Spatial curvature from super-Hubble cosmological fluctuations

Baptiste Blachier,[1,2,*] Pierre Auclair,[1,†] Christophe Ringeval,[1,‡] and Vincent Vennin[3,§]

[1]*Cosmology, Universe and Relativity at Louvain (CURL), Institute of Mathematics and Physics, University of Louvain, 2 Chemin du Cyclotron, 1348 Louvain-la-Neuve, Belgium*

[2]*Department of Physics, École Normale Supérieure de Lyon, 69364 Lyon, France*

[3]*Laboratoire de Physique de l'École Normale Supérieure, ENS, CNRS, Université PSL, Sorbonne Université, Université Paris Cité, 75005 Paris, France*

(Received 6 March 2023; accepted 4 November 2023; published 4 December 2023)

We revisit how super-Hubble cosmological fluctuations induce, at any time in the cosmic history, a nonvanishing spatial curvature of the local background metric. The random nature of these fluctuations promotes the curvature density parameter to a stochastic quantity for which we derive novel non-perturbative expressions for its mean, variance, higher moments, and full probability distribution. For scale-invariant Gaussian perturbations, such as those favored by cosmological observations, we find that the most probable value for the curvature density parameter Ω_{K} today is -10^{-9} and that its mean is $+10^{-9}$, both being overwhelmed by a standard deviation of the order of 10^{-5}. We then discuss how these numbers would be affected by the presence of large super-Hubble non-Gaussianities or if inflation lasted for a very long time. In particular, we find that substantial values of Ω_{K} are obtained if inflation lasts for more than a billion e-folds.

DOI: 10.1103/PhysRevD.108.123510

I. INTRODUCTION

Cosmic structures in the Universe are understood to be seeded by some preexisting super-Hubble cosmological fluctuations. Their gravitational collapse starts when their size becomes smaller than the Hubble radius, an inevitable outcome in any decelerating Friedmann-Lemaître spacetime. Observational evidence of this mechanism is present in the cosmic microwave background (CMB) data by the correlation patterns associated with the polarization and temperature angular power spectra [1,2], as well as in the statistics of the large-scale structures observed at lower redshifts [3,4].

Cosmic inflation, an early era of accelerated cosmic expansion, is the prime candidate to explain the origin of the super-Hubble fluctuations. They are of quantum origin, stretched to length scales much larger than the Hubble radius during inflation [5–16]. At the same time, inflation smooths out any preexisting inhomogeneity, and one of the historical motivations for cosmic inflation is that the spatial curvature of spacetime, Ω_{K}, should be exponentially small at the end of inflation (at most e^{-60}). This prediction is compatible with the current bound $|\Omega_{\mathrm{K}_0}| < 3 \times 10^{-3}$ today, coming from the Planck CMB data and baryon acoustic oscillation measurements.

Intuitively, the existence, today, of Hubble-sized curvature fluctuations suggests that these could be confused with a small nonvanishing spatial curvature of the local background metric. In particular, these modes are expected to induce a limitation on our ability to measure very small values of the curvature density parameter [17–21]. More than being a nuisance, we will show that super-Hubble (hence, "conserved") fluctuations do create spatial curvature.

In order to deal with fluctuations over a background metric when both are intertwined, we can start from the inhomogeneous metric proposed in Refs. [22–25]:

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a^2(\tau) e^{2\zeta(\tau,\boldsymbol{x})} \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{1}$$

This metric is not fully general, as inhomogeneities are all contained in one scalar function ζ. However, as discussed in Refs. [22–25], this is the most generic metric in the absence of vector- and tensor-type inhomogeneities and in the gauge where fixed time slices have uniform energy density and fixed spatial worldlines are comoving with matter. At super-Hubble scales, this reduces to the synchronous gauge supplemented by some additional conditions that fix it uniquely. The quantity $\zeta(\tau, \boldsymbol{x})$ can be shown to be "conserved" at large distances. As such, it provides a nonlinear generalization of the constant-energy-density curvature perturbation [26,27].

*baptiste.blachier@ens-lyon.fr
†pierre.auclair@uclouvain.be
‡christophe.ringeval@uclouvain.be
§vincent.vennin@ens.fr

 © 2023 American Physical Society

Historically, this metric has been intensively discussed in the attempts to explain the acceleration of the Universe by the backreaction of super-Hubble inhomogeneities [28,29]. But, as realized soon after [30–32], the effects of super-Hubble fluctuations onto the background evolution are to modify the spatial curvature. Let us notice that, on top of the background evolution, other observable signatures are possible [33–35]. To our knowledge, the only works having addressed how super-Hubble modes affect the spatial curvature are Refs. [32,36,37], based, however, on perturbative gradient expansions or linear perturbation theory only. When the nonperturbative terms of our derivation can be neglected, we recover some of their results.

The paper is organized as follows. In Sec. II, we derive an exact expression for the curvature density parameter Ω_{K} in terms of the nonlinear curvature perturbation ζ. This promotes Ω_{K} to a stochastic quantity, and in Sec. III we calculate its moments as well as its probability density function, assuming Gaussian statistics for ζ. Finally, we conclude by discussing how the statistics of the curvature density parameter is modified in the presence of non-Gaussian super-Hubble fluctuations or if inflation lasted for a very long time.

