Comparative study of criticality conditions for anomalous dimensions using exact results in an $N=1$ supersymmetric gauge theory

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Two of the conditions that have been suggested to determine the lower boundary of the conformal window in asymptotically free gauge theories are the linear condition, $\gamma_{\bar{\psi}\psi,\mathrm{IR}} = 1$, and the quadratic condition, $\gamma_{\bar{\psi}\psi,\mathrm{IR}}(2-\gamma_{\bar{\psi}\psi,\mathrm{IR}}) = 1$, where $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ is the anomalous dimension of the operator $\bar{\psi}\psi$ at an infrared fixed point in a theory. We compare these conditions as applied to an $\mathcal{N}=1$ supersymmetric gauge theory with gauge group G and N_f pairs of massless chiral superfields Φ and $\tilde{\Phi}$ transforming according to the respective representations $\mathcal{R}$ and $\bar{\mathcal{R}}$ of G. We use the fact that $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and the value $N_f = N_{f,\mathrm{cr}}$ at the lower boundary of the conformal window are both known exactly for this theory. In contrast to the case with a nonsupersymmetric gauge theory, here we find that in higher-order calculations, the linear condition provides a more accurate determination of $N_{f,\mathrm{cr}}$ than the quadratic condition when both are calculated to the same finite order of truncation in a scheme-independent expansion.

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I. INTRODUCTION

There has been considerable interest in asymptotically free gauge theories that have matter content such that they exhibit renormalization-group flows from the deep ultraviolet (UV) to infrared (IR) fixed points (IRFPs) [1,2]. At the infrared fixed point, the beta function vanishes, so the theory is scale-invariant and is inferred to be conformally invariant [3], whence the term "conformal window." With no loss of generality, one may restrict to massless matter fields, since if a matter field had a nonzero mass m_0, one would integrate it out of the effective low-energy theory that is relevant for momentum scales below m_0 in the flow to the infrared limit. The properties of a theory at an infrared fixed point in this conformal window are of fundamental interest. Among these are the scaling dimensions $D_{\mathcal{O}}$ of various (gauge-invariant) local operators, $\mathcal{O}$, such as $\bar{\psi}\psi$ and $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, where ψ and $F_{\mu\nu}$ denote fermion and gauge field-strength operators. Owing to the gauge interactions, the scaling dimension of an operator $\mathcal{O}$ differs from its free-field value, $D_{\mathcal{O},\mathrm{free}}$: $D_{\mathcal{O}} = D_{\mathcal{O},\mathrm{free}} - \gamma_{\mathcal{O}}$, where $\gamma_{\mathcal{O}}$ is the anomalous dimension of $\mathcal{O}$. Higher-loop calculations of anomalous dimensions at an IR fixed point in the conformal window have been performed in a number of works, including [4–16], using both conventional series expansions in powers of the gauge coupling at the IR fixed point and in powers of a scheme-independent expansion variable. Inputs for renormalization-group functions utilized in this work included those in [17–20]. Extensive measurements of anomalous dimensions have been carried out using lattice simulations; some of these works are [21–32].

As one decreases the matter content, the value of the gauge coupling at the IRFP, α_{IR}, increases, and eventually the theory changes qualitatively with the disappearance of this conformal IR fixed point. A commonly studied example is a non-Abelian gauge theory (in $d=4$ spacetime dimensions at zero temperature) with gauge group G and N_f copies ("flavors") of massless Dirac fermions transforming according to a representation R of G. One arranges that N_f is smaller than an upper (u) bound, $N_{f,u}$, depending on G and R, so that the theory is asymptotically free. As N_f decreases below $N_{f,u}$, the theory exhibits the aforementioned conformal IRFP, and the lower boundary of the conformal window occurs as N_f decreases through a critical value denoted $N_{f,\mathrm{cr}}$ [33]. Generalizations of this with several fermions transforming according to different representations have also been studied [13–16,28–30], but here it will be sufficient for our analysis to restrict our consideration to the case of matter fields transforming according to a single representation of the gauge group.

