Conservative binary dynamics from gravitational tail emission processes

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We reanalyze the far zone contribution to the two-body conservative dynamics arising from interaction between radiative and longitudinal modes, the latter sourced by mass and angular momentum, which in the mass case is known as tail process. We verify the expected correspondence between two loop self-energy amplitudes and the gluing of two classical (one leading order, one at one loop) emission amplitudes, with focus on the Ward identities. As part of our analysis, we originally compute emission and self-energy processes with the longitudinal mode sourced by angular momentum for generic electric and magnetic multipoles and we highlight the role of the contribution from source interaction with two gravitational fields.

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I. INTRODUCTION

The recent advent of gravitational wave (GW) astronomy opened a new field not only for observing the universe, but also for investigating the fundamental nature of gravity in the most profound way so far possible.

GW detections [1] by the interferometric LIGO [2] and Virgo [3] observatories have collected signals from compact coalescing binaries in the three completed science runs, with the fourth one presently ongoing. Moreover, a third generation of terrestrial detectors and a space detector are already planned for the next decade, expecting to reach signal-to-noise ratios at $O(10^3)$ [4–6].

Besides representing a new way to observe (or rather *listen to*) the cosmos, such GW signals are privileged windows to investigate the nature of gravity at unprecedented strongly interacting level. Detections require correlation of data with precomputed waveforms [7,8], whose accuracy is crucial in maximizing the physics output of observations, and which, in turn, depends on precise knowledge of the two-body dynamics.

In view of deepening our analytic insight into the two body dynamics, we (re)investigate in the present work the processes arising from the scattering of radiation off the static curvature produced by the same sources of radiation. Such processes, besides representing corrections to the emission process at one (classical) loop, also play a role in two-loop self-energy diagrams contributing to the binary dynamics. In particular the scattering of radiation off the static curvature sourced by the mass of the system is known as *tail* process [9] (M-tail in this paper), because of its phenomenological property of traveling *inside* the light cone, rather than on it, giving rise to a metaphorical “tail strike” in the emitted wave. Tails represent one class of *hereditary* processes, i.e., processes relating the field at the observer to the entire history of the source, rather than its instantaneous state at retarded time. Another class of hereditary processes is represented by *memory* ones [10], where radiation scatters onto itself, and whose investigation we reserve to a successive study.

The scattering of radiation off the static curvature generated by the angular momentum can be described in complete analogy with the tail at the fundamental level, but

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its phenomenological effect is different as it gives rise to an instantaneous effect in the waveform, hence it has been dubbed *failed* tail [11], which will be referred to as *L-ftail* henceforth.

Within conservative dynamics, separation of processes into those involving the exchange of potential modes only, and those involving also radiative modes, is a standard procedure of perturbative computations. This separation is realized by the *method of regions* applied to particle physics [12,13] and the standard near/far zone distinction in traditional gravitational computations [14].

Restricting here to the spin-less case, among the perturbative approaches used to investigate the conservative dynamics, we highlight the post-Minkowskian (PM) and post-Newtonian (PN) approximation schemes, which are best suited for studying respectively unbound scatterings and bound systems. The expansion parameters of the former is GM/b, being G Newton's constant, M the total mass of the two-body system, b the impact parameter, and for the latter $v^2 \sim GM/r$, being v the relative velocity of binary constituents and r the size of the orbit.

In the context of PM approximation, dynamics has been completed at 4PM level [15–18], considering the standard *Feynman* prescription for Green functions of radiative internal modes, which is consistent with the *principal value* prescription, corresponding to time-symmetric Green functions, adopted in [19]. Note that, as far as conservative dynamics is concerned, for self-energy processes involving *up to two* internal radiative Green functions it is indeed equivalent to using Feynman or retarded/advanced Green functions. However a treatment in terms of the in-in formalism [20] is necessary for the memory processes [21–23].

Results obtained with amplitude or effective field theory (EFT) methods within the PM formalism, see also [24,25], can be framed in an elegant and compact form in terms of two-body scattering angle χ, whose PM perturbative expansion has a simple and distinctive scaling with the symmetric mass ratio $\eta \equiv m_1 m_2/M^2$, being $m_{1,2}$ the individual binary constituent masses. It has been shown in [26] that the m-PM contribution to the scattering angle, χ_m, scales with η as $\chi_m \sim \eta^{\lfloor (m-1)/2 \rfloor}$. In particular [25] included nontime symmetric radiation reaction effects, but still does not return the correct scaling of χ_4 with η.

Far zone processes in conservative two-body dynamics have been already considered by several studies, see e.g., [11,21–23], up to 5PN level, i.e., considering tail, L-ftail *and* memory effects, but their result is still inconsistent with the above-mentioned η scaling of χ_4, requiring further work to solve the discrepancy [27,28].

Here, we investigate how tail processes can be analyzed in terms of *generalized unitarity* [29,30], i.e., how self-energy diagrams can be described by gluing a pair of emission diagrams in the context of the nonrelativistic general relativity (NRGR) EFT approach [31] to the gravitational two-body problem within the PN approximating scheme. See [32,33] for an application to generalized unitarity use in NRGR.

With the goal of shedding light on the EFT side of the problem, we revisited the study of processes involving M-tail and L-ftail, showing that the computation procedure adopted so far [11,21,34] leads to an incorrect result for the L-ftail. In particular in the case of the electric quadrupole this is due to having overlooked the process involving a source-graviton-graviton interaction vertex (henceforth quadratic-interaction), negligence leading to an apparent violation of the Ward identities, or equivalently a violation to the gauge fixing condition. In the case of the magnetic quadrupole, the Ward identities can be restored by adding to the action a local term altering the equations of motion and leading to solutions satisfying the gauge condition and the energy-momentum conservation, along a procedure first used in the multipolar PM approach in [35]. For higher order multipoles, the gauge condition is automatically satisfied.

Our new result confirms the independent result of [36] and the known computation of the angular momentum flux ascribable to the L-ftail process in [37].

The paper is structured as follows. In Sec. II we set up notations and write down emission amplitudes for generic multipoles, first at the leading order, then at next-to-leading in G for M-tails and L-ftails, the latter being derived here for the first time for generic electric and magnetic multipoles. We show that the quadratic-interaction process gives a critical contribution to the L-ftail involving the electric quadrupole. The section is ended by a discussion of a residual violation of the Lorentz gauge, happening in the magnetic quadrupole case. In Sec. III we investigate the relation between self-energy diagrams and (square of) emission processes, and how the former are impacted by the Ward identities issue affecting the L-ftail. We use angular momentum balance equation to confirm our new value for the L-ftail self-energy diagram involving the electric quadrupole, and derive for the first time the corresponding values for all electric and magnetic multipoles. Section IV contains our conclusions and prospects, while some technical derivations are detailed in the appendices.

II. EMISSION AMPLITUDES IN EFT

A. One point functions and leading order emission amplitudes

Our starting point is the following multipolar action which gives the linear coupling of matter to the gravitational field:

$$S_{\text{source}} = \int_t \Bigg[\frac{1}{2} E h_{00} - \frac{1}{2} J^{k|l} h_{0k,l} - \sum_{r\geq 0} \left(c_r^{(I)} I^{ijR} \partial_R \mathcal{R}_{0i0j} + \frac{c_r^{(J)}}{2} J^{k|iRl} \partial_R \mathcal{R}_{0ilk} \right) \Bigg], \quad (1)$$

with

$$c_r^{(I)} = \frac{1}{(r+2)!}, \qquad c_r^{(J)} = \frac{2(r+2)}{(r+3)!}, \tag{2}$$

and where $J^{k|iRl}$ are the d-dimensional generalizations [38] of the 3-dimensional magnetic-type multipoles $J^{ijR} = \frac{1}{2}\epsilon_{kl(i}J^{k|jR)l}|_{d=3}$ (symmetrized over the indices ijR) and I^{ijR} the standard electric-type ones.[1] In expression (1), $\mathcal{R}_{\mu\nu\rho\sigma}$ are the components of the Riemann curvature tensor, while R denotes the collective symmetric trace-free index, $R = i_1 \ldots i_r$, with $r = 0$ standing for the quadrupole, $r = 1$ for the octupole, and so on.