II. CURVATURE DENSITY PARAMETER

When spatial curvature is included, the Friedmann-Lemaître-Robertson-Walker (FLRW) line element reads

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a^2(\tau)\frac{\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j}{(1+\frac{K}{4}\delta_{mn}x^m x^n)^2}, \tag{2}$$

where K is a constant, and its Ricci scalar is given by

$$R = 6\frac{\dot{a}^2}{a^2} + 6\frac{\ddot{a}}{a} + \frac{6}{a^2}K. \tag{3}$$

The metric (1) can be viewed as an inhomogeneous generalization of a flat, i.e., $K=0$, FLRW spacetime having a space-dependent scale factor

$$b(\tau,\boldsymbol{x}) \equiv a(\tau)e^{\zeta(\tau,\boldsymbol{x})}, \tag{4}$$

from which one can derive the Ricci scalar

$$R = 6\frac{\dot{b}^2}{b^2} + 6\frac{\ddot{b}}{b} + 2\frac{(\boldsymbol{\nabla}b)^2}{b^4} - 4\frac{\Delta b}{b^3}. \tag{5}$$

We now split $\zeta(\tau,\boldsymbol{x}) = \xi(\boldsymbol{x}) + \zeta_{\mathrm{s}}(\tau,\boldsymbol{x})$ into a conserved part ξ (super-Hubble) and time-dependent fluctuations ζ_{s} (sub-Hubble). Expanding in the (presumably small) short-length part, one has

$$b(\tau,\boldsymbol{x}) = a(\tau)e^{\xi(\boldsymbol{x})}[1+\zeta_{\mathrm{s}}(\tau,\boldsymbol{x})+\cdots], \tag{6}$$

and upon defining

$$\tilde{a}(\tau,\boldsymbol{x}) = a(\tau)e^{\xi(\boldsymbol{x})} \tag{7}$$

one is led to

$$R = 6\frac{\dot{\tilde{a}}^2}{\tilde{a}^2} + 6\frac{\ddot{\tilde{a}}}{\tilde{a}} + \frac{6}{\tilde{a}^2}\left[-\frac{2}{3}\Delta\xi - \frac{1}{3}(\boldsymbol{\nabla}\xi)^2\right] + \cdots. \tag{8}$$

The omitted terms in this expression are the ones appearing in the linear theory of cosmological perturbations, in the synchronous gauge, completed by all possible nonlinear corrections involving powers of $\zeta_{\mathrm{s}}(\tau,\boldsymbol{x})$ and products with $\tilde{a}(\tau,\boldsymbol{x})$ [38]. The mixed terms involving both $\tilde{a}(\tau,\boldsymbol{x})$ and powers of $\zeta_{\mathrm{s}}(\tau,\boldsymbol{x})$ were precisely the ones discussed in the early works on backreaction and are non-observable [30–32]. As can be checked in Eq. (8), the terms we have kept are invariant by a constant shift of $\xi(\boldsymbol{x})$, up to a redefinition of $a(\tau)$.

Since $\xi(\boldsymbol{x})$ varies on super-Hubble length scales only, so does $\tilde{a}(\tau,\boldsymbol{x})$; hence, any observer will identify $\tilde{a}(\tau,\boldsymbol{x})$ as the FLRW scale factor of their local Hubble patch. Let us notice that, in the gauge we work in, the Hubble radius is the same for all observers, since [24,39,40]

$$\tilde{H} \equiv \frac{\dot{\tilde{a}}}{\tilde{a}} = \frac{\dot{a}}{a} = H, \tag{9}$$

which does not depend on $\boldsymbol{x}$. An important remark is that Eqs. (3) and (8) coincide upon identifying

$$K = -\frac{2}{3}\Delta\xi - \frac{1}{3}(\boldsymbol{\nabla}\xi)^2, \tag{10}$$

which is indeed constant, since ξ is conserved, and whose measurable curvature density parameter reads

$$\Omega_{\mathrm{K}} = -\frac{K}{\tilde{a}^2\tilde{H}^2} = -\frac{Ke^{-2\xi}}{a^2H^2}. \tag{11}$$

Let us stress that Eq. (10) is exact in the sense that all the terms omitted involve $\zeta_{\mathrm{s}}(\tau,\boldsymbol{x})$; hence, they are time dependent and cannot be absorbed in K. Equation (10) makes also explicit that only gradients of super-Hubble inhomogeneities have a nontrivial effect.

III. STATISTICS

Current cosmological measurements [41] imply that ζ has Gaussian statistics and can, thus, be treated as a random Gaussian field, with vanishing mean and higher-point correlation functions entirely determined by the power spectrum

$$\langle\zeta(\boldsymbol{k})\zeta(\boldsymbol{k}')\rangle = (2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{k}')P_\zeta(k). \tag{12}$$

This is also in agreement with the most favored inflationary scenarios, where the mean values are identified with vacuum expectation values of quantum operators in the Bunch-Davis vacuum. Later on, we will also use the spherical power spectrum $\mathcal{P}_\zeta(k)$ defined by

$$\mathcal{P}_\zeta(k) = \frac{k^3}{2\pi^2} P_\zeta(k) \simeq \mathcal{P}_*, \tag{13}$$

where the last approximation holds for a scale-invariant power spectrum.

From Eqs. (10) and (11), Ω_K can, therefore, also be seen as a stochastic quantity, though its nonlinear dependence on ξ, and, thus, on ζ, implies that it does not feature Gaussian statistics. In particular, its expectation value does not necessarily vanish.