In addition to its importance in the context of formal quantum field theory, a determination of $N_{f,\mathrm{cr}}$ is important for the analysis of gauge theories with N_f slightly less than $N_{f,\rm cr}$, since in choosing such a theory to study, one needs to know at least the approximate value of $N_{f,\rm cr}$. A theory with N_f slightly below $N_{f,\rm cr}$ has a gauge coupling that runs slowly over a large range of momentum scales, due to an approximate IR zero in the beta function, but eventually becomes large enough to produce spontaneous chiral symmetry breaking and associated dynamical breaking of the approximate dilatation invariance. As a result, these theories (often called "walking" or quasiconformal theories) feature an approximate Nambu-Goldstone boson, the dilaton, as has been confirmed by lattice simulations [24,26,27]. Since the mass of a Nambu-Goldstone boson is protected against large radiative corrections, models incorporating this physics thus have the potential to address the Higgs mass hierarchy problem [34].

Two of the conditions that have been suggested to determine the lower boundary of the conformal window in asymptotically free gauge theories are the linear critical condition (γCC), $\gamma_{\bar\psi\psi,\rm IR} = 1$, and the quadratic critical condition, $\gamma_{\bar\psi\psi,\rm IR}(2-\gamma_{\bar\psi\psi,\rm IR}) = 1$ [35–38]. As is evident from the fact that the quadratic critical condition can be rewritten equivalently as $(\gamma_{\bar\psi\psi,\rm IR}-1)^2 = 0$, it has a double root at $\gamma_{\bar\psi\psi,\rm IR} = 1$ and hence is formally identical to the linear γCC. However, these two critical conditions yield different predictions for $N_{f,\rm cr}$ when using, as input, a finite-order series expansion for $\gamma_{\bar\psi\psi,\rm IR}$. In nonsupersymmetric gauge theories, the quadratic condition has been found to converge faster as a function of the order to which this series for $\gamma_{\bar\psi\psi,\rm IR}$ is computed [14,15]. An interesting question concerns how general this difference is; i.e., is it the case that the quadratic critical condition will also yield more rapid convergence than the linear critical condition in other theories?

In this paper we investigate this question, using as our theoretical laboratory an $\mathcal{N}=1$ supersymmetric gauge theory with gauge group G and N_f pairs of massless chiral superfields Φ and $\tilde\Phi$ transforming according to the respective representations $\mathcal{R}$ and $\bar{\mathcal{R}}$ of G. We take advantage of the key fact that for this theory one has exact results for $\gamma_{\bar\psi\psi,\rm IR}$ and $N_{f,\rm cr}$ [39–42].

This paper is organized as follows. In Sec. II we review some relevant background concerning the $\mathcal{N}=1$ supersymmetric gauge theory and our calculational methods. Section III contains a discussion of the linear and quadratic critical conditions on $\gamma_{\bar\psi\psi,\rm IR}$. Our calculational results on the comparison of these conditions for the supersymmetric theory are presented in Sec. IV. Our conclusions are summarized in Sec. V.

II. BACKGROUND ON THE $\mathcal{N}=1$ SUPERSYMMETRIC GAUGE THEORY AND CALCULATIONAL METHODS

In this section we briefly review some relevant background and our calculational methods. We consider a vectorial $\mathcal{N}=1$ supersymmetric gauge theory (in $d=4$ spacetime dimensions) with gauge group G and matter content consisting of N_f flavors of massless chiral superfields Φ and $\tilde\Phi$ transforming according to the respective representations $\mathcal{R}$ and $\bar{\mathcal{R}}$ of G (with color and flavor labels implicit here). In terms of component fields, the chiral superfield Φ has the decomposition

$$\Phi = \phi + \sqrt{2}\psi\theta + F\theta\theta, \tag{2.1}$$

where ψ is taken as a left-handed Weyl fermion, θ is an anticommuting Grassmann variable, and F is a nondynamical auxiliary field.