The dynamics of the gravitational field is dictated by the bulk action, given by the Einstein-Hilbert action plus a gauge-fixing term:

$$S_{\rm bulk} = 2\Lambda^2 \int \mathrm{d}^{d+1}x\sqrt{-g}\left[\mathcal{R}(g) - \frac{1}{2}\Gamma_\mu\Gamma^\mu\right], \tag{3}$$

with $\Lambda \equiv (32\pi G_d)^{-1/2}$, with G_d being Newton's constant in $d+1$-dimensions, $\Gamma^\mu \equiv \Gamma^\mu_{\nu\rho}g^{\nu\rho}$, being $\Gamma^\mu_{\nu\rho}$ the standard Christoffel connection, and the metric $g_{\mu\nu}$ will be eventually expanded around Minkowski background as per $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

The gauge-fixing term implies that the theory we are solving for falls back into GR only for $\Gamma^\mu = 0$, which implies the Lorentz condition $\partial^\nu \bar{h}_{\mu\nu} = 0$ at linearized level. An overbar denotes the trace-reversed field $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, being $h \equiv \eta^{\mu\nu}h_{\mu\nu}$. The addition to the Lagrangian of the gauge-fixing term modifies the linearized Einstein equations precisely to $\Box\bar{h}_{\mu\nu} = 0$ outside the source.

From the quadratic term in Eq. (3) we can derive the Green's functions for the gravitational perturbation field $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$:

$$P[h_{\mu\nu}, h^{\alpha\beta}] = -\frac{i}{\mathbf{k}^2 - \omega^2}\frac{\mathcal{P}_{\mu\nu}{}^{\alpha\beta}}{\Lambda^2},$$
$$\mathcal{P}_{\mu\nu}{}^{\alpha\beta} \equiv \frac{1}{2}\left(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu - \frac{2}{d-1}\eta_{\mu\nu}\eta^{\alpha\beta}\right), \tag{4}$$

where so far we have not specified the Green's function boundary conditions. GWs correspond to the transverse traceless spatial component of the metric perturbations, and for direction propagation $\hat{\mathbf{n}}$ they can be selected by applying the following projector operator

$$\Lambda^{TT}_{ij,kl}(\hat{\mathbf{n}}) \equiv P_{ik}P_{jl} - \frac{1}{d-1}P_{ij}P_{kl}, \qquad P_{ij}(\hat{\mathbf{n}}) \equiv \delta_{ij} - \hat{n}_i\hat{n}_j. \tag{5}$$

Beside gauge ones, the other components parametrize longitudinal degrees of freedom, which play a role for checking energy-momentum conservation, as it will be discussed below.

We denote by $i\mathcal{A}_{\alpha\beta}(\omega,\mathbf{k})h^{*\alpha\beta}(\omega,\mathbf{k})$ the probability amplitude for the emission of the generic field $h_{\alpha\beta}$, which can be computed by deriving the appropriate Feynman rules from (1) and (3). The classical field at a spacetime position x, given by the one-point function $\langle h_{\mu\nu}(x)\rangle$, is then related to $\mathcal{A}_{\mu\nu}$ by:

$$\langle h_{\mu\nu}(x)\rangle = \int \mathcal{D}h\, e^{iS[h]}h_{\mu\nu}(x)$$
$$= \int_{\mathbf{k}} \frac{\mathrm{d}\omega}{2\pi}\frac{e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 - (\omega + i\mathrm{a})^2}\frac{\mathcal{P}_{\mu\nu}{}^{\alpha\beta}}{\Lambda^2}\mathcal{A}_{\alpha\beta}(\omega,\mathbf{k}), \tag{6}$$

where the correct retarded boundary condition has been selected.[2]

The linearized Lorentz gauge condition translates into "Ward" identities for the classical process consisting of the emission of a single gravitational mode[3]

$$\partial^\mu\langle\bar{h}_{\mu\nu}(x)\rangle = 0 \Leftrightarrow k^\mu\mathcal{A}_{\mu\nu}(\omega,\mathbf{k}) = 0. \tag{7}$$

As it happens in the multipolar PM approach [14], the diagrammatic expansion used in our EFT setup provides a solution of the perturbative form of Einstein's equations

$$\Box\bar{h}_{\mu\nu} = \Lambda_{\mu\nu}, \tag{8}$$

where $\Lambda_{\mu\nu}$ is a source term given by the source energy-momentum tensor plus nonlinear terms in $h_{\mu\nu}$. The gauge condition (7), which is equivalent to the conservation of the pseudoenergy momentum tensor $\Lambda_{\mu\nu}$, is however not automatically satisfied and it has to be checked, and *eventually fixed*, on a case-by-case basis.

It is not uncommon in field theory that loop interactions can break a symmetry which is present in the free theory, causing the symmetry to be *anomalous*. In our case GR invariance under diffeomorphism is broken in the Lagrangian by the gauge fixing term, implying that GR is recovered only when $\Gamma^\mu = 0$, which makes the gauge-fixing term a *double zero*. We will see in Sec. II B that relation (7) may not hold at interacting level, but in a *consistent*, or *integrable* way, i.e., it is possible to add to the action functional a local term restoring the Lorentz condition (7) at the level of the equations of motion.

[1]We adopted the notation $\int_x \equiv \int \mathrm{d}x$, $\int_{\mathbf{k}} \equiv \int \frac{\mathrm{d}^dk}{(2\pi)^d}$. Our metric convention is "mostly plus": $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$.

[2]We displace the Green's function pole by $\pm i\mathrm{a}$, as ϵ is already used to denote $d-3$.

[3]Note that the trace reversion operator $\hat{\mathcal{P}}^{\alpha\beta}_{\mu\nu} \equiv \frac{1}{2}(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu - \eta_{\mu\nu}\eta^{\alpha\beta})$, which turns $h_{\alpha\beta}$ into $\bar{h}_{\mu\nu}$, is identical to its inverse $\mathcal{P}_{\mu\nu}{}^{\alpha\beta}$ only for $d=3$.

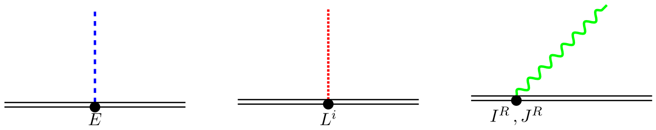

FIG. 1. Leading order emission amplitudes. Blue dashed line refers to a h_{00} polarization, dotted red to h_{0i}, wavy green to h_{ij}.

The conserved multipoles (mass and angular momentum) of action (1), see the first two diagrams of Fig. 1, contribute to (6) as:

$$\mathcal{A}_{00}^{(c)}(\omega,\mathbf{k}) = \frac{1}{2}E(\omega),$$
$$\mathcal{A}_{0k}^{(c)}(\omega,\mathbf{k}) = \frac{1}{4}ik_j\epsilon_{ijk}L_i(\omega),$$
$$\mathcal{A}_{kl}^{(c)}(\omega,\mathbf{k}) = 0, \tag{9}$$

where the finiteness of the amplitude for $d=3$ allowed us to use the standard expression of the $L_i = \frac{1}{2}\epsilon_{ijk}J^{j|k}$. The Ward identities follow from

$$\omega E(\omega) = 0, \qquad \omega L_i(\omega) = 0, \tag{10}$$

and are trivially satisfied at this perturbative order by admitting that E and L_i are conserved.

The leading order electric and magnetic multipole emission amplitudes, see last diagram in Fig. 1, are finite for $d=3$ and read

$$\left.\begin{array}{l} i\mathcal{A}_{00}^{(e)}(\omega,\mathbf{k}) \\ i\mathcal{A}_{0k}^{(e)}(\omega,\mathbf{k}) \\ i\mathcal{A}_{kl}^{(e)}(\omega,\mathbf{k}) \end{array}\right\} = \frac{1}{2}c_r^{(I)}(-i)^{r+1}k_R I^{ijR}(\omega) \times \left\{\begin{array}{l} k_ik_j \\ -\omega k_j\delta_{ik}\,, \\ \omega^2\delta_{ik}\delta_{jl} \end{array}\right. \tag{11}$$

and

$$\left.\begin{array}{l} i\mathcal{A}_{00}^{(m)}(\omega,\mathbf{k}) \\ i\mathcal{A}_{0k}^{(m)}(\omega,\mathbf{k}) \\ i\mathcal{A}_{kl}^{(m)}(\omega,\mathbf{k}) \end{array}\right\} = \frac{1}{2}c_r^{(J)}(-i)^{r+1}\epsilon_{imn}k_nk_R J^{jmR}(\omega)$$
$$\times\left\{\begin{array}{l} 0 \\ -\frac{1}{2}k_j\delta_{ik}\,, \\ \omega\delta_{i(k}\delta_{l)j} \end{array}\right. \tag{12}$$

whose relative Ward identities are trivially satisfied.