Let us make the decomposition $\zeta(\tau, \boldsymbol{x}) = \xi(\boldsymbol{x}) + \zeta_\mathrm{s}(\tau, \boldsymbol{x})$ explicit in Fourier space:

$$\begin{aligned}\zeta(\tau, \boldsymbol{x}) = &\frac{1}{(2\pi)^3} \int \mathrm{d}^3\boldsymbol{k} \Theta(k_\sigma - k) \zeta(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ &+ \frac{1}{(2\pi)^3} \int \mathrm{d}^3\boldsymbol{k} \Theta(k - k_\sigma) \zeta(\tau, \boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}},\end{aligned} \tag{14}$$

where we have introduced a wave number k_σ below which all Fourier modes $\zeta(\tau, k < k_\sigma) = \zeta(k)$ can be approximated as time independent. Based on the theory of cosmological perturbations, and its generalizations [26,27], this wave number is at most of the order of the conformal Hubble parameter at the observer's time, say, τ_0; namely, $k_\sigma \lesssim \tilde{a}(\tau_0)\tilde{H}(\tau_0)$. Let us remark the presence of $\tilde{a}(\tau_0, \boldsymbol{x})$, instead of $a(\tau_0)$, in this expression. *A priori*, this would induce an extra dependence on $\boldsymbol{x}$ in Eq. (14), where one should write $k_\sigma(\boldsymbol{x})$. In order to circumvent this issue, we can, for now, simply choose the cutoff k_σ to be sufficiently small such that it encompasses all possible spatial modulations of $\tilde{a}(\tau_0, \boldsymbol{x})$. In other words, we define

$$k_\sigma \equiv \sigma a_0 H_0, \tag{15}$$

where, in principle, $\sigma < e^{\min_x(\xi)}$. As such, we can identify the conserved quantity with

$$\xi(\boldsymbol{x}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\boldsymbol{k} \Theta(k_\sigma - k) \zeta(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{16}$$

Let us remark that σ also quantifies the possible ambiguities in separating the background, made of the time-independent $\xi(\boldsymbol{x})$, from the modes which contribute to the perturbations, the time-dependent $\zeta_\mathrm{s}(\tau, \boldsymbol{x})$.

A. Mean value

The mean value of the curvature density parameter reads

$$\langle \Omega_\mathrm{K} \rangle = -\frac{\langle K e^{-2\xi} \rangle}{a^2 H^2}, \tag{17}$$

where ξ is given by Eq. (16). The curvature scalar K, given in Eq. (10), can be split into two terms $K = K_1 + K_2$ with

$$K_1 \equiv -\frac{2}{3}\Delta\xi, \qquad K_2 \equiv -\frac{1}{3}(\boldsymbol{\nabla}\xi)^2. \tag{18}$$

Therefore, one needs the Laplacian and the squared gradient of ξ. They read, respectively,

$$\Delta\xi = -\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \Theta(k_\sigma - k) k^2 \zeta(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{19}$$

and

$$\begin{aligned}(\boldsymbol{\nabla}\xi)^2 = -\int &\frac{\mathrm{d}^3\boldsymbol{p}\mathrm{d}^3\boldsymbol{q}}{(2\pi)^6} \Theta(k_\sigma - p)\Theta(k_\sigma - q) \\ &\times \boldsymbol{p}\cdot\boldsymbol{q}\zeta(\boldsymbol{p})\zeta(\boldsymbol{q}) e^{i(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{x}},\end{aligned} \tag{20}$$

from which one can immediately calculate

$$\begin{aligned}\langle K \rangle = \langle K_2 \rangle &= -\frac{1}{3}\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \Theta(k_\sigma - k) k^2 P_\zeta(k) \\ &= -\frac{1}{3}\int_0^{k_\sigma} \mathrm{d}k k \mathcal{P}_\zeta(k) \simeq -\frac{1}{6} k_\sigma^2 \mathcal{P}_*,\end{aligned} \tag{21}$$

the rightmost equality holding only for a scale-invariant power spectrum.

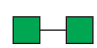
The term $e^{-2\xi}$ appearing in Eq. (17) can be expressed in terms of $\zeta(\boldsymbol{k})$ by using the series representation

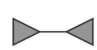
$$e^{-2\xi} = \sum_{n=0}^{+\infty} \frac{(-2)^n}{n!} \xi^n, \tag{22}$$

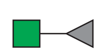
with

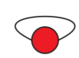
$$\xi^n = \int \frac{\mathrm{d}^3\boldsymbol{k}_1 \ldots \mathrm{d}^3\boldsymbol{k}_n}{(2\pi)^{3n}} \left[\prod_{j=1}^n \Theta(k_\sigma - k_j)\zeta(\boldsymbol{k}_j)\right] e^{i\boldsymbol{x}\cdot\sum_j \boldsymbol{k}_j}. \tag{23}$$

As can be seen in Eq. (17), the mean value of the curvature density parameter requires the explicit determination of an infinite number of terms, the nonvanishing ones being of the form $\langle K_1 \xi^{2p+1} \rangle$ and $\langle K_2 \xi^{2p} \rangle$. From Eqs. (12), (18), and (23), one can make extensive use of the Wick theorem to reduce all the expectation values to a few two-point functions with the following diagrammatic rules:

$\langle K_1 K_1 \rangle \equiv$  $= \langle K_1^2 \rangle,$

$\langle \xi\xi \rangle \equiv$  $= \langle \xi^2 \rangle,$ (24)

$\langle K_1 \xi \rangle \equiv$  $= -2\langle K \rangle,$

$\langle K_2 \rangle \equiv$  $= \langle K \rangle.$

Let us notice that, due to the inner product structure of Eq. (20), the K_2 vertices have two "legs" that can connect only to other K_2 vertices. From Eq. (19), one has

$$\langle K_1^2 \rangle = \frac{4}{9}\int_0^{k_\sigma} \mathrm{d}k k^3 \mathcal{P}_\zeta(k) \simeq \frac{1}{9} k_\sigma^4 \mathcal{P}_*, \tag{25}$$

which allows us to express the second moment of the curvature scalar as

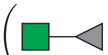
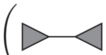
$$\langle K^2 \rangle = \langle K_1^2 \rangle + \frac{5}{3}\langle K \rangle^2 \simeq \frac{1}{9} k_\sigma^4 \mathcal{P}_* \left(1 + \frac{5}{12}\mathcal{P}_*\right). \tag{26}$$

In Eq. (24), we also need the variance of the conserved quantity ξ. It can be determined from Eq. (16) and reads

$$\langle \xi^2 \rangle = \int_{k_\varepsilon}^{k_\sigma} \mathrm{d}k \frac{\mathcal{P}_\zeta(k)}{k} \simeq \mathcal{P}_* \ln\left(\frac{k_\sigma}{k_\varepsilon}\right) \simeq \mathcal{P}_* N_{\mathrm{inf}}, \tag{27}$$

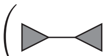
where we have introduced an expected infrared cutoff k_ε. Indeed, in the context of cosmic inflation, the ratio between the largest and shortest lengths being amplified is precisely given by the total amount of stretching generated by the accelerated expansion, the so-called total number of e-folds N_{inf}. For the measured value of $\mathcal{P}_* = 2.1 \times 10^{-9}$ [42], and a not too long inflationary era $N_{\mathrm{inf}} \ll 10^9$, $\langle \xi^2 \rangle$ is a small quantity.

