Axionlike dark matter model involving two-phase structure and two-particle composites

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Within the self-gravitating Bose-Einstein condensate (BEC) model of dark matter (DM), we argue that the axionlike self-interaction of ultralight bosons ensures the existence of both rarefied and dense phases in the DM halo core of (dwarf) galaxies. In fact, this stems from two independent solutions of the Gross-Pitaevskii equation corresponding to the same model parameters. The existence of the two-phase structure did also appear in previously studied models with polynomial self-interactions, which actually involve the truncated expansion series of the axionlike self-interaction. For a small number of particles, this structure disappears along with the gravitational interaction, and the Gross-Pitaevskii equation reduces to the stationary sine-Gordon equation, the one-dimensional antikink solution of which mimics a single-phase DM radial distribution in the halo core. Quantum mechanically, this solution corresponds to a zero-energy bound state of two particles in a closed scattering channel formed by the domain-wall potential with a finite asymptotics. To produce a two-particle composite with low positive energy and a finite lifetime, we appeal to the resonant transition of one asymptotically free particle of a pair from an open channel (with a model scattering potential) to the closed channel. Using the Feshbach resonance concept, the problem of two-channel quantum mechanics is solved in the presence of a small external influence which couples the two channels, and an analytical solution is obtained in the first approximation. Analyzing the dependence of scattering data on interaction parameters, we reveal a long-lived two-particle composite (dimer) possessing a lifetime of millions of years. This result is rather surprising and supposes important implications of dimers being involved in forming large DM structures. It is shown that the dimers' appearance is related with the regime of infinite scattering length due to resonance. The revealed dependence of the DM scattering length a on the parameters of interactions can theoretically justify variation of a in the DM dominated galaxies and its role for large DM structures.

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I. INTRODUCTION

Axionlike bosons with low mass and periodic self-interaction belong to the most popular candidates for the role of dark matter (DM) particles [1]. Axions were hypothetically introduced by Peccei-Quinn (PQ) [2,3] as pseudo-Goldstone bosons to resolve the problem of strong charge parity (CP) in quantum chromodynamics (QCD). Two nonthermal mechanisms for the axion production in the early Universe were soon proposed, namely vacuum misalignment [4–6] and cosmic string decay [7]. In fact, the thermalization of axions, which were created by the vacuum realignment in the PQ scenarios with either broken or unbroken symmetry, seems irrelevant at the early stage because of their initial coherence. On the other hand, it was argued that the axion component, which appeared during the decay of topological defects, is thermalized due to self-interactions [1]. Besides, gravitational scattering can lead to the rethermalization of the QCD axions in the era of radiation dominance, as suggested in [8,9]. This indicates that a system of axions can have huge occupation numbers and be treated nonrelativistically. In this regard, the discussion in the literature [8,10–12] on the existence of an axion Bose-Einstein condensate (BEC) led to identifying any condensate regime with a classical field, regardless of whether the axions are in the ground state or not. Thereby, it is recognized that the axions in an occupied state are coherent, so their distribution should be considered from the point of view of classical field theory or quantum mechanics (in a spirit of Gross and Pitaevskii) [13]. Moreover, the wave-based approach does prevail over the corpuscular one as the particle mass decreases.

It is argued that axions with masses predicted by QCD [14,15] are not capable to form giant BECs comparable in size to the DM halos [16,17]. Indeed, the first attempt to describe galactic halos formed by bosons [18], either in their ground state (BEC) or in an appropriate

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isothermal state, resulted in an extremely small mass of the order of 10^{-24} eV$/c^2$. A similar mass estimate was also obtained in [19] when studying the rotation curves by considering a giant system of self-gravitating Bose liquid. This means that there may exist other types of axions with a very small mass, called ultralight axions [1]. We also call these (dark) particles *axionlike*. Note that the difference between the masses of QCD axions and ultralight axions can reach tens of orders of magnitude.

In view of significant mass spread, it is easy to miss the emergence of composites (complexes of axionlike particles) due to specific nature of their interaction. In order to fill the gap, we are studying the problem of axion dimer formation in this paper. It is expected that interaction characteristics (e.g., scattering length) differ greatly for individual axions and their composites (like dimers).

Noteworthy, a large number of theoretical works are devoted to the study of the one-sort BEC DM models at vanishing absolute temperature (see, for instance, [11–13,16,17,19–26]). Their being aimed at the description of astrophysical phenomena should borrow the chiral cosinelike self-interaction [2,27] (or at least its truncated part), and also requires engaging the gravity (usually treated separately from the unification theory), the account of which leads to breaking the inherent symmetry. Note also that there are works, including experimental ones (see [1,28,29]), that study the interaction of axions with other substances and their transformation (say, due to the Primakoff effect, see [30,31]).

The condensate properties of the gravitating axionlike DM, with both the leading pairwise contributions to the self-interaction [24–26] and the next three-particle corrections [17,32,33] being taken into account, are promising for further exploration and application of axionlike particles. Having got a number of characteristics, the dilute and dense phases along with the first-order phase transitions are theoretically predicted in BEC [17,32,33]. Besides, analysis of the effects and ways to better describe the observables suggest the existence of moleculelike composites [34] and the relevance of deformation-based description [35–37]. Theoretically, these possibilities are considered as very important, especially when dealing with the dark sector. There are also indications that quantum entanglement may be involved [33,38].

Thus, there are arguments concerning both the first-order phase transition at zero temperature with changing the interaction parameters and its influence on the rotation curves of the DM-dominated galaxies [33]. The distinct phases could also apparently affect the state of DM bosonic stars [17], the merging of which may produce gravitational waves [39,40] as an alternative to black hole merging [41]. Physically, the phase transition in BEC DM is associated with quantum fluctuations in regions with a relatively high number density of ultralight particles, where the three-particle effects become significant.

Anyway, the choice of the self-interaction potential is decisive. Having gained an idea of the nontrivial phase structure of DM with two- and three-particle interactions and its manifestations in observables [32,33], we want to show here that the model with the cosinelike interaction mentioned above should also lead to similar consequences. Obviously, the already used self-interactions of the polynomial form now may be treated as the expansion terms of the total potential. It is important that the cosinelike generalization not only complicates the form of interaction, but also reduces the number of independent parameters. We focus on revealing different phases of the DM with axionlike interaction in the spherically symmetric case, when the main function we find is the spatial distribution of particles in the BEC.

In general, it is natural to assume that the DM also consists of particles of different sorts, including composites. At first glance, composites ("molecules") would be produced in a dense environment. However, as was shown earlier [34], high particle density leads to particle disintegration triggered by frequent collisions. But the production of composites may be caused by a large scattering length of particle interaction. The very possibility of changing the scattering length is able to shed new light on the properties and behavior of the initial (elementary) DM particles, the nature of which have not yet been identified. What is observed and described, including in the present study, is mainly the result of self-interaction (and that of gravitation), that is, a steady state with a vanishingly small scattering length, confirmed by numerous models based on the Gross-Pitaevskii equation [17,25,33,34]. Therefore, we need to explain these distinct particle states separated by an energy gap.

Within qualitative treatment, formation of the simplest "molecules" of two and three particles is explored in Ref. [34]. Here, we turn to scattering processes, using certain analogy with nuclear processes, and also with experimentally studied phenomena in atomic BECs in the laboratory [42]. Assuming this to be admissible in the DM as well, we appeal to the quantum mechanical formation of a dimer of two particles, borrowing the ideas of the Feshbach resonance and using two scattering channels [42–45]. Although the different channels may be associated with configurations of internal degrees of freedom of DM particles (a detailed analysis of which is an independent task), we include auxiliary influences in our consideration to disclose a plausible mechanism for the formation of bound states during the Universe evolution. Thereby, we emphasize the fact that one good potential is not sufficient to form DM composites in space.

The starting point for studying the dimer formation is the bound state of two particles held by the axionlike interaction. Then, omitting the gravitational interaction between a few *ultralight* particles, the Gross-Pitaevskii equation reduces, in fact, to the stationary sine-Gordon equation.

In order to gain more analytical results, we may restrict ourselves to the one-dimensional case that enables reproducing its (anti)kink solution at zero energy [46]. In the absence of gravity, the two-phase structure should disappear, leaving us with one (mixed) phase. Thanks to the analytical solution, the sine-Gordon potential rewritten as a function of space is, of course, a domain wall with finite asymptotics. This means that a pair of particles is in a closed channel with zero energy, and we are faced with the task of bringing an asymptotically free particle with low energy into this trap.

As mentioned above, our solution is to admit the existence of an open channel in which an incident particle is scattered by another interaction (one of those in which DM particles may participate due to internal properties). If the particle in this channel has a small positive energy close to the energy of the bound state (i.e., zero), one can expect a resonant transition between the open and closed channels in the spatial region of the trap under a small external influence/force. This mechanism leads to the appearance of an isolated positive energy level of a two-particle (compound) system, which becomes possessing finite lifetime [44]. We treat this state as a dimer, whose characteristics are important for understanding its role in the formation of the DM halo. According to the Feshbach resonance concept [43], the dimer emergence at resonance is accompanied by an infinite scattering length, which eventually depends on the interaction parameters. This fact may also affect the elucidation of other processes. Indeed, it was previously assumed that the so-called unitary regime (of infinite scattering length) could induce instability of the BEC DM halo, similar to what is observed in the laboratory BEC [47,48]. But this phenomenon occurs at high density, which we exclude.

The paper is organized as follows. In Sec. II we show the existence of two phases of the BEC DM halo core on the base of a pair of distinct solutions of the stationary Gross-Pitaevskii equation with axionlike periodic and gravitational interactions at fixed values of the coupling constants and chemical potential. The reduction of the Gross-Pitaevskii equation to the stationary sine-Gordon equation is carried out in Sec. III for the case of a small number of (ultralight) particles, when the gravitational interaction is negligible. The exact one-dimensional solution is also discussed there, and its further use outlined. The Feshbach resonance concept is applied to describe the axion dimer formation in Sec. IV. Therein, divided in five subsections for reader's convenience, the problem is solved and its various aspects are highlighted. The necessary preliminaries and the model formulation are given in Sec. IV A—it introduces the basic concepts and tools. The most significant analytical part of study is presented in Sec. IV B, where the quantum mechanics equations of the two-channel problem are solved. Therein, an isolated energy level of a compound system of two particles is uncovered and discussed. Also, analytical expressions for the wave functions are given in the first approximation. The dependence of the scattering length on the interaction parameters is studied in Sec. IV C. The free parameters of the model are fixed there, and the Feshbach resonance at zero energy is analyzed. In Sec. IV D, when considering resonant scattering, the information about the resonance (and dimer) involving incident particle with nonzero energy is numerically extracted. The physical characteristics of the dimer in the context of DM halo are given and discussed in Sec. IV E. It is revealed and emphasized that *the lifetime of the dimer is of the order of millions of years*. The final Sec. V is devoted to discussion of implications as well as concluding remarks.