We denote the running gauge coupling as $g = g(\mu)$, where μ is the Euclidean energy or momentum scale at which this coupling is measured, and define $\alpha(\mu) = g(\mu)^2/(4\pi)$. As noted above, we restrict consideration of this theory to the range of N_f where it is asymptotically free. Owing to this, its properties can be computed perturbatively in the UV limit at large μ, where $\alpha(\mu)\to 0$. The dependence of $\alpha(\mu)$ on μ is described by the renormalization-group beta function:

$$\beta = \frac{d\alpha(\mu)}{d\ln\mu}. \tag{2.2}$$

The argument μ will generally be suppressed in the notation. The series expansion of β in powers of α is

$$\beta = -2\alpha\sum_{\ell=1}^{\infty} b_\ell a^\ell, \tag{2.3}$$

where

$$a \equiv \frac{g^2}{16\pi^2} = \frac{\alpha}{4\pi} \tag{2.4}$$

and b_ℓ is the ℓ-loop coefficient. We restrict here to mass-independent, supersymmetry-preserving regularization and renormalization schemes and to gauge-independent scheme transformations. The first two coefficients in (2.3) are [43]

$$b_1 = 3C_A - 2T_f N_f \tag{2.5}$$

and [44–46]

$$b_2 = 6C_A^2 - 4(C_A + 2C_f)T_f N_f, \tag{2.6}$$

where C_A, T_f, and C_f are group invariants [47]. These coefficients b_1 and b_2 are scheme-independent, while the b_ℓ with $\ell \geq 3$ are scheme-dependent. With an overall minus sign extracted, as in Eq. (2.3), the condition of asymptotic freedom is that $b_1 > 0$, and thus $N_f < N_{f,u}$, where the upper bound on N_f is

$$N_{f,u} = \frac{3C_A}{2T_f}. \tag{2.7}$$

Note that if $N_f = N_{f,u}$ so that $b_1 = 0$, then the two-loop coefficient has the negative value $b_2 = -12C_f C_A$, so [with the minus sign prefactor in Eq. (2.3)] the theory is not asymptotically free. This is the reason that we require the strict inequality $N_f < N_{f,u}$ for asymptotic freedom rather than the condition $N_f \le N_{f,u}$.

A number of additional exact results have been established about the IR phase structure of the theory [39–42]. We briefly summarize some relevant properties here. For a general gauge group G and representation $\mathcal{R}$, if N_f is in the conformal-window (CW) interval

$$\text{CW}: N_{f,\text{cr}} \le N_f < N_{f,u}, \quad \text{i.e.,} \quad \frac{N_{f,u}}{2} \le N_f < N_{f,u}, \tag{2.8}$$

where

$$N_{f,\text{cr}} = \frac{3C_A}{4T_f} = \frac{N_{f,u}}{2}, \tag{2.9}$$

the theory flows from the UV to an IR fixed point of the renormalization group. (The CW interval is also commonly called the non-Abelian Coulomb phase.)

In general, the expressions in Eqs. (2.7) and (2.9) for $N_{f,u}$ and $N_{f,\text{cr}}$ are not necessarily integers. In cases where $N_{f,u}$ or $N_{f,\text{cr}}$ is not an integer, one implicitly treats it as a formal result applicable in the framework in which one generalizes N_f from the non-negative integers to the non-negative real numbers. This will not be important for our present analysis, which focuses on a comparison of the relative accuracies of linear and quadratic γ critical conditions when used with finite-order perturbative anomalous-dimension inputs. However, for reference, we give some illustrative examples for the case $G = \text{SU}(N_c)$. If $\mathcal{R} = F$, the fundamental representation, then $N_{f,\text{cr}} = (3/2)N_c$, which is integral if and only if N_c is even. If $\mathcal{R} = \text{Adj}$, the adjoint representation, then $N_{f,u} = 3/2$ and $N_{f,\text{cr}} = 3/4$. Finally, if $\mathcal{R}$ is the rank-2 symmetric or antisymmetric tensor representation (denoted S_2 and A_2, respectively), then $N_{f,u} = 2N_{f,\text{cr}} = 3N_c/(N_c \pm 1)$, where the upper (lower) sign applies for S_2 and A_2.

With $b_1 > 0$ for asymptotic freedom, the condition that this two-loop beta function should have an IR zero is that $b_2 < 0$, which is that $N_f > N_{f,b2z}$, where

$$N_{f,b2z} = \frac{3C_A^2}{2T_f(C_A + 2C_f)}. \tag{2.10}$$

As we discussed in [48] (see also [49,50]), $N_{f,b2z}$ may be larger or smaller than $N_{f,\text{cr}}$, depending on the chiral superfield representation $\mathcal{R}$.