B. Tail-corrected emission amplitudes

At next-to-leading order in gravitational interactions one has to consider processes of the type represented in Fig. 2. The first one, involving the scattering of radiation off the background curvature sourced by the system total mass, gives a contribution to the waveform arriving later than the wavefront, which propagates at the speed of light, while the tail propagates inside the light cone.

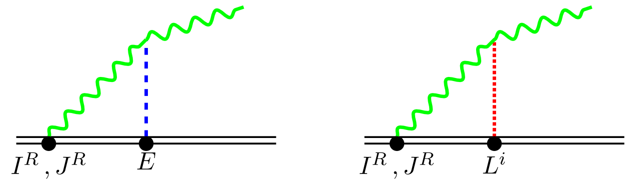

FIG. 2. Next-to-leading order emission processes involving the scattering of radiation off the background static curvature sourced by energy and angular momentum.

The second process in Fig. 2, involving the angular momentum, gives a purely local term in the waveform, hence not giving rise to a *tail* effect. However its diagrammatic analogy with the tail process suggests to lump it together with the M-tail, and we refer to it in this work as L-ftail.

The M-tail amplitude is divergent for $d\to 3$ and its radiative, transverse-traceless, on-shell ($\mathbf{k}=\omega\hat{\mathbf{n}}$) part is [39,40]

$$\begin{aligned} &i(\mathcal{A}_{ij}^{(e-M\text{-tail})})^{TT}(\omega,\omega\hat{\mathbf{n}}) \\ &\quad= (-i)^{r+1}\omega^2c_r^{(I)}(iGE\omega)\Lambda_{ij,kl}^{TT}k_RI^{klR}(\omega) \\ &\qquad\times\left(\frac{1}{\epsilon}-\kappa_{r+2}+\frac{\log x}{2}\right), \end{aligned} \tag{13}$$

$$\begin{aligned} &i(\mathcal{A}_{ij}^{(m-M\text{-tail})})^{TT}(\omega,\omega\hat{\mathbf{n}}) \\ &\quad= (-i)^{r+1}\omega c_r^{(J)}(iGE\omega)\Lambda_{ij,kl}^{TT}k_Rk_nJ^{n|kRl}(\omega) \\ &\qquad\times\left(\frac{1}{\epsilon}-\pi_{r+2}+\frac{\log x}{2}\right), \end{aligned} \tag{14}$$

$$\kappa_l = \frac{2l^2+5l+4}{l(l+1)(l+2)}+\sum_{i=1}^{l-2}\frac{1}{i}, \qquad \pi_l = \frac{l-1}{l(l+1)}+\sum_{i=1}^{l-1}\frac{1}{i}, \tag{15}$$

where $\epsilon\equiv d-3$, $x\equiv -e^\gamma\omega^2/\mu\pi$, and the inverse length scale μ is implicitly defined by $G_d = G\mu^{-\epsilon}$, with G denoting the standard $3+1$-dimensional Newton's constant.

The Ward identities $k^\mu\mathcal{A}_{\mu\nu}^{(e(m)-M\text{-tail})}(\omega,\omega\hat{\mathbf{n}})=0$ are conveniently found by taking the divergene of $\mathcal{A}_{\mu\nu}^{(e(m)-M-tail)}$ before performing the loop integration, as reported in Appendix A.

The L-ftail amplitudes are finite and local and the expressions of their radiative TT part for generic electric and magnetic multipoles are originally given here

$$i(\mathcal{A}_{ij}^{(e-L\text{-ftail})})^{TT}(\omega,\omega\hat{\mathbf{n}}) = (-i)^r c_r^{(I)} G\Lambda^{TT}_{ij,kl}\epsilon_{mnq}\frac{i\omega^2 L^q I^{pR(k}(\omega)}{(r+1)(r+2)(r+3)(r+4)} \times \{k_n(2[6+r(r+4)]\delta_{l)m}k_p k_R - r(10+r(r+5))\delta_{l)p}\delta_{i_1 m}k_{R-1}\omega^2) + 24\delta_{0r}\delta_{l)m}\delta_{np}k_R\omega^2\}, \quad (16)$$

$$i(\mathcal{A}_{ij}^{(m-L-\text{ftail})})^{TT}(\omega,\omega\hat{\mathbf{n}}) = (-i)^r c_r^{(J)} G\Lambda^{TT}_{ij,kl}\frac{i\omega^3 L^q J^{pR(k}(\omega)}{(r+1)(r+2)(r+3)(r+4)(r+5)} \times \{\delta_{l)i_1}[r(r+1)(r^2+8r+19)k_p k_q - (r^4+9r^3+27r^2+39r+68(1-\delta_{0r}))\omega^2\delta_{pq}]k_{R-1} + [(r+3)(r^2+5r+16)\delta_{l)p}k_q - (r^4+12r^3+53r^2+102r+84)\delta_{l)q}k_p]k_R\}, \quad (17)$$

with their un-integrated form also reported in Appendix A.

This time the spatial Ward identities are *not* satisfied

$$k^\mu \mathcal{A}_{\mu l}^{(e-L\text{-ftail})}(\omega,\omega\hat{\mathbf{n}}) = (-i)^{r+1}\frac{c_r^{(I)}}{2\Lambda^2}\left(\frac{\omega}{4}\right)k_j\omega^2\epsilon_{ijk}L^k I^{iRl}(\omega)\int_{\mathbf{q}}\frac{q_R}{(\mathbf{q}^2-\omega^2)} \overset{d\to 3}{=} -\delta_{r0}\frac{G}{2}k_j\omega^4\epsilon_{ijk}L^k I^{il}(\omega). \quad (18)$$

As the $\mathbf{q}$-integral gives a result proportional to the symmetric, traceless combination of Kronecker deltas with R indices, expression (18) vanishes except for $r = 0$, meaning that the L-ftail emission process involving the electric quadrupole violates the spatial components of the Ward identity. Analogously, for the magnetic part one finds

$$k^\mu \mathcal{A}_{\mu l}^{(m\text{-}L\text{-tail})}(\omega,\omega\hat{\mathbf{n}}) \overset{d\to 3}{=} -\delta_{r0}\frac{4}{15}G\omega^5 L^k J^{lk}(\omega). \quad (19)$$

These two Ward identity violations are resolved in two different ways, as it will be shown in the next subsections.

C. Amplitudes involving a quadratic-interaction vertex

Processes like the ones shown in Fig. 3 must be also considered, as they are of the same G order as the (f)tail ones. It is straightforward to observe that such diagrams involve time derivatives of the multipole linearly coupled to gravity, implying that the right diagram in Fig. 3 is actually vanishing.

We then focus on the left diagram and first consider the multipole to be I^{ij}. The angular momentum interaction vertex at quadratic order in the gravitational field is [41]:

$$S^{\text{spin}}_{\text{source}} = -\int_t \frac{1}{2}J^{k|l}\left[h_{0k,l} + \frac{1}{4}h_{l\lambda}\dot{h}^\lambda_k + \frac{1}{2}h^\lambda_l h_{0\lambda,k} - \frac{1}{2}h^\lambda_l h_{0k,\lambda} + \mathcal{O}(h^3)\right], \quad (20)$$

where λ is a space-time index, lowered and raised with $\eta_{\mu\nu}$.

A straightforward evaluation gives the following amplitude (here again $\mathbf{k} = \omega\hat{\mathbf{n}}$)

$$i\mathcal{A}_{00}^{(e-L-\text{quad})}(\omega,\omega\hat{\mathbf{n}}) = 0,$$
$$i\mathcal{A}_{0l}^{(e-L-\text{quad})}(\omega,\omega\hat{\mathbf{n}}) = -i\frac{G}{2}\omega^3 k_k L^j \varepsilon^{ij[k}I^{l]i}(\omega),$$
$$i\mathcal{A}_{kl}^{(e-L-\text{quad})}(\omega,\omega\hat{\mathbf{n}}) = -i\frac{G}{2}\omega^4 L^j \varepsilon^{ij(k}I^{l)i}(\omega), \quad (21)$$

whose divergence is exactly opposite to the one reported in (18). By replacing I_{ij} with any higher multipole of electric type, one obtains instead a vanishing result because of tracelessness of the multipoles themselves (the amplitude can have at most two free indices and there are not enough k^j factors to contract with all the multipole indices).

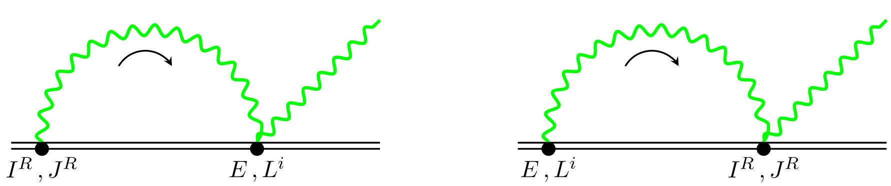

FIG. 3. Emission amplitudes involving quadratic-interaction vertices. The arrow indicate the direction of the retarded propagator in the loop, as dictated by the in-in formalism. The processes described by the right diagram have vanishing amplitude because E and L^i are conserved quantities.

It follows that the amplitude combination $\mathcal{A}^{(e\text{-}L\text{-tot})} \equiv \mathcal{A}^{(e-L\text{-ftail})} + \mathcal{A}^{(e\text{-}L\text{-quad})}$ satisfies the Ward identities $k^\mu \mathcal{A}_{\mu\nu}^{(e\text{-}L\text{-tot})} = 0$ even for $r = 0$, and its expression is

$$i(\mathcal{A}_{ij}^{(e-L\text{-tot})})^{TT}(\omega, \omega\hat{\mathbf{n}}) = (-i)^r c_r^{(I)} G\Lambda_{ij,kl}^{TT}\epsilon_{mnq}\frac{i\omega^2 L^q I^{pR(k}(\omega)}{(r+1)(r+2)(r+3)(r+4)} \times \{k_n(2[6+r(r+4)]\delta_{l)m}k_p k_R - r(10+r(r+5))\delta_{l)p}\delta_{i_1 m}k_{R-1}\omega^2)\}. \tag{22}$$

All the other cases, that is when E and/or the magnetic multipoles J^R are involved, give vanishing results, again because virtue of the tracelessness of the multipole moments. To summarize, the class of processes represented in Fig. 3 give a non-vanishing contribution only in the case they involve L^i and I^{ij}, which fixes the Ward identity violation of the corresponding L-ftail process.

D. Ward-fixing amplitude correction

The only Ward-violating amplitude left is the L-ftail involving the magnetic quadrupole. Emitted waveforms which do not fulfill the Lorentz gauge condition have already been treated in the standard multipolar PM formalism [14,35].

The fix consists in finding a particular (nonunique) solution $\bar{h}_{\mu\nu}^{(W)}$ of the homogeneous, linearized Einstein equations

$$\Box\bar{h}_{\mu\nu}^{(W)} = 0, \tag{23}$$

whose divergence compensates the previously encountered Ward identity violation. Once added to the previous anomalous solution $\bar{h}_{\mu\nu}$, one will have

$$\Box(\bar{h}_{\mu\nu} + \bar{h}_{\mu\nu}^{(W)}) = \Lambda_{\mu\nu}, \qquad \partial^\mu(\bar{h}_{\mu\nu} + \bar{h}_{\mu\nu}^{(W)}) = 0. \tag{24}$$

The tedious but straightforward check of the full, nonlinearized Lorentz condition is relegated to Appendix C.

In the case we are interested in, the amplitude corresponding to the compensating terms $\bar{h}_{\mu\nu}^{(W)}$ is

$$\begin{aligned}\mathcal{A}_{00}^{(W,m-L\text{-ftail})}(\omega,\omega\hat{\mathbf{n}}) &= -\frac{4}{15}G\omega^3 k^l L^k J^{lk}(\omega),\\ \mathcal{A}_{0l}^{(W,m-L\text{-ftail})}(\omega,\omega\hat{\mathbf{n}}) &= \frac{4}{15}G\omega^4 L^k J^{lk}(\omega),\\ \mathcal{A}_{kl}^{(W,m-L\text{-ftail})}(\omega,\omega\hat{\mathbf{n}}) &= 0;\end{aligned} \tag{25}$$

the TT part of the amplitude in Eq. (17) is not affected by the anomaly fixing term (25), whose space-space part is vanishing. Interesting enough, if we did not include the quadratic-interaction amplitude and tried to fix also the electric quadrupole L-ftail following the multipolar PM formalism procedure, we would have obtained a correcting amplitude exactly equal to Eq. (21), as it is done in [36].

We have also checked that $\mathcal{A}^{(e\text{-}L\text{-tot})}$ and $\mathcal{A}^{(m\text{-}L\text{-tot})} \equiv \mathcal{A}^{(m-L\text{-ftail})}$ return the correct contributions to the radiative multipole moments U_R, V_R, as they are defined and explicated in [42] for $r \leq 2$ and for $r \leq 1$ in the electric and magnetic cases, respectively.

III. RELATION BETWEEN SELF-ENERGY DIAGRAMS AND EMISSION AMPLITUDES

The self-energy diagrams can be factorized into products of emission diagrams or, equivalently, emission amplitudes can be glued together to form self-energy amplitudes. Note that self-energy amplitudes contribute to the dynamics of the source [43], hence, their consistent computation is crucial to obtaining the correct effective Lagrangian ruling the source dynamics.

For instance, the expression for the simplest self-energy diagram, as computed with the usual EFT Feynman rules [11]

$$\begin{aligned}i\mathcal{S}^{(I^2)} = \frac{i}{64\Lambda^2}\int\frac{\mathrm{d}\omega}{2\pi}\omega^4 I_{ij}(\omega)I_{kl}^*(\omega)\int_{\mathbf{k}}\frac{1}{\mathbf{k}^2-\omega^2}\\ \times\Bigg[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{(d-1)}\delta_{ij}\delta_{kl}\\ + \frac{2}{(d-1)\omega^2}(k_i k_j\delta_{kl} + k_k k_l\delta_{ij})\\ - \frac{1}{\omega^2}(k_i k_k\delta_{jl} + k_i k_l\delta_{jk} + k_j k_k\delta_{il} + k_j k_l\delta_{ik})\\ + \frac{4}{c_d\omega^4}k_i k_j k_k k_l\Bigg],\end{aligned} \tag{26}$$

with $c_d \equiv 2(d-1)/(d-2)$, is identically equivalent to

$$\begin{aligned}i\tilde{\mathcal{S}}^{(I^2)} &= \frac{1}{2\Lambda^2}\int_{\mathbf{k}}\frac{d\omega}{2\pi}\mathcal{A}_{\mu\nu}^{(e,r=0)}(\omega,\mathbf{k})\\ &\quad\times\mathcal{P}[h_{\mu\nu},h_{\rho\sigma}]\mathcal{A}_{\rho\sigma}^{(e,r=0)}(-\omega,-\mathbf{k})\\ &= \frac{1}{2}\int_{\mathbf{k}}\frac{d\omega}{2\pi}(\mathcal{A}_{ij}^{(e,r=0)})^{TT}(\omega,\mathbf{k})\\ &\quad\times\mathcal{P}[h_{ij},h_{kl}](\mathcal{A}_{kl}^{(e,r=0)})^{TT}(-\omega,-\mathbf{k}),\end{aligned} \tag{27}$$

with $(\mathcal{A}_{ij}^{(e)})^{TT} \equiv \Lambda_{ij,ab}^{TT}(\mathbf{n})\mathcal{A}_{ab}^{(e)}$, and the standard in-out formalism with *Feynman's* prescription for the Green's functions is understood. As a consequence of the Ward identities, the TT part alone of the emission amplitude is sufficient to reconstruct the self-energy diagram. The equality between $\mathcal{S}$ and $\tilde{\mathcal{S}}$ holds for all the electric and magnetic multipoles, and it applies also to generic tail diagrams, see Appendix D).