II. GROSS-PITAEVSKII EQUATION FOR AXIONLIKE DM AND ITS SOLUTION

To start with, we formulate stationary macroscopic model of gravitating Bose-Einstein condensate (BEC) of ultralight bosons with axionlike interaction V_{SI}, restricting ourselves to the spherical symmetry and the absence of hydrodynamic flows. Let the BEC with a constant chemical potential $\tilde{\mu}$ be described by real function $\psi(r)$ of radial variable $r = |\mathbf{r}|$, with $\psi^2(r)$ defining a local particle density. The behavior of $\psi(r)$ in the ball $B = \{\mathbf{r} \in \mathbb{R}^3 | |\mathbf{r}| \leq R\}$ is determined by the vanishing variation of the energy functional Γ with respect to the variation of $\psi(r)$ along with the Poisson equation for the gravitational potential $V_{\mathrm{gr}}(r)$:

$$\frac{\Gamma}{4\pi} = \int_0^R \left[\frac{\hbar^2}{2m}(\partial_r\psi)^2 - \tilde{\mu}\psi^2 + mV_{\mathrm{gr}}\psi^2 + V_{\mathrm{SI}}\right] r^2 \mathrm{d}r, \quad (1)$$

$$\Delta_r V_{\mathrm{gr}} = 4\pi G m \psi^2. \quad (2)$$

The radial part of Laplace operator Δ_r and its inverse Δ_r^{-1} of variable r are

$$\Delta_r f(r) = \partial_r^2 f(r) + \frac{2}{r}\partial_r f(r), \quad (3)$$

$$\Delta_r^{-1} f(r) = -\frac{1}{r}\int_0^r f(s)s^2\mathrm{d}s - \int_r^R f(s)s\mathrm{d}s; \quad (4)$$

R is the radius of the ball where matter is concentrated.

The instantonic axionlike self-interaction is chosen here to be [2,27]

$$V_{\mathrm{SI}} = \frac{U}{v}\left[1 - \cos\left(\sqrt{v}\psi\right)\right] - \frac{U}{2}\psi^2 \quad (5)$$

$$= U\left[-\frac{v}{4!}\psi^4 + \frac{v^2}{6!}\psi^6 - \ldots\right], \quad (6)$$

where the axion field φ is related to the wave function by $|\varphi|^2 = (\hbar^2/m)|\psi|^2$ in the nonrelativistic limit [17]. Note also that there exists the effective axionic potential [49,50].

Thus, from the series expansion of $\cos(\sqrt{v}\psi)$, we see that the first term in (6) corresponds to a two-particle self-interaction, and the next term ψ^6 implies a three-particle self-interaction.

Note that the works [24,25,37,51–53] (see also references therein) intensively studied the BEC DM models with only pair interaction for $v = -|v|$, in the Thomas-Fermi approximation, when $\hbar \to 0$ in Eq. (1). Switching on the three-particle interaction enables to observe the two-phase structure of BEC DM that is studied in [17,32,33]. For this reason, we expect a similar effect to occur here.

The axionlike cosine-interaction (5) is characterized by two constants U and v, which have the dimensions of energy and volume, respectively. In the particle physics [49,50], they are related with the axion mass and decay constant f_a as $U = mc^2$, $v = \hbar^3 c/(m f_a^2)$. Their relativistic nature is noted in [17] in the context of nonrelativistic model of axion stars. However, in astrophysical applications, these constants may have other meaning and values, which will be discussed below.

To analyze the general properties of the model, we reformulate it in dimensionless variables:

$$\xi = \frac{\sqrt{mU}}{\hbar} r, \qquad \chi(\xi) = \sqrt{v}\psi(r), \tag{7}$$

$$\xi_B = \frac{\sqrt{mU}}{\hbar} R, \qquad A = 4\pi \frac{G\hbar^2 m}{U^2 v}, \tag{8}$$

$$u = \frac{\tilde{\mu}}{U}, \qquad \nu = 1 + 2u + 2A\Phi_0, \tag{9}$$

where ν plays the role of effective chemical potential, which absorbs the constant term of axion interaction and the gravitational potential at the origin $\xi = 0$, namely

$$\Phi_0 = -\int_0^{\xi_B} \chi^2(\xi)\xi d\xi. \tag{10}$$

In our study, ν is regarded as a free variable parameter, due to arbitrariness of u.

The model equations in terms of the wave-function $\chi(\xi)$ and auxiliary gravitational potential $\Phi(\xi)$ read

$$(\Delta_\xi + \nu)\chi - 2A\Phi\chi - \sin\chi = 0, \tag{11}$$

$$\Phi(\xi) = -\frac{1}{\xi}\int_0^\xi \chi^2(s) s^2 ds + \int_0^\xi \chi^2(s) s ds, \tag{12}$$

where $\Delta_\xi \Phi(\xi) = \chi^2(\xi)$ is satisfied, and $\Phi(0) = 0$.

To obtain a finite and stable solution $\chi(\xi)$, the nonlinear Eqs. (11) and (12) should be (numerically) integrated under the following conditions: $\chi(0) < \infty$, $\chi'(0) = 0$, $\chi''(0) < 0$. For given A and ν, the finite initial value $\chi(0) = z$ should be positive solution of the transcendental equation

$$2Az^2 + \left(\nu - \frac{\sin z}{z}\right)(\nu - \cos z) = 0 \tag{13}$$

which is derived by substituting $\chi(\xi) = \chi(0) + \chi''(0)\xi^2/2$ into (11) and (12) for $\xi \to 0$ and by finding $\chi''(0)$. Note that the requirement $\chi''(0) < 0$ is equivalent to imposing $\nu > \sin z/z$ for positive ν.

The absence of a solution z for a given pair (A, ν) means that $\chi(\xi) = 0$ everywhere. It happens for $\nu > \nu_{\max}$, where $\nu_{\max}(A)$ is also found numerically from (13). For $\nu < \nu_{\max}$, two branches of $\chi_0(\nu)$ can occur, which indicate the existence of two regimes and a first-order phase transition in the model.

Let us emphasize that the magnitude of parameter A plays a crucial role for subsequent implications. Assuming that gravity is weaker than the axion self-interaction, we take the parameter $A \gtrsim 10^{-3}$. Then, Eq. (13) leads to two independent solutions for $z > \pi$. Indeed, choosing A as in Fig. 1, the upper branch of $\chi_0(\nu)$ corresponds to $\chi_0 \in [\chi_s; 5\pi/2]$, while the lower branch of $\chi_0(\nu)$ gives us values of χ_0 within the interval $[3\pi/2; \chi_s]$, where the separating value $\chi_s = \chi_0(\nu_{\max}) \lesssim 2\pi$.

Therefore, there exist two independent solutions $\chi^{(\alpha)}(\xi)$, $\alpha = 1, 2$, of the set of Eqs. (11)–(13) for the same parameters A and ν. They are characterized by $\chi_0^{(\alpha)}$ and $\xi_B^{(\alpha)}$, which belong to different branches ($\alpha = 1, 2$) as in Fig. 1.

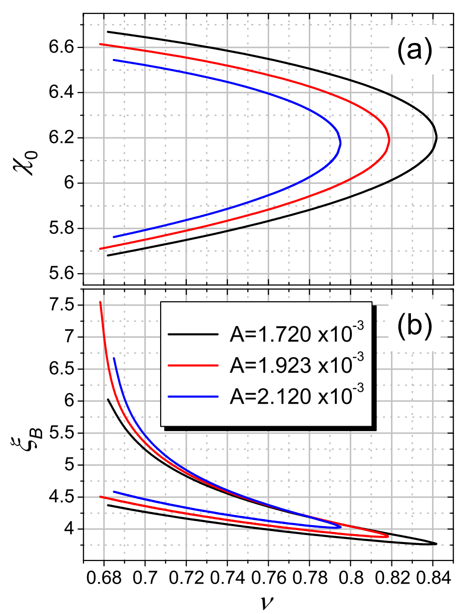

FIG. 1. The initial value $\chi_0 = \chi(0)$ (a) and the first zero ξ_B (b) of $\chi(\xi)$ versus parameter ν for different A. The curves are limited from the right by the values $\nu_{\max}^{\text{black}} = 0.841954176$, $\nu_{\max}^{\text{red}} = 0.818967072$, $\nu_{\max}^{\text{blue}} = 0.795228679$. Each rightmost point $\chi_s = \chi_0(\nu_{\max})$ divides the corresponding curve into upper and lower branches identified with different phases. At $\nu > \nu_{\max}$ one has $\chi_0 = 0$ and $\chi(\xi) = 0$.

This means that any space average $F(\nu)$ in the statistical description, for instance, *mean particle density*

$$\sigma(\nu) = \frac{3}{\xi_B^3}\int_0^{\xi_B}\chi^2(\xi)\xi^2 d\xi, \tag{14}$$

for fixed A and variable $\nu \leq \nu_{\max}$, also consists of two branches $F^{(1)}(\nu)$ and $F^{(2)}(\nu)$, e.g.

$$\sigma^{(\alpha)}(\nu) = 3(\xi_B^{(\alpha)})^{-3}\int_0^{\xi_B^{(\alpha)}}\left[\chi^{(\alpha)}(\xi)\right]^2\xi^2 d\xi. \tag{15}$$

Then the graph of the function $F(\nu)$ is the union of the graphs for $F^{(1)}(\nu)$ and $F^{(2)}(\nu)$ so that $F^{(1)}(\nu_{\max}) = F^{(2)}(\nu_{\max})$ by construction. But it is convenient for us to continue the use of the definition (14), keeping in mind the need to take into account different branches.

Contrary to the expectation of a weak axion field ($\chi_0 < \pi$) near the true vacuum [27], the situation looks different in the nonrelativistic model of DM with Newtonian interaction. Also, there is no invariance there under the global transformation $\chi \to \chi + 2\pi$ (it is violated by gravitation). Besides, such a discrepancy is related with the consideration of the condensate in a finite volume (of galactic DM halo). We might expect some distinctions when describing gravity as a space-time geometry.

Indeed, Fig. 1(b) demonstrates the value of (first) zero ξ_B of oscillating function $\chi(\xi)$ (that is, $\chi(\xi_B) = 0$), which limits the system size in our model and is found by integrating Eqs. (11) and (12) for given A and ν.