For a general gauge group G, the $\mathcal{N} = 1$ theory under consideration here, with N_f flavors of chiral superfields Φ and $\tilde{\Phi}$ in the representations $\mathcal{R}$ and $\bar{\mathcal{R}}$, respectively, is invariant under a classical continuous global (cgb) symmetry

$$\begin{aligned} G_{cgb} &= \text{U}(N_f) \otimes \text{U}(N_f) \otimes \text{U}(1)_R \\ &= \text{SU}(N_f) \otimes \text{SU}(N_f) \otimes \text{U}(1)_V \otimes \text{U}(1)_A \otimes \text{U}(1)_R, \end{aligned} \tag{2.11}$$

where the first and second $\text{U}(N_f)$ groups consist of operators acting on Φ^j and $\tilde{\Phi}_i$, respectively, with $i, j = 1, \ldots, N_f$, and the R-symmetry group $\text{U}(1)_R$ is defined by the following commutation relations:

$$[Q_\alpha, R] = Q_\alpha, \qquad [Q_\alpha^\dagger, R] = -Q_\alpha^\dagger, \tag{2.12}$$

where the Q_α and $Q_\alpha^\dagger$ are the generators of the supersymmetry transformations (with α spinor index here). The $\text{U}(1)_A$ symmetry is anomalous, due to instantons, so the actual nonanomalous continuous global symmetry of the theory is

$$G_{gb} = \text{SU}(N_f) \otimes \text{SU}(N_f) \otimes \text{U}(1)_V \otimes \text{U}(1)_R. \tag{2.13}$$

This symmetry is exact at an IR fixed point in the conformal window. The representations of the matter chiral superfields under the gauge and global symmetry groups are listed in Table I for the generic case in which the representation $\mathcal{R}$ is complex.

We will focus on the gauge-invariant quadratic operator products of the "meson" type,

$$M_i^j = \tilde{\Phi}_i \Phi^j, \tag{2.14}$$

where, as above, i and j are flavor indices and the group indices are implicit, with it being understood that they are contracted in such a way as to yield a singlet under the gauge group G. As a holomorphic product of chiral superfields, M_i^j is again a chiral superfield. The bilinear fermion operator product in M_i^j is $\tilde{\psi}_i \psi^j \equiv \tilde{\psi}_{i,L}^T C \psi_L^j$, where C is the conjugation Dirac matrix, and we use the convention of writing $\tilde{\psi}_{i,L}$ and ψ_L^j as left-handed Weyl fermions. Because the global symmetry (2.13) is exact in the conformal window, the meson-type quadratic chiral superfields transform according to (irreducible) representations of the group G_{gb}. The anomalous dimension of this

TABLE I. Matter content of a vectorial $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group G and matter content consisting of N_f massless chiral superfields Φ and $\tilde{\Phi}$ transforming according to the representations $\mathcal{R}$ and $\bar{\mathcal{R}}$, respectively. The symmetry groups correspond to those in Eq. (2.13).

	$\text{SU}(N_c)$	$\text{SU}(N_f)$	$\text{SU}(N_f)$	$U(1)_V$	$U(1)_R$
Φ	$\mathcal{R}$	$\square$	1	1	$1 - [C_A/(2T_f N_f)]$
$\tilde{\Phi}$	$\bar{\mathcal{R}}$	1	$\bar{\square}$	−1	$1 - [C_A/(2T_f N_f)]$

operator is independent of the flavor indices i and j [51], so in [52] and here, we denote its value at the superconformal IRFP simply as $\gamma_{M,\mathrm{IR}}$. Using the fact that $\tilde{\psi}_{i,L} = (\psi^i_R)^c$, the fermion bilinear in $\tilde{\Phi}_i\Phi^i$ can be rewritten in the standard form $\bar{\psi}_i\psi^i$ of a mass term in a nonsupersymmetric theory. Denoting $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ as the anomalous dimension of the latter bilinear, it follows that

$$\gamma_{M,\mathrm{IR}} = \gamma_{\bar{\psi}\psi,\mathrm{IR}}. \tag{2.15}$$

A closed-form expression for the beta function of this theory was derived by Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) [39]:

$$\beta_{\mathrm{NSVZ}} = -\frac{\alpha^2}{2\pi}\left[\frac{b_1 - 2N_f T_f \gamma_M}{1 - \frac{C_A\alpha}{2\pi}}\right]. \tag{2.16}$$