Actually, this LO self-energy diagram is purely dissipative: after $\mathbf{k}$-integration in (26), (27), one finds purely imaginary ω-integrand in $\mathcal{S}^{(I^2)}$. Via standard optical theorem they can be related to a probability loss, or, after multiplying the ω-integrand by $|\omega|$, to energy emission.[4]

Using more appropriately the *in-in* formalism, i.e., doubling the degrees of freedom and applying retarded/advanced Green's functions [20,44,45], one finds the same real part of the self-energy action as with the in-out (after appropriate identification of the doubled degrees of freedom, see Appendix B) and the imaginary part of the ω-integrand is the Fourier transform of a real functional, which can be used to generate non-conservative equations of motion.

In summary, as mentioned in the introduction and derived in [46], as far as self-energy diagrams with *only two* radiative Green's functions are concerned, one can consistently use the in-out formalism with Feynman Green's functions to obtain the conservative part of the equations of motion, and the optical theorem for the emitted energy. The in-in formalism is, however, necessary to derive directly the dissipative part of the equations of motion, see Appendix B for details.

One can verify that analogous relations hold for the M-tail process,

$$\begin{aligned} i\mathcal{S}^{(e-M\text{-tail})} &= -G^2E\frac{2^{r+2}(r+3)(r+4)}{(r+1)(r+2)(2r+5)!}\times\int\frac{d\omega}{2\pi}(\omega^2)^{r+3}I^{ijR}(\omega)I^{ijR}(-\omega)\left[\frac{1}{\tilde{\varepsilon}}-\gamma_r^{(e)}\right],\\ &=\frac{1}{2\Lambda^2}\int_{\mathbf{k}}\frac{\mathrm{d}\omega}{2\pi}(\mathcal{A}_{ij}^{e\text{-}M\text{-tail}})^{TT}(\omega,\mathbf{k})P[h_{ij},h_{kl}](\mathcal{A}_{kl}^{(e)})^{TT}(-\omega,-\mathbf{k}), \end{aligned} \tag{28}$$

$$\begin{aligned} i\mathcal{S}^{(m-M\text{-tail})} &= -G^2E\frac{2^{r+2}(r+2)^2(r+4)(r!)^2}{(2r+1)(2r+3)(2r+5)(2r)![(r+3)!]^2}\\ &\quad\times\int\frac{\mathrm{d}\omega}{2\pi}J^{b|iRj}(\omega)J^{b'|kR'l}(-\omega)\omega^{2r+6}[\delta_{bb'}\delta_{ik}+(r+1)\delta_{ib'}\delta_{kb}]\left[\frac{1}{\tilde{\varepsilon}}-\gamma_r^{(m)}\right],\\ &=\frac{1}{2\Lambda^2}\int_{\mathbf{k}}\frac{\mathrm{d}\omega}{2\pi}(\mathcal{A}_{ij}^{m\text{-}M\text{-tail}})^{TT}(\omega,\mathbf{k})P[h_{ij},h_{kl}](\mathcal{A}_{kl}^{(m)})^{TT}(-\omega,-\mathbf{k}), \end{aligned} \tag{29}$$

with

$$\begin{aligned} \frac{1}{\tilde{\varepsilon}} &\equiv \frac{1}{d-3}+\log x-i\pi\mathrm{sgn}(\omega),\\ \gamma_r^{(e)} &\equiv \frac{1}{2}(H_{r+\frac{5}{2}}-H_{\frac{1}{2}}+2H_r+1)+\frac{2}{(r+2)(r+3)},\\ \gamma_r^{(m)} &= \frac{2}{r+3}+\frac{1}{2r+5}-\frac{1}{r+2}-\frac{1}{r+4}+H_{r+1}+\frac{1}{2}H_{r+\frac{3}{2}}+\log 2, \end{aligned} \tag{30}$$

which are the same numbers, although written in different form, obtained in [22] via direct computation of the self-energy diagrams. Incidentally, this allows one to derive explicit relations between the $\gamma_r^{(e,m)}$ coefficients appearing in (28), (29) and the finite terms in the emission amplitudes, κ_{r+2}, π_{r+2} of (13), 14):

$$\gamma_r^{(e)} = \kappa_{r+2}-\left(\frac{1}{2}+\frac{1}{r+3}+\frac{1}{r+4}-\frac{1}{2}H_{r+\frac{5}{2}}-\log 2\right), \tag{31}$$

$$\gamma_r^{(m)} = \pi_{r+2}-\left(\frac{1}{2}+\frac{1}{r+3}+\frac{1}{r+4}-\frac{1}{2}H_{r+\frac{5}{2}}-\log 2\right)+\frac{r+5}{2(r+3)}. \tag{32}$$

[4]When computing the self-energy diagram with Feynman Green's functions, the ω integrand is complex and it is *not* the Fourier transform of a real function in direct space.

For the L-ftails we find[5]:

$$i\mathcal{S}^{(e\text{-}L\text{-tot})} = G^2 \frac{(12+50r+35r^2+10r^3+r^4)}{(r+1)^2(r+2)^2(r+3)!(2r+5)!!} \epsilon^{ikl} L^l \int \frac{\mathrm{d}\omega}{2\pi} I^{ijR}(\omega) I^{kjR}(-\omega)\omega^{7+2r}$$
$$= \frac{1}{2\Lambda^2}\int_{\mathbf{k}} \frac{\mathrm{d}\omega}{2\pi} (\mathcal{A}_{ij}^{e\text{-}L\text{-tot}})^{TT}(\omega,\mathbf{k}) P[h_{ij},h_{kl}] (\mathcal{A}_{kl}^{(e)})^{TT}(-\omega,-\mathbf{k}), \quad (33)$$

$$i\mathcal{S}^{(m\text{-}L-\text{tot})} = -G^2 \frac{4(36+50r+32r^2+10r^3+r^4)}{(r+1)^2(r+3)^2(r+3)!(2r+5)!!} \epsilon^{ikl} L^l \int \frac{\mathrm{d}\omega}{2\pi} J^{ijR}(\omega) J^{kjR}(-\omega)\omega^{7+2r}$$
$$= \frac{1}{2\Lambda^2}\int_{\mathbf{k}} \frac{\mathrm{d}\omega}{2\pi} (\mathcal{A}_{ij}^{m\text{-}L-\text{tot}})^{TT}(\omega,\mathbf{k}) P[h_{ij},h_{kl}] (\mathcal{A}_{kl}^{(m)})^{TT}(-\omega,-\mathbf{k}). \quad (34)$$

In the electric $r=0$ case one has

$$i\mathcal{S}^{(LI^2)} \equiv i\mathcal{S}^{(e\text{-}L\text{-tot})}|_{r=0} = \frac{1}{30} G^2 \epsilon^{ikl} L^l \times \int \frac{\mathrm{d}\omega}{2\pi} I^{ij}(\omega) I^{jk}(-\omega)\omega^7, \quad (35)$$

also in agreement with [36], once the corresponding quadratic-interaction process is also added in the self-energy calculation. Note that the numerical factor $1/30$ corrects the value $8/15$ obtained via the incomplete computation in [11] which did not take into account the process with quadratic interaction in Fig. 3.

One can check the result (35) by computing the radiated energy and angular momentum calculable from the L-ftail quadrupolar emission amplitude corrected by the quadratic-interaction process, and comparing the result with the mechanical energy and angular momentum loss derivable from the equations of motion generated by the functional $\mathcal{S}^{(LI^2)}$, which are expected to agree with the former modulo total derivatives, or *Schott* terms [47,48]. It turns out that the contribution to the energy emission is a total derivative, thus not being useful for our purposes; we then focus on angular momentum emission.

Starting from standard textbook formula, see, e.g., Eq. (2.61) of [49], for the *emitted* angular momentum

$$\epsilon_{ijq}\dot{L}^q = \frac{r^2}{32\pi G}\int d\Omega \langle \dot{h}_{kl}^{TT} x_i \partial_j h_{kl}^{TT} - 2\dot{h}_{kj}^{TT} h_{ki}^{TT}\rangle - i \leftrightarrow j, \quad (36)$$

and using the standard quadrupole formula for GWs one obtains the leading order (LO) term

$$\epsilon_{ijq}\dot{L}^q|_{LO} = \frac{2G}{5}(\langle \dddot{I}_{ik}\ddot{I}_{jk}\rangle - \langle \dddot{I}_{jk}\ddot{I}_{ik}\rangle), \quad (37)$$

which matches the mechanical angular momentum loss obtained using the Burke-Thorne acceleration $a_i^{(BT-I)} = -\frac{2G}{5} x^j I_{ij}^{(5)}$ [50], modulo Schott terms.

Using the emission amplitude one has

$$(h_{ij}^{(e\text{-}L\text{-tot})})^{TT} = \frac{2G}{r}\Lambda_{ij,kl}(\ddot{I}_{kl} - G\dddot{I}_{a(k}\epsilon_{l)bq} L^q \hat{n}^a \hat{n}^b), \quad (38)$$

which plugged into (36) enables us to compute its contribution to the angular momentum emission rate

$$\epsilon_{ijq}\dot{L}^q|_{LI^2} = \frac{2G^2}{15} L^q I_{jk}^{(3)} I_{kl}^{(4)} \epsilon_{ilq} - i \leftrightarrow j, \quad (39)$$

which matches, again modulo Schott terms, the mechanical angular momentum loss obtained from the modified Burke-Thorne acceleration

$$a_i^{(BT-LI)} = \frac{2G^2}{15} L^q (x^j I_{jk}^{(7)} \epsilon_{ikq} - x^j I_{ik}^{(7)} \epsilon_{jkq}). \quad (40)$$

Acceleration (40) can be obtained from the in-in version of the effective action (35), see Appendix B for details. Note that (39) agrees also with standard results, see, e.g., Eq. (2.7) of [37], giving further confirmation that the $1/30$ coefficient in the expression for $\mathcal{S}^{(LI^2)}$ is indeed correct.

Equation (39) is the leading order term of a series of angular momentum flux contributions by L-ftails involving electric and magnetic multipoles of all orders, which can be straightforwardly derived either from the corresponding emission amplitudes (22), (17), or from the effective actions (33), (34).

[5]Notice that the term carrying the δ_{0r} in the magnetic TT emission amplitude happens to give a vanishing contribution to the self-energy.

IV. CONCLUSION AND DISCUSSION

We have analyzed next-to-leading order far-zone diagrams contributing to both conservative and dissipative two-body dynamics. The interaction studied are of the tail-type, i.e., due to emission of radiation which subsequently interacts with the quasistatic curvature generated by the mass and angular momentum of the source.

We applied field theory methods within the framework of NRGR which makes use of standard gauge-fixed path integral formulation to derive the classical effective action. In analogy with what has been found in the classical approach of the multipolar post-Minkowskian formalism, we found that, in some cases, the gauge condition chosen to make the kinetic term of the gravitational field invertible is not respected by loop corrected classical solutions, giving origin to anomalous scattering amplitudes. We have then shown how such apparent anomalies are canceled, respectively in the electric and magnetic case, by the inclusion of a quadratic-interaction process, and by the standard multipolar PM correction procedure.

Then, we showed the consequence of fixing Ward identity in emission diagrams for self-energy ones, by suitably obtaining the latter by glueing the former, i.e., via *generalized unitarity*. As a natural prosecution of the present work, we plan to study in an analogous approach the memory process, which involves the emission of radiation scattering off another radiative mode, which has both analogies and differences with respect to the tail and tail-like processes studied in the present work.

While a violation and the recovery of the gauge-fixing condition is also expected in the memory case, as per the results of [35], the presence of three radiative degrees of freedom in the memory self-energy amplitude requires a thorough treatment within the in-in formalism which will be the subject of a future investigation, together with the fundamental origin of the violation of the energy momentum conservation.

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APPENDIX A: UNINTEGRATED TAIL EMISSION AMPLITUDES

$$i\mathcal{A}_{\mu\nu}^{(e\text{-}M\text{-tail})}(\omega,\mathbf{k}) = \frac{(-i)^{r+1}c_r^{(I)}}{\Lambda^2} EI^{ijR}(\omega)\int_{\mathbf{q}} \frac{(k+q)_R}{[(\mathbf{k}+\mathbf{q})^2-\omega^2]\mathbf{q}^2} f_{\mu\nu}^{(e-M)}, \tag{A1}$$

$$i\mathcal{A}_{\mu\nu}^{(m\text{-}M\text{-tail})}(\omega,\mathbf{k}) = \frac{(-i)^{r+1}c_r^{(J)}}{\Lambda^2} EJ^{b|iRa}(\omega)\int_{\mathbf{q}} \frac{(k+q)_b(k+q)_R}{[(\mathbf{k}+\mathbf{q})^2-\omega^2]\mathbf{q}^2} f_{\mu\nu}^{(m-M)}. \tag{A2}$$

$$f_{00}^{(e-M)} = \left(-\frac{i}{4}\right)\left[\frac{1}{c_d}(k+q)_i(k+q)_j(\mathbf{k}+\mathbf{q})\cdot\mathbf{q} + (k_ik_j+k_iq_j+q_iq_j)\omega^2\right], \tag{A3}$$

$$f_{0k}^{(e-M)} = \left(\frac{i\omega}{4}\right)\left\{\frac{1}{c_d}(k+q)_i(k+q)_jq_k + (k+q)_i[(\mathbf{k}+\mathbf{q})\cdot\mathbf{k}\delta_{jk} - k_jq_k + k_kq_j] - \omega^2\delta_{jk}q_i\right\}, \tag{A4}$$

$$f_{kl}^{(e-M)} = \left(-\frac{i}{4}\right)\left\{\omega^4\delta_{ik}\delta_{jl} + \omega^2(k+q)_i[q_j\delta_{kl} - (q_k\delta_{jl}+q_l\delta_{jk})] + \frac{1}{c_d}(k+q)_i(k+q)_j[k_kq_l + k_lq_k + 2q_kq_l - (\mathbf{k}+\mathbf{q})\cdot\mathbf{q}\delta_{kl}]\right\}, \tag{A5}$$

$$f_{00}^{(m-M)} = \left(-\frac{1}{8}\right)\omega(k_aq_i + k_iq_a), \tag{A6}$$

$$f_{0k}^{(m-M)} = \left(\frac{1}{8}\right)[\omega^2(q_i\delta_{ak} + q_a\delta_{ik}) - (k+q)_i(k+q)_c(k_c\delta_{ak} - k_a\delta_{ck})], \tag{A7}$$

$$f_{kl}^{(m-M)} = \left(\frac{1}{8}\right)\omega[\omega^2(\delta_{ak}\delta_{il} + \delta_{ik}\delta_{al}) - (k+q)_i(q_l\delta_{ak} + q_k\delta_{al}) + \delta_{kl}(k+q)_iq_a], \tag{A8}$$

$$i\mathcal{A}_{\mu\nu}^{(e\text{-}L\text{-tail})}(\omega,\mathbf{k}) = \frac{(-i)^r c_r^{(I)}}{\Lambda^2}J^{b|a}I^{ijR}(\omega)\int_{\mathbf{q}}\frac{q_a(k+q)_R}{[(\mathbf{k}+\mathbf{q})^2-\omega^2]\mathbf{q}^2}f_{\mu\nu}^{(e-L)}, \tag{A9}$$

$$i\mathcal{A}_{\mu\nu}^{(m\text{-}L\text{-tail})}(\omega,\mathbf{k}) = \frac{(-i)^r c_r^{(J)}}{\Lambda^2}J^{s|t}J^{b|iRa}(\omega)\int_{\mathbf{q}}\frac{q_t(k+q)_{b'}(k+q)_R}{[(\mathbf{k}+\mathbf{q})^2-\omega^2]\mathbf{q}^2}f_{\mu\nu}^{(m-L)}, \tag{A10}$$

$$f_{00}^{(e-L)} = \left(-\frac{i\omega}{8}\right)\{\delta_{jb}\omega^2q_i + (k+q)_j[k_b(2k_i - q_i) + 3(\mathbf{k}+\mathbf{q})\cdot\mathbf{q}\delta_{ib}]\}, \tag{A11}$$

$$\begin{aligned} f_{0k}^{(e-L)} = \left(\frac{i}{8}\right)&\{(k+q)_i(k+q)_j(\mathbf{k}\cdot\mathbf{q}\delta_{bk} - k_bq_k) \\ &+[\mathbf{k}\cdot\mathbf{q}\delta_{ib}\delta_{jk} - \delta_{bk}q_iq_j + q_j(k+q)_k\delta_{ib} + (2k+q)_jk_b\delta_{ik}]\omega^2\}, \end{aligned} \tag{A12}$$

$$\begin{aligned} f_{kl}^{(e-L)} = \left(\frac{i\omega}{16}\right)&(\delta_{kc}\delta_{ld} + \delta_{kd}\delta_{lc})\{-[q_j\delta_{ib}\delta_{cd} + 2(q_i\delta_{bc} - q_c\delta_{ib} + k_b\delta_{ic})\delta_{jd}]\omega^2 \\ &+ (k+q)_j[2(k+q)_cq_i\delta_{bd} - k_bq_i\delta_{cd} - 2(k+q)_cq_d\delta_{ib} + (\mathbf{k}+\mathbf{q})\cdot\mathbf{q}\delta_{cd}\delta_{ib} \\ &+2k_bq_d\delta_{ic} - 2(\mathbf{k}+\mathbf{q})\cdot\mathbf{q}\delta_{bd}\delta_{ic}]\}, \end{aligned}$$

$$f_{00}^{(m-L)} = \left(-\frac{1}{16}\right)[\omega^2(q_i\delta_{as} - k_a\delta_{is}) + (k+q)_i(k_a(3k+2q)_s + 3(\mathbf{k}+\mathbf{q})\cdot\mathbf{q}\delta_{as})], \tag{A13}$$

$$\begin{aligned} f_{0k}^{(m-L)} = \left(\frac{\omega}{16}\right)&[(2k+q)_ik_s\delta_{ak} + (k+q)_kq_i\delta_{as} + (k_ak_s + \mathbf{k}\cdot\mathbf{q}\delta_{as})\delta_{ik} \\ &+(\mathbf{k}\cdot\mathbf{q}\delta_{ak} - k_a(k+q)_k)\delta_{is} + 2k_aq_i\delta_{ks}], \end{aligned} \tag{A14}$$

$$\begin{aligned} f_{kl}^{(m-L)} = \left(\frac{1}{16}\right)&\{(k+q)_i[(k_kq_l + k_lq_k)\delta_{as} + 2q_kq_l\delta_{as} + k_a(k+q)_l\delta_{sk} \\ &+ (\mathbf{k}+\mathbf{q})\cdot\mathbf{q}(\delta_{al}\delta_{sk} + \delta_{ak}\delta_{sl}) + k_a(k+q)_k\delta_{sl} - k_s(q_k\delta_{al} + q_k\delta_{al})] \\ &- \omega^2[(q_k\delta_{il} + q_l\delta_{ik})\delta_{as} + (q_k\delta_{al} + q_l\delta_{ak})\delta_{is} - q_i(\delta_{ak}\delta_{sl} + \delta_{al}\delta_{sk})] \\ &+ \delta_{kl}[\omega^2(q_i\delta_{as} - k_a\delta_{is}) - k_i\delta_{as}(\mathbf{k}+\mathbf{q})\cdot\mathbf{q} - q_i(\mathbf{k}+\mathbf{q})\cdot\mathbf{q}\delta_{as} - (k+q)_ik_ak_s] \\ &+ k_a(\delta_{ik}\delta_{sl} + \delta_{il}\delta_{sk}) - 2k_s(\delta_{al}\delta_{ik} + \delta_{ak}\delta_{il})\}. \end{aligned} \tag{A15}$$

APPENDIX B: MECHANICAL ANGULAR MOMENTUM FLUX AND IN-IN FORMALISM

The real parts of the effective actions associated to tail and ftail processes via Feynman Green's functions carry information about the time-symmetric part of the equations of motion. Their imaginary parts carry the information of the probability loss, from which it is possible to recover the energy and angular momentum loss with standard methods [51,52].

However, adopting the in-in formalism [44,45], it is possible to derive time-asymmetric equations of motion by writing a functional $\mathcal{S}_{in-\text{in}}$ for a degree of freedom propagating forward in time "1" and one backward "2":

$$\mathcal{S}_{\text{in-in}} = \int dt(\mathcal{L}_1 - \mathcal{L}_2). \tag{B1}$$

The in-in functional is then written in terms of the Keldysh+, − variables defined, for a generic dynamical variable x, as $x_+ \equiv (x_1 + x_2)/2$, $x_- \equiv x_1 - x_2$, in terms of which, e.g., (35) can be recast into its in-in counterpart

$$\mathcal{S}^{(LI^2)}_{\text{in-in}} = \frac{G^2}{30}\int \mathrm{d}t(\dddot{I}_{+ik}\ddot{I}_{-jk} + \dddot{I}_{-ik}\ddot{I}_{+jk})\epsilon_{ijq}L^q, \quad \text{(B2)}$$

and the equation of motion (40) can be derived from $\frac{\delta S^{(LI^2)}_{in-in}}{\delta x_-}|_{x_-=0} = 0$, and by taking the physical limit $x_+ \to x$, leading to

$$\epsilon_{ijq}\dot{L}^q|_{(BT-LI^2)} = -\frac{2G^2}{15}L^q I_{jk}(I^{(7)}_{kl}\epsilon_{ilq} - I^{(7)}_{il}\epsilon_{klq}) - i \leftrightarrow j. \quad \text{(B3)}$$

The angular momentum loss involving the magnetic quadrupole L-ftail is derived along the same lines.

APPENDIX C: NONLINEARITIES IN THE GAUGE CONDITIONS

The gauge we have been using to compute amplitudes is the harmonic gauge, defined by

$$\Gamma^\alpha_{\mu\nu}g^{\mu\nu} = 0. \quad \text{(C1)}$$

When expanded to first order in the fields, we obtain the condition

$$\partial^\mu \bar{h}_{\mu\nu} = \partial^\mu\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = 0. \quad \text{(C2)}$$

This condition holds only for the leading-order processes. For higher orders, on the other hand, like the tails and L-ftails studied in this paper, we have to solve Eq. (C1) iteratively in G. In this case, the general structure of the problem can be organized as

$$\begin{aligned}
&\partial^\mu \bar{h}^{(1)}_{\mu\nu} = 0,\\
&\partial^\mu \bar{h}^{(2)}_{\mu\nu} = \lambda^{(2)}_{\mu\nu}(h^{(1)}, h^{(1)}),\\
&\partial^\mu \bar{h}^{(3)}_{\mu\nu} = \lambda^{(3)}_{\mu\nu}(h^{(1)}, h^{(1)}, h^{(1)}) + \gamma^{(3)}_{\mu\nu}(h^{(2)}, h^{(1)}),\\
&\qquad \cdots,
\end{aligned} \quad \text{(C3)}$$

where $\bar{h}^{(n)}_{\mu\nu}$ represents processes of order G^n and $\lambda^{(n)}_{\mu\nu}$, $\gamma^{(n)}_{\mu\nu}$, etc., are functions of the lower-order perturbations $h^{(n-1)}, h^{(n-2)}, \ldots, h^{(1)}$. In particular, at order G^2, we have

$$\partial^\mu \bar{h}^{(2)}_{\mu\alpha} = h^{(1)\mu\nu}\left(h^{(1)}_{\alpha\mu,\nu} - \frac{1}{2}h^{(1)}_{\mu\nu,\alpha}\right). \quad \text{(C4)}$$

