leftrightarrow 0 = T\delta S + \Phi\delta Q. \tag{3.15}$$

Since the black hole temperature does not depend on the parameter q, one can combine the relation $T\delta S = \delta(TS)$ with $\Phi\delta Q = Q\delta\Phi$ to rewrite the first law as

$$\delta\left(TS + \frac{1}{2}\Phi Q\right) = 0. \tag{3.16}$$

This relation can be obtained from a hairy deformation of the *Smarr formula* [21]

$$TS + \frac{1}{2}\Phi Q - \frac{\sigma l}{8\pi G} = 0, \tag{3.17}$$

where the third term is an extra contribution, independent of the solution parameter q.

IV. THE MT BLACK HOLE IN TELEPARALLEL GRAVITY

In the framework of PG, the teleparallel theory of gravity (TG) is defined by the condition of vanishing curvature [6]. The TG dynamics is naturally defined by a Lagrangian which is quadratic in torsion,

$$L_T \coloneqq T^{i\star}(a_1{}^{(1)}T^i + a_2{}^{(2)}T^i + a_3{}^{(3)}T^i), \tag{4.1}$$

where ${}^{(n)}T^i$ are irreducible components of the torsion. From the physical point of view, of particular importance is a *one-parameter family* of TG Lagrangians, defined by

$$(a_1, a_2, a_3) = a_0 \times (1, -2, -1/2 + \gamma), \tag{4.2}$$

which is empirically indistinguishable from GR.

By adopting the relation $\omega^{ij} = 0$ as a gauge fixing condition for local Lorentz symmetry, torsion takes the simplified form $T^i = d\vartheta^i$. To examine thermodynamic properties of the MT solution, we use the tetrad (2.8) to obtain

$$T^0 = -N'C\vartheta^0\vartheta^1, \qquad T^2 \coloneqq \frac{NC}{r}\vartheta^1\vartheta^2,$$
$$T^3 = \frac{1}{r}(\coth\theta\vartheta^2\vartheta^3 + NC\vartheta^1\vartheta^3). \tag{4.3}$$

Since ${}^{(3)}T^i = 0$, the torsion covariant momentum $H_i = 2a_0{}^\star({}^{(1)}T^i - 2{}^{(2)}T^i)$ does not depend on the parameter γ:

$$H_0 = \frac{2a_0}{r}(\coth\theta\vartheta^1\vartheta^3 - 2NC\vartheta^2\vartheta^3),$$
$$H_1 = -\frac{2a_0}{r}\coth\theta\vartheta^0\vartheta^3,$$
$$H_2 = \frac{2a_0C}{r}(Nr)'\vartheta^0\vartheta^3,$$
$$H_3 = -\frac{2a_0C}{r}(Nr)'\vartheta^0\vartheta^2. \tag{4.4}$$

In TG, the general boundary term (3.1b) is reduced to

$$\delta B(\xi) = (\xi\rfloor\vartheta^i)\delta H_i + \delta\vartheta^i(\xi\rfloor H_i) + \text{matter terms}. \tag{4.5}$$

Since the matter part remains the same as in GR, all we need to calculate is the gravitational contribution, the nonvanishing part of which reads

$$\delta B_G = \vartheta^0{}_t\delta H_0 + \delta\vartheta^2 H_{2t} + \delta\vartheta^3 H_{3t}. \tag{4.6}$$

Explicit calculation gives the following gravitational boundary terms at infinity and horizon:

$$(\delta\Gamma_G)_\infty = \int_{S_\infty}\delta B_G = -12a_0\sigma\frac{r}{\ell^2}q\delta q + \mathcal{O}_1, \tag{4.7a}$$

$$(\delta\Gamma_G)_H = \int_{S_H}\delta B_G = 2a_0\kappa\sigma\delta r_+^2 = T\delta S. \tag{4.7b}$$

Thus, the final values of the gravitational boundary terms in GR and in teleparallel gravity, presented respectively in Eqs. (3.4) and (4.7), coincide. Hence, energy, entropy and the first law of the MT black hole in the one-parameter TG coincide with the corresponding GR results.

V. CONCLUSIONS

In this paper, we used the general Hamiltonian approach proposed in Ref. [9] to study energy, entropy and the first law of the MT black hole in two complementary geometric settings.

First, we showed that our canonical approach, originally designed for black holes in PG, can be successfully applied to the electrically charged black hole with scalar hair, the solution found by Martinez and Troncoso [16] as a *Riemannian* solution of GR.

With vanishing energy and constant temperature, the first law (3.15) is associated to a hairy deformation of the Smarr formula (3.17).

Then, the MT solution is reinterpreted as an exact solution of the *teleparallel gravity*. Although the original analytic expressions (3.3) and (4.6) for the gravitational boundary terms in GR and TG are different, their final values coincide. Thus, different geometries can have the same dynamical content.

We expect that the present analysis can be consistently extended to other hairy black holes.

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APPENDIX: GRAVITATIONAL FIELD EQUATION (2.4c)

To prove the gravitational field equation (2.4c), we find it convenient to express all the energy-momentum currents in terms of the corresponding energy-momentum tensors.

For the scalar field, the energy-momentum current τ_i defines the corresponding energy-momentum tensor $\tau^k{}_i$ by

$$\tau_i \coloneqq h_i \rfloor L_\phi - (h_i \rfloor d\phi) H_\phi = -\hat{\epsilon}_k \tau^k{}_i,$$
$$\tau^k{}_i \coloneqq \partial^k \phi \partial_i \phi - \delta^k_i \mathcal{L}_\phi, \qquad \mathcal{L}_\phi \coloneqq -{}^\star L_\phi. \tag{A1}$$

An analogous procedure in the electromagnetic sector yields

$$\mathcal{T}_i \coloneqq h_i \rfloor L_{\rm em} - (h_i \rfloor F) H_{\rm em} = -\hat{\epsilon}_k \mathcal{T}^k{}_i,$$
$$\mathcal{T}^k{}_i \coloneqq -\frac{1}{4\pi}\left(F^{km}F_{im} - \frac{1}{4}\delta^k_i F^{mn}F_{mn}\right), \tag{A2}$$

and finally, the gravitational energy-momentum current is defined as

$$E_i \coloneqq h_i \rfloor L_G - (h_i \rfloor R^{mn}) H_{mn} = \hat{\epsilon}_k E^k{}_i,$$
$$E^k{}_i \coloneqq 2a_0 G^k{}_i = 2a_0\left(R^k{}_i - \frac{1}{2}\delta^k_i R\right). \tag{A3}$$

Then, the gravitational field equation (2.4c) can be written in an equivalent tensorial form as

$$2a_0 G^k{}_i = \tau^k{}_i + \mathcal{T}^k{}_i. \tag{A4}$$

For $k = 12a_0$, explicit calculation confirms the validity of this equation.

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