It is convenient to introduce the notation

$$x \equiv \frac{N_f}{N_{f,u}} \tag{2.17}$$

and

$$x_{\mathrm{cr}} \equiv \frac{N_{f,\mathrm{cr}}}{N_{f,u}} = \frac{1}{2}. \tag{2.18}$$

Thus, the conformal window is the interval

$$\frac{1}{2} \le x < 1. \tag{2.19}$$

One can express the anomalous dimension of an operator such as a fermion bilinear $\bar{\psi}\psi$ in a gauge theory as a series expansion in the squared gauge coupling,

$$\gamma_{\bar{\psi}\psi} = \sum_{\ell=1}^{\infty} c_\ell a^\ell, \tag{2.20}$$

where c_ℓ is the ℓ-loop coefficient. As noted above, the value of this anomalous dimension at an IRFP is written as $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$. The one-loop coefficient c_1 is scheme-independent, while the c_ℓ with $\ell \ge 2$ are scheme-dependent.

Physical quantities such as anomalous dimensions at an IRFP clearly must be scheme-independent. In conventional computations of these quantities, one first writes them as series expansions in powers of the coupling, as in (2.20), and then evaluates these series expansions with α set equal to α_{IR}, calculated to a given loop order. However, a (finite-order) series expansion of this type is scheme-dependent beyond the leading terms. Scheme dependence is also present in higher-order perturbative calculations in quantum chromodynamics (QCD), and its effects have been routinely addressed in studies comparing perturbative QCD predictions with experimental data. Formally speaking, these studies were on scheme dependence in the vicinity of the UV fixed point at zero coupling in QCD. Studies of scheme dependence in the different context of an IR fixed point located away from zero coupling have been carried out in [50,53–58]. For perturbative series calculations of anomalous dimensions, it is desirable to use a formalism in which results calculated to each order are scheme-independent.

Since $\alpha_{\mathrm{IR}} \to 0$ as $b_1 \to 0$ at the upper end of the conformal window, it follows that one can reexpress the series expansion for $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ in terms of a variable that is proportional to b_1, namely, the scheme-independent variable [2,59]

$$\Delta_f = N_{f,u} - N_f. \tag{2.21}$$

In the present theory,

$$\Delta_f = \frac{b_1}{2T_f}. \tag{2.22}$$

Scheme-independent calculations of anomalous dimensions of various operators at an IRFP were carried out in [8–12] for nonsupersymmetric gauge theories, and results were compared with measured values from lattice simulations. In [52] we carried out corresponding scheme-independent calculations of anomalous dimensions of several composite superfield operator products in the present $\mathcal{N} = 1$ supersymmetric theory. In general, the scheme-independent series expansion for a (gauge-invariant) operator $\mathcal{O}$ at an IRFP in the conformal window can be written as

$$\gamma_{\mathcal{O},\mathrm{IR}} = \sum_{j=1}^{\infty} \kappa_{\mathcal{O},j}\Delta_f^j. \tag{2.23}$$

The truncation of this series to $O(\Delta_f^p)$ inclusive is denoted $\gamma_{\mathcal{O},\mathrm{IR},\Delta_f^p}$:

$$\gamma_{\mathcal{O},\mathrm{IR},\Delta_f^p} = \sum_{j=1}^{p} \kappa_{\mathcal{O},j}\Delta_f^j. \tag{2.24}$$

Thus, for the operator M we write

$$\gamma_{M,\mathrm{IR},\Delta_f^p} = \gamma_{\bar{\psi}\psi,\mathrm{IR},\Delta_f^p} = \sum_{j=1}^{p} \kappa_{M,j}\Delta_f^j \tag{2.25}$$

with

$$\kappa_{M,j} = \kappa_{\bar{\psi}\psi,j}. \tag{2.26}$$

It is convenient to define the reduced scheme-independent expansion variable

$$y \equiv \frac{\Delta_f}{N_{f,u}} = 1 - \frac{N_f}{N_{f,u}} = 1 - x. \tag{2.27}$$