Below we show that, for processes of order G^2, the functions appearing on the right-hand side of Eqs. (C3) vanish on shell, and therefore, do not contribute for self-energy diagrams of tail-like processes. To show this, consider two processes of order G, given generically by

$$\begin{aligned}
h^{(1)}_{\mu\nu} &= G\int\frac{d\omega}{2\pi}A^R(\omega)\int_{\mathbf{k}}\frac{e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 - \omega^2}K_L, \quad \text{and}\\
h'^{(1)}_{\mu\nu} &= G\int\frac{d\omega'}{2\pi}B^{R'}(\omega')\int_{\mathbf{q}}\frac{e^{-i\omega' t + i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2 - \omega'^2}Q_{L'},
\end{aligned} \quad \text{(C5)}$$

where $A^R(\omega)$ and $B^{R'}(\omega')$ represent the integrand for arbitrary multipoles, including the ones related to conserved multipoles, by making, e.g., $A^R(\omega) \to E\delta(\omega)$. K_L represents any combination of the momenta k's, and likewise for $Q_{L'}$. By plugging this into the right-hand side of Eq. (C4), we encounter the following behavior:

$$\begin{aligned}
\partial^\mu \bar{h}_{\mu\nu} &\sim G^2\partial^\mu\left[\int\frac{d\omega}{2\pi}A^R(\omega)\int_{\mathbf{k}}\frac{e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 - \omega^2}K_L\right.\\
&\qquad\left.\times\int\frac{d\omega'}{2\pi}B^{R'}(-\omega')\int_{\mathbf{q}}\frac{e^{i\omega' t - i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2 - \omega^2}Q_{L'}\right]\\
&\to \int_{\mathbf{k}}\frac{d\omega}{2\pi}\frac{e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 - (\omega + i\mathrm{a})^2}\times\frac{[k^\mu \bar{\mathcal{A}}_{\mu\nu}]}{\Lambda^2},
\end{aligned} \quad \text{(C6)}$$

where

$$\begin{aligned}
k^\mu \bar{\mathcal{A}}_{\mu\nu} &= (\omega^2 - \mathbf{k}^2)\int\frac{d\omega'}{2\pi}A^R(\omega + \omega')B^{R'}(-\omega')\\
&\quad\times\int_{\mathbf{q}}\frac{(KQ)_{LL'}}{[(\mathbf{k}+\mathbf{q})^2 - (\omega + \omega')^2](\mathbf{q}^2 - \omega'^2)}.
\end{aligned} \quad \text{(C7)}$$

Notice that this expression is always vanishing on-shell, and hence, will not play any role in the construction of self-energy diagrams for tail-like processes, see the appendix below. This justifies the use of the linearized Lorentz condition $\partial^\mu \bar{h}_{\mu\nu} = 0$ for processes of order G^2.

APPENDIX D: CUTTING AND GLUING AMPLITUDES

We present in this section a heuristic derivation of generalized unitarity applied to tail-like self-energy diagrams. The gluing of emission amplitudes can be written as

$$\begin{aligned}
i\tilde{\mathcal{S}}^{(\text{tail})} &= -\frac{i}{\Lambda^2}\int_{\mathbf{k}}\frac{d\omega}{2\pi}\frac{\mathcal{A}^{(\text{tail})TT}_{ij}(\omega,\mathbf{k})\mathcal{A}^{(LO)TT}_{ij}(-\omega,-\mathbf{k})}{\mathbf{k}^2 - \omega^2}\\
&= -\frac{i}{16\pi^2\Lambda^2}\int \mathrm{d}\Omega\int\frac{d\omega}{2\pi}(i\omega)\mathcal{A}^{(\text{tail})TT}_{ij}(\omega,\omega\hat{\mathbf{n}})\\
&\quad\times\mathcal{A}^{(LO)TT}_{ij}(-\omega,-\omega\hat{\mathbf{n}})\\
&= -\frac{i}{16\pi^2\Lambda^2}\int_t\int \mathrm{d}\Omega\dot{\mathcal{A}}^{(\text{tail})\mathrm{TT}}_{ij}(t,\hat{\mathbf{n}})\mathcal{A}^{(LO)TT}_{ij}(t,\hat{\mathbf{n}}),
\end{aligned} \quad \text{(D1)}$$

the first passage holding because of the useful identity:

$$\int_{\mathbf{k}} \frac{k_{i_1}\dots k_{i_{2l}}}{\mathbf{k}^2-(\omega\pm i\mathrm{a})^2} = \left(\mp i\frac{\omega}{4\pi}\right)\delta_{i_1\dots i_{2l}}\frac{\omega^{2l}}{(2l+1)!!} = \left(\mp i\frac{\omega}{16\pi^2}\right)\omega^{2l}\int \mathrm{d}\Omega\,\hat{n}_{i_1}\dots\hat{n}_{i_{2l}}, \tag{D2}$$

which can be inserted into (D1) as $\mathcal{A}^{(\mathrm{tail})}, \mathcal{A}^{(LO)}$ have no poles in $\mathbf{k}$. To derive the non time-symmetric equations of motion, it is necessary to recast the action (D1) into its in-in counterpart, as described in Appendix B, to obtain the mechanical energy loss

$$-\mathbf{a}\cdot\mathbf{v} = \frac{1}{16\pi^2\Lambda^2}\int \mathrm{d}\Omega \dddot{\mathcal{A}}_{ij}^{(\mathrm{tail})\mathrm{TT}}(t,\hat{\mathbf{n}})(\mathcal{A}_{ij}^{(LO)TT}(t,\hat{\mathbf{n}}))^{(-1)}, \tag{D3}$$

where a double integration by parts over $\mathcal{A}^{(LO)}$, which is linear in $\ddot{I}_{ij}$, introduces the "antiderivative" $(\mathcal{A}_{ij}^{(LO)})^{(-1)} \propto \dot{I}_{ij}$.

Had we used the Feynman boundary condition in (D2), we would have obtained $-i|\omega|$ instead of $\mp i\omega$ in the first parentheses, giving rise to the standard optical theorem relationship between self-energy imaginary part and emission probability, which can be related to the energy loss by multiplying the ω-integrand by $|\omega|$. While a consistent use of retarded/advanced Green's functions requires the in-in formalism, see Appendix B [53], if we limit ourselves to the conservative dynamics and the computation of the energy flux one can use Feynman Green's functions.

The energy loss of Eq. (D3) agrees, modulo Schott terms, with the one obtained by direct computation of the gravitational luminosity at infinity $\mathcal{F}$, via the asymptotic GW waveform

$$h_{ij}^{TT} \simeq -\frac{1}{4\pi r}\int_\omega \left(-\frac{1}{\Lambda^2}\right)\mathrm{e}^{-i\omega t_{\mathrm{ret}}}\dot{\mathcal{A}}_{ij}^{TT}(\omega,\mathbf{n}\omega) = \frac{1}{4\pi r\Lambda^2}\dot{\mathcal{A}}_{ij}^{TT}(\mathbf{n},t_{\mathrm{ret}}), \tag{D4}$$

as

$$\mathcal{F} = \Lambda^2 r^2\int \mathrm{d}\Omega \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT} = \frac{1}{16\pi^2\Lambda^2}\int \mathrm{d}\Omega\,\dot{\mathcal{A}}_{ij}^{TT}(t_{\mathrm{ret}},\hat{\mathbf{n}})\dot{\mathcal{A}}_{ij}^{TT}(t_{\mathrm{ret}},\hat{\mathbf{n}}), \tag{D5}$$

with

$$\mathcal{A}_{ij}^{TT} \simeq \mathcal{A}_{ij}^{(LO)TT} + \mathcal{A}_{ij}^{(\mathrm{tail})TT}, \tag{D6}$$

and expanding at NLO.

Note that, considering the extra i provided by the integration over $\mathbf{k}$, see Eq. (D2), one has that the real part of $\mathcal{A}^{(\mathrm{tail})}/\mathcal{A}^{(LO)}$ contributes to the probability and energy loss, the imaginary part to the conservative dynamics,[6] and the self-energy action is completely determined by the emission amplitude.

[6]Note that the Fourier transform of a direct space real function is in general complex, here by real part of the ω integrand $A(\omega)$ of the effective action we mean a function satisfying $A^*(\omega) = A(-\omega)$.

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