Denoting by $W_{2p} = (2p)!/(p!2^p)$ the number of Wick contractions between p pairs, one obtains

$\langle K_1 \xi^{2p+1} \rangle = (2p+1)\Big($  $\Big) \times W_{2p} \Big($  $\Big)^p$

$$= -\frac{(2p+1)!}{p!\, 2^{p-1}} \langle K \rangle \langle \xi^2 \rangle^p, \tag{28}$$

and

$\langle K_2 \xi^{2p} \rangle =$  $\times W_{2p} \Big($  $\Big)^p$

$$= \frac{(2p)!}{p!\, 2^p} \langle K \rangle \langle \xi^2 \rangle^p. \tag{29}$$

The infinite series obtained by combining Eqs. (28), (29), and (22) can be resummed, and one gets the exact expression

$$\langle \Omega_{\mathrm{K}} \rangle = -\frac{5}{a^2 H^2} \langle K \rangle e^{2\langle \xi^2 \rangle}. \tag{30}$$

Making use of Eqs. (21) and (27), for a scale-invariant power spectrum, Eq. (30) simplifies to

$$\langle \Omega_{\mathrm{K}_0} \rangle \simeq \frac{5}{6} \frac{k_\sigma^2}{a_0^2 H_0^2} \mathcal{P}_* e^{2\mathcal{P}_* N_{\mathrm{inf}}} \simeq \frac{5}{6} \sigma^2 \mathcal{P}_*, \tag{31}$$

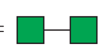
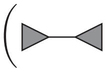
which saturates for $\sigma = 1$ at $\langle \Omega_{\mathrm{K}_0} \rangle \simeq 1.7 \times 10^{-9}$, a barely open universe were we to interpret this number within a FLRW metric with trivial topology.

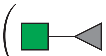
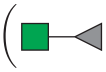
B. Variance

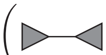
There is little hope to measure such a small value of $\langle \Omega_{\mathrm{K}} \rangle$, but, Ω_{K} being a stochastic variable, its realizations are also dictated by the higher moments, the second one being given by

$$\langle \Omega_{\mathrm{K}}^2 \rangle = \frac{\langle K^2 e^{-4\xi} \rangle}{a^4 H^4} = \frac{\langle (K_1^2 + 2K_1 K_2 + K_2^2) e^{-4\xi} \rangle}{a^4 H^4}. \tag{32}$$

Using again a series representation for the exponential, Eq. (32) can be expanded in an infinite sum requiring the calculation of the nonvanishing terms $\langle K_1^2 \xi^{2p+2} \rangle$, $\langle K_1 K_2 \xi^{2p+1} \rangle$, and $\langle K_2^2 \xi^{2p} \rangle$, with $p \geq 0$. Using the diagrammatic rules of Eq. (24), one gets

$\langle K_1^2 \xi^{2p+2} \rangle =$  $\times W_{2p+2} \Big($  $\Big)^{p+1}$

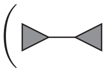
$+ (2p+1) \Big($  $\Big) \times 2p \Big($  $\Big)$

$\times W_{2p} \Big($  $\Big)^p$

$$= \frac{(2p+1)!}{p!\, 2^p} \langle K_1^2 \rangle \langle \xi^2 \rangle^{p+1} + 4 \frac{(2p+2)!}{p!\, 2^p} \langle K \rangle^2 \langle \xi^2 \rangle^p, \tag{33}$$

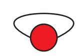
together with

$\langle K_1 K_2 \xi^{2p+1} \rangle =$  $\times (2p+1) \Big($  $\Big)$

$\times W_{2p} \Big($  $\Big)^p$

$$= -2 \frac{(2p+1)!}{p!\, 2^p} \langle K \rangle^2 \langle \xi^2 \rangle^p, \tag{34}$$

and

$\langle K_2^2 \xi^{2p} \rangle = \Big[\Big($  $\Big)^2 + 2 \Big($  $\Big) \Big]$

$\times W_{2p} \Big($  $\Big)^p = \frac{5}{3} \frac{(2p)!}{p!\, 2^p} \langle K \rangle^2 \langle \xi^2 \rangle^p. \quad (35)$

Summing all the terms coming from the expansion of Eq. (32) gives the exact expression

$$\langle \Omega_{\mathrm{K}}^2 \rangle = \frac{1}{a^4 H^4} (\langle K^2 \rangle + 80 \langle K \rangle^2) e^{8\langle \xi^2 \rangle}. \tag{36}$$

For a scale-invariant power spectrum, using Eqs. (21), (27), and (26), one obtains

$$\langle \Omega_{\mathrm{K}_0}^2 \rangle \simeq \frac{1}{9}\frac{k_\sigma^4}{a_0^4 H_0^4}\mathcal{P}_*\left(1+\frac{245}{12}\mathcal{P}_*\right)e^{8\mathcal{P}_* N_{\mathrm{inf}}} \simeq \frac{1}{9}\sigma^4\mathcal{P}_*. \quad (37)$$

Using Eq. (31) for $\sigma = 1$, the standard deviation of Ω_{K_0} is given by

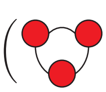
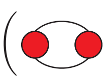
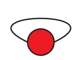
$$\sqrt{\langle \Omega_{\mathrm{K}_0}^2\rangle - \langle \Omega_{\mathrm{K}_0}\rangle^2} \simeq \frac{\sigma^2}{3}\sqrt{\mathcal{P}_*} \simeq 1.5\times 10^{-5}. \quad (38)$$

In summary, Eqs. (30) and (36) show that, in a Universe filled with cosmological fluctuations stretched over super-Hubble scales, the curvature density parameter is not vanishingly small but is promoted to a stochastic variable. At any time in the cosmic history, we therefore expect an observer to measure a realization of Ω_{K} dominated by its standard deviation, i.e., at about 1.5×10^{-5}. However, Eq. (10) makes explicit that K is a nonlinear functional of ξ. As such, even if ξ is of Gaussian statistics, the probability distribution of Ω_{K} is, *a priori*, non-Gaussian. The rarity of extreme values of Ω_{K} could, therefore, be affected by the higher moments, and we now turn to their calculation.

C. Higher moments

All the higher moments $\langle \Omega_{\mathrm{K}}^n\rangle$ with $n > 2$ can be explicitly calculated with the same method as the one employed for the mean value and the variance. Expanding the exponential in series and using the binomial expansion of $(K_1 + K_2)^n$ shows that one has to determine the mean value of combinations of the form $\langle K_1^p K_2^q \xi^m\rangle = \langle K_2^q\rangle\langle K_1^p \xi^m\rangle$. Those can all be expressed in terms of powers of $\langle \xi^2\rangle$, $\langle K\rangle$, and $\langle K^2\rangle$ by using the diagrammatic rules of Eq. (24).

The only new subtlety consists in evaluating the terms in $\langle K_2^q\rangle$ that need to be decomposed into "self-cycles." For instance, the third moment requires one to evaluate

$\langle K_2^3\rangle = 2^2 2!\Big($  $\Big) + 2\Big($  $\Big)\times 3\Big($  $\Big)$
$+\Big($  $\Big)^3$, (39)

and one obtains

$$\langle \Omega_{\mathrm{K}}^3\rangle = -\frac{\langle K\rangle}{a^6 H^6}\left(39\langle K^2\rangle + \frac{19430}{9}\langle K\rangle^2\right)e^{18\langle \xi^2\rangle}. \quad (40)$$

Similarly, the fourth moment is given by

$$\langle \Omega_{\mathrm{K}}^4\rangle = \frac{1}{a^8 H^8}\left(3\langle K^2\rangle^2 + 1728\langle K^2\rangle\langle K\rangle^2 + \frac{736682}{9}\langle K\rangle^4\right)e^{32\langle \xi^2\rangle}, \quad (41)$$

and so on and so forth. These expressions are not particularly illuminating, but the leading-order terms of all the moments are diagrammatically tractable, and one can show that, for a scale-invariant power spectrum, the standardized moments $\tilde{\mu}_n$ (the moments divided by the nth power of the standard deviation) verify

$$\tilde{\mu}_{n=2p} \simeq W_n e^{(2n^2-4n)\langle \xi^2\rangle},$$
$$\tilde{\mu}_{n=2p+1} \simeq nW_{n-1}(1+4n)\frac{\sqrt{\mathcal{P}_*}}{2}e^{(2n^2-4n)\langle \xi^2\rangle}. \quad (42)$$

All odd standardized moments are suppressed by the factor $\sqrt{\mathcal{P}_*}$ with respect to the even ones. Moreover, provided the exponential terms in Eq. (42) are close to unity, i.e., for $n^2\langle \xi^2\rangle \ll 1$, the even moments exactly match the ones associated with a Gaussian probability distribution. As such, $\langle \Omega_{\mathrm{K}}^n\rangle$ shows significant deviations compared to the Gaussian expectations only for large values of $n^2 \gtrsim 1/\langle \xi^2\rangle$. To better assess the effect of these higher moments, we next turn our attention to the functional form of the Ω_{K}'s probability distribution.

D. Probability distribution

The probability density function of Ω_{K} can be determined by noticing that Eqs. (10) and (11) imply that Ω_{K} can be seen as a nonlinear functional over five stochastic Gaussian variables, $\boldsymbol{\Xi} \equiv (\xi, \Delta\xi, \boldsymbol{\nabla}\xi)$. As such, defining $\bar{\Omega}_{\mathrm{K}} \equiv (a^2H^2/k_\sigma^2)\Omega_{\mathrm{K}}$ and marginalizing over the five-dimensional space associated with $\boldsymbol{\Xi}$, one has

$$P(\bar{\Omega}_{\mathrm{K}}) = \int \frac{\mathrm{d}^5\boldsymbol{\Xi}}{(2\pi)^{5/2}}\delta\left(\bar{\Omega}_{\mathrm{K}} + \frac{K}{k_\sigma^2}e^{-2\xi}\right)\frac{e^{-\frac{1}{2}\boldsymbol{\Xi}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Xi}}}{\sqrt{\det\boldsymbol{\Sigma}}}, \quad (43)$$

where the five-dimensional covariance matrix $\boldsymbol{\Sigma}$ is completely determined by the diagrammatic rules of Eq. (24). All but one integral appearing in Eq. (43) can be analytically reduced, and, after some algebra, one obtains