The typical solutions for the wave function $\chi(\xi)$ are presented in Fig. 2(a), where $\xi \in [0; \xi_B]$. They describe a core with a finite magnitude of the particle (and mass) density at the center $\xi = 0$. Besides, an additional information may be extracted through introducing the effective potential W [see Fig. 2(b)] as a function of radial variable ξ, namely

$$W(\xi) = 2A\Phi_0 + 2A\Phi(\xi) + \frac{\sin\chi(\xi)}{\chi(\xi)}, \tag{16}$$

$$\left[-\Delta_\xi + W(\xi)\right]\chi(\xi) = \varepsilon\chi(\xi), \tag{17}$$

when Eq. (11) is rewritten in the form of Schrödinger equation with the "energy" $\varepsilon = 1 + 2u$, see Eq. (9).

Figure 2(b) shows the forms of potential $W(\xi)$. The black curve describes the effective potential of the mixed (single-phase) state at $\nu = \nu_{\max}$. The blue and red curves correspond to the effective potentials of two phases for the same parameters A and $\nu < \nu_{\max}$, but obtained from the solutions $\chi(\xi)$ with different $\chi_0(\nu)$ and $\xi_B(\nu)$ belonging to different branches in Fig. 1. The different positions and depths of the minima of these potentials confirm the existence of two macroscopic states in the axion system. The particles being in one of the two phases is conditioned by the applied factors, e.g., pressure [32,33].

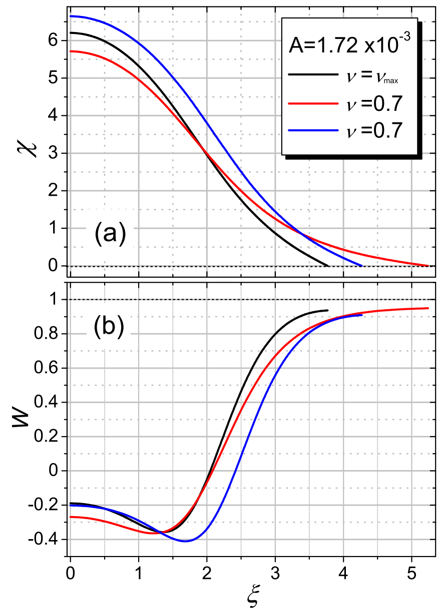

FIG. 2. A particular form of the wave function $\chi(\xi)$ (a) and the effective potential $W(\xi)$ (b) for the fixed parameters A and ν. Blue and red lines are obtained with the same A and ν, but differ in the initial values χ_0, see Fig. 1(a). The value of χ_0 for blue curves belongs to the upper branch of $\chi_0(\nu)$ in Fig. 1(a), while the red lines are plotted using χ_0 of the lower branch of $\chi_0(\nu)$. Black curves correspond to a single state at $\nu = \nu_{\max}$.

Outside the system at $\xi > \xi_B$, where the matter is absent, $W(\xi)$ is continuously extended by the gravitational potential of the form $1 - 2A\mathcal{N}/\xi$, where $\mathcal{N}$ is the total number of particles in the ball $\xi \leq \xi_B$.

Let us estimate the characteristics of our model, which are related with the dimensionless quantities (7)–(9). For this aim, we refer to the model from [32], which dictates to separate the approaches to describing the core and the tail of the DM halo due to different role of self-interaction in relatively dense and rarefied regions. It is worth noting that recent astronomical measurements indicate spatial fluctuations in the density of cold DM (around the quasar) on the scale of 10 kpc [54], which may be associated with an oscillating tail of the condensate wave function, rather than the mentioned smooth tail.

Focusing on the phenomena in the core, we need to reproduce the size scale r_0 and the central mass density ρ_0, which define $r = r_0\xi$ and $\rho(\xi) = \rho_0\chi^2(\xi)/\chi_0^2$. At the same time, this is needed to control the parameters U and v in (5).

As it was stated in Ref. [17], models for describing compact objects (such as axion stars) and cosmological models lead to different parametrizations of axions. First of all, this concerns the different mass ranges. While cosmological models constrain the axion mass as 10^{-7} eV $< mc^2 < 10^{-2}$ eV, another kind of models suggests that $mc^2 \sim 10^{-22}$ eV, commonly attributed to fuzzy DM [21,55]. Physically, the choice of a smaller particle mass ensures the formation of certain structures in the Universe [16], some of which we are trying to describe. It motivates us to fix $m \sim 10^{-22}$ eVc^{-2}. Although the parameter U in the chiral models is set proportional to mc^2 [50], we do not declare such an identity in the nonrelativistic case, and admit only that the value of U provides the predominance of axion repulsion over gravitation at relatively large magnitude of axion field.

Besides, we have to determine f_a in order to specify $\upsilon = \hbar^3 c/(m f_\mathrm{a}^2)$. Although 10^{18} eV $< f_\mathrm{a} < 10^{21}$ eV in cosmology [4], the decay constant f_a may be appearing larger in the models with ultralight particles [17].

Thus, combining the definition $\upsilon = \hbar^3 c/(m f_\mathrm{a}^2)$ and the relation $m\chi_0^2 = \upsilon\rho_0$ resulting from (7), we obtain that

$$f_\mathrm{a} \simeq 3.304 \times 10^{19}\ \mathrm{eV}\left[\frac{\rho_0}{10^{-19}\ \mathrm{kg\,m^{-3}}}\right]^{1/2} \times \left[\frac{mc^2}{10^{-22}\ \mathrm{eV}}\right]^{-1}\left[\frac{\chi_0}{2\pi}\right]^{-1}, \tag{18}$$

where the central mass density ρ_0 is taken to be of the order of 10^{-19} kg m^{-3}, while a mean mass density is assumed to be of the order of 10^{-20} kg m^{-3} as usual [17,25].

Using the second relation of (8) and $m\chi_0^2 = \upsilon\rho_0$, we find:

$$U \simeq 2.145 \times 10^{-29}\ \mathrm{eV}\left[\frac{\rho_0}{10^{-19}\ \mathrm{kg\,m^{-3}}}\right]^{1/2} \times \left[\frac{A}{2\times 10^{-3}}\right]^{-1/2}\left[\frac{\chi_0}{2\pi}\right]^{-1}. \tag{19}$$

The characteristic scale is defined here as $r_0 = \hbar/\sqrt{mU}$ and equals to

$$r_0 \simeq 0.138\ \mathrm{kpc}\left[\frac{\rho_0}{10^{-19}\ \mathrm{kg\,m^{-3}}}\right]^{-1/4}\left[\frac{mc^2}{10^{-22}\ \mathrm{eV}}\right]^{-1/2} \times \left[\frac{A}{2\times 10^{-3}}\right]^{1/4}\left[\frac{\chi_0}{2\pi}\right]^{1/2}. \tag{20}$$

Such r_0 is appropriate for estimating the size of the central part of the DM halo as $R = r_0\xi_B$, but should be fitted together with the total mass M.

To justify the first-order phase transition in the model, one has to develop a statistical approach, which is omitted here. Nevertheless, the discontinuous change in particle density (14) at zero temperature is expected to be caused by a change in the long-wave part of pressure [32]

$$\Pi = -\frac{3}{\xi_B^3}\int_0^{\xi_B}\left[(\partial_r\chi)^2 - \nu\chi^2\right]\xi^2\mathrm{d}\xi, \tag{21}$$

which also consists of two branches, as stated above.

Clearly, the effect of different DM phases on the observables, as well as on the rotation curves, also deserves a separate study.

III. SINE-GORDON EQUATION AND A BOUND STATE

Let us analyze the ground state of the DM halo core by turning to a one-dimensional model with the coordinate $\xi \in [0; +\infty)$, when gravity is absent and only the axion self-interaction plays a key role. This means that Eq. (17) under the simplifications

$$A = 0, \qquad \varepsilon = 0, \qquad \Delta_\xi \to \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \tag{22}$$

reduces to the stationary sine-Gordon equation:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + W\right)\chi = 0, \qquad W = \frac{\sin\chi}{\chi}, \tag{23}$$

which is in the form of Schrödinger equation with the axionlike potential W.

Equation (23) is invariant under the global transformation $\chi(\xi) \mapsto \chi(\xi) + 2\pi n$, $n \in \mathbb{Z}$, while Eq. (11) is not. Its general solution is easily derived by integration, which is carried out in numerous works (see, for instance, Sec. V.3 in [46]). For physical reasons, we write down and exploit the stationary antikink solution

$$\chi_\mathrm{ak}(\xi) = 4\arctan \mathrm{e}^{-\xi}, \qquad \xi \geq 0. \tag{24}$$

We also consider the solution of the form $\chi_\mathrm{ak}(\xi - L)$ with an arbitrary constant L. Altogether these solutions at dimensionless energy $\varepsilon = 0$ describe the ground state and, moreover, devoid any nodes, in contrast to the oscillating solutions in Sec. II. According to (23), they determine the potential W in terms of the coordinate ξ.

It is clear that the solution (24) can be also obtained from Eq. (23) with the potential W depending directly on ξ. Using the auxiliary formula

$$\sin 4z = 4\frac{1-\tan^2 z}{(1+\tan^2 z)^2}\tan z, \tag{25}$$

one arrives at W of the form

$$W_\mathrm{ak}(\xi) = \frac{\tanh\xi}{2\cosh\xi \arctan \mathrm{e}^{-\xi}}. \tag{26}$$

Note that, replacing χ with $4\arctan\varphi$, the sine-Gordon potential reduces to the form $\sin\chi = 4\varphi\cos_{\mu=1}\varphi$ accordingly to (25), where $\cos_\mu z$ is the μ-deformed cosine-function [56,57] taken at $\mu = 1$. Moreover, $\cos_{\mu=1}\xi$ is used in [56] to simulate the potential of two coupled axions at the quantum mechanical level.

To reproduce (24) by solving Eq. (23), we have to take $\chi_\mathrm{ak}(0) = \pi$ and $\chi'_\mathrm{ak}(0) = -2$. The same approach relates the potential $W_\mathrm{ak}(\xi - L)$ with the solution $\chi_\mathrm{ak}(\xi - L)$. At $\xi = 0$, we set $\chi_\mathrm{ak} = 4\arctan \mathrm{e}^L$ and $\chi'_\mathrm{ak} = -2/\cosh L$.

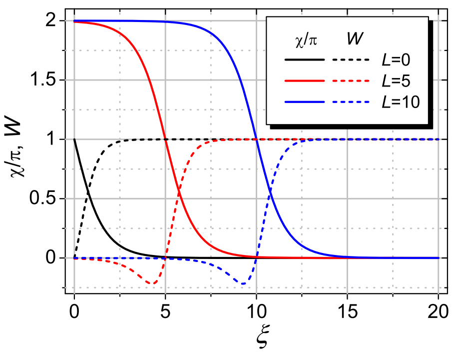

FIG. 3. Antikink solution $\chi(\xi - L)$ (solid lines) and corresponding potential $W(\xi - L)$ (dashed lines) for different L.

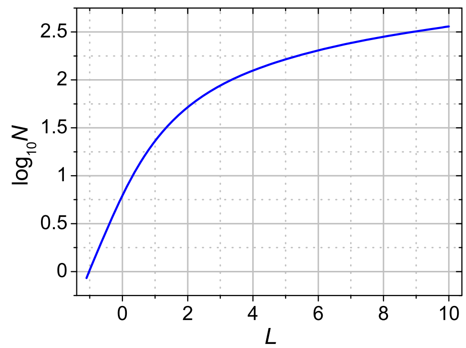

FIG. 4. Normalization (27) associated with the number of particles as a function of length parameter L.

The behavior of $\chi_{\rm ak}(\xi - L)$ and $W_{\rm ak}(\xi - L)$ is shown in Fig 3. For fixed L, we see that $\chi_{\rm ak} \to 0$ and $W_{\rm ak} \to 1$ for $\xi \to \infty$ as long as $\chi_{\rm ak}$ reaches its maximum at $\xi = 0$. For large L, the value of $\chi_{\rm ak}(\xi - L)$ at $\xi = 0$ tends to 2π, which is similar to the single-phase solution in Fig. 2(a), colored in black. We can deduce that the profile of $\chi_{\rm ak}$ qualitatively depicts the DM halo core due to the potential $W_{\rm ak}$, which results in an infinite scattering length a in the Born approximation and forms a closed scattering channel for particles with zero total energy ε (or ν), which are unable to overcome the potential barrier/domain wall.

Although the gravitational interaction of a large number of relatively fast particles modifies this barrier [cf. Fig. 2(b)], the mechanism of injection of a slow particle into a closed channel is of special interest. One possibility which we further explore is the transfer of particle between different scattering channels using the Feshbach resonance stimulated by an additional impact.

Let us calculate the integral over the entire (one-dimensional) space:

$$N(L) \equiv \int_0^\infty \chi_{\rm ak}^2(\xi - L)\mathrm{d}\xi \tag{27}$$

$$= 4\int_0^\alpha \frac{z^2}{\sin z}\mathrm{d}z; \qquad \alpha = 2\arctan \mathrm{e}^L. \tag{28}$$

Note that $2\alpha = \chi_{\rm ak}(\xi - L)$ at $\xi = 0$, that is the (maximal) value of axion field at the origin.

On integrating, the result is presented in differing forms:

$$\begin{aligned} N &= 2\alpha^2 + 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n-1}-1}{(n+1)(2n)!}B_{2n}\alpha^{2n+2} \\ &= 2\mathrm{e}^{\mathrm{i}\alpha}\left[\Phi\left(\mathrm{e}^{2\mathrm{i}\alpha}, 3, \frac{1}{2}\right) - 2\mathrm{i}\alpha\Phi\left(\mathrm{e}^{2\mathrm{i}\alpha}, 2, \frac{1}{2}\right)\right] \\ &\quad + 4\alpha^2\left(\ln\tan\frac{\alpha}{2} - \mathrm{i}\frac{\pi}{2}\right) - 14\zeta(3), \end{aligned} \tag{29}$$

where B_{2n} is the Bernoulli number; $\Phi(z, s, a)$ is the Lerch transcendent; $\zeta(3) = 1.20205\ldots$ is the particular value of Riemann zeta-function; at last, $\mathrm{i} = \sqrt{-1}$. The first expression is valid for $\alpha < \pi$ and given by Eq. (1.5.44.1) in [58]. Behavior of $N(L)$ is also shown in Fig. 4.

By construction, $N(L)$ is related with the number of particles in nonlinear problem (23). This can be useful, for instance, to control the effect of gravitation, as mentioned above. It is obvious that $N(L)$ is distinct from the (anti)kink topological charge [46].

On the other hand, when formulating the linear Schrödinger equation with potential $W_{\rm ak}(\xi - L)$, we can use $N(L)$ to normalize $\chi_{\rm ak}(\xi - L)$, the fact that will be applied below in the quantum mechanical setting.

Thus, we summarize that, neglecting gravity when considering a small number of axions, their (ground) bound state is still revealed even in the one-dimensional case, which nevertheless retains main physical properties of the model we need and only simplifies the mathematical description. Besides, the model (field) equation is reformulated in an equivalent quantum-mechanical form through introducing an effective potential depending on space.

IV. FESHBACH RESONANCE

We would like to consider in more detail the mechanism of the transition of a DM particle into the bound state described above, noting that the presence of one good potential is apparently not enough for this. We appeal to resonance scattering, which implies the existence of an isolated (quasidiscrete) energy level.

Our approach inherits the ideas of Feshbach resonance [43], which uses at least two channels of scattering, one open and one closed channels with distinct Hamiltonians. Coupling these channels enables to create an isolated level and bind the scattered particle. This approach seems to be appropriate, since it is difficult to directly transfer a zero-energy particle through the domain wall into the trap of sine-Gordon potential.

Let us assume that a spinless particle is able to jump between open and closed channels. When the total energy exceeds the open channel threshold ($E = 0$), the open channel becomes both an incoming and an outgoing channel. The Feshbach resonance occurs when the energy of the bound state in the closed channel is close to the threshold of the open channel. Due to the coupling of the channels, the unperturbed bound state of the closed channel becomes dressed. This dressed state is treated as a quasibound state of the scattered particle. The scattered particle temporarily passes into the quasibound state with positive energy and returns to the open channel after a typical time delay $\tau = 2/\Gamma$, determined by the decay width Γ of the quasibound state. We associate this state of two interacting particles with a composite (dimer).

A. Preliminaries and model formulation

Formulating our model on the base of stationary Schrödinger equation, we consider in fact a slowly moving particle which is assumed to be in one of the two channels. One channel is *closed* and is described in the absence of external interaction by the sine-Gordon equation in the ground state with energy $E^{(1)} = 0$, as mentioned earlier in Sec. III. An *open* (entrance) channel, the existence of which we assume, corresponds to elastic scattering due to another interaction, where a wave/particle initially has low energy $E^{(2)} = k^2 > 0$ determined by momentum k, which we also treat as the relative momentum of the pair of interacting particles. Then, there is an energy gap between these channels $Q = E^{(2)} - E^{(1)} > 0$, which is further affected by external impact. Thus, such a model involves three different interactions, and $E^{(1)}$ and $E^{(2)}$ are not energy levels of the same Hamiltonian. For this reason, the interaction parameters must be tuned to obtain the desired effect of resonant transition.

We describe the motions of a particle in two channels, adopting the matrix representation [45]:

$$\mathbb{H}X = EX, \qquad \mathbb{H} = \begin{pmatrix} H_{\rm bs} & \Omega \\ \Omega^\dagger & H_{\rm wv} \end{pmatrix}, \tag{30}$$

where $H_{\rm bs}$ and $H_{\rm wv}$ are the Hamiltonians of the bound state (in closed channel) and the scattered wave (in open channel), respectively. The coupling between channels is represented by Ω and is associated with extra force, which is turned on and starts to act *after* fixing the gap Q. In other words, $E^{(1)}$ and $E^{(2)}$ are given at $\Omega = 0$, while energy E is determined by switching Ω.

We account for the energy gap Q in Eq. (30) by defining $H_{\rm bs} = H_{\rm ak} + Q$, where

$$H_{\rm ak} = -\frac{{\rm d}^2}{{\rm d}\xi^2} + W_{\rm ak}(\xi - L) \tag{31}$$

with the potential $W_{\rm ak}(\xi)$ from Eq. (26), and $W_{\rm ak}(0) = 0$.

Hence, we use the dependence on parameter $L > 0$, which plays an important role in further constructions.

In a sense, the system is doubly degenerate at $\Omega = 0$ due to existing two independent wave functions for the same eigenvalue E:

$$X^{(1)} = \begin{pmatrix} \chi^{(1)} \\ 0 \end{pmatrix}, \qquad X^{(2)} = \begin{pmatrix} 0 \\ \chi^{(2)} \end{pmatrix}, \tag{32}$$

which are evidently orthogonal in this representation. We identify $\chi^{(1)}$ with $\chi_{\rm ak}(\xi - L)$ from Eq. (24). In principle, we need to write $X = X^{(1)}\cos\alpha + X^{(2)}\sin\alpha$ with some α in order to normalize the total wave function X with respect to the matrix representation.

As shown in Fig. 3, the spatial interval $\xi \in [0; L]$ is most significant for the manifestation of a bound state. Therefore, essential processes should be related with this region, which defines the *resonance zone*. For this reason, we concentrate there on the external force, which is parametrized by ω as

$$\Omega(\xi) = -\omega^2\theta(L - \xi), \qquad \Omega^\dagger = \Omega. \tag{33}$$

Here θ is the Heaviside step-function.

Similarly, we define the interaction in the open channel by the square-well potential:

$$V_{\rm sq}(\xi) = -V\theta(L - \xi), \qquad V > 0. \tag{34}$$

The strength V along with ω^2 are the variable parameters of the model.

Since we are studying the mechanism of the emergence of resonance and two-particle composite, the refinement of the nature and form of these extra interactions remains for further consideration. Here we only use their simplest version and discuss their origin after the computations performed.

By construction, all spatial functions in such a model are divided into two components belonging either to the interval $\xi \in [0; L]$ or to the interval $\xi \in [L; \infty)$, which we label by "<" and ">" relative to the separating point $\xi = L$. Then, the wave functions for the channels are numbered by $\alpha = 1, 2$ and decomposed as

$$\chi^{(\alpha)}(\xi) = \theta(L - \xi)\chi^{(\alpha)}_<(\xi) + \theta(\xi - L)\chi^{(\alpha)}_>(\xi). \tag{35}$$

We connect the functions at separating point $\xi = L$ by the matching condition

$$\frac{\rm d}{{\rm d}\xi}\ln\chi^{(\alpha)}_<(\xi)\bigg|_{\xi=L} = \frac{\rm d}{{\rm d}\xi}\ln\chi^{(\alpha)}_>(\xi)\bigg|_{\xi=L}, \tag{36}$$

to guarantee the equality of derivatives and proportionality of the functions in the left and right sides of (36).

Before proceeding further, we recall the known results for the open channel ($\alpha = 2$) in the absence of coupling Ω. The scattering characteristics result from the equation

$$H_{\mathrm{wv}}\chi^{(2)} \equiv \left(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + V_{\mathrm{sq}}(\xi)\right)\chi^{(2)} = E\chi^{(2)}. \quad (37)$$

The solution to Eq. (37) is given as

$$\chi^{(2)}(\xi) = \theta(L-\xi)\chi^{(2)}_{<}(\xi) + \theta(\xi - L)\chi^{(2)}_{>}(\xi), \quad (38)$$

$$\chi^{(2)}_{<}(\xi) = \sin K\xi, \qquad \chi^{(2)}_{>}(\xi) = \sin KL \frac{\sin(k\xi+\delta)}{\sin(kL+\delta)}, \quad (39)$$

where $K = \sqrt{E+V}$ and $k = \sqrt{E}$, that is, $E = k^2$. Note that $\chi^{(2)}_{>}$ behaves as $\exp(-\sqrt{|E|}\xi)$ when $E < 0$.

The phase shift δ is derived from the relation (36):

$$K\cot(KL) = k\cot(kL+\delta). \quad (40)$$

Then, computing the scattering length as

$$a = -\lim_{k\to 0} \frac{\tan\delta(k)}{k}, \quad (41)$$

one finds its expression for potential $V_{\mathrm{sq}}(\xi)$:

$$a_V = L\left(1 - \frac{\tan(\sqrt{V}L)}{\sqrt{V}L}\right). \quad (42)$$

Note that a_V demonstrates discontinuous behavior for $\sqrt{V}L = (2n-1)\pi/2$ and $n \in \mathbb{N}$. We omit the detailed consideration of this zero-energy resonance.

Let us emphasize the essential difference between the physical consequences of the zero and divergent scattering lengths a_V in one and three dimensions, despite the formal similarity of the presented expressions to the three-dimensional case [59]. While the vanishing a_V means complete transparency in three dimensions, the opposite effect occurs in one dimension: the reflection coefficient becomes equal to unity, which leads to complete opacity. Transparency in one dimension is achieved when a_V diverges. Nevertheless, the divergence of the scattering length reveals a zero-energy bound state in both three-dimensional and one-dimensional cases [60].

In one dimension (see [61]), the scattering matrix $S = \mathrm{e}^{2\mathrm{i}\delta}$ and amplitude f are

$$S = \mathrm{e}^{-2\mathrm{i}kL}\frac{K\cot(KL) + \mathrm{i}k}{K\cot(KL) - \mathrm{i}k}, \qquad f = \frac{1}{2}(\mathrm{e}^{2\mathrm{i}\delta} - 1). \quad (43)$$

These formulas tell us how to extract scattering data in the open channel.

B. Two-channel quantum mechanics

Thus, we admit a single bound state in the closed channel and a continuum of waves with momentum k in the open channel. Taking into account the complexity of the problem involving antikink potential, we intend to analytically describe the Feshbach resonance between the channels in the first approximation.

As mentioned above, the initial set of equations in entire space is

$$(H_{\mathrm{ak}} + Q - E)\chi^{(1)} + \Omega\chi^{(2)} = 0, \quad (44)$$

$$(H_{\mathrm{wv}} - E)\chi^{(2)} + \Omega^{\dagger}\chi^{(1)} = 0. \quad (45)$$

For convenience, we will use the bra- and ket-vectors to simplify the notation of matrix elements.

In the first approximation, we put [44]:

$$|\chi^{(1)}\rangle = \frac{\lambda}{N(L)}|\chi_{\mathrm{ak}}\rangle, \qquad \langle\xi|\chi_{\mathrm{ak}}\rangle = \chi_{\mathrm{ak}}(\xi - L), \quad (46)$$

where λ is a complex constant which should be found; $N(L)$ is given by Eq. (27).

Acting by $\langle\chi_{\mathrm{ak}}|$ on Eq. (44), one has

$$\lambda = \frac{\langle\chi_{\mathrm{ak}}|\Omega|\chi^{(2)}_{<}\rangle}{E - Q}, \quad (47)$$

where it has been used that $H_{\mathrm{ak}}|\chi_{\mathrm{ak}}\rangle = 0$, the normalization $\langle\chi_{\mathrm{ak}}|\chi_{\mathrm{ak}}\rangle = N(L)$, and the equality $\langle\chi_{\mathrm{ak}}|\Omega|\chi^{(2)}\rangle = \langle\chi_{\mathrm{ak}}|\Omega|\chi^{(2)}_{<}\rangle$ due to the form of $\Omega(\xi)$. At this stage, the coefficient λ still depends on the unknown function $\chi^{(2)}_{<}$.

Introducing the auxiliary Hamiltonian

$$H_2 = -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - K^2, \qquad K^2 = E + V, \quad (48)$$

which is defined in the region $\xi \in [0; L]$, the equations for the open channel take the form

$$H_2|\chi^{(2)}_{<}\rangle + \Omega^{\dagger}|\chi^{(1)}\rangle = 0, \quad (49)$$

$$-\frac{\mathrm{d}^2\chi^{(2)}_{>}}{\mathrm{d}\xi^2} - k^2\chi^{(2)}_{>} = 0. \quad (50)$$

Solution to Eq. (49) can be written as

$$\begin{aligned}|\chi^{(2)}_{<}\rangle &= |\tau_0\rangle - G^{(+)}_2\Omega^{\dagger}|\chi^{(1)}\rangle \\ &= |\tau_0\rangle - \frac{\lambda}{N(L)}G^{(+)}_2\Omega^{\dagger}|\chi_{\mathrm{ak}}\rangle, \end{aligned} \quad (51)$$

where unperturbed wave function $\tau_0(\xi) = \langle\xi|\tau_0\rangle$ coincides with $\chi^{(2)}_{<}(\xi)$ from Eq. (39) and is such that

$$H_2\tau_0(\xi) = 0, \qquad \tau_0(\xi) = \sin K\xi. \quad (52)$$

The Hamiltonian H_2 determines also the Green's operator $G_2^{(+)} = (H_2 - \mathrm{i}\epsilon)^{-1}$ that contains the shifted energy $E + \mathrm{i}\epsilon$ at $\epsilon \to 0$. The corresponding Green's function is

$$G_2^{(+)}(\xi, \zeta; K) = \frac{\sin K\xi_1 \cos K\xi_2}{K} + \mathrm{i}\frac{\sin K\xi \sin K\zeta}{K},$$
$$\xi_1 = \min(\xi, \zeta), \qquad \xi_2 = \max(\xi, \zeta), \tag{53}$$

which serves for finding the outgoing wave under the boundary condition $G_2^{(+)}(0, \zeta; K) = 0$.

To express λ in terms of the known solutions τ_0 and χ_{ak}, let us operate by $\langle\chi_{\mathrm{ak}}|\Omega$ on Eq. (51). Then we obtain

$$\lambda = \frac{\langle\chi_{\mathrm{ak}}|\Omega|\tau_0\rangle}{E - Q + N^{-1}(L)\langle\chi_{\mathrm{ak}}|\Omega G_2^{(+)}\Omega^\dagger|\chi_{\mathrm{ak}}\rangle}. \tag{54}$$

The condition of vanishing of the denominator reveals the isolated (quasidiscrete) energy level of the dressed state [43,44].

Having introduced the notations

$$\omega^4\Delta_L(K) = N^{-1}(L)\mathrm{Re}\langle\chi_{\mathrm{ak}}|\Omega G_2^{(+)}\Omega^\dagger|\chi_{\mathrm{ak}}\rangle, \tag{55}$$

$$\omega^4\gamma_L(K) = N^{-1}(L)\mathrm{Im}\langle\chi_{\mathrm{ak}}|\Omega G_2^{(+)}\Omega^\dagger|\chi_{\mathrm{ak}}\rangle, \tag{56}$$

let us sketch how this works for a fixed $Q > 0$. We first imagine the situation when $V \gg E$ for $E \to 0$, and the denominator of λ vanishes at some complex value of the energy $E' - \mathrm{i}\Gamma'/2$, thereby making the magnitude of the wave functions extremely large. The resonance energy $E = E'$ and decay width Γ' may be simply determined:

$$E' = Q - \omega^4\Delta_L(\sqrt{V}), \qquad \Gamma' = 2\omega^4\gamma_L(\sqrt{V}). \tag{57}$$

For relatively small ω^2 and positive $\Delta_L(\sqrt{V})$, we can achieve that $Q > E' > 0$ due to two additional interactions. Besides, for positive $\gamma_L(\sqrt{V})$, the lifetime of particle in such a state is $\tau = 2/\Gamma'$, in dimensionless units.

In general case we write

$$\lambda = \frac{\langle\chi_{\mathrm{ak}}|\Omega|\tau_0\rangle}{E - Q + \omega^4\Delta_L(K) + \mathrm{i}\omega^4\gamma_L(K)}. \tag{58}$$

Consider the overlap integral that defines $\langle\chi_{\mathrm{ak}}|\Omega|\tau_0\rangle$:

$$B_L(K) \equiv 4\int_0^L \arctan\, \mathrm{e}^{L-\xi} \sin K\xi \mathrm{d}\xi. \tag{59}$$

It can be transformed to the form

$$B_L(K) = \frac{1}{2\mathrm{i}}\left[\phi_L(L, \mathrm{i}K) - \phi_L(L, -\mathrm{i}K)\right] - \frac{1}{2\mathrm{i}}\left[\phi_L(0, \mathrm{i}K) - \phi_L(0, -\mathrm{i}K)\right]. \tag{60}$$

Using the Lerch transcendent $\Phi(z, s, a)$, we introduce

$$\phi_L(\xi, a) \equiv \frac{\mathrm{e}^{a\xi}}{a}\left[4\arctan\, \mathrm{e}^{L-\xi} - 2\mathrm{e}^{L-\xi}\Phi\left(-\mathrm{e}^{2(L-\xi)}, 1, \frac{1-a}{2}\right)\right]. \tag{61}$$

This function is such that

$$\partial_\xi\phi_L(\xi, a) = \mathrm{e}^{a\xi}\chi_{\mathrm{ak}}(\xi - L). \tag{62}$$

Thus, we have $\langle\chi_{\mathrm{ak}}|\Omega|\tau_0\rangle = -\omega^2 B_L(K)$.

As seen above, the first correction to the wave function $\chi_<^{(2)}(x)$ is determined by $G_2^{(+)}\Omega^\dagger|\chi_{\mathrm{ak}}\rangle$. To find the complex function $\langle\xi|G_2^{(+)}\Omega^\dagger|\chi_{\mathrm{ak}}\rangle = -\omega^2\tau_1(\xi)$, we turn to the solution of the inhomogeneous equation

$$H_2\tau_1(\xi) = \chi_{\mathrm{ak}}(\xi - L), \tag{63}$$

$$\tau_1(\xi) = \int_0^L G_2^{(+)}(\xi, \zeta; K)\chi_{\mathrm{ak}}(\zeta - L)\mathrm{d}\zeta. \tag{64}$$

On computing, the summands of $\tau_1(\xi) = \tau_1^{\mathrm{R}}(\xi) + \mathrm{i}\tau_1^{\mathrm{I}}(\xi)$ are written as

$$\tau_1^{\mathrm{R}}(\xi) = \frac{\mathrm{e}^{-\mathrm{i}K\xi}}{2\mathrm{i}K}\phi_L(\xi, \mathrm{i}K) - \frac{\mathrm{e}^{\mathrm{i}K\xi}}{2\mathrm{i}K}\phi_L(\xi, -\mathrm{i}K) + \left[\phi_L(L, \mathrm{i}K) + \phi_L(L, -\mathrm{i}K)\right]\frac{\sin(K\xi)}{2K} + \mathrm{i}\left[\phi_L(0, \mathrm{i}K) - \phi_L(0, -\mathrm{i}K)\right]\frac{\cos(K\xi)}{2K}, \tag{65}$$

$$\tau_1^{\mathrm{I}}(\xi) = B_L(K)\frac{\sin(K\xi)}{K}. \tag{66}$$

Taking into account the form of $\tau_1^{\mathrm{I}}(\xi)$ and the definition of $B_L(K)$, we specify the function $\gamma_L(K)$ [see Eq. (56)]:

$$\gamma_L(K) = \frac{B_L^2(K)}{KN(L)} \geq 0. \tag{67}$$

At the same time, the real part $\tau_1^{\mathrm{R}}(\xi)$ determines also the deviation $\Delta_L(K)$ so that

$$\Delta_L(K) = N^{-1}(L)\int_0^L \chi_{\mathrm{ak}}(\xi - L)\tau_1^{\mathrm{R}}(\xi)\mathrm{d}\xi. \tag{68}$$

This integral is not simple to be calculated analytically, instead we present the numerical result in Fig. 5.

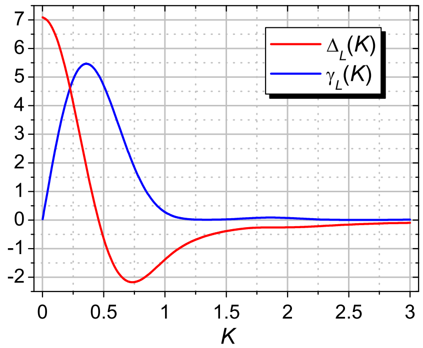

FIG. 5. The functions $\Delta_L(K)$ and $\gamma_L(K)$ which determine $\langle\chi_{\rm ak}|\Omega G_2^{(+)}\Omega^\dagger|\chi_{\rm ak}\rangle$ at $L=5$. There is $K_\Delta \simeq 0.4563077$ such that $\Delta_L(K_\Delta)=0$.

Let us note the similarity of the behavior of $\gamma_L(K)$ and $\Delta_L(K)$ with analogous functions from Ref. [56], which are calculated for another potential in three dimensions. *This means that the number of spatial dimensions does not affect main physical aspect of the problem.* By combining, the closed channel wave function in the first approximation reads

$$\chi^{(1)}(\xi) = -\frac{\omega^2}{D_L(K)}\sqrt{\frac{K\gamma_L(K)}{N(L)}}\chi_{\rm ak}(\xi - L), \tag{69}$$

where we have used the notation $D_L(K)$ introduced as

$$\begin{aligned} D_L(K) &= K^2 - V - Q + \omega^4\Delta_L(K) + \mathrm{i}\omega^4\gamma_L(K) \\ &= D_L^{\rm R}(K) + \mathrm{i}D_L^{\rm I}(K). \end{aligned} \tag{70}$$

Solution (69) vanishes at $\omega = 0$ and describes a short-lived state, that is seen by restoring for a moment the time dependence due to decaying factor $\exp(-\mathrm{i}Et)$ determined by the complex $E = Q - \omega^4\Delta_L(K) - \mathrm{i}\omega^4\gamma_L(K)$.

Combining the terms with $\tau_0(\xi)$ and $\tau_1^{\rm I}(\xi)$ due to proportionality $\tau_1^{\rm I}(\xi) \propto \tau_0(\xi)$, we present the open channel wave function in the resonance zone as

$$\chi_<^{(2)}(\xi) = \frac{D_L^{\rm R}(K)}{D_L(K)}\tau_0(\xi) - \frac{\omega^4}{D_L(K)}\sqrt{\frac{K\gamma_L(K)}{N(L)}}\tau_1^{\rm R}(\xi). \tag{71}$$

To find the phase shift δ for the function $\chi_>^{(2)}(\xi)$ namely

$$\begin{aligned} \chi_>^{(2)}(\xi) &= \chi_<^{(2)}(L)\frac{\sin(k\xi+\delta)}{\sin(kL+\delta)}; \\ \chi_<^{(2)}(L) &= \frac{D_L^{\rm R}(K)\sin KL - D_L^{\rm I}(K)\cos KL}{D_L(K)}, \end{aligned} \tag{72}$$

we appeal to the matching condition (36). We obtain

$$K\frac{D_L^{\rm R}(K)\cos KL + D_L^{\rm I}(K)\sin KL}{D_L^{\rm R}(K)\sin KL - D_L^{\rm I}(K)\cos KL} = k\cot(kL+\delta).$$

This relation can be rewritten as

$$K\cot(KL - \delta_{\rm rs}) = k\cot(kL+\delta), \tag{73}$$

$$\delta_{\rm rs}(K) = \arctan\frac{D_L^{\rm I}(K)}{D_L^{\rm R}(K)}, \tag{74}$$

where $\delta_{\rm rs}$ is the phase shift caused by interactions in the resonance zone. At $\delta_{\rm rs} \equiv 0$, only potential scattering with $V_{\rm sq}(x)$ remains in the open channel.

Now it is easy to extract the total phase shift:

$$\begin{aligned} \delta &= -kL + \arctan\left[\frac{k}{K}\tan(KL - \delta_{\rm rs})\right] \\ &\simeq k\left[\frac{\tan(KL-\delta_{\rm rs})}{K} - L\right], \end{aligned} \tag{75}$$

where the second expression is used for $k \to 0$.

C. Feshbach phenomenon

Let us dwell on the effects at zero energy and momentum k. It is reasonable to study the properties of the scattering length a of the particles in the open channel, having got the phase shift δ in (75) and using the Eq. (41). For our purposes, we represent a in the form

$$a(\omega^2) = a_{\rm bg}\left(1 + \frac{\mathcal{D}}{\omega^4 - \omega_c^4}\right). \tag{76}$$

In this formula, we have explicitly taken into account the dependence on the magnitude of the external influence ω^2 [see (33)] and shown the presence of critical value ω_c^2:

$$\omega_c^2 = \frac{\sqrt{Q}}{\sqrt{\Delta_L(K_V) + \gamma_L(K_V)\tan(K_V L)}}; \quad K_V \equiv \sqrt{V}. \tag{77}$$

This is determined by the energy gap $Q > 0$ between the two channels, which actually coincides with the kinetic energy of the incident particle outside the resonance zone at $\xi > L$. Avoiding here the zero-energy resonances in the open channel at $K_V L = (2n-1)\pi/2$ for $n \in \mathbb{N}$, we require $0 < \Delta_L(K_V) + \gamma_L(K_V)\tan(K_V L) < \infty$ to ensure a real value of ω_c^2.

The remaining characteristics are given as

$$a_{\rm bg} = a_V + \ell, \qquad \mathcal{D} = \omega_c^4\frac{\ell}{a_{\rm bg}}, \tag{78}$$

$$\ell = \frac{1}{K_V}\frac{\gamma_L(K_V)[1+\tan^2(K_V L)]}{\Delta_L(K_V) + \gamma_L(K_V)\tan(K_V L)} > 0, \tag{79}$$

where a_V is defined by (42), while $a_{\rm bg}$ is the so-called background scattering length. Note that the dependence of a on ω^2 vanishes at $Q = 0$ ($\omega_c^2 = 0$) so that $a = a_V + \ell$.

At $V \to 0$, we obtain $a_V = 0$, and

$$a = \ell_0 \frac{\omega^4}{\omega^4 - \omega_c^4}, \qquad \omega_c^4 = \frac{Q}{\Delta_L(0)}, \tag{80}$$

$$\ell_0 = \frac{1}{N(L)\Delta_L(0)} \lim_{K\to 0} \left(\frac{B_L(K)}{K}\right)^2, \tag{81}$$

where $B_L(K)$ is defined by Eq. (59), and Eq. (67) is used.

Thus, we gain the relation $a = \ell_0(L) \propto L$ when both $Q = 0$ and $V = 0$.

In general, the Feshbach phenomenon which is in our focus, results from the *dependence of the scattering length a on the interaction parameters*, that allows us to detect bound states by the divergence of a. Our model assumes that the change in a depends on Q (defining ω_c^2) and on ω^2, while $V > 0$ is given.

Analyzing Eq. (76), let us fix $L = 5$, $K_V = 0.2$ that provides $K_V < K_\Delta$ and $\Delta_L(K_V) > 0$ (see Fig. 5), and $Q = 0.15$. It leads to the following characteristics in dimensionless units: $a_V \simeq -2.787039$, $a_{\rm bg} \simeq 3.467758$, $\omega_c^2 \simeq 0.1128185$, and $\mathcal{D} \simeq 0.0229575$. The dependence of a on the external impact strength ω^2 is shown in Fig. 6, which confirms that the bound state does indeed occur at $\omega^2 = \omega_c^2$. Critical value ω_c^2 determines the threshold of the production of shallow dimers at $a(\omega^2) \gg L$ with the binding energy [62]

$$E_{\rm bind} = -\varkappa^2 \propto -\frac{1}{a^2(\omega^2)}. \tag{82}$$

This follows from considering the case of small negative energy $E \to 0^-$, when $\chi_>^{(2)} \propto \exp[-\varkappa(\xi - L)]$ for $\xi > L$, and Eq. (36) yields

$$-\varkappa = \lim_{k\to 0} \frac{\mathrm{d}}{\mathrm{d}\xi} \ln \chi_<^{(2)}(\xi)\Big|_{\xi=L} = -\frac{1}{a(\omega^2) - L}. \tag{83}$$

Thus, the Feshbach phenomenon justifies the need for a large scattering length in the formation of composites

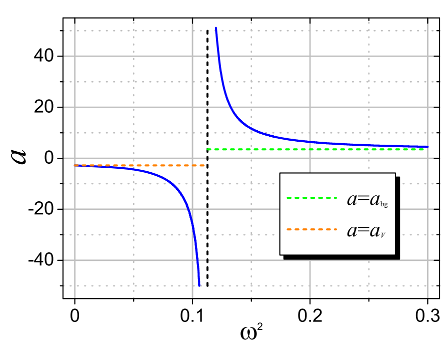

FIG. 6. Scattering length a as the function of external impact ω^2. The dashed straight lines are asymptotics. Black line corresponds to $\omega^2 = \omega_c^2$, while green and orange lines are for $a = a_{\rm bg}$ and $a = a_V$, respectively.

(of at least two particles), as predicted in Ref. [34] using phenomenological approaches. This is due to the fact that the Feshbach phenomenon as a zero-energy effect is valid in a different number of spatial dimensions, although there are distinct physical implications of zero and diverging scattering lengths in one and three dimensions [60].

In principle, the energy gap Q can be maintained by a spatially homogeneous interaction that induces the energy predominance of one configuration of the system of particles over another. In this regard, we mention experiments with alkali atoms, the energy configurations of which are determined by the spin and the applied magnetic field. Therefore, there, the Feshbach phenomenon is related with a resonant transition between configurations with a change of the magnetic field [42,62].

D. Resonance scattering

To reveal the newly formed bound state (of two axions), we also investigate the resonance scattering of an incident particle with a nonzero energy $E = k^2$.

The scattering matrix element $S = \mathrm{e}^{2\mathrm{i}\delta}$ for the open channel [cf. Eq. (43)] is

$$S = \mathrm{e}^{-2\mathrm{i}kL} \frac{K\cot(KL - \delta_{\rm rs}) + \mathrm{i}k}{K\cot(KL - \delta_{\rm rs}) - \mathrm{i}k}. \tag{84}$$

Denoting its denominator as

$$F(k) = K\cot(KL - \delta_{\rm rs}) - \mathrm{i}k; \qquad K = \sqrt{k^2 + V}, \tag{85}$$

the condition $F(k) = 0$ determines the pole of S and the resonance point (although, not every pole of S is related with the compound system existence).

Usually, a resonance is observed in narrow region of energy E, which covers the resonant value $E_{\rm res} = E_0 - \mathrm{i}\Gamma_0/2$ with some $E_0 > 0$ and $\Gamma_0 > 0$. Positivity of E_0 makes this level unstable, whose lifetime is $\tau = 2/\Gamma_0$, in dimensionless units.

Here, we find $E_{\rm res} = K_{\rm res}^2 - V$ for given Q, V, ω, and L by solving the equation $F(\sqrt{K_{\rm res}^2 - V}) = 0$ in an appropriate form. After identical transformation is performed, the following equation should be solved numerically at $\omega^4 \ll 1$ by using an iterative procedure with the initial value $K = \sqrt{V + Q}$:

$$K_{m+1}^2 = V + Q - \omega^4 \left[\Delta_L(K_m) + \mathrm{i}\gamma_L(K_m) \frac{k_m \cot K_m L - \mathrm{i}K_m}{K_m \cot K_m L - \mathrm{i}k_m}\right],$$
$$k_m = \sqrt{K_m^2 - V}. \tag{86}$$

Omitting the indexes m and $m+1$ restores the equation equivalent to $F(\sqrt{K^2 - V}) = 0$.

To extract the resonance part of S-matrix, we expand the complex function $F(k)$ of a real k in vicinity of complex root $k_{\rm res} = \sqrt{E_{\rm res}} = k_r - {\rm i}\kappa_r$ as

$$F(k) \simeq C(k - k_r + {\rm i}\kappa_r), \qquad F(k_{\rm res}) = 0, \tag{87}$$

where $C \equiv F'(k_{\rm res})$ is a complex constant.

Since $k = k^*_{\rm res}$ solves conjugate equation $F^*(k) = 0$, and $F^*(k) \simeq C^*(k - k_r - {\rm i}\kappa_r)$ near the resonance, we define the phase shift δ_0 related with potential scattering so that

$$e^{2{\rm i}\delta_0} = e^{-2{\rm i}kL}\frac{C^*}{C}. \tag{88}$$

The calculated ingredients enable to write down the scattering matrix and the total phase shift in the form:

$$S \simeq e^{2{\rm i}\delta_0}\frac{k - k_r - {\rm i}\kappa_r}{k - k_r + {\rm i}\kappa_r}, \tag{89}$$

$$\delta = \delta_0 - \arctan\frac{\kappa_r}{k - k_r}. \tag{90}$$

Representing these quantities in the conventional form in terms of E, we must expand $F(\sqrt{E})$ in powers of E:

$$F(\sqrt{E}) \simeq \frac{C}{2k_{\rm res}}(E - E_{\rm res}), \tag{91}$$

where C is as above. It leads to redefinition of phase shift δ_0 because of the relation:

$$\arctan\frac{\kappa_r}{k - k_r} - \frac{1}{2}\arctan\frac{\kappa_r}{k_r} = \arctan\frac{\Gamma_0}{2(E - E_0)}. \tag{92}$$

Thus, we can see that the phase shift δ experiences a jump $\delta(k_r - 0) - \delta(k_r + 0) = \pi$, which reveals a resonance. Besides, δ determines the cross section σ in accordance with the optic theorem in one dimension [61,63]:

$$\sigma(k) = 2\sin^2\delta(k). \tag{93}$$

Iterating Eq. (86) for $L = 5$, $V = 0.04$, $Q = 0.15$, and $\omega_1^4 = \omega_c^4$, used to test the Feshbach phenomenon above, we obtain the solution:

$$E_{\rm res} = 0.1752355035 - {\rm i}0.07423053097. \tag{94}$$

The behavior of $\delta(k)$ and $\sigma(k)$ is depicted in Fig. 7 and shows that the incident particle with momentum $k = \sqrt{Q}$ can be bound, if k is within the interval $(k_r - \kappa_r/2; k_r + \kappa_r/2)$ for $k_r = {\rm Re}\sqrt{E_{\rm res}} > 0$ and $\kappa_r = -{\rm Im}\sqrt{E_{\rm res}} > 0$. Note that the asymmetric form of the resonant peak in Fig. 7(b) is due to the term $-kL$ in the phase shift δ_0.

The obtained formulas and results describe, in general, the mechanism of the occurrence of resonance (associated with two-particle complex—dimer) without specifying the extra interactions used. Although the model needs to be refined in accordance with specific physical conditions, this formalism remains applicable to various studies.

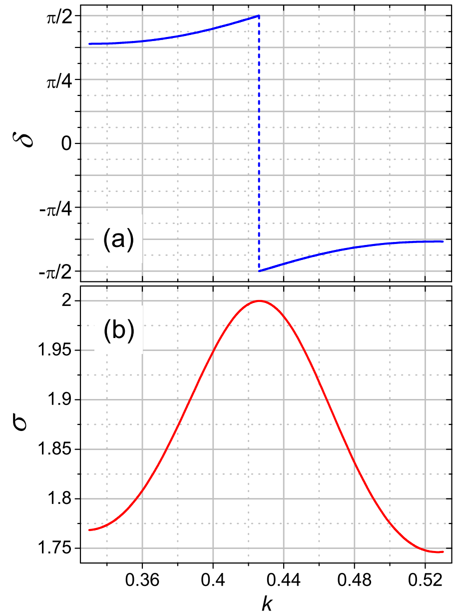

FIG. 7. Phase shift δ (a) and cross section σ (b) as functions of incident momentum k. Jump and peak in the graphs indicate the resonance and appear at $k_r = {\rm Re}\sqrt{E_{\rm res}} \simeq 0.4275189270$.

E. Analysis of DM dimer

Let us now convert dimensionless characteristics into physical units. The main scale we need is the length scale r_0 associated with the interaction radius. In principle, its refinement requires additional considerations that are beyond the scope of this study. Anyway, we apply r_0 which is given by Eq. (20) and is a measure of the size of the DM halo core. Taking into account the relation to dimensionless variable in Sec. II, the nonrelativistic scales for energy and time are

$$\varepsilon_0 = \frac{\hbar^2}{2mr_0^2}, \qquad \tau_0 = \frac{\hbar}{2\varepsilon_0}, \tag{95}$$

where m is the mass of axionlike particle, and τ_0 is determined using the (minimum) uncertainty principle.

Substituting the typical values of $m = 10^{-22}$ eV$/c^2$ and $r_0 = 0.138$ kpc from Sec. II, we arrive at the estimates

$$\varepsilon_0 \simeq 1.074 \times 10^{-29}\ {\rm eV}, \qquad \tau_0 \simeq 9.715 \times 10^5\ {\rm yrs}. \tag{96}$$

Analyzing, the resonance energy $\varepsilon_0{\rm Re}E_{\rm res} \simeq 1.882 \times 10^{-30}$ eV turns out to be tens of orders of magnitude lower than the critical temperature $T_c^{(d=3)}$ of free-boson BEC in three dimensions. It results from substituting the axionlike particle concentration $n^{(d=3)} = \rho_0/m \simeq 5.61 \times 10^{38}$ m^{-3} at typical parameters in Sec. II into the known expression written for d dimensions:

$$T_c^{(d)} = \frac{2\pi\hbar^2}{m}\left(\frac{n^{(d)}}{\zeta(d/2)}\right)^{2/d}, \quad (97)$$

where $\zeta(s)$ is the Riemann zeta-function.

In fact, we get the same result in the one-dimensional case, defining the concentration as $n^{(d=1)} \sim (n^{(d=3)})^{1/3}$.

Defining the resonance lifetime as $t = 2\tau_0/\Gamma_0$ and accounting for $\Gamma_0 \simeq 0.148$ as in the test above, we deduce compound system stability over a period

$$t \simeq 1.313 \times 10^7 \text{ yrs}. \quad (98)$$

We associate the resonance with a dimer (two-axion composite), whose one-dimensional wave function $\chi_{\mathrm{D}}(\xi)$ and the binding energy E_{bind} in dimensionless units are expected to be [see Eqs. (82) and (83)]

$$\chi_{\mathrm{D}}(\xi) \propto \exp\left(-\frac{\xi}{a}\right), \qquad E_{\mathrm{bind}} = -\frac{1}{a^2}. \quad (99)$$

Here a is the large scattering length given by Eq. (76); $\xi \gg L$ is a dimensionless distance between particles; besides, the dimensionless binding energy has to be converted into physical units using the scale ε_0.

These formulas are valid for $a \gg L$, where L is related with the dimensionless radius of axionlike interaction (see Sec. III). This indicates the regime of large scattering length a, which is achieved for a coupling ω^2 near its critical value ω_c^2 (see Fig. 6). Thereby, it confirms the hypothesis about the need for large a made in Ref. [34].

Note that the three-dimensional wave function of a dimer in the spherically symmetric case behaves as $\exp(-\mu r)/r$, where r is the distance between particles, and is often encountered for a simplified description of composites in nuclear physics, for example, as deuteron in the ground state [59,62]. Zoo of diverse (two-particle) states in this field of physics also enables to discover analogs of molecules, the formation of which is viewed within the Feshbach resonance concept. An important guide for confirming the existence of axion dimers can be dipion molecules. This follows from the common nature of axions and pions [2], that may also lead to dark analogs of pions and their molecules.

In any case, the long-term resonance in our scenario suggests consideration of DM as multicomponent environment due to the participation of composites. The presence of composites affects BEC DM properties and stimulates a detailed study of aspects of the composites formation.

Moreover, the dependence of the pair scattering length on interactions gives us a theoretical possibility to explain a vanishingly small a observed in the BEC models with two-particle interaction [25,32,33], as well as its variation in the DM halo of dwarf galaxies. Indeed, this can be done on the basis of Eq. (80). For the sake of correctness, it requires specifying the interaction parameters V and ω^2. Perhaps, the gravity also plays a certain role, which we explicitly do not take into account here when studying the Feshbach resonance.

V. CONCLUSION

In an attempt to describe the axionlike DM, we have involved a periodic chiral self-interaction that initially possesses $U(1)$ symmetry, which is broken in the massive system due to including Newtonian gravity. While the expected two different phases in a self-gravitating BEC DM are mostly revealed by a jump in the particle density, the desired multispecies matter, according to the current view, can be formed by both axionlike particles and their composites created in the processes that are similar to those in high energy (particle) physics. These phenomena can be helpful for identifying DM particles.

We develop the effects within the quantum mechanics, finding both the BEC wave function and the wave functions of both the composite (dimer) and its constituents. This means that we stay aside the quantized fields with a variable number of particles, appealing to the quantum mechanical formation and decay of DM dimers. Since the Gamow's tunneling seems unsuitable even for describing dimer decay, the Feshbach resonance theory appears to be relevant. Typically, the Feshbach resonance requires different scattering channels with their own interaction potentials, so one may be faced with assigning internal degrees of freedom to DM particles, the configuration of which determines each channel. Ignoring this issue as for now, we only focus on the problem with additional model potentials. This way, in particular, may be sufficient to describe the spontaneous creation and decay of compound particles in nuclear physics [45]. On the other hand, the detection of Feshbach dimers in atomic BECs in the laboratory [42,62] made it possible to trace the resonance mechanism in detail, motivating us to apply it to the axionlike DM model and outline its implementation. An important property of the Feshbach resonance is the increase of the scattering length to infinity, which coincides with the condition for the creation of composites, disclosed by the use of effective models [34].

Thus, in our theoretical study we combine the models of axionlike particles, including the sine-Gordon equation with its soliton solutions, and the Feshbach resonance concept. We focused on the quantum-mechanical formation of axion dimers in scattering processes and discussed in this regard the multicomponent DM consisting of axions and their composites, which were previously predicted in the qualitative study in Ref. [34]. Besides, having obtained the crucially important scattering length dependence on the interaction parameters, we got an idea of how to ensure its very small but different values when describing the DM halo of various (dwarf) galaxies on the base of the Gross-Pitaevskii-Poisson equations with pair interaction [17,25,34].

Using the cosinelike self-interaction derived for QCD axions [2,27], we formulated in Sec. II the gravitating BEC DM model based on the modified Gross-Pitaevskii-Poisson equations (11)–(12) at zero temperature. This type of nonlinearity generalizes the polynomial interactions that were used in the preceding models and are looking as its truncation [17,25,32,33]. On the other hand, the gained properties motivated us, first, to look for distinct phases of the axionlike BEC DM. Indeed, the existence of both rarefied or dilute (gaseous) and dense (liquidlike) phases immediately results from two independent solutions to the Gross-Pitaevskii equation even without developing a statistical description. But, the first-order phase transition in this model, the dynamics of which has yet to be detailed, would not be independently controlled by the strengths of pair and three-particle interactions as previously. This situation is in contrast with that considered in Ref. [33]. In any case, the phase transition between states must be stimulated by pressure/compression induced by long-wavelength quantum fluctuations [32]. The predominance of one or another phase in the DM halo of a certain galaxy can be inferred from the characteristics determined by fitting rotation curves and other observables. For instance, the DM gaseous (dilute) phase dominates in galaxy M81dwB, according to Ref. [33].

Regarding composites of DM, we note that the dense BEC phase is unfavored for composites because of their probable destruction caused by frequent collisions, as shown in [34]. On the other hand, for their appearance in a rarefied phase, a large scattering length is needed and has to be argued. Although an interaction potential with nonzero asymptotics often leads to an extremely large scattering length in the Born approximation [59], the scattering length also diverges due to zero-energy resonance, when the scattered particle goes into a bound state. Choosing the latter option, the bound state associated with a composite of at least two particles must be characterized by an isolated (quasidiscrete) level of positive energy and a finite lifetime. This is dictated, in particular, by the Feshbach resonance concept [43–45].

Intending to get more analytical results, we turn to the one-dimensional case. Then, focusing on the problem of a few interacting axions in the ground state, the three-dimensional Gross-Pitaevskii equation reduces to the stationary sine-Gordon equation with its antikink solution as in Sec. III. Comparing Figs. 2 and 3, we see that the effective potential W in Fig. 3 basically inherits the behavior of the gravitation-modified potentials for the two phases in Fig. 2(b), while the antikink solution mimics the DM distribution in the DM halo core. The discrepancy between the particle energies on the right-hand sides of Eq. (17) and Eq. (23) means that the distribution profile in Fig. 3 is formed by axions in the state of zero energy. Therefore, we are faced with the need to explain the appearance of axions with zero energy in a one-dimensional trap W, despite the presence of a domain wall.

To resolve this problem for at least two particles, we use two scattering channels: closed and open. A closed channel is represented by a bound state induced by the potential W with asymptotics $W \to 1$. An open channel implies elastic and asymptotically free scattering with a tiny positive energy. Let a particle perform transit between the channels coupled by an external impact. Given both the scattering potential (34) in an open channel and the coupling (33), the two-channel quantum-mechanical problem is formulated in Sec. IV. Though the square-well potentials are used therein, another form of them is also allowed. Then, with a certain adjustment of the parameters of extra interactions, an intermediate level appears, called the "dressed" state, which enables it to overcome the initial energy gap $Q > 0$ between the two channels. In a sense, such a level appearance is similar to the result of splitting, within a degeneracy problem under the action of perturbation.

The most significant processes take place in the resonance zone, which is a finite region of space bounded by a common radius of interaction L (in dimensionless units) for all potentials. To infer the information about processes far from the resonance zone, we resort to scattering theory and, thereby, extract data from the phase shift δ of the wave function of outgoing particle in an open channel, after leaving the resonance zone. Although the basic scattering characteristics in one and three dimensions do differ, we relate the scattering length a with δ by Eq. (41) as usual [63].

The analytical solution (69)–(72) of the two-channel problem is obtained in the first approximation, by taking into account potential scattering in an open channel with square well (34). This also comprises the characteristics of the dressed state that occurs when a particle hops between channels with close energies. Possessing positive energy, the dressed state has a finite lifetime and the resonance property to decay. If we imagined two interacting particles, one of which is pinned at the origin, then the dressed state would be seen as a compound system or an excited dimer. Besides, one justifies a nonzero decay width at zero energy due to the dependence of dressed state characteristics (57) on the magnitude V of the potential (34).

At zero energy, we consider the Feshbach phenomenon to reveal a bound state by the divergence of the scattering length a at certain (critical) value of the external influence. Parametrizing the external interaction (33) by ω^2, the critical value ω_c^2 is determined by $\sqrt{Q}$ in Eq. (77). That means that the scattering length $a(\omega^2)$ behaves as $a(\omega_c^2 \pm \epsilon^2) \to \pm\infty$ at $\epsilon \to 0$, that is shown in Fig. 6, and confirms the existence of a bound (dressed) state. This is valid in both one and three dimensions, although the divergent and zero scattering lengths have quite opposite effects on reflectance and transparency in one and three dimensions [60].

Note also the similarity of this phenomenon with that for alkali atoms in the laboratory, when ω^2 is replaced by a magnetic field B. Thus, one can expect the formation of shallow dimers with binding energy $E_{\rm bind} = -1/a^2$ in dimensionless units at large scattering length $a \gg L$.

On the other hand, having got the dependence of a on the interaction parameters, one could reproduce the vanishingly small values of a that take place in the BEC DM models with pair interaction [17,25,34]. Although we provide formulas for this, a detailed analysis is omitted.

To complete the study, the resonance scattering at nonzero energy is considered, and we find the complex value of resonant energy $E_{\rm res} = E_0 - {\rm i}\Gamma_0/2$ as a pole of the scattering matrix for some fixed values of the parameters: $L = 5$, $\sqrt{V} = 0.2$, and $\omega^2 = \omega_c^2$ at $Q = 0.15$. One gets $E_0 > 0$ and $\Gamma_0 > 0$, in contrast to the typical bound state with $E_0 < 0$ and $\Gamma_0 = 0$. We conclude that an incident particle with energy $E = Q$ and momentum $k = \sqrt{Q}$ participates in the resonance in Fig. 7, because $E_0 - \Gamma_0/2 \leq Q \leq E_0 + \Gamma_0/2$.

To estimate the dimer lifetime $t = 2\tau_0/\Gamma_0$, we use the timescale $\tau_0 = m r_0^2/\hbar$ for nonrelativistic axions with mass $m \simeq 10^{-22}$ eV$/c^2$. Substituting the scale for the DM halo core $r_0 \simeq 0.138$ kpc found in Sec. II, jointly with the numerically obtained value $\Gamma_0 \simeq 0.148$, *we get the encouraging value of lifetime, namely* $t \simeq 1.3 \times 10^7$ yrs. This may be sufficient for dimers participation in forming large DM structures. But, the fate of dimers depends on the potentials used, which can be of gravitational, stochastic, and even (dark) electromagnetic (due to the desired Primakoff effect [30]) nature. A reasonable justification for the extra scattering channel(s) needs the existence of internal degrees of freedom in DM particles, allowing them to interact differently. In this way, two channels in laboratory experiments with atoms result from energetically different spin configurations in an applied magnetic field. Meanwhile, in the case of DM particles, the internal degrees of freedom may be isospin or something else.

Although in DM-dominated galaxies the scattering length takes on different values [25], caused in our model by the effect of additional interactions with situationally distinct characteristics, its extremely large value, leading to the formation of dimers, does require fine tuning of certain conditions. Probably, such tuning may not always occur in all galaxies and appears to be spontaneous. However, the emergence conditions for the (unitary) regime of infinite scattering length may be fulfilled during the formation of galaxies along with changes in the parameters of extra interactions (similar to a magnetic field change in the laboratory).

In addition, we would like to note the need to take into account long-lived dimers in the formation of multicomponent DM halos, which ensures BEC stability according to the results of Ref. [64]. Clearly, these issues requires further study, and likewise the production of dimers and other composites in the environment [65,66].

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