Since $1/2 \le x < 1$ in the conformal window [cf. Eq. (2.19)], it follows that in the conformal window y takes values in the range

$$\text{CW: } 0 < y \le \frac{1}{2}, \tag{2.28}$$

and we denote $y_{cr} = 1 - x_{cr} = 1/2$ at the lower end of the conformal window.

In the conformal window, the anomalous dimension at the IRFP in the conformal window, the exact expression for $\gamma_{\bar\psi\psi,IR} = \gamma_{M,IR}$, is

$$\gamma_{M,IR} = \frac{3C_A}{2T_f N_f} - 1 = \frac{N_{f,u}}{N_f} - 1 = \frac{1}{x} - 1. \tag{2.29}$$

This can be seen, for example, by solving for γ_M at the IR zero of the NSVZ beta function in Eq. (2.16), which is thus $\gamma_{M,IR}$. (Another derivation makes use of the R charges of the Φ and $\tilde\Phi$ chiral superfields, as discussed in [52].) This anomalous dimension $\gamma_{M,IR}$ can be expressed in terms of y as follows:

$$\gamma_{M,IR} = \frac{\Delta_f}{N_f} = \frac{\Delta_f}{N_{f,u} - \Delta_f} = \frac{y}{1-y} = \sum_{j=1}^{\infty} y^j. \tag{2.30}$$

Thus, the coefficient $\kappa_{M,j}$ in Eq. (2.25) has the value

$$\kappa_{M,j} = \kappa_{\bar\psi\psi,j} = \frac{1}{(N_{f,u})^j}. \tag{2.31}$$

The finite sum (2.25) was evaluated in our previous work [52], yielding

$$\gamma_{M,IR,\Delta_f^p} = \gamma_{\bar\psi\psi,IR,\Delta_f^p} = y\left(\frac{y^p - 1}{y - 1}\right). \tag{2.32}$$

Note that the numerator of the expression on the right-hand side of Eq. (2.32) contains a factor $(y-1)$ which cancels the denominator in Eq. (2.32), so that the resulting expression is a polynomial, as is clear from its definition (2.25) or from Eq. (2.30).

In [11] we showed that, for a given N_f in the conformal window, γ_{M,IR,Δ_f^p} approaches the exact result in Eqs. (2.29) and (2.30) exponentially rapidly [see Eqs. (2.37)–(2.41) in [11]]. We recall this result, since it is relevant here. As in [11], we define the fractional difference

$$\epsilon_p \equiv \frac{\gamma_{M,IR} - \gamma_{M,IR,\Delta_f^p}}{\gamma_{M,IR}}. \tag{2.33}$$

Using $\gamma_{M,IR}$ from Eqs. (2.29) or (2.30) and γ_{M,IR,Δ_f^p} from Eq. (2.32), this is

$$\epsilon_p = y^p. \tag{2.34}$$

Since $y^p = e^{-p\ln(1/y)}$ and $0 < y \le 1/2$ in the conformal window, this fractional difference evidently approaches zero exponentially rapidly as a function of the truncation order, p. This is true for any value of y in the conformal window, and, as a special case, it is true in the limit $y \to y_{cr} = 1/2$.

III. ANOMALOUS DIMENSION CONDITIONS IN CONFORMAL WINDOW

From analyses of the Schwinger-Dyson equation for the fermion propagator, of operator product expansions, and other arguments [35–38], it has been suggested that the upper bound

$$\gamma_{\bar\psi\psi,IR} \le 1 \tag{3.1}$$

applies for an IRFP in the conformal window. Since $\gamma_{\bar\psi\psi,IR}$ increases as one decreases N_f throughout the conformal window, it follows that the lower end of this conformal regime occurs when the inequality (3.1) is saturated, i.e., when the following condition holds:

$$\gamma_{\bar\psi\psi,IR} = 1. \tag{3.2}$$

That is, