

Starobinsky inflation with a quadratic Weyl tensor

Antonio De Felice,^{1,*} Ryodai Kawaguchi^{2,†}, Kotaro Mizui^{2,‡} and Shinji Tsujikawa^{2,§}

¹*Center for Gravitational Physics and Quantum Information, Yukawa Institute for Theoretical Physics, Kyoto University, 606-8502 Kyoto, Japan*

²*Department of Physics, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan*



(Received 4 September 2023; accepted 14 November 2023; published 12 December 2023)

In Starobinsky inflation with a Weyl squared Lagrangian $-\alpha C^2$, where α is a coupling constant, we study the linear stability of cosmological perturbations on a spatially flat Friedmann-Lemaître-Robertson-Walker background. In this theory, there are two dynamical vector modes propagating as ghosts for $\alpha > 0$. This condition is required to avoid tachyonic instabilities of vector perturbations during inflation. The tensor sector has four propagating degrees of freedom, among which two of them correspond to ghost modes. However, the tensor perturbations approach constants after the Hubble radius crossing during inflation, and hence the classical instabilities are absent. In the scalar sector, the Weyl curvature gives rise to a ghost mode coupled to the scalaron arising from the squared Ricci scalar. We show that two gauge-invariant gravitational potentials, which are both dynamical in our theory, are subject to exponential growth after the Hubble radius crossing. There are particular gauge-invariant combinations like the curvature perturbations whose growth is suppressed, but it is not possible to remove the instability of other propagating degrees of freedom present in the perturbed metric. This violent and purely classical instability present in the scalar sector makes the background unviable. Furthermore, the presence of such classical instability makes the quantization of the modes irrelevant, and the homogeneous inflationary background is spoiled by the Weyl curvature term.

DOI: [10.1103/PhysRevD.108.123524](https://doi.org/10.1103/PhysRevD.108.123524)

I. INTRODUCTION

The inflationary paradigm [1–4] can successfully resolve several shortcomings in big bang cosmology, e.g., the horizon and flatness problems. Moreover, it can explain the origin of large-scale structures in the Universe by stretching quantum fluctuations over super-Hubble scales during the accelerated expansion [5–9]. The spectra of scalar and tensor perturbations predicted in standard slow-roll inflation are consistent with the observed cosmic microwave background (CMB) temperature anisotropies. After the data release of WMAP [10] and Planck [11] satellites, we have been able to distinguish between many different models of inflation. In particular, the first model advocated by Starobinsky [5] is still one of the best-fit models to the Planck CMB data [12].

In the Starobinsky model, inflation is driven by a quadratic Ricci scalar term βR^2 , where β is a positive coupling constant. The period of cosmic acceleration ends when the βR^2 term drops below the Ricci scalar R [13–17]. The quadratic curvature scalar gives rise to a new

propagating degree of freedom (d.o.f.) dubbed the “scalaron” [5] with a mass squared $m_s^2 = 1/(6\beta)$ on the Minkowski background [18–22]. Indeed, the $f(R)$ gravity given by the Lagrangian $f(R) = R + \beta R^2$ is equivalent to Brans-Dicke theory [23] with a scalaron potential arising from the gravitational sector [24–26]. From the observed amplitude of CMB temperature fluctuations, the scalaron mass is constrained to be $m_s \simeq 10^{-5} M_{\text{pl}}$, where M_{pl} is the reduced Planck mass [15,27–29]. The Starobinsky model predicts the scalar spectral index $n_s \simeq 1 - 2/N$ and the tensor-to-scalar ratio $r \simeq 12/N^2$ [29–31], where N is the number of e -foldings counted backward from the end of inflation. On scales relevant to the CMB observations ($N \simeq 55 \sim 60$), the theoretical predictions of n_s and r are well-consistent with the Planck data combined with other data [12].

From the viewpoint of an ultraviolet completion of gravity, there are also other quadratic curvature contributions to the Lagrangian constructed from scalar products of the Riemann tensor $R_{\mu\nu\rho\sigma}$ and the Ricci tensor $R_{\mu\nu}$ [32]. Given that the Gauss-Bonnet curvature invariant $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is a topological term that does not affect the field equations of motion [33], the general quadratic-order Lagrangian can be expressed in the form $L_2 = -\alpha C^2 + \beta R^2$, where $C^2 = 2R_{\mu\nu}R^{\mu\nu} - 2R^2/3 + \mathcal{G}$ is a squared Weyl curvature. This quadratic theory of gravity,

*antonio.defelice@yukawa.kyoto-u.ac.jp

†ryodai0602@fuji.waseda.jp

‡mizmiz8612@akane.waseda.jp

§tsujikawa@waseda.jp

which was originally advocated by Stelle [34], is renormalizable and also asymptotically free [35]. However, the Weyl curvature generally gives rise to ghost d.o.f.s associated with derivative terms higher than second order in the field equations of motion [36].

Albeit the appearance of ghosts in Weyl gravity with the Lagrangian $-\alpha C^2$, the perturbative expansion about the Minkowski vacuum shows that all the linear perturbations in scalar, vector, and tensor sectors propagate with the speed of light [37–39]. This means that, in the absence of additional matter sources, the perturbations are not subject to Laplacian instabilities. Hence, on Minkowski, the Weyl ghosts can be of “soft” types [40], i.e., even in the presence of the ghosts, the classical perturbations do not grow by either Laplacian or tachyonic instabilities on the given background. However, this situation should be different by introducing some matter fields or by taking into account the βR^2 term on curved backgrounds. The latter corresponds to Stelle’s quadratic curvature theory mentioned above, in which case the scalaron field arising from the βR^2 term can be gravitationally coupled to the Weyl ghost.

In this paper, we would like to address the stability of linear perturbations on the inflationary background realized by Stelle’s theory. We note that there are some related works in which the dynamics of cosmological perturbations during inflation were discussed in the presence of the Weyl curvature term [41–45]. Most of those papers assumed the existence of a canonical scalar field besides the Weyl and Einstein-Hilbert terms. Since the squared Weyl curvature is conformally invariant, the Lagrangian $L = R - \alpha C^2 + \beta R^2$ of Stelle’s theory can be transformed to that in the Einstein frame with kinetic and potential energies of the scalaron field as well as the Weyl term [46–49].

The analysis of tensor perturbations on a spatially-flat Friedmann-Lemaître-Robertson-Walker (FLRW) background [41] showed that the tensor ghosts can be soft during inflation in that they do not grow by classical instabilities. In the vector sector, there are two dynamical propagating d.o.f.s arising from the Weyl curvature term $-\alpha C^2$ [42]. The ghosts do not appear if $\alpha < 0$, but vector perturbations are subject to tachyonic instabilities. For $\alpha > 0$ the ghosts are present, but vector perturbations decay during inflation. Thus, despite the presence of ghosts for $\alpha > 0$, both tensor and vector perturbations are not prone to classical Laplacian or tachyonic instabilities. These results were already recognized in Refs. [41,42] according to the analysis in the Einstein frame, but we will study whether a similar property holds in the Jordan frame. Indeed, for $\alpha > 0$, the classical instabilities are absent for both the tensor and vector sectors.

In the scalar sector, the analysis of Ref. [42] in the Einstein frame of Stelle’s theory showed that gravitational potentials in a Newtonian gauge exhibit rapid growth after the Hubble radius crossing during inflation. On the other hand, the same paper also found that the curvature

perturbation in a comoving gauge remains bounded. In Ref. [44], it was claimed that the former growth of gravitational potentials is a gauge artifact and that scalar perturbations are not prone to real instabilities. So far, it is not yet clear whether the instability in the scalar sector induced by the Weyl term corresponds to a real, physical one. To clarify this issue, we need to scrutinize whether the instability of scalar perturbations generally persists or not independent of the gauge choices.

In this paper, we will study the evolution of cosmological perturbations during inflation in the Jordan frame of Stelle’s theory by paying particular attention to the classical stability of the scalar modes. For this purpose, we choose several different physical gauges and analytically derive the closed fourth-order differential equations for the gravitational potentials as well as other gauge-invariant variables like curvature perturbations. We will explicitly show that two dynamical propagating d.o.f.s arise from the βR^2 term (i.e., scalaron) and the Weyl curvature, one of which always behaves as a ghost mode. Therefore, in general, there are four independent initial conditions necessary to uniquely specify the classical evolution of the scalar sector, and that is the reason why the system can be described in terms of a closed fourth-order differential equation for one single scalar mode, or evidently, by two second-order differential equations for two independent fields.

We will show that the two gravitational potentials Ψ and Φ , which are both propagating d.o.f.s, exponentially grow after the Hubble radius crossing. This instability of Ψ and Φ occurs independently of the gauge choice made to study their dynamics. Among other relevant gauge-invariant variables, we also find that the curvature perturbation is a specific variable approaching a constant in the large-scale limit. However, the exponential increase of at least one dynamical scalar d.o.f. appearing in the perturbed line element does not allow the FLRW spacetime to be a stable cosmological background. Thus, the inflationary background is violated by this real, physical, and classical instability of scalar perturbations induced by the Weyl ghost coupled to the scalaron. Therefore, we conclude the propagating ghost d.o.f. in the scalar sector is not of the soft type. Well before the end of inflation, the cosmological background is spoiled by the classical instability and it changes to a highly inhomogeneous Universe. In such a context, we believe that this lack of a homogeneous background makes the quantization of perturbations irrelevant. If the ghost modes were soft, then the quantization procedure would acquire relevance and the results could be interesting. However, this is not the case for inflation in quadratic gravity with the Weyl term.

This paper is organized as follows. In Sec. II, we briefly review the background dynamics of inflation realized in Stelle’s theory. In Sec. III, we revisit how vector perturbations propagate as truly dynamical d.o.f.s and show that the absence of tachyonic instabilities requires the condition

$\alpha > 0$. In Sec. IV, we see that, despite the appearance of ghosts arising from the Weyl term, the four dynamical d.o.f.s of tensor perturbations approach constants after the Hubble radius crossing. In Sec. V, we study the evolution of scalar perturbations by choosing several different gauges and show that, independently of the gauge choices, two modes propagate and the FLRW background is spoiled by the presence of instabilities of at least one of the dynamical d.o.f.s present in the perturbed line element. Although the classical instability itself is present for any nonzero initial conditions of scalar modes, we confirm in Sec. VI its presence by numerically integrating the perturbation equations of motion with proper initial conditions. Section VII is devoted to conclusions.

II. INFLATION IN QUADRATIC GRAVITY

The action in quadratic gravity contains scalar products of two contractions of the Riemann tensor $R_{\mu\nu\rho\sigma}$, Ricci tensor $R_{\mu\nu}$, and Ricci scalar R . On using the property that the Gauss-Bonnet term $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is topological, the Riemann products $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ can be eliminated from the action. Taking the Einstein-Hilbert term $M_{\text{pl}}^2 R/2$ into account, the action of quadratic gravity can be expressed in the form [34]

$$S = \frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{-g} (R - \alpha C^2 + \beta R^2), \quad (2.1)$$

where g is a determinant of the metric tensor $g_{\mu\nu}$, α , and β are constants, and C^2 is the Weyl tensor squared given by

$$C^2 = 2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2 + \mathcal{G}. \quad (2.2)$$

Up to boundary terms, the action (2.1) can be expressed as

$$S = \frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{-g} \left[R - 2\alpha R_{\mu\nu}R^{\mu\nu} + \left(\frac{2}{3}\alpha + \beta \right) R^2 \right]. \quad (2.3)$$

We consider a spatially flat FLRW background described by the line element

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad (2.4)$$

where $a(t)$ is a time-dependent scale factor, and $N(t)$ is a lapse function. Varying the action (2.3) with respect to $N(t)$ and $a(t)$, respectively, and setting $N(t) = 1$ at the end, it follows that

$$H^2 + 6\beta(6H^2\dot{H} - \dot{H}^2 + 2H\ddot{H}) = 0, \quad (2.5)$$

$$2\dot{H} + 3H^2 + 6\beta(18H^2\dot{H} + 9\dot{H}^2 + 12H\ddot{H} + 2\ddot{H}) = 0, \quad (2.6)$$

where $H = \dot{a}/a$ is the Hubble expansion rate, with a dot being the derivative with respect to t . At the background level, the Weyl curvature term does not contribute to the field equations of motion. This is an outcome of the conformal invariance of the Weyl curvature term, whose components vanish for the conformally flat background. Taking the time derivative of (2.5) and combining it with Eq. (2.5), we obtain the same equation as (2.6). This means that there is only a single independent equation of motion, Eq. (2.5), governing the background dynamics.

During inflation, the Hubble expansion rate is nearly constant, and hence the last two terms in the parenthesis of Eq. (2.5) are suppressed relative to the term $6H^2\dot{H}$. Then, there is the approximate relation

$$\dot{H} \simeq -\frac{1}{36\beta} = -\frac{m_S^2}{6}, \quad (2.7)$$

where m_S^2 corresponds to the mass squared of a scalaron field given by [18–22]

$$m_S^2 = \frac{1}{6\beta}. \quad (2.8)$$

Provided that

$$\beta > 0, \quad (2.9)$$

there is no tachyonic instability arising from the negative value of m_S^2 . Under the condition (2.9), the Hubble parameter (2.7) also decreases during inflation. We will impose the condition (2.9) throughout the discussion below.

From Eq. (2.7), we obtain the following integrated solutions:

$$H(t) \simeq H_i - \frac{m_S^2}{6}(t - t_i), \quad (2.10)$$

$$a(t) \simeq a_i \exp \left[H_i(t - t_i) - \frac{m_S^2}{12}(t - t_i)^2 \right], \quad (2.11)$$

where H_i and a_i are the values of H and a at the onset of inflation, respectively. We introduce the slow-roll parameter ϵ , as

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \simeq \frac{m_S^2}{6H^2}. \quad (2.12)$$

The end of inflation is characterized by the Hubble parameter H_f when ϵ becomes equivalent to 1, so that $H_f \simeq m_S/\sqrt{6}$. As we will see below, H_i is larger than m_S . Then, by using Eq. (2.10), we can approximately estimate the time t_f at the end of inflation, as $t_f \simeq t_i + 6H_i/m_S^2$.

The number of e -foldings acquired during inflation is given by

$$N \equiv \int_{t_i}^{t_f} H dt \simeq H_i(t_f - t_i) - \frac{m_S^2}{12}(t_f - t_i)^2 \simeq \frac{3H_i^2}{m_S^2}. \quad (2.13)$$

Taking $N = 60$ as a typical minimal e -folding number required to address the horizon and flatness problems, we obtain the initial Hubble parameter $H_i \simeq 4.5m_S$. This value translates to

$$\beta H_i^2 \simeq 3.4, \quad (2.14)$$

and hence βH^2 is of order 1 during inflation. From the viewpoint of ultraviolet completion of gravity, it is natural to consider the value of $|\alpha|$ same order as β .

We note that quadratic gravity given by the action (2.1) corresponds to the $f(R) = R + \beta R^2$ theory with the Weyl squared term $-\alpha C^2$. Under a conformal transformation of the metric tensor $g_{\mu\nu}$, the theory can be transformed to a metric frame described by Einstein's gravity in the presence of a scalaron field with the potential and the Weyl squared term [46–49]. In this Einstein frame, the dynamics of cosmological perturbations during inflation were carried out in Refs. [41–45] without necessarily relating the scalaron potential with the one arising from the original Lagrangian $f(R) = R + \beta R^2$. In this paper, we will perform all the analysis in the physical Jordan frame. To study the dynamics of perturbations during inflation, we do not need to take into account additional matter sources to the Jordan-frame action (2.1).

Around the background (2.4) with $N(t) = 1$, we consider metric perturbations which depend on the cosmic time t and spatial coordinates x^i . The perturbed line element is given by

$$ds^2 = -(1 + 2A)dt^2 + 2a(t)(\partial_i B + V_i)dt dx^i + a^2(t) \times [(1 + 2\psi)\delta_{ij} + 2\partial_i \partial_j E + \partial_i F_j + \partial_j F_i + h_{ij}]dx^i dx^j, \quad (2.15)$$

where we used the notation $\partial_i = \partial/\partial x^i$, and the Latin indices represent spatial coordinates. The four quantities A, B, ψ , and E correspond to scalar perturbations, while V_i and F_i are vector perturbations satisfying the divergence-free conditions $\delta^{ij}\partial_j V_i = 0$ and $\delta^{ij}\partial_j F_i = 0$. The intrinsic tensor perturbation is given by h_{ij} , which satisfies the traceless and transverse conditions $h_i^i = 0$ and $\delta^{ik}\partial_k h_{ij} = 0$.

III. VECTOR PERTURBATIONS

In quadratic gravity given by the action (2.3), we first study the dynamics of vector perturbations during inflation. Since there is the residual gauge d.o.f., we choose the following gauge condition

$$F_i = 0, \quad (3.1)$$

where $i = 1, 2, 3$. Then, the perturbed line element in the vector sector is given by

$$ds^2 = -dt^2 + 2a(t)V_i dt dx^i + a^2(t)\delta_{ij}dx^i dx^j. \quad (3.2)$$

For practical computations, it is convenient to choose the vector-field configuration in the form

$$V_i = [V_1(t, z), V_2(t, z), 0], \quad (3.3)$$

which satisfies the divergence-free condition $\partial^i V_i = 0$. Expanding the action (2.3) up to quadratic order in vector perturbations, integrating it by parts, and using the background Eq. (2.5), the second-order action yields

$$\mathcal{S}_v^{(2)} = -\frac{M_{\text{Pl}}^2}{2}\alpha \sum_{i=1,2} \int d^4x \times a \left\{ \dot{U}_i^2 - \frac{U_i'^2}{a^2} - \left[\frac{1}{2\alpha} + \frac{6\beta}{\alpha}(2H^2 + \dot{H}) \right] U_i^2 \right\}, \quad (3.4)$$

where a prime represents the derivative with respect to z , and $U_i \equiv V_i'$. The Weyl curvature term gives rise to two dynamical vector perturbations U_1 and U_2 . For $\alpha > 0$, it is clear from the action (3.4) that both U_1 and U_2 behave as ghosts. On the other hand, the vector ghosts are absent if $\alpha < 0$.

Varying the action (3.4) with respect to U_i (with $i = 1, 2$), we obtain their equations of motion in real space. Then, we perform the Fourier transformation

$$U_i = \frac{1}{(2\pi)^{1/2}} \int dk \tilde{U}_i(t, k) e^{ikz}, \quad (3.5)$$

where k is a comoving wave number, and \tilde{U}_i is a function of t and k . Omitting the tilde from $\tilde{U}_i(t, k)$ in the following, we obtain the vector perturbation equations of motion in Fourier space, as

$$\ddot{U}_i + H\dot{U}_i + \left[\frac{k^2}{a^2} + m_W^2 + \frac{6\beta}{\alpha}(2H^2 + \dot{H}) \right] U_i = 0, \quad (3.6)$$

where

$$m_W^2 \equiv \frac{1}{2\alpha}, \quad (3.7)$$

is the mass squared arising from the Weyl curvature [36–39]. We note that the Lagrangian βR^2 contributes to the vector mass through the term $(6\beta/\alpha)(2H^2 + \dot{H})$. During inflation

($|\dot{H}| \ll H^2$), the effective mass squared of vector perturbations is approximately given by

$$m_{\text{eff}}^2 \simeq m_W^2 + \frac{12\beta}{\alpha} H^2 = \frac{1 + 24\beta H^2}{2\alpha}. \quad (3.8)$$

Since we are considering a positive coupling β , we have $m_{\text{eff}}^2 > 0$ if $\alpha > 0$ and $m_{\text{eff}}^2 < 0$ if $\alpha < 0$.

For the Weyl coupling constant in the range $|\alpha| \lesssim \beta$, the effective mass squared (3.8) is at least of order H^2 . Then, after $(k/a)^2$ drops below $|m_{\text{eff}}^2|$ during inflation, Eq. (3.6) approximately reduces to

$$\ddot{U}_i + H\dot{U}_i + m_{\text{eff}}^2 U_i \simeq 0. \quad (3.9)$$

Since H and m_{eff}^2 can be approximated as constants during inflation, we obtain the following solution

$$U_i = c_1 \exp\left(\frac{-H + \sqrt{H^2 - 4m_{\text{eff}}^2}}{2} t\right) + c_2 \exp\left(\frac{-H - \sqrt{H^2 - 4m_{\text{eff}}^2}}{2} t\right), \quad (3.10)$$

where c_1 and c_2 are integration constants.

For $\alpha > 0$ (i.e., $m_{\text{eff}}^2 > 0$), if H is initially in the range $H > 2m_{\text{eff}}$, the amplitude of vector perturbations first decreases in proportion to $U_i \propto \exp[(-H + \sqrt{H^2 - 4m_{\text{eff}}^2})t/2]$. After H decreases below $2m_{\text{eff}}$, the vector perturbation starts to oscillate with the decreasing amplitude ($|U_i| \propto e^{-Ht/2}$). This means that, even though the two vector ghosts are present for $\alpha > 0$, vector perturbations decay exponentially during inflation.

For $\alpha < 0$, the negative mass squared m_{eff}^2 leads to the tachyonic growth of U_i . During inflation, the first term on the right-hand side of Eq. (3.10) corresponds to a growing mode. The increase of U_i is particularly prominent after H^2 drops below the order of $-m_{\text{eff}}^2$, during which $U_i \propto \exp(\sqrt{|m_{\text{eff}}^2|}t)$. Then, for $\alpha < 0$, the FLRW background is destroyed by the growth of vector perturbations. To avoid such an instability problem, we will focus on the coupling in the range

$$\alpha > 0, \quad (3.11)$$

in the following.

We recall that, in the above discussion, we have focused on the case $|\alpha| \lesssim \beta$. In the coupling range $|\alpha| \gg \beta$ with βH_i^2 of order 1, $|m_{\text{eff}}^2|$ is much smaller than H^2 during inflation. Taking the limit $|m_{\text{eff}}^2|/H^2 \rightarrow 0$ in Eq. (3.10), the rapid growth of U_i is absent. In Sec. V, however, we will show that scalar perturbations are subject to exponential instabilities in the coupling range $|\alpha| \gg \beta$.

IV. TENSOR PERTURBATIONS

We proceed to study the dynamics of tensor perturbations h_{ij} with the perturbed line element given by

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j, \quad (4.1)$$

with the traceless and transverse conditions $h_i^i = 0$ and $\partial^i h_{ij} = 0$. Without loss of generality, we can consider gravitational waves propagating along the z direction, whose nonvanishing components are

$$h_{11} = -h_{22} = \frac{h_1(t, z)}{\sqrt{2}}, \quad h_{12} = h_{21} = \frac{h_2(t, z)}{\sqrt{2}}, \quad (4.2)$$

where h_1 and h_2 are functions of t and z . These components of h_{ij} automatically satisfy the traceless and transverse conditions mentioned above.

Expanding the action (2.3) up to second order in h_1 and h_2 and integrating the action by parts, we obtain the following quadratic-order action

$$\begin{aligned} \mathcal{S}_t^{(2)} = & \frac{M_{\text{Pl}}^2}{4} \int d^4 x a^3 \sum_{i=1,2} \left[-\alpha \ddot{h}_i^2 - \frac{\alpha}{a^4} h_i'^2 + \frac{2\alpha}{a^2} \dot{h}_i^2 \right. \\ & + \left. \left\{ \frac{1}{2} + (\alpha + 6\beta)(2H^2 + \dot{H}) \right\} \dot{h}_i^2 \right. \\ & \left. - \left\{ \frac{1}{2} + 6\beta(2H^2 + \dot{H}) \right\} \frac{h_i'^2}{a^2} \right], \quad (4.3) \end{aligned}$$

where we also used the background Eq. (2.5). The presence of the Weyl curvature term gives rise to the fourth-order differential equation for h_i . We perform the Fourier transformation of h_i , as

$$h_i = \frac{1}{(2\pi)^{1/2}} \int dk \tilde{h}_i(t, k) e^{ikz}, \quad (4.4)$$

where \tilde{h}_i depends on t and the wave number k . In Fourier space, the second-order action $\tilde{\mathcal{S}}_t^{(2)} = \int dt d^3 k L_t$ can be obtained under the replacements $\ddot{h}_i^2 \rightarrow \ddot{\tilde{h}}_i^2$, $h_i'^2 \rightarrow k^4 \tilde{h}_i^2$, $\dot{h}_i^2 \rightarrow k^2 \tilde{h}_i^2$, $\dot{\tilde{h}}_i^2 \rightarrow \dot{\tilde{h}}_i^2$, and $h_i'^2 \rightarrow k^2 \tilde{h}_i^2$ in Eq. (4.3). Omitting the tilde from \tilde{h}_i , the second-order Lagrangian density in Fourier space is given by

$$\begin{aligned} L_t = & \frac{a^3 M_{\text{Pl}}^2}{4} \sum_{i=1,2} \left[-\alpha \ddot{h}_i^2 \right. \\ & + \left. \left\{ \frac{1}{2} + (\alpha + 6\beta)(2H^2 + \dot{H}) + \frac{2\alpha k^2}{a^2} \right\} \dot{h}_i^2 \right. \\ & \left. - \frac{k^2}{a^2} \left\{ \frac{1}{2} + 6\beta(2H^2 + \dot{H}) + \frac{\alpha k^2}{a^2} \right\} h_i^2 \right]. \quad (4.5) \end{aligned}$$

To understand the appearance of ghost d.o.f.s, we introduce Lagrangian multipliers χ_i (with $i = 1, 2$) such that

$$\bar{L}_t = L_t + \frac{a^3 M_{\text{Pl}}^2 \alpha}{4} \sum_{i=1,2} (\ddot{h}_i + c_1 \dot{h}_i + c_2 h_i - c_3 \chi_i)^2, \quad (4.6)$$

where c_i 's are time-dependent coefficients. We note that the coefficient $a^3 M_{\text{Pl}}^2 \alpha / 4$ in front of \ddot{h}_i^2 has been introduced to cancel the first term in Eq. (4.5). We fix c_i 's in Eq. (4.6) to obtain the Lagrangian density containing a canonical form

$$\begin{aligned} \bar{L}_t = & \frac{1}{2} a^3 M_{\text{Pl}}^2 \sum_{i=1,2} \left(\dot{h}_i^2 + 2\dot{h}_i \dot{\chi}_i + \left[3\beta(H^2 - \dot{H})(2H^2 + \dot{H}) - \frac{5H^2}{4} + \frac{\dot{H}}{2} - \frac{k^2}{a^2} + \frac{9\{1 - 4\beta(2H^2 + \dot{H})\}^2}{32\alpha} \right. \right. \\ & + \frac{\alpha}{12} \left. \left\{ 6H^2(H^2 - 2\dot{H}) - \frac{1}{\beta}(H^2 - \dot{H}) - 12\dot{H} \left(\frac{2k^2}{a^2} - 5\dot{H} \right) \right\} \right] h_i^2 + \frac{2}{\alpha} \chi_i^2 \\ & - \left. \left\{ 2H^2 - 2\dot{H} + \frac{2k^2}{a^2} + \frac{3[4\beta(2H^2 + \dot{H}) - 1]}{2\alpha} \right\} h_i \chi_i \right). \end{aligned} \quad (4.8)$$

From this expression, we find that there are four dynamical perturbations h_1 , h_2 , χ_1 , and χ_2 in the tensor sector, in agreement with the analysis of Refs. [38,41,42]. Terms containing the product of time derivatives in Eq. (4.8) can be expressed in the form

$$(\bar{L}_t)_K = \sum_{i=1,2} \dot{\vec{\psi}}_i \mathbf{K} \dot{\vec{\psi}}_i^T, \quad (4.9)$$

where $\vec{\psi}_i = (h_i, \chi_i)$, and \mathbf{K} is a 2×2 symmetric matrix whose components are

$$K_{11} = \frac{1}{2} a^3 M_{\text{Pl}}^2, \quad K_{12} = K_{21} = \frac{1}{2} a^3 M_{\text{Pl}}^2, \quad K_{22} = 0. \quad (4.10)$$

The absence of ghosts requires the following conditions

$$K_{11} = \frac{1}{2} a^3 M_{\text{Pl}}^2 > 0, \quad \text{and} \quad -K_{12}^2 = -\frac{1}{4} a^6 M_{\text{Pl}}^4 > 0. \quad (4.11)$$

While the former is satisfied, the latter is always violated. Hence the two ghosts are present, besides the other two no-ghost modes.

In the following, we will study the propagation of tensor perturbations during inflation. Varying the Lagrangian density (4.6) with respect to χ_i , it follows that:

$$\begin{aligned} \chi_i = & \frac{1}{2} \alpha \ddot{h}_i + \frac{3}{2} \alpha H \dot{h}_i \\ & + \frac{1}{2} \left[\frac{\alpha k^2}{a^2} + \alpha(H^2 - \dot{H}) + 3\beta(2H^2 + \dot{H}) - \frac{3}{4} \right] h_i, \end{aligned} \quad (4.12)$$

of the kinetic terms \dot{h}_i and $\dot{\chi}_i$, without the product $\dot{h}_i \dot{\chi}_i$. For this purpose, we choose

$$\begin{aligned} c_1 = & 3H, \\ c_2 = & \frac{k^2}{a^2} + \frac{4\alpha(H^2 - \dot{H}) + 12\beta(2H^2 + \dot{H}) - 3}{4\alpha}, \\ c_3 = & \frac{2}{\alpha}. \end{aligned} \quad (4.7)$$

After integration by parts, the Lagrangian density (4.6) reduces to

where we used the coefficients (4.7). We also vary Eq. (4.8) with respect to h_i and eliminate χ_i and their time derivatives by exploiting Eq. (4.12). Then, we obtain the following fourth-order differential equation

$$\begin{aligned} \dots \ddot{h}_i = & -6H \ddot{h}_i - \left[\frac{2k^2}{a^2} + \frac{(4\alpha + 6\beta)\dot{H}}{\alpha} + \frac{1 + (22\alpha + 24\beta)H^2}{2\alpha} \right] \ddot{h}_i \\ & - \left[\frac{2Hk^2}{a^2} + \frac{(\alpha + 6\beta)\dot{H}^2}{2\alpha H} + \frac{4(\alpha + 6\beta)H\dot{H}}{\alpha} \right. \\ & + \left. \frac{H\{72\beta(\alpha + 6\beta)H^2 - \alpha + 12\beta\}}{12\alpha\beta} \right] \dot{h}_i \\ & - \left[\frac{k^4}{a^4} + \left(\frac{1 + 24\beta H^2}{2\alpha} + \frac{6\beta\dot{H}}{\alpha} \right) \frac{k^2}{a^2} \right] h_i. \end{aligned} \quad (4.13)$$

We note that the same equation also follows by directly varying the original Lagrangian density (4.5) with respect to h_i . Equation (4.13) governs the dynamics of tensor perturbations.

We first solve Eq. (4.13) in the high-momentum regime, namely, for the modes deep inside the Hubble radius ($k/a \gg H$). After the Hubble radius crossing during inflation, the perturbations enter the super-Hubble region $k/a < H$. The evolution of h_i in the latter large-scale regime will be discussed later. Keeping only the most dominant terms for sub-Hubble perturbations and expressing the Fourier components as $h_i(t, k) = \tilde{h}_i(t) e^{-i \int \omega(t, k) dt}$, where \tilde{h}_i is a function of t , and ω depends on t and k , the Wentzel-Kramers-Brillouin (WKB) approximation gives the relation $\dot{h}_i \simeq -i\omega h_i$. In this WKB regime, we also

have the inequality $|\dot{\omega}| \ll \omega^2$. Then, Eq. (4.13) approximately reduces to

$$\omega^4 + 6iH\omega^3 - \frac{2k^2\omega^2}{a^2} - \frac{2iHk^2\omega}{a^2} + \frac{k^4}{a^4} \simeq 0. \quad (4.14)$$

We search for solutions of the kind $\omega = c_i k/a$, where c_i is the tensor propagation speed. Substituting this dispersion relation into Eq. (4.14) and taking the small-scale limit $k/(aH) \gg 1$, it follows that

$$(c_i^2 - 1)^2 \frac{k^4}{a^4} \left[1 + \mathcal{O}\left(\frac{aH}{k}\right) \right] = 0. \quad (4.15)$$

At leading order in the expansion of the small parameter aH/k , we obtain

$$c_i^2 = 1, \quad (4.16)$$

for all the four dynamical modes h_1, h_2, χ_1 , and χ_2 . Since the propagation speeds are luminal, there are no classical Laplacian instabilities in the tensor sector for perturbations deep inside the Hubble radius.

Let us consider the evolution of super-Hubble tensor modes after the Hubble radius crossing, i.e., $k/a \ll H$. Since $|\dot{H}| \ll H^2$ during slow-roll inflation, Eq. (4.13) approximately reduces to

$$\ddot{h}_i + 6H\ddot{h}_i + 12\lambda_1 H^2 \dot{h}_i + 36\lambda_2 H^3 h_i \simeq 0, \quad (4.17)$$

where

$$\lambda_1 = \frac{(22\alpha + 24\beta)H^2 + 1}{24\alpha H^2},$$

$$\lambda_2 = \frac{72\beta(\alpha + 6\beta)H^2 - \alpha + 12\beta}{432\alpha\beta H^2}. \quad (4.18)$$

The difference between λ_1 and λ_2 is given by

$$\lambda_2 - \lambda_1 = -\frac{3}{4} - \frac{1}{72\alpha H^2} - \frac{1}{432\beta H^2}. \quad (4.19)$$

For the coupling constants α and β with $\alpha H^2 \gtrsim \mathcal{O}(1)$ and $\beta H^2 \gtrsim \mathcal{O}(1)$ during inflation, we can neglect the last two terms in Eq. (4.19) relative to $-3/4$. On using the approximate relation $\lambda_2 \simeq \lambda_1 - 3/4$ in this case and assuming that H is constant during inflation, the solution to Eq. (4.17) can be expressed as

$$h_i = A_i + B_i e^{-3Ht} + C_i e^{-[3 - \sqrt{45 - 48\lambda_1}]Ht/2} + D_i e^{-[3 + \sqrt{45 - 48\lambda_1}]Ht/2}, \quad (4.20)$$

where A_i, B_i, C_i , and D_i are integration constants, and

$$\lambda_1 \simeq \frac{11}{12} + \frac{\beta}{\alpha} > \frac{11}{12}. \quad (4.21)$$

In the last inequality, we exploited the fact that both α and β are positive.¹ For $11/12 < \lambda_1 \leq 15/16$, the last three terms in Eq. (4.20) decrease exponentially. For $\lambda_1 > 15/16$, the last two terms in Eq. (4.20) exhibit damped oscillations with a decreasing amplitude proportional to $e^{-3Ht/2}$. This means that h_i approaches the constant value A_i .

If we consider the small Weyl coupling constant $\alpha \ll \beta$ with $\beta H^2 \gtrsim \mathcal{O}(1)$, then we have $\lambda_1 \simeq \lambda_2 \simeq \beta/\alpha \gg 1$. In the limit that $\beta/\alpha \rightarrow \infty$, the solution to Eq. (4.17) is given by

$$h_i = A_i + B_i e^{-3Ht} + C_i e^{-(3+i\Omega)Ht/2} + D_i e^{-(3-i\Omega)Ht/2}, \quad (4.22)$$

where Ω is a constant. In this case, the amplitude of h_i decreases in time as well and finally reaches a constant, A_i .

We have thus shown that, despite the appearance of ghosts, the tensor perturbation does not exhibit rapid growth during inflation. In other words, the higher-order derivatives of h_i appearing in the action (4.3) hardly affect the standard conservation property of h_i , after the Hubble radius crossing.

V. SCALAR PERTURBATIONS

Let us next study the evolution of scalar perturbations for the perturbed line element given by

$$ds^2 = -(1 + 2A)dt^2 + 2a(t)\partial_i B dt dx^i + a^2(t)[(1 + 2\psi)\delta_{ij} + 2\partial_i \partial_j E] dx^i dx^j. \quad (5.1)$$

We consider an infinitesimal-gauge transformation

$$\tilde{t} = t + \xi^0, \quad \tilde{x}^i = x^i + \delta^{ij} \partial_j \xi, \quad (5.2)$$

from one coordinate $x^\mu = (t, x^i)$ to another coordinate $\tilde{x}^\mu = (\tilde{t}, \tilde{x}^i)$, where ξ^0 and ξ are scalar quantities. Then, the four scalar perturbations A, B, ψ , and E transform, respectively, as [50–53]

$$\tilde{A} = A - \dot{\xi}^0, \quad \tilde{B} = B + \frac{\xi^0}{a} - a\dot{\xi},$$

$$\tilde{\psi} = \psi - H\xi^0, \quad \tilde{E} = E - \xi. \quad (5.3)$$

The gauge-invariant gravitational potentials are defined by [50]

¹In the limit $\alpha \gg \beta$ and $\beta H^2 \gtrsim \mathcal{O}(1)$, we find that $h_i = A_i + B_i e^{-3Ht} + C_i e^{-Ht} + D_i e^{-2Ht}$, so that h_i tends to A_i also in this case.

$$\begin{aligned}\Psi &= A + \frac{d}{dt}[a(B - a\dot{E})], \\ \Phi &= \psi + aH(B - a\dot{E}).\end{aligned}\quad (5.4)$$

We can also construct the following gauge-invariant variables

$$\mathcal{A} \equiv A - \frac{d}{dt}\left(\frac{\psi}{H}\right), \quad \mathcal{B} \equiv aB + \frac{\psi}{H} - a^2\dot{E}. \quad (5.5)$$

While \mathcal{B} is related to Φ as $\mathcal{B} = \Phi/H$, $\mathcal{A} = \Psi - (d/dt)(\Phi/H)$ is not proportional to Ψ .

It is known that $f(R)$ gravity with nonlinear functions of R gives rise to a scalar d.o.f. $\phi = df/dR$ [5,31]. The quadratic action (2.1) contains the function $f(R) = R + \beta R^2$, in which case $\phi = 1 + 2\beta R$. Then, the perturbation of the new scalar d.o.f. is equivalent to $\delta\phi = 2\beta\delta R$. We can construct a gauge-invariant quantity $\zeta = \psi - H\delta\phi/\phi$ [54,55], or, equivalently,²

$$\zeta = \psi - \frac{H}{R}\delta R. \quad (5.6)$$

There is also the following combination analogous to the Mukhanov-Sasaki variable [56,57]:

$$\delta R_f = \delta R - \frac{\dot{R}}{H}\psi. \quad (5.7)$$

which is related to ζ , as $\delta R_f = -\dot{R}\zeta/H$.

We expand the action (2.3) up to quadratic order in scalar perturbations without fixing gauges and then derive the field equations of motion by varying the second-order action with respect to A , B , ψ , and E . These perturbation equations of motion are written in a gauge-ready form [58–60], in that they are ready for the reader to choose a particular gauge. To fix the spatial part of the gauge transformation vector ξ^μ , we choose the gauge

$$E = 0. \quad (5.8)$$

In the following, we will work in Fourier space with the three dimensional comoving wave number \mathbf{k} . We omit a tilde for perturbed quantities in the Fourier space. Then, the perturbation of the Ricci scalar is given by

$$\begin{aligned}\delta R &= 6(\ddot{\psi} + 4H\dot{\psi}) + \frac{4k^2}{a^2}\psi - 6H\dot{A} - 12(2H^2 + \dot{H})A \\ &\quad + \frac{2k^2}{a^2}A + \frac{2}{a}k^2(\dot{B} + 3HB),\end{aligned}\quad (5.9)$$

where $k = |\mathbf{k}|$. For the temporal part of ξ^μ , we can consider several different gauge choices, including (A) Newtonian gauge ($B = 0$), (B) flat gauge ($\psi = 0$), and (C) unitary gauge ($\delta R = 0$). The physical results, including the stability conditions and the evolution of scalar perturbations, are independent of the gauge choices.

We first study conditions for the absence of ghosts and then proceed to address the dynamics of gauge-invariant perturbations for several different gauge choices.

A. No-ghost conditions

In the flat gauge with $\psi = 0$ and $E = 0$, the gauge-invariant variables in Eq. (5.5) reduce to $\mathcal{A} = A$ and $\mathcal{B} = aB$, respectively. We expand the action (2.3) up to quadratic order in two perturbations \mathcal{A} and \mathcal{B} . In Fourier space, the second-order perturbed scalar Lagrangian density L_s contains products of the time derivatives of \mathcal{A} and \mathcal{B} in the form

$$L_s \supset K_{11}\dot{\mathcal{A}}^2 + K_{22}\dot{\mathcal{B}}^2 + 2K_{12}\dot{\mathcal{A}}\dot{\mathcal{B}}, \quad (5.10)$$

with the coefficients

$$\begin{aligned}K_{11} &= 18a^3\beta H^2 M_{\text{pl}}^2, & K_{22} &= -\frac{2(\alpha - 3\beta)M_{\text{pl}}^2 k^4}{3a}, \\ K_{12} &= -6a\beta H M_{\text{pl}}^2 k^2.\end{aligned}\quad (5.11)$$

Thus, both the fields \mathcal{A} and \mathcal{B} propagate as dynamical perturbations. The absence of ghosts requires the following two conditions

$$K_{11} = 18a^3\beta H^2 M_{\text{pl}}^2 > 0, \quad (5.12)$$

$$K_{11}K_{22} - K_{12}^2 = -12a^2\alpha\beta H^2 M_{\text{pl}}^4 k^4 > 0. \quad (5.13)$$

The kinetic term $K_{11}\dot{\mathcal{A}}^2$ corresponds to that of the scalaron perturbation \mathcal{A} , which does not behave as a ghost for $\beta > 0$. Under the no-ghost condition $\beta > 0$ of the scalaron, the second equality (5.13) is always violated for $\alpha > 0$. Hence the other propagating DOF corresponds to a ghost mode, which is induced by the presence of the Weyl curvature term. To see this property more explicitly, we define the following fields that make the kinetic matrix diagonal

$$\mathcal{A}_1 \equiv \mathcal{A} + \frac{K_{12}}{K_{11}}\mathcal{B}, \quad \mathcal{B}_1 \equiv \mathcal{B}. \quad (5.14)$$

²Note that the field ζ is not well-defined on an exact de Sitter space where H is constant. In the background for this model, the Hubble expansion rate varies due to the slow-roll evolution of the scalaron d.o.f., so we can still introduce ζ .

Then, the products of the time derivative of the new fields can be expressed in the form

$$\begin{aligned} L_s &\supset K_{11}\dot{\mathcal{A}}_1^2 + \frac{K_{11}K_{22} - K_{12}^2}{K_{11}}\dot{\mathcal{B}}_1^2 \\ &= a^3 \left(18\beta H^2 M_{\text{Pl}}^2 \dot{\mathcal{A}}_1^2 - \frac{2M_{\text{Pl}}^2 \alpha k^4}{3a^4} \dot{\mathcal{B}}_1^2 \right). \end{aligned} \quad (5.15)$$

For positive values of α and β , we also introduce the canonically normalized fields

$$\mathcal{A}_2 \equiv 6M_{\text{Pl}}H\sqrt{\beta}\mathcal{A}_1, \quad \mathcal{B}_2 \equiv M_{\text{Pl}}\sqrt{\frac{4\alpha k^2}{3a^2}}\mathcal{B}_1, \quad (5.16)$$

so that the kinetic products of the Lagrangian density can be expressed as

$$(L_s)_K = a^3 \left(\frac{1}{2}\dot{\mathcal{A}}_2^2 - \frac{1}{2}\dot{\mathcal{B}}_2^2 \right). \quad (5.17)$$

From this expression, it is clear that \mathcal{A}_2 and \mathcal{B}_2 are the canonically normalized perturbations corresponding to the no-ghost scalaron field and the Weyl scalar ghost, respectively.

In the above discussion we have chosen the flat gauge, but independent of the gauge choices, the scalar sector contains one ghost and the other no-ghost d.o.f.s. In summary, for positive values of α and β , there are two vector ghosts, two tensor ghosts, and one scalar ghost among the total eight propagating d.o.f.s.³ In the following, we study the evolution of scalar perturbations in detail by choosing three different gauges.

B. Newtonian gauge

The Newtonian gauge corresponds to setting $B = 0$ and $E = 0$. In this case the gauge-invariant gravitational potentials in Eq. (5.4) reduce to $\Psi = A$ and $\Phi = \psi$, so the perturbed line element is given by

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 + 2\Phi)\delta_{ij}dx^i dx^j. \quad (5.18)$$

In the Newtonian gauge, the coordinate transformation vector ξ^μ is fixed on the FLRW background without any singularity. For this gauge choice, both Ψ and Φ play the role of two dynamical perturbations. From Eqs. (5.6) and (5.9), the relation between ζ and the gravitational potentials is

³The recent analysis of black hole perturbations in Weyl gravity without the βR^2 term [61] shows that there are seven dynamical d.o.f.s on a static and spherically symmetric background. Adding the βR^2 term gives rise to one scalar d.o.f., so the total dynamical d.o.f.s match each other on two different backgrounds.

$$\begin{aligned} \zeta = \Phi - \frac{H}{R} &\left[6(\ddot{\Phi} + 4H\dot{\Phi}) + \frac{4k^2}{a^2}\Phi - 6H\dot{\Psi} \right. \\ &\left. - 12(2H^2 + \dot{H})\Psi + \frac{2k^2}{a^2}\Psi \right]. \end{aligned} \quad (5.19)$$

Varying the second-order action of scalar perturbations with respect to A , B , ψ , and E , we obtain the four perturbation equations of motion, respectively, for which we express in the form

$$\mathcal{E}_A = 0, \quad \mathcal{E}_B = 0, \quad \mathcal{E}_\psi = 0, \quad \mathcal{E}_E = 0, \quad (5.20)$$

After setting $B = 0 = E$ in the end, two of the above equations are independent, but the other two equations can be also used to obtain the closed differential equations for Ψ and Φ .

For example, the fourth-order differential equation for Ψ can be derived by the following procedure. We first solve the two equations $\mathcal{E}_A = 0$ and $\mathcal{E}_B = 0$ for $\ddot{\Phi}$ and $\dot{\Phi}$. Taking the time derivative of the latter and combining it with the former, we can eliminate the term $\ddot{\Phi}$ to obtain the other equation containing $\dot{\Phi}$. Then, we can solve for $\dot{\Phi}$ and Φ by combining the two equations containing $\dot{\Phi}$. Performing a similar procedure further, we can express $\dot{\Phi}$ and Φ in terms of the derivatives of Ψ up to third order. Taking the time derivative of the Φ equation and eliminating the $\dot{\Phi}$ term, we obtain the fourth-order differential equation for Ψ in the form

$$\ddot{\Psi} + \mu_1 H \ddot{\Psi} + \mu_2 H^2 \dot{\Psi} + \mu_3 H^3 \Psi + \mu_4 H^4 \Psi = 0, \quad (5.21)$$

where $\mu_{1,2,3,4}$ are time-dependent dimensionless functions containing the k dependence. Due to its complexities, we do not write the explicit forms of these coefficients.

Similarly, we can derive the fourth-order differential equation for Φ in the form

$$\ddot{\Phi} + \nu_1 H \ddot{\Phi} + \nu_2 H^2 \dot{\Phi} + \nu_3 H^3 \Phi + \nu_4 H^4 \Phi = 0, \quad (5.22)$$

where the functions ν_i ($i = 1, 2, 3, 4$) are not the same as μ_i , respectively.

For the modes deep inside the Hubble radius ($k \gg aH$), the coefficients in Eqs. (5.21) and (5.22) reduce to

$$\begin{aligned} \mu_1 \simeq \nu_1 \simeq 6, \quad \mu_2 \simeq \nu_2 \simeq \frac{2k^2}{(aH)^2}, \\ \mu_3 \simeq \nu_3 \simeq \frac{2k^2}{(aH)^2}, \quad \mu_4 \simeq \nu_4 \simeq \frac{k^4}{(aH)^4}. \end{aligned} \quad (5.23)$$

In the WKB regime, we substitute the solutions $\Psi = \Psi_0 e^{-i \int \omega dt}$ and $\Phi = \Phi_0 e^{-i \int \omega dt}$ into Eqs. (5.21) and (5.22) with the coefficients (5.23), where Ψ_0 and Φ_0 are

constants. This process leads to the same relation as Eq. (4.14). Writing the dispersion relation as $\omega = c_s k/a$ and taking the limit $k/a \gg H$, we obtain the squared propagation speeds

$$c_s^2 = 1, \quad (5.24)$$

for both Ψ and Φ . This means that, for the modes deep inside the Hubble radius, the Laplacian instabilities are absent for the two gravitational potentials.

For super-Hubble modes, we take the limit $k/(aH) \ll 1$ in the coefficients μ_i and ν_i . Moreover, we also take the limit where the slow-roll parameter $\epsilon = -\dot{H}/H^2$ goes to 0. Then, the coefficients are simplified to

$$\mu_1 \simeq \frac{864\beta^2(\alpha + \beta)H^4 + 18\beta(\alpha + 4\beta)H^2 - \alpha + 3\beta}{6\beta H^2[48\beta(\alpha + 3\beta)H^2 + \alpha + 12\beta]}, \quad (5.25)$$

$$\mu_2 \simeq -\frac{288(\alpha - 12\beta)\beta^2(\alpha + 3\beta)H^4 - 6\beta(7\alpha^2 + 36\alpha\beta + 216\beta^2)H^2 + \alpha^2 - 6\alpha\beta - 36\beta^2}{6\alpha\beta H^2[48\beta(\alpha + 3\beta)H^2 + \alpha + 12\beta]}, \quad (5.26)$$

$$\mu_3 \simeq -\frac{1728\beta^2(\alpha^2 - 15\alpha\beta - 24\beta^2)H^6 - 12\beta(17\alpha^2 + 120\alpha\beta + 216\beta^2)H^4 - \alpha(7\alpha + 36\beta)H^2 + \alpha - 3\beta}{12\alpha\beta H^4[48\beta(\alpha + 3\beta)H^2 + \alpha + 12\beta]}, \quad (5.27)$$

$$\mu_4 \simeq \frac{41472\beta^4(\alpha + \beta)H^6 - 96\beta^2(\alpha^2 + 33\alpha\beta + 36\beta^2)H^4 + 6\beta(\alpha^2 - 38\alpha\beta - 72\beta^2)H^2 - \alpha(\alpha - 3\beta)}{24\alpha\beta^2 H^4[48\beta(\alpha + 3\beta)H^2 + \alpha + 12\beta]}, \quad (5.28)$$

and

$$\nu_1 \simeq \frac{36\beta(\alpha + \beta)H^2 + 2\alpha + 3\beta}{12\beta(\alpha + 3\beta)H^2}, \quad (5.29)$$

$$\nu_2 \simeq -\frac{2\beta(\alpha - 12\beta)H^2 - \alpha - \beta}{2\alpha\beta H^2}, \quad (5.30)$$

$$\nu_3 \simeq -\frac{144\beta^2(\alpha^2 - 15\alpha\beta - 24\beta^2)H^4 + 4\beta(2\alpha^2 - 15\alpha\beta - 36\beta^2)H^2 - 2\alpha^2 - 7\alpha\beta - 6\beta^2}{48\alpha\beta^2(\alpha + 3\beta)H^4}, \quad (5.31)$$

$$\nu_4 \simeq \frac{864\beta^2(\alpha + \beta)H^4 + 72\beta(\alpha + \beta)H^2 + 2\alpha + 3\beta}{24\alpha\beta(\alpha + 3\beta)H^4}. \quad (5.32)$$

Let us first consider the case in which the inequality

$$\beta H^2 \gg 1 \quad (5.33)$$

is satisfied during inflation. Then, Eqs. (5.25)–(5.32) approximately reduce to

$$\begin{aligned} \mu_1 \simeq \nu_1 &\simeq \frac{3(\alpha + \beta)}{\alpha + 3\beta}, & \mu_2 \simeq \nu_2 &\simeq -\frac{\alpha - 12\beta}{\alpha}, \\ \mu_3 \simeq \nu_3 &\simeq -\frac{3(\alpha^2 - 15\alpha\beta - 24\beta^2)}{\alpha(\alpha + 3\beta)}, \\ \mu_4 \simeq \nu_4 &\simeq \frac{36\beta(\alpha + \beta)}{\alpha(\alpha + 3\beta)}. \end{aligned} \quad (5.34)$$

On using these approximate coefficients and exploiting the approximation that H is constant during inflation, the

solutions to Eqs. (5.21) and (5.22) in the super-Hubble regime $k/(aH) \ll 1$ are given by

$$\begin{aligned} \Psi &= C_1 e^{-Ht} + C_2 e^{-\frac{3(\alpha+\beta)}{\alpha+3\beta}Ht} + C_3 e^{(1+\sqrt{1-48\beta/\alpha})Ht/2} \\ &\quad + C_4 e^{(1-\sqrt{1-48\beta/\alpha})Ht/2}, \end{aligned} \quad (5.35)$$

$$\begin{aligned} \Phi &= D_1 e^{-Ht} + D_2 e^{-\frac{3(\alpha+\beta)}{\alpha+3\beta}Ht} + D_3 e^{(1+\sqrt{1-48\beta/\alpha})Ht/2} \\ &\quad + D_4 e^{(1-\sqrt{1-48\beta/\alpha})Ht/2}, \end{aligned} \quad (5.36)$$

where C_i 's and D_i 's are constants. For $\alpha > 0$ and $\beta > 0$, the third and fourth terms in Eqs. (5.35) and (5.36) grow in time, while the first and second terms decay. Depending on the values of α and β , the amplitudes of Ψ and Φ increase as

$$\{|\Psi|, |\Phi|\} \propto e^{Ht/2} \quad \text{for } \alpha < 48\beta, \quad (5.37)$$

$$\{|\Psi|, |\Phi|\} \propto e^{(1+\sqrt{1-48\beta/\alpha})Ht/2} \quad \text{for } \alpha > 48\beta. \quad (5.38)$$

In the coupling range (5.37), Ψ and Φ exhibit oscillations with the growing amplitudes. In the other coupling regime (5.38), Ψ and $|\Phi|$ increase even faster than $e^{Ht/2}$. In the large Weyl coupling limit $\alpha \gg 48\beta$, the gravitational potentials grow rapidly in proportion to e^{Ht} .

The above results are valid for βH^2 exceeding the order 1 during inflation. In particular, for the couplings in the ranges $\alpha \ll \beta$ and $\beta H^2 \gtrsim \mathcal{O}(1)$, the solutions to Ψ and Φ correspond to the limits $\beta/\alpha \rightarrow \infty$ in Eqs. (5.35) and (5.36). This means that even a small Weyl coupling constant α induces the exponential growth of gravitational potentials. Then, the homogeneous FLRW background is violated by the rapid growth of Ψ and Φ in the perturbed metric (5.18). In the above analytic estimation we used the approximation $\beta H^2 \gg 1$, but in Sec. VIC, we will show that, even for $\beta H^2 = \mathcal{O}(1)$, both the amplitudes of Ψ and Φ increase exponentially. The fact that Ψ and Φ are subject to exponential growth does not depend on the gauge choices either. Since all scalar perturbations are always determined through two independent modes, at least one of them needs to be unstable. It should be also pointed out that this instability is purely classical.

Besides Ψ and Φ , there are also other gauge-invariant scalar perturbations. Let us consider the evolution of the curvature perturbation ζ defined in Eq. (5.6). In the Newtonian gauge, we will describe a method to find a closed differential equation for ζ , where ζ is given by Eq. (5.19). We can think of Eq. (5.19) as an equation that sets, on shell, ζ as a function of the other fields. Therefore, we proceed by adding a term to the second-order scalar Lagrangian density L_s , as follows:

$$\begin{aligned} \bar{L}_s = L_s + b_1(t) \left\{ \Phi - \frac{H}{\dot{R}} \left[6(\ddot{\Phi} + 4H\dot{\Phi}) + \frac{4k^2}{a^2} \Phi - 6H\dot{\Psi} \right. \right. \\ \left. \left. - 12(2H^2 + \dot{H})\Psi + \frac{2k^2}{a^2} \Psi \right] - \zeta \right\}^2, \quad (5.39) \end{aligned}$$

where $b_1(t)$ is a function of t . It is clear that at this level ζ is just a Lagrange multiplier, and its equation of motion, algebraic for ζ itself, makes \bar{L}_s reduce to the original Lagrangian density L_s . We choose the coefficient $b_1(t)$ to cancel the term in $\dot{\Psi}^2$, and, by doing so, also the term in $\dot{\Phi}^2$ cancels out from the Lagrangian.

After a few integrations by parts, we see that the field Ψ can be set to be a Lagrangian multiplier, and as such, it is integrated out from the Lagrangian by using its equation of motion. By doing so, we arrive at an equivalent Lagrangian density, $\bar{L}_s = \bar{L}_s(\dot{\zeta}, \dot{\Phi}, \zeta, \Phi)$, which then depends on two propagating d.o.f.s, Φ and ζ , as expected. The reduced Lagrangian density contains the products of kinetic terms of the form

$$\bar{L}_s \supset \bar{K}_{11} \dot{\zeta}^2 + \bar{K}_{22} \dot{\Phi}^2 + 2\bar{K}_{12} \dot{\zeta} \dot{\Phi}. \quad (5.40)$$

For positive values of α and β , we have $\bar{K}_{11} > 0$ and $\bar{K}_{11}\bar{K}_{22} - \bar{K}_{12}^2 < 0$ in the slow-roll limit. Hence there is one ghost mode besides the other no-ghost mode. This property agrees with the no-ghost conditions derived in Sec. VA for the flat gauge.

To derive the closed-form perturbation equation of ζ , we proceed as follows. From the equation of motion for the field Φ , which we write in the form $\mathcal{E}_\Phi = 0$, we find an expression for $\ddot{\Phi}$ that can be inserted into the equation for ζ , i.e., $\mathcal{E}_\zeta = 0$. Now, we take the time derivative of this last equation to obtain $\dot{\mathcal{E}}_\zeta = 0$. We can still substitute this new equation into the expression of $\ddot{\Phi}$, previously found, and solve it for $\dot{\Phi}$. We repeat the step on considering now the equation $\dot{\mathcal{E}}_\zeta = 0$, and after replacing it with the two expressions for $\ddot{\Phi}$ and $\dot{\Phi}$, we can solve it with respect to Φ itself. At this point, we replace all these Φ -related expressions into the equation of motion $\mathcal{E}_\zeta = 0$. Then, we obtain the fourth-order differential equation

$$\zeta^{(4)} + \lambda_1 H \zeta^{(3)} + \lambda_2 H^2 \ddot{\zeta} + \lambda_3 H^3 \dot{\zeta} + \lambda_4 H^4 \zeta = 0, \quad (5.41)$$

where λ_i 's are time-dependent dimensionless coefficients.

Taking the sub-Hubble limit $k/(aH) \gg 1$ with $\epsilon \rightarrow 0$, the coefficients in Eq. (5.41) reduce to

$$\begin{aligned} \lambda_1 \simeq \frac{168\beta H^2 + 1}{12\beta H^2}, \quad \lambda_2 \simeq \frac{2k^2}{(aH)^2}, \\ \lambda_3 \simeq \frac{(120\beta H^2 + 1)k^2}{12\beta H^4 a^2}, \quad \lambda_4 \simeq \frac{k^4}{(aH)^4}. \quad (5.42) \end{aligned}$$

Using the solution $\zeta = \zeta_0 e^{-i \int \omega dt}$ under the WKB approximation $\omega = c_s k/a \gg H$, the leading-order dispersion relation is given by $\omega^4 - 2k^2 \omega^2/a^2 + k^4/a^4 \simeq 0$. Hence the curvature perturbation propagates with the luminal speed for the modes deep inside the Hubble radius.

For super-Hubble perturbations, we take the limits $k/(aH) \rightarrow 0$ and $\epsilon \rightarrow 0$ for the coefficients λ_i . For the coupling in the range $\beta H^2 \gg 1$, we approximately have

$$\begin{aligned} \lambda_1 \simeq \frac{8(8\alpha + 3\beta)}{5\alpha + 3\beta}, \quad \lambda_2 \simeq \frac{157\alpha^2 + 111\alpha\beta + 36\beta^2}{\alpha(5\alpha + 3\beta)}, \\ \lambda_3 \simeq \frac{2(\alpha + 6\beta)(49\alpha + 15\beta)}{\alpha(5\alpha + 3\beta)}, \quad \lambda_4 \simeq 0. \quad (5.43) \end{aligned}$$

In this regime, we can integrate Eq. (5.41) to give

$$\begin{aligned} \zeta = c_1 + c_2 e^{-\frac{(49\alpha+15\beta)Ht}{5\alpha+3\beta}} + c_3 e^{(-3+\sqrt{1-48\beta/\alpha})Ht/2} \\ + c_4 e^{-(3+\sqrt{1-48\beta/\alpha})Ht/2}. \quad (5.44) \end{aligned}$$

For positive values of α and β , the last three terms in Eq. (5.44) decay in time. Hence, the curvature perturbation approaches a constant c_1 after the Hubble radius crossing. In the above estimation, we have used the approximation $\beta H^2 \gg 1$, but we have numerically confirmed that ζ approaches a constant even for $\beta H^2 = \mathcal{O}(1)$.

Despite the exponential increase of two gravitational potentials during inflation, there is a particular gauge-invariant combination ζ that does not grow in the large-scale limit. Unlike the standard single-field slow-roll inflation, however, we have two propagating d.o.f.s in the scalar sector. In the description of Lagrange multipliers explained above, the two dynamical d.o.f.s correspond to the perturbations Φ and ζ . Even though ζ is not enhanced after the Hubble radius crossing, the other dynamical field Φ is subject to exponential growth. Thus, the analysis in the Newtonian gauge shows that the Weyl curvature term violates the homogeneous inflationary background.

C. Flat gauge

In the flat gauge with the gauge conditions $\psi = 0$ and $E = 0$, we have $\mathcal{A} = A$ and $\mathcal{B} = aB$ in Eq. (5.5). Then, the perturbed line element is given by

$$ds^2 = -(1 + 2\mathcal{A})dt^2 + 2\partial_i \mathcal{B} dt dx^i + a^2(t) \delta_{ij} dx^i dx^j. \quad (5.45)$$

On the expanding cosmological background ($H \neq 0$), the coordinate transformation vector ξ^μ is always regular for the flat gauge. The gauge-invariant variables \mathcal{A} and \mathcal{B} are related to Ψ and Φ according to

$$\mathcal{A} = \Psi - \frac{d}{dt} \left(\frac{\Phi}{H} \right), \quad \mathcal{B} = \frac{\Phi}{H}. \quad (5.46)$$

While \mathcal{B} is directly proportional to Φ , \mathcal{A} corresponds to a combination of Ψ and Φ .

After setting $\psi = 0 = E$ in the perturbation equations of motion, the dynamical system in the flat gauge has two propagating d.o.f.s \mathcal{A} and \mathcal{B} (or Φ). To derive the closed differential equation for \mathcal{A} , we solve the two equations $\mathcal{E}_A = 0$ and $\mathcal{E}_B = 0$ for $\ddot{\mathcal{B}}$ and $\dot{\mathcal{B}}$. Following a similar procedure to that performed in the Newtonian gauge, we can express the terms $\dot{\mathcal{B}}$ and \mathcal{B} in terms of \mathcal{A} and its derivatives. Taking the time derivative of the \mathcal{B} equation and combining it with the $\dot{\mathcal{B}}$ equation, we obtain the fourth-order differential equation of \mathcal{A} in the form

$$\ddot{\mathcal{A}} + \tau_1 H \dot{\mathcal{A}} + \tau_2 H^2 \ddot{\mathcal{A}} + \tau_3 H^3 \dot{\mathcal{A}} + \tau_4 H^4 \mathcal{A} = 0, \quad (5.47)$$

where τ_i 's are time-dependent functions.

Taking the sub-Hubble limit $k \gg aH$ with $\epsilon \rightarrow 0$, the coefficients in Eq. (5.47) reduce to

$$\tau_1 \simeq 10, \quad \tau_2 \simeq \frac{2k^2}{(aH)^2}, \quad \tau_3 \simeq \frac{6k^2}{(aH)^2}, \quad \tau_4 \simeq \frac{k^4}{(aH)^4}. \quad (5.48)$$

On using the WKB approximation, it follows that the field \mathcal{A} propagates with the speed of light.

For super-Hubble modes $k \ll aH$ with the coupling $\beta H^2 \gg 1$, taking the slow-roll limit $\epsilon \rightarrow 0$ gives

$$\begin{aligned} \tau_1 &\simeq 6, & \tau_2 &\simeq \frac{11\alpha + 12\beta}{\alpha}, \\ \tau_3 &\simeq \frac{6(\alpha + 6\beta)}{\alpha}, & \tau_4 &\simeq -\frac{\alpha + 6\beta}{3\alpha\beta H^2}. \end{aligned} \quad (5.49)$$

In this regime, Eq. (5.47) can be integrated to give

$$\begin{aligned} \mathcal{A} &= c_1 e^{-(9\alpha\beta H + \sqrt{\Delta_1 - 12\alpha\beta H \sqrt{\Delta_2}})t/(6\alpha\beta)} \\ &+ c_2 e^{-(9\alpha\beta H - \sqrt{\Delta_1 - 12\alpha\beta H \sqrt{\Delta_2}})t/(6\alpha\beta)} \\ &+ c_3 e^{-(9\alpha\beta H + \sqrt{\Delta_1 + 12\alpha\beta H \sqrt{\Delta_2}})t/(6\alpha\beta)} \\ &+ c_4 e^{-(9\alpha\beta H - \sqrt{\Delta_1 + 12\alpha\beta H \sqrt{\Delta_2}})t/(6\alpha\beta)}, \end{aligned} \quad (5.50)$$

where

$$\Delta_1 \equiv 9H^2\alpha\beta^2(5\alpha - 24\beta), \quad (5.51)$$

$$\Delta_2 \equiv 3\beta(\alpha + 6\beta)[3\beta H^2(\alpha + 6\beta) + \alpha]. \quad (5.52)$$

The amplitudes of the first two terms in Eq. (5.50) decrease in proportion to $e^{-3Ht/2}$, while the third term decreases as $\propto e^{-3Ht}$. Taking the limit $\beta H^2 \gg 1$, the leading-order contribution to the term $\Delta_1 + 12\alpha\beta H \sqrt{\Delta_2}$ is $(9\alpha\beta H)^2$. Then, for $\alpha > 0$ and $\beta > 0$, the leading-order contribution to the last term in Eq. (5.50) is the constant c_4 . Picking up the next-to-leading correction, we obtain the following solution

$$\mathcal{A} \simeq c_4 \left(1 + \frac{t}{18\beta H} \right). \quad (5.53)$$

For the number of e -foldings N of order 10, the correction induced by the time-dependent terms in Eq. (5.53) is suppressed compared to the leading-order constant term.

Following a similar procedure performed for the perturbation \mathcal{A} , we can also derive the fourth-order differential equation for \mathcal{B} in the form

$$\ddot{\mathcal{B}} + \eta_1 H \dot{\mathcal{B}} + \eta_2 H^2 \ddot{\mathcal{B}} + \eta_3 H^3 \dot{\mathcal{B}} + \eta_4 H^4 \mathcal{B} = 0, \quad (5.54)$$

where the η_i 's are time-dependent coefficients and are not identical to the ν_i 's in Eq. (5.22). If we use the variable Φ instead of \mathcal{B} , the coefficients of the fourth-order differential equation for Φ exactly coincide with those derived in the Newtonian gauge.

Using the WKB approximation for the modes deep inside the Hubble radius, the perturbation \mathcal{B} obeys

$$\ddot{\mathcal{B}} + \frac{2k^2}{a^2} \dot{\mathcal{B}} + \frac{k^4}{a^4} \mathcal{B} \simeq 0, \quad (5.55)$$

so that \mathcal{B} propagates with the speed of light. Taking the super-Hubble limit ($k \ll aH$) with $\epsilon \rightarrow 0$, the coefficients in Eq. (5.54) reduce to

$$\eta_1 \simeq \frac{36(\alpha + \beta)\beta H^2 + 2\alpha + 3\beta}{12(\alpha + 3\beta)\beta H^2}, \quad (5.56)$$

$$\eta_2 \simeq \frac{2(12\beta - \alpha)H^2 + 1}{2\alpha H^2}, \quad (5.57)$$

$$\eta_3 \simeq -\frac{72\beta(\alpha^2 - 15\alpha\beta - 24\beta^2)H^4 - 2(\alpha^2 + 42\alpha\beta + 36\beta^2)H^2 - 2\alpha - 3\beta}{24\alpha\beta(\alpha + 3\beta)H^4}, \quad (5.58)$$

$$\eta_4 \simeq \frac{5184\beta^3(\alpha + \beta)H^4 + 12\alpha\beta(\alpha + 9\beta)H^2 + \alpha(2\alpha + 3\beta)}{144\alpha\beta^2(\alpha + 3\beta)H^4}. \quad (5.59)$$

For $\beta H^2 \gg 1$, the coefficients $\eta_{1,2,3,4}$ approximately reduce to the values $\nu_{1,2,3,4}$ given in Eq. (5.34), respectively. The field \mathcal{B} is subject to exponential growth during inflation analogous to Φ , see Eqs. (5.37) and (5.38). This means that, even though \mathcal{A} does not grow significantly, the other metric perturbation \mathcal{B} in the line element (5.45) increases rapidly to violate the FLRW background.

In the flat gauge, the perturbation ζ is given by

$$\zeta = -\frac{H}{R}\delta R = -\frac{H}{R}\left[\frac{2k^2\dot{\mathcal{B}}}{a^2} - 6H\dot{\mathcal{A}} + \left(\frac{2k^2}{a^2} - 24H^2 - 12\dot{H}\right)\mathcal{A} + \frac{4\mathcal{B}Hk^2}{a^2}\right]. \quad (5.60)$$

To derive the fourth-order differential equation for ζ , we build the following Lagrangian density

$$\bar{L}_s = L_s + b_2(t)\left\{\zeta + \frac{H}{R}\left[\frac{2k^2\dot{\mathcal{B}}}{a^2} - 6H\dot{\mathcal{A}} + \left(\frac{2k^2}{a^2} - 24H^2 - 12\dot{H}\right)\mathcal{A} + \frac{4\mathcal{B}Hk^2}{a^2}\right]\right\}^2, \quad (5.61)$$

which is equivalent to the original Lagrangian density L_s . The function $b_2(t)$ needs to be chosen so that the kinetic term of \mathcal{A} vanishes identically. After a few integrations by parts, the field \mathcal{A} becomes a Lagrange multiplier which can be integrated out, leaving ζ and \mathcal{B} as two dynamical d.o.f.s. After varying \bar{L}_s with respect to ζ and \mathcal{B} , we can proceed along the same lines as finding the equations of motion for the fields \mathcal{A} and \mathcal{B} . This leads to the closed differential equation for ζ in the form (5.41), where λ_i 's are exactly the same as those derived in the Newtonian gauge. Hence ζ approaches a constant after the Hubble radius crossing. However, the fact that the other dynamical perturbation \mathcal{B} grows exponentially means that the instability of the FLRW background cannot be avoided. We also note that the fourth-order differential equation of Ψ exactly coincides with the one obtained in the Newtonian gauge. Hence the two gravitational potentials Ψ and Φ are unstable in the flat gauge as well, by reflecting the fact that both Ψ and Φ contain the dependence of aB .

D. Unitary gauge

Let us finally discuss the evolution of scalar perturbations in the unitary gauge with $\delta R = 0$. Since the curvature

perturbation ζ is equivalent to ψ , the gauge condition translates to

$$\delta R = 6(\ddot{\zeta} + 4H\dot{\zeta}) + \frac{4k^2}{a^2}\zeta - 6H\dot{\mathcal{A}} - 12(2H^2 + \dot{H})\mathcal{A} + \frac{2k^2}{a^2}\mathcal{A} + \frac{2k^2}{a}(\dot{\mathcal{B}} + 3HB) = 0. \quad (5.62)$$

Using this condition together with the perturbation equations of motion, we can derive the fourth-order differential equation for ζ . We first solve the perturbation equation $\mathcal{E}_A = 0$ for $\ddot{\mathcal{B}}$ and take the time derivative of Eq. (5.62) to obtain the first derivative $\dot{\mathcal{B}}$. Using Eq. (5.62) to solve for $\dot{\mathcal{B}}$, one can express \mathcal{B} and its time derivatives in terms of ζ , \mathcal{A} , and their time derivatives. In this way, all the \mathcal{B} -dependent quantities can be eliminated from the perturbation equations of motion. The next step is to remove the \mathcal{A} -dependent terms. On using the two equations $\mathcal{E}_\psi = 0$ and $\mathcal{E}_E = 0$, we can solve for $\ddot{\mathcal{A}}$ and $\dot{\mathcal{A}}$. Then, following a similar procedure as before, it is possible to express \mathcal{A} in terms of the derivatives of ζ up to third order. Taking the time derivative of this equation and eliminating the $\dot{\mathcal{A}}$ term,

we obtain the fourth-order differential equations of ζ with the exactly same coefficients as $\lambda_{1,2,3,4}$ in Eq. (5.41). Then, the constancy of ζ after the Hubble radius crossing also holds in the unitary gauge. This result is consistent with the analysis of Ref. [42] in the Einstein frame. Similarly, we obtain the same fourth-order differential equation for \mathcal{A} as Eq. (5.47), so the solution in the super-Hubble regime is given by Eq. (5.53).

The closed differential equations for Ψ and Φ can be also obtained by using the following relations

$$A = \Psi - \dot{\mathcal{B}}_u, \quad \zeta = \Phi - H\mathcal{B}_u, \quad (5.63)$$

where

$$\mathcal{B}_u \equiv aB. \quad (5.64)$$

The gauge condition (5.62) and the perturbation equations of motion can be now expressed in terms of the gauge-invariant variables Ψ , Φ , \mathcal{B}_u and their time derivatives. Indeed, the above change of variables automatically removes the \mathcal{B}_u -dependent terms from the two perturbation equations $\mathcal{E}_A = 0$ and $\mathcal{E}_B = 0$. Combining these two, it is straightforward to derive the fourth-order differential equations for Ψ and Φ . Again, we find that they are identical to Eqs. (5.21) and (5.22) derived in the Newtonian gauge, respectively, with the completely same coefficients. Hence the same instabilities of Ψ and Φ are present after the Hubble radius crossing, while the growth of ζ and \mathcal{A} is suppressed.

In terms of the gauge-invariant variables, the perturbed line element in the unitary gauge can be expressed as

$$ds^2 = - \left[1 + 2\mathcal{A} + 2 \frac{d}{dt} \left(\frac{\zeta}{H} \right) \right] dt^2 + \frac{2}{H} (\partial_i \Phi - \partial_i \zeta) dt dx^i + a^2(t) (1 + 2\zeta) \delta_{ij} dx^i dx^j. \quad (5.65)$$

Due to the suppressed growth of \mathcal{A} and ζ , metric perturbations in the g_{00} and g_{ij} components are not subject to classical instabilities. However, the $\partial_i \Phi$ term in the g_{0i} component exhibits an exponential increase after the Hubble radius crossing. Since the $\partial_i \zeta$ term approaches a constant in the super-Hubble regime, the dominance of $\partial_i \Phi$ over $\partial_i \zeta$ in g_{0i} leads to the instability of the FLRW background. Indeed, we numerically confirmed that the gauge-invariant perturbation $\mathcal{B}_u = (\Phi - \zeta)/H$ grows exponentially in the super-Hubble regime due to the enhancement of Φ . We have thus analytically shown that, for any physical gauge choices, the Weyl curvature makes the inflationary FLRW background unstable.

VI. NUMERICAL SIMULATIONS WITH THE DISCUSSION OF INITIAL CONDITIONS

In this section, we will numerically confirm the instability of the FLRW background in Weyl gravity with the βR^2 term. For this purpose, we first discuss the choice of initial conditions of perturbations and then proceed to the numerical analysis.

A. Fourth-order system and initial conditions

We have learned so far that each of the considered perturbation fields obeys a fourth-order differential equation, which, under the WKB approximation, is solved as the solutions describing waves propagating with the speed of light. We will show that it is indeed possible, starting from the reduced action of a single scalar field v possessing the term \ddot{v}^2 , to find an equivalent Lagrangian density of two scalar fields with second-order equations of motion. The discussion in this section can be applied to any dynamical perturbation v with some gauge choices, but in Sec. VI B we will consider the flat gauge for concreteness.

For the modes deep inside the Hubble radius, we should expect the field v in Fourier space to satisfy a fourth-order equation of motion, which can be derived by the following approximate Lagrangian

$$L_v \simeq Q \left(\ddot{v}^2 - 2 \frac{k^2}{a^2} \dot{v}^2 + \frac{k^4}{a^4} v^2 \right). \quad (6.1)$$

Here and in the following, we assume that the function $Q = Q(t, k^2)$ can be either positive or negative. In Sec. VI B, we will see that it is possible to obtain a Lagrangian density⁴ reducing to the form (6.1) in the high- k regime with v related to the perturbations \mathcal{A} and \mathcal{B} . We note that, in the action (6.1), we are assuming the high momentum regime in which the WKB approximation holds for the dynamics of v .

Then, we can introduce an auxiliary field, w , as

$$L_v \simeq Q \left[\dot{v}^2 - 2 \frac{k^2}{a^2} v^2 + \frac{k^4}{a^4} v^2 - (b_2 w + \ddot{v})^2 \right], \quad (6.2)$$

where $b_2(t, k^2)$ is a general function, so far undetermined. By integrating out the field w , we find once more the original Lagrangian density. Therefore, the two Lagrangian densities (6.1) and (6.2) are equivalent to each other, both of which lead to the same dynamics. It is also clear that the term in \ddot{v}^2 vanishes in Eq. (6.2) for the new Lagrangian density. At this level, we can introduce the quantity

$$v = v_2 + \frac{a^2 b_2}{2k^2} w, \quad (6.3)$$

⁴The Lagrangian, in this case, will take the following general form $L_v = Q(\ddot{v}^2 - 2Q_1 \dot{v}^2 + Q_2 v^2)$, where Q , Q_1 , and Q_2 are functions of time and k^2 .

which is meant to diagonalize the kinetic matrix. Then, we perform the other field redefinitions

$$v_2 = \frac{a}{k} v_3, \quad w = \frac{k}{a} w_3, \quad (6.4)$$

as to make the kinetic terms only background dependent, i.e., independent of the wave number k .

Having assumed that $Q \neq 0$, we can further introduce the following field redefinitions

$$v_3 = \frac{a^{3/2}}{2\sqrt{|Q|}} v_4, \quad w_3 = \frac{a^{3/2}}{b_2\sqrt{|Q|}} w_4, \quad (6.5)$$

to obtain canonical kinetic terms for the fields v_4 and w_4 . At this level, the Lagrangian for the modes deep inside the Hubble radius reduces to

$$L_v \simeq -\text{sign}(Q) \left[\frac{a^3}{2} (\dot{v}_4^2 - \dot{w}_4^2) - \frac{k^2 a}{4} (v_4^2 - 3w_4^2 + 2v_4 w_4) \right]. \quad (6.6)$$

This Lagrangian still leads to the dynamics of perturbations propagating with the speed of light. However, the effective mass matrix $C = \frac{1}{4} k^2 a \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$ cannot be diagonalized by any real (finite and constant-in-time) Lorentz transformation that would leave instead the kinetic matrix in the canonical form. Therefore, the two modes v_4 and w_4 are not completely decoupled.

Nonetheless, we can use the WKB approximation and look for solutions of the kind $v_4 \propto e^{-i \int \omega dt}$ and $w_4 \propto e^{-i \int \omega dt}$, for which $\ddot{v}_4 \propto -\omega^2 v_4$ and $\ddot{w}_4 \propto -\omega^2 w_4$. In this case, it follows that the dispersion relation $\omega = k/a$ needs to hold, which means that both modes propagate with the luminal speeds. Furthermore, we obtain the relation $w_4 = v_4$, and hence the dynamics of the mode $w_4 - v_4$ is set to vanish. Still, proper initial conditions need to be imposed on the field v_4 (or w_4).

To better understand the behavior of solutions or the choice of initial conditions, we use the perturbation equations of motion in the high- k regime expressed in terms of the conformal time $\eta = \int a^{-1} dt$,

$$\frac{d^2 v_4}{d\eta^2} \simeq -\frac{k^2}{2} (v_4 + w_4), \quad \frac{d^2 w_4}{d\eta^2} \simeq \frac{k^2}{2} (v_4 - 3w_4). \quad (6.7)$$

Then, we obtain the following general solutions

$$w_4 = c_1 \sin(k\eta) + c_2 \cos(k\eta) + c_3 k\eta \sin(k\eta) + c_4 k\eta \cos(k\eta), \quad (6.8)$$

$$v_4 = c_1 \sin(k\eta) + c_2 \cos(k\eta) + c_3 [k\eta \sin(k\eta) + 4 \cos(k\eta)] + c_4 [k\eta \cos(k\eta) - 4 \sin(k\eta)], \quad (6.9)$$

where c_1, c_2, c_3 , and c_4 are integration constants. The choice of exact plane wave initial conditions corresponds to $c_3 = 0 = c_4$, so that $v_4 = w_4$. As in the usual normalization scheme of the Bunch-Davies vacuum,⁵ we choose a positive frequency solution $v_4 = w_4 = C_0 e^{-ik\eta}$ and impose the conditions $\sqrt{|v_4^2| |(a^3 \dot{v}_4)^2|} = 1/2$ and $\sqrt{|w_4^2| |(a^3 \dot{w}_4)^2|} = 1/2$. This fixes the coefficient C_0 to be $1/(a\sqrt{2k})$, and hence

$$v_4 = w_4 = \frac{1}{a\sqrt{2k}} e^{-i \int \frac{k}{a} dt}. \quad (6.10)$$

In terms of the original perturbation v , we have

$$v = \frac{a^{5/2}}{2k\sqrt{|Q|}} (v_4 + w_4) = \frac{a^{3/2}}{\sqrt{2k^3|Q|}} e^{-i \int \frac{k}{a} dt}. \quad (6.11)$$

We will choose this as the initial condition of v for the modes deep inside the Hubble radius. Note that we will not discuss the quantization of perturbations in our theory. We have already shown analytically the presence of violent classical instabilities for the modes after the Hubble radius crossing. In other words, the scalar ghost is not of the soft type in our theory. The classical instabilities induced by the ghost make the quantization of perturbations irrelevant. Only for a stable classical background, it would be worth investigating the quantization procedure.

B. Initial conditions for the perturbations \mathcal{A} and \mathcal{B}

In Sec. VIA, we assumed the existence of the Lagrangian density (6.1) leading to the closed fourth-order differential equation. In this section, we will prove its existence by considering perturbations in the flat gauge. For this purpose, we will proceed as follows. In the flat gauge, the kinetic Lagrangian for the fields \mathcal{A} and \mathcal{B} was already discussed in Eq. (5.10). We remind the reader about the field redefinitions that were introduced to obtain canonical kinetic terms. The field redefinitions given in Eqs. (5.14) and (5.16) allow us to obtain the Lagrangian density of two canonically normalized fields \mathcal{A}_2 and \mathcal{B}_2 in the form (5.17). The original perturbations \mathcal{A} and \mathcal{B} are related to \mathcal{A}_2 and \mathcal{B}_2 , as

$$\mathcal{A} = \frac{1}{6M_{\text{Pl}} H} \left(\frac{\mathcal{A}_2}{\sqrt{\beta}} + \frac{\sqrt{3}\mathcal{B}_2}{\sqrt{\alpha}} \right), \quad \mathcal{B} = \frac{\sqrt{3}a^2 \mathcal{B}_2}{2M_{\text{Pl}} k^2 \sqrt{\alpha}}. \quad (6.12)$$

⁵From a purely classical point of view, the initial condition representing a plane wave consists of setting $dv_4/d\eta = -ikv_4$, and so on for all other higher derivatives. The normalization of v_4 itself is not established as the field Cv_4 is still a solution of the equations of motion (C is a constant). In this case, the physical quantity to consider is $v_4(\eta)/v_4(\eta_i)$.

From the canonical expression (5.17), we wish to find the Lagrangian density of the form (6.1). We perform the following field redefinitions

$$\mathcal{A}_2 = \frac{1}{\sqrt{2}}(\mathcal{Y} - \mathcal{Z}), \quad \mathcal{B}_2 = \frac{1}{\sqrt{2}}(\mathcal{Y} + \mathcal{Z}), \quad (6.13)$$

so that the kinetic terms are simplified to

$$L_s \ni -a^3 \dot{\mathcal{Y}} \dot{\mathcal{Z}} + \dots = \mathcal{Z} \frac{d}{dt}(a^3 \dot{\mathcal{Y}}) + \dots, \quad (6.14)$$

up to total derivatives. Varying the full Lagrangian density L_s with respect to \mathcal{Z} , it follows that \mathcal{Z} can be expressed in terms of \mathcal{Y} and its first and second-time derivatives. Substituting \mathcal{Z} into the second expression of Eq. (6.14), we find that the Lagrangian density can be expressed in the form

$$L_s = Q(\ddot{\mathcal{Y}}^2 - 2Q_1 \dot{\mathcal{Y}}^2 + Q_2 \mathcal{Y}^2), \quad (6.15)$$

where Q , Q_1 , and Q_2 depend on t and k^2 . It should be noticed that, up to this point, we have not made any approximation for particular wavenumbers. However, if we look for the behavior of the three quantities Q , Q_1 , and Q_2 in the high- k regime, we find

$$Q \simeq \frac{27\alpha\beta a^7 H^2}{(\alpha - 3\beta)(\sqrt{\alpha} - \sqrt{3\beta})^2 k^4}, \quad Q_1 \simeq \frac{k^2}{a^2}, \quad Q_2 \simeq \frac{k^4}{a^4}, \quad (6.16)$$

which means that⁶ $\text{sign}(Q) = \text{sign}(\alpha - 3\beta)$. The behavior of Q_1 and Q_2 makes sure that the propagation of the mode \mathcal{Y} is luminal in the WKB approximation scheme.

Using the discussion given in Sec. VI A, \mathcal{Y} plays the role of the field v with Q given in Eq. (6.16). Then, we can choose the initial condition of \mathcal{Y} , as

$$\begin{aligned} \mathcal{Y} &= \frac{a^{3/2}}{\sqrt{2k^3|Q|}} e^{-i \int_{a_i}^k dt} \\ &\simeq \frac{k^{1/2} \sqrt{|\alpha - 3\beta|} |\sqrt{\alpha} - \sqrt{3\beta}|}{3\sqrt{6\alpha\beta} a^2 H} e^{ik(e^{-N} - 1)/(a_i H_i)}, \end{aligned} \quad (6.17)$$

where $N = \ln(a/a_i)$ is the e -folding number, and we set $N = 0$ at the initial time. Substituting this solution into the relation between \mathcal{Z} and \mathcal{Y} , $\dot{\mathcal{Y}}$, $\ddot{\mathcal{Y}}$, we find

⁶For the special case where $\alpha = 3\beta$, by looking at Eq. (5.10), we can see that the field \mathcal{B} can be easily integrated out in terms of \mathcal{A} and its first and second time derivatives. Alternatively, we can still use the procedure described here, but now $Q \simeq -a^3/(6H^2)$ and $\mathcal{Z} \simeq ik\mathcal{Y}/(aH) \gg \mathcal{Y}$.

$$\begin{aligned} \mathcal{Z} &= \frac{\sqrt{\alpha} + \sqrt{3\beta}}{\sqrt{\alpha} - \sqrt{3\beta}} \mathcal{Y} - \frac{6i\sqrt{3\alpha\beta}}{(\sqrt{\alpha} - \sqrt{3\beta})^2} \frac{aH}{k} \mathcal{Y} \\ &+ \mathcal{O}\left(\frac{a^2 H^2}{k^2}\right) \mathcal{Y}, \end{aligned} \quad (6.18)$$

which is valid for the modes deep inside the Hubble radius.⁷

On using Eqs. (6.12), (6.13), and (6.18), the WKB solutions to \mathcal{A} and \mathcal{B} are

$$\begin{aligned} \mathcal{A} &= \frac{1}{6\sqrt{2}M_{\text{Pl}}H} \left[\frac{1}{\sqrt{\beta}}(\mathcal{Y} - \mathcal{Z}) + \sqrt{\frac{3}{\alpha}}(\mathcal{Y} + \mathcal{Z}) \right] \\ &= \frac{\sqrt{6}ai}{2(\sqrt{\alpha} - \sqrt{3\beta})M_{\text{Pl}}k} \mathcal{Y} \left[1 + \mathcal{O}\left(\frac{aH}{k}\right) \right], \end{aligned} \quad (6.19)$$

$$\begin{aligned} \mathcal{B} &= \frac{\sqrt{6}a^2}{4M_{\text{Pl}}k^2\sqrt{\alpha}}(\mathcal{Y} + \mathcal{Z}) \\ &= \frac{\sqrt{6}a^2}{2(\sqrt{\alpha} - \sqrt{3\beta})M_{\text{Pl}}k^2} \mathcal{Y} \left[1 + \mathcal{O}\left(\frac{aH}{k}\right) \right], \end{aligned} \quad (6.20)$$

where we used the relation (6.18). Recall that \mathcal{Y} is given by Eq. (6.17). The ratio between the leading-order terms to \mathcal{A} and \mathcal{B} is $\mathcal{B}/\mathcal{A} = -ia/k$, so the amplitude $|\mathcal{H}\mathcal{B}/\mathcal{A}|$ is of order $aH/k \ll 1$ for sub-Hubble modes. We will use Eqs. (6.19) and (6.20) as the initial conditions of perturbations for the modes deep inside the Hubble radius.

C. Numerical integration

In the flat gauge, we numerically integrate the closed fourth-order differential equations for \mathcal{A} and \mathcal{B} together with those of Ψ , Φ , and ζ . For this purpose, we introduce the following quantities:

$$\begin{aligned} h &= \frac{H}{H_i}, & K &= \frac{k}{aH_i}, & \bar{\alpha} &= \alpha H_i^2, \\ \bar{\beta} &= \beta H_i^2, & \chi &= H_i \mathcal{B}, \end{aligned} \quad (6.21)$$

where H_i is the Hubble parameter at the onset of integration (with scale factor a_i). We also define the following perturbed variables

⁷Alternatively, we can write the Lagrangian density in the form $L_s = \mathcal{Y}(d/dt)(a^3 \dot{\mathcal{Z}}) + \dots$ and vary the action with respect to \mathcal{Y} and use the equation of motion for \mathcal{Y} to express L_s with respect to \mathcal{Z} and its derivatives. In the high- k regime, the Lagrangian density is given by $L_s \simeq \tilde{Q}[\dot{\mathcal{Z}}^2 - (2k^2/a^2)\dot{\mathcal{Z}}^2 + (k^4/a^4)\mathcal{Z}^2]$, where $\tilde{Q} = 27\alpha\beta a^7 H^2 / [(\alpha - 3\beta)(\sqrt{\alpha} + \sqrt{3\beta})^2 k^4]$. The WKB solution to \mathcal{Z} derived by this procedure is consistent with the leading-order relation of Eq. (6.18).

$$\begin{aligned} \mathcal{A}_k &= \frac{k^{3/2}}{\sqrt{2\pi^2}} \mathcal{A}, & \mathcal{B}_k &= \frac{k^{3/2}}{\sqrt{2\pi^2}} \mathcal{B}, & \chi_k &= H_i \mathcal{B}_k, \\ \Psi_k &= \frac{k^{3/2}}{\sqrt{2\pi^2}} \Psi, & \Phi_k &= \frac{k^{3/2}}{\sqrt{2\pi^2}} \Phi, & \zeta_k &= \frac{k^{3/2}}{\sqrt{2\pi^2}} \zeta. \end{aligned} \quad (6.22)$$

Then, the two gravitational potentials can be expressed as

$$\Psi_k = \mathcal{A}_k + h\chi'_k, \quad \text{and} \quad \Phi_k = h\chi_k, \quad (6.23)$$

where a prime in this section denotes the differentiation with respect to the number of e -foldings $N = \ln(a/a_i)$. Together with solving the perturbation equations of motion, we integrate the following background equations of motion

$$h' = -h\epsilon, \quad \epsilon' = \frac{3}{2}\epsilon^2 - 3\epsilon + \frac{1}{12\bar{\beta}h^2}. \quad (6.24)$$

The initial value of ϵ is chosen to realize the sufficient number of e -foldings ($N > 70$) during inflation. On using Eqs. (6.19) and (6.20), we choose the initial conditions of \mathcal{A}_k and χ_k at $N = 0$, as⁸

$$\begin{aligned} \mathcal{A}_k(0) &= iK_i \frac{|\bar{\alpha} - 3\bar{\beta}|^{1/2} \text{sign}(\sqrt{\bar{\alpha}} - \sqrt{3\bar{\beta}}) H_i}{6\sqrt{2\pi}\sqrt{\bar{\alpha}\bar{\beta}}} \frac{H_i}{M_{\text{Pl}}}, \\ \frac{d^n \mathcal{A}_k}{dN^n}(0) &\simeq (-iK_i)^n \mathcal{A}_k(0), \end{aligned} \quad (6.25)$$

$$\begin{aligned} \chi_k(0) &= \frac{|\bar{\alpha} - 3\bar{\beta}|^{1/2} \text{sign}(\sqrt{\bar{\alpha}} - \sqrt{3\bar{\beta}}) H_i}{6\sqrt{2\pi}\sqrt{\bar{\alpha}\bar{\beta}}} \frac{H_i}{M_{\text{Pl}}}, \\ \frac{d^n \chi_k}{dN^n}(0) &\simeq (-iK_i)^n \chi_k(0), \end{aligned} \quad (6.26)$$

where $K_i = K(0) = k/(a_i H_i)$. Notice that there is a simple relation $\mathcal{A}_k(0) = iK_i \chi_k(0)$ for the leading-order solution. The ratio H_i/M_{Pl} and the couplings α, β determine the initial amplitude of χ_k . The typical Hubble scale for Starobinsky inflation is $H_i/M_{\text{Pl}} = \mathcal{O}(10^{-5})$, so that $|\chi_k(N=0)| = \mathcal{O}(10^{-6})$ for the couplings α and β whose orders are similar to each other. The initial conditions of gravitational potentials are $\Psi_k(0) = \mathcal{A}_k(0) + \chi'_k(0)$ and $\Phi_k(0) = \chi_k(0)$.

From Eq. (5.60), the curvature perturbation ζ_k can be expressed, in terms of \mathcal{A}_k and χ_k , as

$$\begin{aligned} \zeta_k &= \frac{4\bar{\beta}[h(2\chi_k + \chi'_k) + \mathcal{A}_k] K^2}{1 - 6\bar{\beta}\epsilon(\epsilon - 2)h^2} \\ &\quad - \frac{12\bar{\beta}h^2[2\mathcal{A}_k(2 - \epsilon) + \mathcal{A}'_k]}{1 - 6\bar{\beta}\epsilon(\epsilon - 2)h^2}. \end{aligned} \quad (6.27)$$

⁸All the perturbations labeled by k , for instance, χ_k, ζ_k , etc., satisfy the same closed fourth-order differential equation as their unnormalized counterparts, χ, ζ , etc., with different initial conditions only by the factor $k^{3/2}/\sqrt{2\pi^2}$.

From Eq. (6.25), we find that $\mathcal{A}_k(0)$ is proportional to $K_i H_i/M_{\text{Pl}}$ and hence $\zeta_k(0)$ contains a large term $K_i^2 (H_i/M_{\text{Pl}})$ for the sub-Hubble modes $K_i \gg 1$. Because of the relation $\mathcal{A}_k(0) = iK_i \chi_k(0)$, this term exactly cancels the other contribution $\chi'_k(0) K_i^2$ for the leading-order initial condition $\chi'_k(0) = -iK_i \chi_k(0)$. This cancellation implies that we need to take into account terms of order $\mathcal{O}(a^2 H^2/k^2)$ in Eq. (6.18) to estimate the initial value of $\zeta_k(0)$ correctly. We also note that Eq. (6.27) contains the N derivatives of \mathcal{A}_k and χ_k . Since the amplitudes of \mathcal{A}_k and χ_k change in time, we take the N derivatives of these fields without neglecting their time dependence. These precise manipulations show that the leading-order contributions to the first and second terms in Eq. (6.27) cancel each other with respect to the large K_i expansion. Then, the initial value of $\zeta_k(0)$ is typically of order $\bar{\beta} \chi_k(0) K_i$. After deriving the precise numerical value of $\zeta_k(0)$ from Eq. (6.27) without using the approximation, the N derivatives of ζ_k at $N = 0$ can be estimated as $(d^n \zeta_k/dN^n)(0) \simeq (-iK_i)^n \zeta(0)$.

As we will show in the numerical calculation below, the perturbation χ_k is unstable, but the decrease of the K^2 term (proportional to a^{-2}) in Eq. (6.27) suppresses the growth of $\chi_k K^2$. Therefore, after the Hubble radius crossing, ζ_k depends mostly on \mathcal{A}_k and its N -derivative. Hence, if the growth of \mathcal{A}_k is insignificant, this is also the case for ζ_k . Furthermore, this shows that, in the super-Hubble regime, ζ_k is related only to \mathcal{A}_k and vice versa. Thus, the whole scalar sector, which consists of two independent dynamical d.o.f.s, cannot be described by ζ_k and \mathcal{A}_k alone in the regime $k/a \ll H$.

In Fig. 1, we plot the evolution of $|\Phi_k| = |h\chi_k|$, $|\Psi_k|$, $|\mathcal{A}_k|$, and $|\zeta_k|$ for $\bar{\alpha} = 1$, $\bar{\beta} = 6$, and $K(0) = 100$ with the slow-roll parameter $\epsilon(0) = 0.0047$. The initial conditions for χ_k are instead chosen as to fulfill Eq. (6.26), with $H_i = 10^{-5} M_{\text{Pl}}$. We solved the fourth-order differential equations of χ_k and \mathcal{A}_k and computed Φ_k , Ψ_k , and ζ_k by exploiting the relations (6.23) and (6.27). We also performed the direct integration of the fourth-order differential equations for Φ_k , Ψ_k , and ζ_k by implementing the initial conditions for each of them in terms of those given for $\chi_k(0)$ and found that the results are in perfect agreement with those computed from χ_k and \mathcal{A}_k .

In Fig. 1, we observe that both the amplitudes of $\Phi_k = H\mathcal{B}_k$ and Ψ_k grow in proportion to $e^{Ht/2} \simeq e^{N/2}$. We recall that we used the approximation $\beta H^2 \gg 1$ to derive the analytic solutions (5.37), but the numerical results show that this estimation is valid even for βH^2 of order 1. In Fig. 1, the gravitational potentials exceed the order 1 around the e -foldings $N = 30 \sim 35$ after the onset of inflation. Thus, the exponential growth of \mathcal{B} in the perturbed line element (5.45) invalidates the FLRW background. Due to the uncertainty principle, the initial value of the perturbation χ_k has a nonvanishing value related to the energy scale H_i/M_{Pl} during inflation. Since H_i/M_{Pl} should

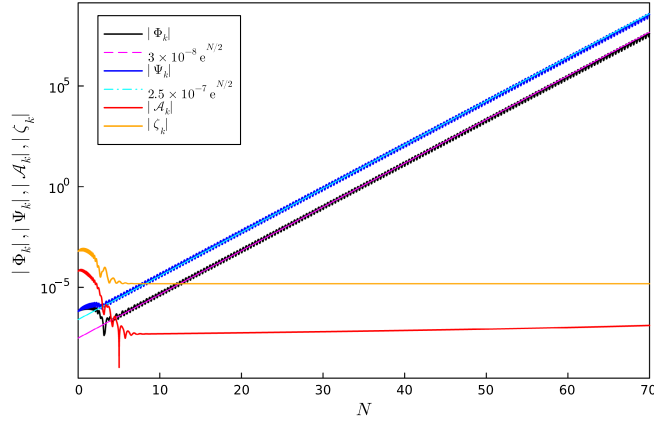


FIG. 1. Exponential increase of $|\Phi_k| = |h\chi_k|$ and $|\Psi_k|$ for the couplings $\bar{\alpha} = 1$ and $\bar{\beta} = 6$. In the regime $k \ll aH$, the growth of both $|\Phi_k|$ and $|\Psi_k|$ can be well-fitted by an exponential function proportional to $e^{N/2}$. The initial conditions are chosen to be $h(0) = 1$, $\epsilon(0) = 0.0047$, $H_i/M_{\text{Pl}} = 10^{-5}$, and $\Phi_k(0) = \chi_k(0) \simeq -6.31 \times 10^{-7}$ [which is determined by Eq. (6.26)] for the sub-Hubble mode $K(0) = 100$. We also show two exponential functions proportional to $e^{N/2}$ as a dashed pink line and a dash-dotted light blue line, which fit well with the numerical solutions of $|\Phi_k|$ and $|\Psi_k|$, respectively. The initial conditions of \mathcal{A}_k and ζ_k are known from Eqs. (6.25) and (6.27), as $\mathcal{A}_k(0) \simeq -6.34 \times 10^{-7} - 6.32 \times 10^{-5}i$ and $\zeta_k(0) \simeq 1.3 \times 10^{-4} - 5.9 \times 10^{-4}i$. To show the evolution of ζ_k , we have solved its own closed fourth-order differential equation. We see that $|\zeta_k|$ approaches a constant after the Hubble radius crossing, while \mathcal{A}_k exhibits very mild growth.

not be much smaller than 10^{-5} , the gravitational potentials reach order 1 after the amplification of $e^{N/2}$ times with $N > 30$. As we estimated analytically in Sec. V, the perturbation ζ_k approaches a constant after the Hubble radius crossing, while \mathcal{A}_k shows very mild growth. We recall that ζ_k and \mathcal{A}_k are related to each other in the super-Hubble regime and that they are not sufficient to describe the dynamics of scalar perturbations with two propagating d.o.f.s. Indeed, we cannot eliminate the instability of the other dynamical perturbation $\mathcal{B}_k = \Phi_k/H$.

In Fig. 2, we show the evolution of gauge-invariant perturbations by keeping the same initial conditions and model parameters as those in Fig. 1, except for $\bar{\alpha}$ which is set to a smaller value ($\bar{\alpha} = 1/400$) and for the initial values of the fields χ_k , \mathcal{A}_k , and ζ_k . Even with this small value of the Weyl coupling, both $|\Phi_k|$ and $|\Psi_k|$ increase in proportion to $e^{Ht/2} \simeq e^{N/2}$ after the Hubble radius crossing, while the growth of $|\zeta_k|$ and $|\mathcal{A}_k|$ is suppressed. Indeed, this behavior is expected according to the analytic estimations of $\Phi_k = H\mathcal{B}_k$, Ψ_k , \mathcal{A}_k , and ζ_k . For small values of α , the last two terms in Eqs. (5.35) and (5.36) grow in proportion to $e^{Ht/2}$ with oscillations.⁹ Note that Starobinsky inflation

⁹In Fig. 2, the oscillations of $|\Psi_k|$ and $|\Phi_k|$ are not clearly seen at large N , but we confirmed that they are present by enlarging the figure.

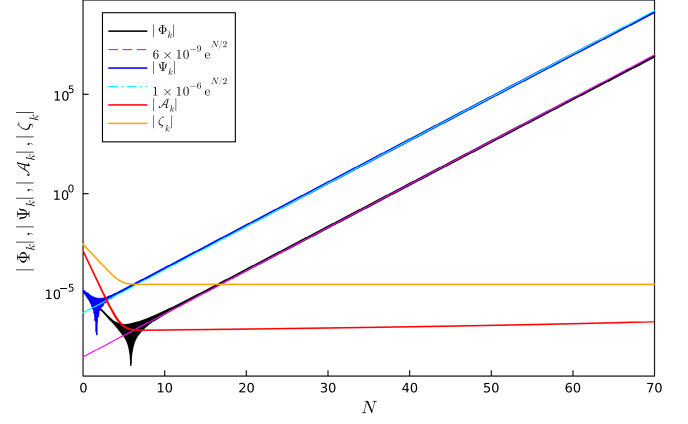


FIG. 2. The same as Fig. 1, but for the Weyl coupling $\bar{\alpha} = 1/400$, $\chi_k(0) \simeq -1.3 \times 10^{-5}$, $\mathcal{A}_k(0) \simeq -1.3 \times 10^{-5} - 0.0013i$, and $\zeta_k(0) \simeq 0.0028 - 2.86 \times 10^{-5}i$. Also for this small coupling constant $\bar{\alpha}$, the gravitational potentials grow as $e^{N/2}$.

without the Weyl term ($\alpha = 0$) cannot be recovered by simply taking the limit $\alpha \rightarrow 0$ in our theory. For $\alpha = 0$ there is only a single scalar d.o.f. arising from the βR^2 term, in which case the exponential growth of gravitational potentials is absent. However, as we saw for the vector and tensor perturbations, the squared mass of the extra modes, when the Weyl-squared term is present, is typically of order α^{-1} . Then, in the limit $\alpha \rightarrow 0$, the extra modes become very massive and we should expect them to acquire a mass larger than an ultraviolet cutoff scale of order M_{Pl} . In this case, they may be integrated out from the theory. On the other hand, the instability of scalar perturbations persists for the Weyl coupling constant in the range $\alpha \gtrsim M_{\text{Pl}}^{-2}$.

In Fig. 3, the evolution of gauge-invariant perturbations is plotted for the large Weyl coupling $\bar{\alpha} = 400$ with $\bar{\beta} = 6$. In this case, the condition $\alpha > 48\beta$ is satisfied and hence the two gravitational potentials grow as Eq. (5.38) without oscillations. Indeed, our numerical results in Fig. 3 demonstrate that, after the Hubble radius crossing, the growth of $|\Phi_k|$ and $|\Psi_k|$ occurs faster than in the case $\bar{\alpha} = 1$. The suppressed growth of the other perturbations \mathcal{A}_k and ζ_k also agrees with the analytic estimation in the regime $\beta H^2 \gg 1$.

In general, out of two exponentially growing modes Φ_k and Ψ_k , one can find some linear combinations of them, like ζ_k , whose growth is suppressed in the super-Hubble regime. In this theory, however, two scalar perturbations determine the stability of the background and not only one. Therefore, the stability of one linear combination is not sufficient for guaranteeing the stability of the whole dynamical system. In the flat gauge, the instability of the field \mathcal{B} appearing in the perturbed metric (5.45) is enough to make the whole background unstable in the super-Hubble regime. We also note that, in the flat gauge, the Weyl tensor component C^0_{i0j} in real space can be expressed as

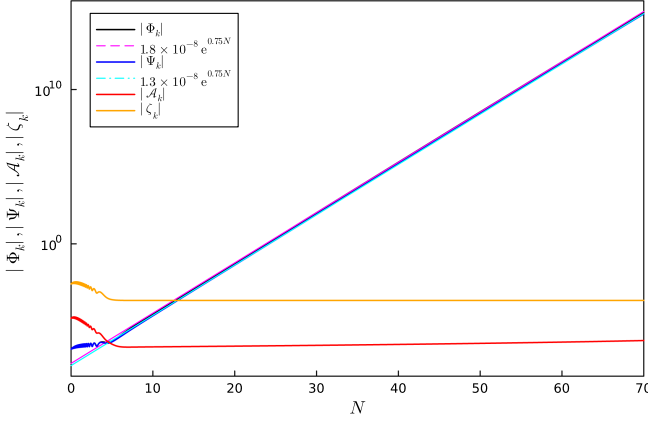


FIG. 3. The same as Fig. 1, except for $\bar{\alpha} = 400$, $\chi_k(0) \simeq 1.5 \times 10^{-7}$, $\mathcal{A}_k(0) \simeq 1.5 \times 10^{-7} + 1.5 \times 10^{-5}i$, $\zeta_k(0) \simeq -3.2 \times 10^{-5} - 0.0025i$, and for the Weyl coupling $\bar{\alpha} = 400$. For this large coupling constant $\bar{\alpha}$, the gravitational potentials grow as fast as $e^{3N/4}$.

$$C^0_{i0j} = -\frac{1}{2} \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) (\mathcal{A} + \dot{\mathcal{B}} - H\mathcal{B}), \quad (6.28)$$

which vanishes on the FLRW background. In Fourier space, we can also consider the evolution of the perturbation $C_k \equiv \mathcal{A}_k + \dot{\mathcal{B}}_k - H\mathcal{B}_k$ to see the departure from the background. While \mathcal{A}_k does not exhibit the exponential growth, the amplitude of $\mathcal{B}_k = \Phi_k/H$ evolves as $\mathcal{B}_k = b_0 e^{\lambda Ht}$ after the Hubble radius crossing, where b_0 and λ are nonzero constants and H is assumed to be constant during inflation. From Eqs. (5.37) and (5.38), the power λ is in the range $1/2 \leq \lambda < 1$. Since the amplitude of $\dot{\mathcal{B}}_k - H\mathcal{B}_k$ has the time dependence $\dot{\mathcal{B}}_k - H\mathcal{B}_k = b_0(\lambda - 1)H e^{\lambda Ht} \neq 0$ for super-Hubble modes, the perturbation C_k grows exponentially to spoil the FLRW background. Indeed, we have numerically found the exponential growth of the amplitude of C_k .