$$P(\bar{\Omega}_{\mathrm{K}}) = \frac{k_\sigma^2}{4\pi}\sqrt{\frac{27\sqrt{2}\Sigma_\omega}{\langle \xi^2\rangle|\langle K\rangle|^3}}\int_{-\infty}^{+\infty}\mathrm{d}x e^{-\frac{x^2}{2\langle \xi^2\rangle}+2x} \times e^{-\frac{1}{2}\bar{\omega}^2(\bar{\Omega}_{\mathrm{K}},x)}H_{-\frac{3}{2}}\left[\frac{3\Sigma_\omega}{\sqrt{8}|\langle K\rangle|} - \frac{\bar{\omega}(\bar{\Omega}_{\mathrm{K}},x)}{\sqrt{2}}\right], \quad (44)$$

where we have defined

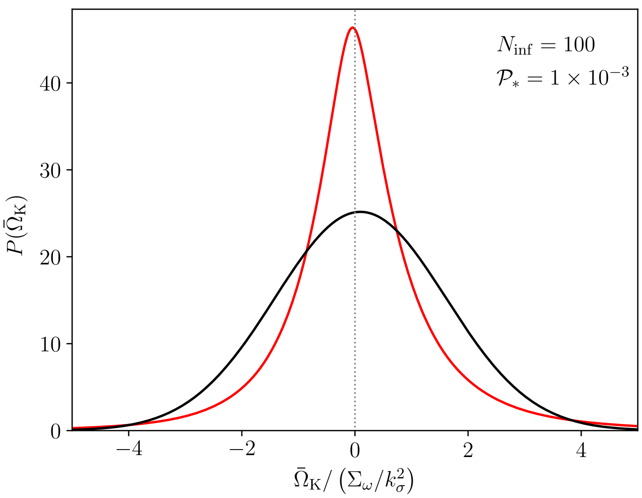


FIG. 1. Probability distribution function for $\bar{\Omega}_{\rm K} = (aH/k_\sigma)^2\Omega_{\rm K}$ (red curve) for unrealistically large values of $\mathcal{P}_* = 10^{-3}$ (and $N_{\rm inf} = 100$), compared to a Gaussian of the same mean and variance (black curve). Notice that the most probable value of $\bar{\Omega}_{\rm K}$ is slightly negative, whereas the mean value remains slightly positive.

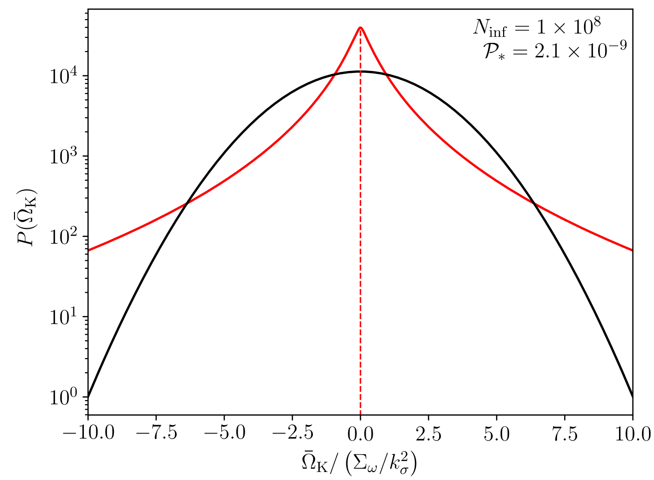


FIG. 2. Probability distribution function for $\bar{\Omega}_{\rm K} = (aH/k_\sigma)^2\Omega_{\rm K}$ (red curve) for the currently favored value of $\mathcal{P}_* = 2.1 \times 10^{-9}$ and for a large number of e-folds $N_{\rm inf} = 10^8$. The variance $\langle\xi^2\rangle$ is no longer a small quantity, and the distribution acquires heavy tails. Even though the width at half maximum is $\mathcal{O}(\sqrt{\mathcal{P}_*})$, substantial values of $|\bar{\Omega}_{\rm K}|$ are not rare anymore. For comparison, the black curve shows a Gaussian of the same mean and variance.

$$\Sigma_\omega^2 \equiv \langle K_1^2\rangle - 4\frac{\langle K\rangle^2}{\langle\xi^2\rangle},$$
$$\bar{\omega}(\bar{\Omega}_{\rm K}, x) \equiv e^{2x}\frac{\bar{\Omega}_{\rm K}}{(\Sigma_\omega/k_\sigma^2)} + \frac{2|\langle K\rangle|}{\langle\xi^2\rangle\Sigma_\omega}x. \tag{45}$$

In Eq. (44), $H_\nu(x)$ stands for the generalized Hermite polynomial of fractional order, defined from the parabolic cylinder functions [43] as $H_\nu(x) \equiv 2^{\nu/2}e^{x^2/2}D_\nu(\sqrt{2}x)$. This distribution shows that, for $\langle\xi^2\rangle \simeq \mathcal{P}_* N_{\rm inf} \ll 1$, one can use the approximation

$$e^{-\frac{x^2}{2\langle\xi^2\rangle}+2x} \simeq \sqrt{2\pi\langle\xi^2\rangle}e^{2\langle\xi^2\rangle}\delta(x - 2\langle\xi^2\rangle) \tag{46}$$

to simplify the integral over x in Eq. (44). Remarking that, in this limit, the argument of the Hermite function is dominated by the first term, which is a constant scaling as $1/\sqrt{\mathcal{P}_*}$, $P(\bar{\Omega}_{\rm K})$ is, therefore, close to a Gaussian distribution over the quantity $\bar{\omega}(\bar{\Omega}_{\rm K}, 2\langle\xi^2\rangle)$. In other words, for $\langle\xi^2\rangle \ll 1$, the distribution of $\bar{\Omega}_{\rm K}$ is almost Gaussian, with a width given by $\Sigma_\omega/k_\sigma^2 \simeq \sqrt{\mathcal{P}_*}/3$ and a peak located at a very small *negative* value:

$$\bar{\Omega}_{\rm K}|_{\rm max} \simeq \frac{4}{k_\sigma^2}\langle K\rangle e^{-4\langle\xi^2\rangle} \simeq -\frac{2}{3}\mathcal{P}_*. \tag{47}$$

For the curvature parameter today, one would get the most probable value at $\Omega_{\rm K_0}|_{\rm max} \simeq -1.4 \times 10^{-9}$, a barely closed universe were we to interpret this number within a FLRW metric with trivial topology. Let us notice the different sign than the mean value of Eq. (31); the distribution is indeed slightly skewed by the Hermite function. This can be seen in Fig. 1, where we have plotted $P(\bar{\Omega}_{\rm K})$ for an unrealistically large value of $\mathcal{P}_* = 10^{-3}$. These distortions are also apparent in the odd moments of Eq. (42) which are, as already noted, all proportional to $\sqrt{\mathcal{P}_*}$.

When $\langle\xi^2\rangle \simeq \mathcal{P}_* N_{\rm inf}$ increases, Eq. (46) is no longer accurate, and all the terms of Eq. (44) are relevant. The distribution now acquires heavy tails, kicking in at increasingly smaller values of $|\bar{\Omega}_{\rm K}|$ and erasing the Gaussian profile in the neighborhood of $\bar{\Omega}_{\rm K}|_{\rm max}$. In Fig. 2, we have plotted $P(\bar{\Omega}_{\rm K})$, in logarithmic scales, for $\mathcal{P}_* = 2.1 \times 10^{-9}$ and for a large number of e-folds $N_{\rm inf} = 10^8$. These heavy tails imply that large values of $|\Omega_{\rm K_0}|$ are (much) more likely than what a Gaussian profile would imply. Their existence is also manifest in the moments of Eq. (42) through the exponential coefficients involving $\langle\xi^2\rangle$. Such an effect is reminiscent of the nonlinear mapping of vacuum quantum fluctuations encountered in the context of stochastic inflation [44,45].

Finally, let us mention that numerical computations of $\langle\Omega_{\rm K}\rangle$ and $\langle\Omega_{\rm K}^2\rangle$ based on using the distribution of Eq. (44) do match the values we can get from Eqs. (30) and (36).

IV. DISCUSSION

If inflation lasts for a long period, then substantial values of $\Omega_{\rm K_0}$ might be produced. Indeed, letting $\sigma^2 \simeq e^{-\langle\xi^2\rangle}$ to implement the condition stated below Eq. (15), Eq. (37) becomes $\langle\Omega_{\rm K_0}^2\rangle^{1/2} \simeq \sqrt{\mathcal{P}_*}e^{3\mathcal{P}_* N_{\rm inf}}/3$. For this value not to exceed the current observational bound $|\Omega_{\rm K_0}| < 3 \times 10^{-3}$, with $\mathcal{P}_* = 2.1 \times 10^{-9}$ this leads to $N_{\rm inf} < 7 \times 10^8$. On the one hand, this suggests that scenarios leading to phases of inflation lasting for more than a billion e-folds might be disfavored by current cosmological data. On the other hand, future cosmological surveys, such as the ones using the neutral hydrogen line at 21 cm, may possibly detect a nonvanishing curvature if inflation actually lasted slightly less than a billion e-folds [46]. Notice that the aforementioned bound becomes more stringent if one accounts for the slightly red observed spectral index.

Let us note, however, that when the above bound on $N_{\rm inf}$ is saturated, $\langle \xi^2 \rangle \simeq 1.5$. *A priori*, our nonlinear formulas do not require $\langle \xi^2 \rangle$ to be small; hence, they can still be used in that case. In particular, although one can see that all the moments are becoming exponentially large with $\langle \xi^2 \rangle$, Eq. (44) shows that $P(\bar{\Omega}_{\rm K})$ remains well defined. Nonetheless, the fact that the scale k_σ must be set in a way that accommodates potentially large values of ζ suggests that our formalism may not be best suited in that case, and the upper bound we have obtained on $N_{\rm inf}$ must be taken with care. Moreover, for large $\langle \xi^2 \rangle$, possible backreaction effects on super-Hubble scales could also induce deviations from Gaussianity.

If inflation lasts even longer, $\Omega_{\rm K}$ gets even larger, and our formalism needs to be extended in at least two ways. First, when $|\Omega_{\rm K}|$ becomes of the order of unity, or more, the metric associated with Eq. (1) is not acceptable anymore. For instance, a large negative curvature density parameter would imply a compact manifold, and this demands another coordinate system than the one in Eq. (1). Second, when $|\Omega_{\rm K}|$ becomes sizable, it opens up a channel of backreaction of the curvature perturbation onto the background dynamics, which, in turn, alters the inflationary amplification of the curvature perturbations themselves [47,48]. This mechanism might be tractable in an extended stochastic-inflation formalism [49–53], which we plan to develop in a future work.

Finally, let us insist that our derivation of the statistics of $\Omega_{\rm K}$ is not rooted in any perturbative expansion of metric coefficients. The assumptions made are that ζ is of Gaussian statistics and conserved on super-Hubble scales. As such, our results would be modified if curvature perturbations are non-Gaussian at nonobservably large scales. This is, strictly speaking, not excluded, although it would require very specific early-Universe models for which curvature perturbations are Gaussian at observable scales today (in order to satisfy the tight constraints on non-Gaussianities [41]) and non-Gaussian at larger scales. Another hypothesis that could be broken is that ξ is conserved by adiabaticity. The presence of entropic modes today could invalidate this assumption, but, as for non-Gaussianities, their presence during inflation is also disfavored by current data.

ACKNOWLEDGMENTS

This work is supported by the “Fonds de la Recherche Scientifique—FNRS” under Grant No. T.0198.19 as well as by the Wallonia-Brussels Federation Grant ARC No. 19/24-103.

[1] W. Hu, D. N. Spergel, and M. J. White, Phys. Rev. D **55**, 3288 (1997).
[2] N. Aghanim *et al.* (Planck Collaboration), Astron. Astrophys. **641**, A1 (2020).
[3] J. Fonseca, S. Camera, M. Santos, and R. Maartens, Astrophys. J. Lett. **812**, L22 (2015).
[4] I. Tutusaus *et al.* (EUCLID Collaboration), Astron. Astrophys. **643**, A70 (2020).
[5] A. A. Starobinsky, JETP Lett. **30**, 682 (1979).
[6] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[7] A. H. Guth, Phys. Rev. D **23**, 347 (1981).
[8] A. D. Linde, Phys. Lett. **108B**, 389 (1982).
[9] A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).
[10] A. D. Linde, Phys. Lett. **129B**, 177 (1983).
[11] V. F. Mukhanov and G. V. Chibisov, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 549 (1981); [JETP Lett. **33**, 532 (1981)].
[12] V. F. Mukhanov and G. V. Chibisov, Zh. Eksp. Teor. Fiz. **83**, 475 (1982); [Sov. Phys. JETP **56**, 258 (1982)].
[13] A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982).
[14] A. H. Guth and S. Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982).
[15] S. W. Hawking, Phys. Lett. **115B**, 295 (1982).
[16] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).
[17] T. P. Waterhouse and J. P. Zibin, arXiv:0804.1771.
[18] T. Buchert and M. Carfora, Classical Quantum Gravity **25**, 195001 (2008).
[19] M. Vardanyan, R. Trotta, and J. Silk, Mon. Not. R. Astron. Soc. **397**, 431 (2009).
[20] C. D. Leonard, P. Bull, and R. Allison, Phys. Rev. D **94**, 023502 (2016).
[21] S. Anselmi, M. F. Carney, J. T. Giblin, S. Kumar, J. B. Mertens, M. O'Dwyer, G. D. Starkman, and C. Tian, J. Cosmol. Astropart. Phys. 02 (2023) 049.
[22] D. S. Salopek and J. R. Bond, Phys. Rev. D **42**, 3936 (1990).
[23] P. Creminelli and M. Zaldarriaga, J. Cosmol. Astropart. Phys. 10 (2004) 006.
[24] E. W. Kolb, S. Matarrese, A. Notari, and A. Riotto, Mod. Phys. Lett. A **20**, 2705 (2005).

[25] D. H. Lyth, K. A. Malik, and M. Sasaki, J. Cosmol. Astropart. Phys. 05 (2005) 004.
[26] G. I. Rigopoulos and E. P. S. Shellard, J. Cosmol. Astropart. Phys. 10 (2005) 006.
[27] D. Langlois and F. Vernizzi, Phys. Rev. Lett. **95**, 091303 (2005).
[28] E. W. Kolb, S. Matarrese, A. Notari, and A. Riotto, arXiv: hep-th/0503117.
[29] E. Barausse, S. Matarrese, and A. Riotto, Phys. Rev. D **71**, 063537 (2005).
[30] C. M. Hirata and U. Seljak, Phys. Rev. D **72**, 083501 (2005).
[31] E. W. Kolb, S. Matarrese, and A. Riotto, New J. Phys. **8**, 322 (2006).
[32] G. Geshnizjani, D. J. H. Chung, and N. Afshordi, Phys. Rev. D **72**, 023517 (2005).
[33] L. P. Grishchuk and I. B. Zeldovich, Sov. Astron. **22**, 125 (1978).
[34] J. Garcia-Bellido, A. R. Liddle, D. H. Lyth, and D. Wands, Phys. Rev. D **52**, 6750 (1995).
[35] A. L. Erickcek, S. M. Carroll, and M. Kamionkowski, Phys. Rev. D **78**, 083012 (2008).
[36] R. H. Brandenberger and C. S. Lam, arXiv:hep-th/0407048.
[37] M. Kleban and M. Schillo, J. Cosmol. Astropart. Phys. 06 (2012) 029.
[38] P. Carrilho and K. A. Malik, J. Cosmol. Astropart. Phys. 02 (2016) 021.
[39] G. Geshnizjani and R. Brandenberger, Phys. Rev. D **66**, 123507 (2002).
[40] S. Matarrese, M. A. Musso, and A. Riotto, J. Cosmol. Astropart. Phys. 05 (2004) 008.
[41] Y. Akrami *et al.* (Planck Collaboration), Astron. Astrophys. **641**, A9 (2020).
[42] Y. Akrami *et al.* (Planck Collaboration), Astron. Astrophys. **641**, A10 (2020).
[43] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1980).
[44] C. Pattison, V. Vennin, H. Assadullahi, and D. Wands, J. Cosmol. Astropart. Phys. 10 (2017) 046.
[45] J. M. Ezquiaga, J. García-Bellido, and V. Vennin, J. Cosmol. Astropart. Phys. 03 (2020) 029.
[46] A. Witzemann, P. Bull, C. Clarkson, M. G. Santos, M. Spinelli, and A. Weltman, Mon. Not. R. Astron. Soc. **477**, L122 (2018).
[47] W. Handley, Phys. Rev. D **100**, 123517 (2019).
[48] M. I. Letey, Z. Shumaylov, F. J. Agocs, W. J. Handley, M. P. Hobson, and A. N. Lasenby, arXiv:2211.17248.
[49] A. A. Starobinsky, Lect. Notes Phys. **246**, 107 (1986).
[50] A. S. Goncharov, A. D. Linde, and V. F. Mukhanov, Int. J. Mod. Phys. A **02**, 561 (1987).
[51] A. A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357 (1994).
[52] V. Vennin and A. A. Starobinsky, Eur. Phys. J. C **75**, 413 (2015).
[53] J. Grain and V. Vennin, Eur. Phys. J. C **81**, 132 (2021).