We also solved the perturbation equations in the Newtonian gauge and obtained the same numerical solutions for the gauge-invariant fields Ψ_k , Φ_k , ζ_k , and \mathcal{A}_k as those in the flat gauge. In the Newtonian gauge, the violation of the FLRW background occurs by the growth of two gravitational potentials Ψ and Φ in the perturbed line element (5.18). For this gauge choice, the Weyl tensor component C^0_{i0j} in real space is given by [62]

$$C^0_{i0j} = -\frac{1}{2} \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) (\Psi - \Phi). \quad (6.29)$$

As we can see in Figs. 1 and 2, the gravitational potentials Ψ_k and Φ_k in Fourier space are generally different from each other. Hence, the amplitude of the combination $\Psi_k - \Phi_k$ also grows exponentially after the Hubble radius crossing. For increasing α relative to β , the difference between Ψ_k and Φ_k tends to be smaller (see Fig. 3). Provided that α is finite, however, the exponential growth

of C^0_{i0j} always occurs by reflecting the fact that the amplitude of C^0_{i0j} is proportional to $|\lambda - 1| H e^{\lambda Ht}$ with $1/2 \leq \lambda < 1$, as we discussed for the flat gauge above. Indeed, irrespective of the values of α , we numerically confirmed the exponential increase of $|\Psi_k - \Phi_k|$ in the Newtonian gauge.

In the unitary gauge, we numerically observed the same exponential growth of Φ_k and the constancy of ζ_k after the Hubble radius crossing. In this case, the gauge-invariant combination $\mathcal{B}_u = (\Phi - \zeta)/H$ appearing in the perturbed metric (5.65) is subject to the exponential growth, thereby invalidating the FLRW background.

VII. CONCLUSIONS

In this paper, we studied the dynamics of cosmological perturbations during inflation in quadratic gravity containing the Weyl term $-\alpha C^2$ besides the Ricci squared term βR^2 in the action. Although the Weyl curvature does not affect the background inflationary dynamics driven by the βR^2 term, the evolution of perturbations is modified by the presence of derivatives higher than second order. Since these higher-order derivatives can give rise to ghosts, it is of interest to explore whether or not the ghosts can lead to instabilities of the FLRW background.

As we discussed in Sec. II, geometric inflation is realized by the βR^2 term with $\beta > 0$, where the coupling constant β is related to the mass squared m_S^2 of a new scalar d.o.f. (scalaron) as $\beta = 1/(6m_S^2)$. To realize the number of e -foldings larger than 60, we require that the Hubble parameter H_i at the onset of inflation is in the range $\beta H_i^2 \gtrsim \mathcal{O}(1)$. If we transform the action (2.1) to that in the Einstein frame, the quadratic gravity can be interpreted as the conformally invariant Weyl theory in the presence of a canonical scalaron field with the potential. Unlike the past related works [41–45], we have carried out all the analysis in the physical Jordan frame.

In Sec. III, we showed that the Weyl term gives rise to two dynamical vector d.o.f.s propagating with the speed of light. For the Weyl coupling $\alpha > 0$ the two ghosts are present with the positive mass squared $m_W^2 = 1/(2\alpha)$, while, for $\alpha < 0$, there are no ghosts. In the latter case, however, the negative value of m_W^2 leads to the tachyonic instability of vector perturbations for $|\alpha|$ at most of order β . To avoid such an instability which violates the inflationary FLRW background, we demand the condition $\alpha > 0$ at the expense of admitting the existence of ghosts.

In Sec. IV, we derived the second-order action of tensor perturbations and introduced Lagrange multiplier fields χ_i ($i = 1, 2$) associated with higher-order time derivatives. There are four dynamical d.o.f.s in the tensor sector, two of which behave as ghosts. Using the WKB approximation for the modes deep inside the Hubble radius ($k/a \gg H$), the speed of tensor perturbations is equivalent to 1 with vanishing masses. Despite the presence of the Weyl ghost,

the classical perturbations are not subject to either Laplacian or tachyonic instabilities for subhorizon modes. In the super-Hubble regime ($k/a \ll H$), tensor perturbations h_i obey the fourth-order differential Eq. (4.17). Provided that the couplings α and β are in the ranges $\alpha > 0$ and $\beta H^2 \gtrsim 1$, we showed that h_i 's approach constants after the Hubble radius crossing. This means that, despite the presence of ghosts, tensor perturbations are subject to neither Laplacian nor tachyonic instabilities.

In Sec. V, we studied the stability and evolution of scalar perturbations by choosing several different gauge conditions. There are two dynamical propagating d.o.f.s in the scalar sector arising from the Lagrangians $-\alpha C^2$ and βR^2 . For $\alpha > 0$ and $\beta > 0$, the scalaron is not a ghost, but the other dynamical mode behaves as a ghost. To study the dynamics of perturbations, we also introduced several gauge-invariant perturbations such as those defined in Eqs. (5.4)–(5.6). We chose the Newtonian, flat, and unitary gauges and derived the closed differential equations for Ψ , Φ , \mathcal{A} , $\mathcal{B} = \Phi/H$, and ζ . We found that the coefficients of these differential equations are uniquely fixed independent of the gauge choices. We showed that, after the Hubble radius crossing, both Ψ and Φ grow exponentially, while \mathcal{A} and ζ approach constants.

In the Newtonian gauge given by the perturbed line element (5.18), the exponential growth of Ψ and Φ occurs in the g_{00} and g_{ii} metric components. This violates the stability of the FLRW background after the perturbations cross the Hubble radius during inflation. For the flat-gauge line element (5.45) the growth of \mathcal{A} is suppressed, but the exponential increase of $\mathcal{B} = \Phi/H$ occurs together with the enhancement of Φ . We have also numerically confirmed this behavior for the gauge-invariant perturbations in the numerical simulations of Figs. 1–3 performed in Sec. VI. In the unitary gauge, the perturbed line element (5.65) also contains the instability mode Φ in the g_{0i} metric component. We stress that these instabilities are the

physical ones arising from the gravitational interaction between the scalaron and the other ghost d.o.f.

We have thus shown that the inflationary FLRW background realized by the βR^2 term is violated by the presence of the Weyl term. In other words, the Universe becomes highly inhomogeneous during inflation, being incompatible with the observations of CMB temperature anisotropies. This instability of scalar perturbations is present for the wide coupling range $\alpha \gtrsim M_{\text{Pl}}^{-2}$ in which the mass term $1/\sqrt{\alpha}$ associated with the Weyl term does not exceed the ultraviolet scale of order M_{Pl} . Unless the scalar ghost arising from the Weyl term is suitably eliminated as a physical propagating d.o.f. and the classical instability of the background is removed, the quadratic curvature theory with $\alpha \neq 0$ is excluded as a viable model of inflation (or at most, the coupling α must be so small that the mass of the extra modes becomes larger than the cutoff of the theory). Related to the ghost issue, there is an approach of “fakeon” where the ghost does not appear as a physical state after quantizing it as a fake d.o.f. [63–65]. There are also some approaches to the ghost problem in quantum field theory by keeping its physical status intact [66–69]. In such approaches to the ghost problem, it will be of interest to study the stability of cosmological perturbations and resulting observational consequences in detail.

ACKNOWLEDGMENTS

The work of A.D.F. was supported by the Japan Society for the Promotion of Science Grants-in-Aid for Scientific Research No. 20K03969 and by Grant No. PID2020–118159 GB-C41 funded by MCIN/AEI/10.13039/501100011033. S.T. was supported by the Grant-in-Aid for Scientific Research Fund of the JSPS No. 22K03642 and Waseda University Special Research Project No. 2023C-473.

-
- [1] A. A. Starobinsky, *Phys. Lett.* **91B**, 99 (1980).
 - [2] K. Sato, *Mon. Not. R. Astron. Soc.* **195**, 467 (1981).
 - [3] D. Kazanas, *Astrophys. J. Lett.* **241**, L59 (1980).
 - [4] A. H. Guth, *Phys. Rev. D* **23**, 347 (1981).
 - [5] A. A. Starobinsky, *JETP Lett.* **30**, 682 (1979).
 - [6] A. A. Starobinsky, *Phys. Lett.* **117B**, 175 (1982).
 - [7] S. W. Hawking, *Phys. Lett.* **115B**, 295 (1982).
 - [8] A. H. Guth and S. Y. Pi, *Phys. Rev. Lett.* **49**, 1110 (1982).
 - [9] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, *Phys. Rev. D* **28**, 679 (1983).
 - [10] D. N. Spergel *et al.* (WMAP Collaboration), *Astrophys. J. Suppl. Ser.* **148**, 175 (2003).
 - [11] P. A. R. Ade *et al.* (Planck Collaboration), *Astron. Astrophys.* **571**, A16 (2014).
 - [12] Y. Akrami *et al.* (Planck Collaboration), *Astron. Astrophys.* **641**, A10 (2020).
 - [13] A. A. Starobinsky, *JETP Lett.* **42**, 152 (1985).
 - [14] A. Vilenkin, *Phys. Rev. D* **32**, 2511 (1985).
 - [15] M. B. Mijic, M. S. Morris, and W.-M. Suen, *Phys. Rev. D* **34**, 2934 (1986).
 - [16] S. Tsujikawa, K.-i. Maeda, and T. Torii, *Phys. Rev. D* **60**, 123505 (1999).
 - [17] A. A. Starobinsky, S. Tsujikawa, and J. Yokoyama, *Nucl. Phys.* **B610**, 383 (2001).
 - [18] G. J. Olmo, *Phys. Rev. D* **72**, 083505 (2005).

- [19] V. Faraoni, *Phys. Rev. D* **72**, 124005 (2005).
- [20] T. Chiba, T. L. Smith, and A. L. Erickcek, *Phys. Rev. D* **75**, 124014 (2007).
- [21] I. Navarro and K. Van Acoleyen, *Phys. Lett. B* **622**, 1 (2005).
- [22] L. Amendola and S. Tsujikawa, *Phys. Lett. B* **660**, 125 (2008).
- [23] C. Brans and R. H. Dicke, *Phys. Rev.* **124**, 925 (1961).
- [24] J. O'Hanlon, *Phys. Rev. Lett.* **29**, 137 (1972).
- [25] T. Chiba, *Phys. Lett. B* **575**, 1 (2003).
- [26] T. P. Sotiriou and V. Faraoni, *Rev. Mod. Phys.* **82**, 451 (2010).
- [27] V. F. Mukhanov, L. A. Kofman, and D. Y. Pogosian, *Phys. Lett. B* **193**, 427 (1987).
- [28] D. S. Salopek, J. R. Bond, and J. M. Bardeen, *Phys. Rev. D* **40**, 1753 (1989).
- [29] J.-c. Hwang and H. Noh, *Phys. Lett. B* **506**, 13 (2001).
- [30] V. F. Mukhanov and G. V. Chibisov, *JETP Lett.* **33**, 532 (1981).
- [31] A. De Felice and S. Tsujikawa, *Living Rev. Relativity* **13**, 3 (2010).
- [32] A. Salvio, *Front. Phys.* **6**, 77 (2018).
- [33] D. Lovelock, *J. Math. Phys. (N.Y.)* **12**, 498 (1971).
- [34] K. S. Stelle, *Phys. Rev. D* **16**, 953 (1977).
- [35] E. S. Fradkin and A. A. Tseytlin, *Nucl. Phys.* **B201**, 469 (1982).
- [36] K. S. Stelle, *Gen. Relativ. Gravit.* **9**, 353 (1978).
- [37] A. Hindawi, B. A. Ovrut, and D. Waldram, *Phys. Rev. D* **53**, 5583 (1996).
- [38] C. Bogdanos, S. Capozziello, M. De Laurentis, and S. Nesseris, *Astropart. Phys.* **34**, 236 (2010).
- [39] K. Hinterbichler and M. Saravani, *Phys. Rev. D* **93**, 065006 (2016).
- [40] A. V. Smilga, *Nucl. Phys.* **B706**, 598 (2005).
- [41] T. Clunan and M. Sasaki, *Classical Quantum Gravity* **27**, 165014 (2010).
- [42] N. Deruelle, M. Sasaki, Y. Sendouda, and A. Youssef, *J. Cosmol. Astropart. Phys.* **03** (2011) 040.
- [43] Y. S. Myung and T. Moon, *Mod. Phys. Lett. A* **30**, 1550172 (2015).
- [44] M. M. Ivanov and A. A. Tokareva, *J. Cosmol. Astropart. Phys.* **12** (2016) 018.
- [45] D. Anselmi, F. Fruzza, and M. Piva, *Classical Quantum Gravity* **38**, 225011 (2021).
- [46] B. Whitt, *Phys. Lett.* **145B**, 176 (1984).
- [47] J. D. Barrow and S. Cotsakis, *Phys. Lett. B* **214**, 515 (1988).
- [48] K.-i. Maeda, *Phys. Rev. D* **39**, 3159 (1989).
- [49] S. Gottlober, H. J. Schmidt, and A. A. Starobinsky, *Classical Quantum Gravity* **7**, 893 (1990).
- [50] J. M. Bardeen, *Phys. Rev. D* **22**, 1882 (1980).
- [51] H. Kodama and M. Sasaki, *Prog. Theor. Phys. Suppl.* **78**, 1 (1984).
- [52] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, *Phys. Rep.* **215**, 203 (1992).
- [53] B. A. Bassett, S. Tsujikawa, and D. Wands, *Rev. Mod. Phys.* **78**, 537 (2006).
- [54] V. N. Lukash, *Sov. Phys. JETP* **52**, 807 (1980).
- [55] D. H. Lyth, *Phys. Rev. D* **31**, 1792 (1985).
- [56] V. F. Mukhanov, *JETP Lett.* **41**, 493 (1985).
- [57] M. Sasaki, *Prog. Theor. Phys.* **76**, 1036 (1986).
- [58] J.-c. Hwang and H.-r. Noh, *Phys. Rev. D* **65**, 023512 (2002).
- [59] L. Heisenberg, R. Kase, and S. Tsujikawa, *Phys. Rev. D* **98**, 123504 (2018).
- [60] R. Kase and S. Tsujikawa, *Int. J. Mod. Phys. D* **28**, 1942005 (2019).
- [61] A. De Felice and S. Tsujikawa, *J. Cosmol. Astropart. Phys.* **10** (2023) 004.
- [62] R. Durrer, *The Cosmic Microwave Background* (Cambridge University Press, Cambridge, England, 2020).
- [63] D. Anselmi, *J. High Energy Phys.* **06** (2017) 086.
- [64] D. Anselmi, *J. High Energy Phys.* **02** (2018) 141.
- [65] D. Anselmi, E. Bianchi, and M. Piva, *J. High Energy Phys.* **07** (2020) 211.
- [66] C. M. Bender and P. D. Mannheim, *Phys. Rev. Lett.* **100**, 110402 (2008).
- [67] J. F. Donoghue and G. Menezes, *Phys. Rev. D* **100**, 105006 (2019).
- [68] A. Salvio, *Int. J. Mod. Phys. A* **36**, 2130006 (2021).
- [69] J. Kubo, J. Kuntz, J. Rezacek, and P. Saake, *J. Cosmol. Astropart. Phys.* **11** (2022) 049.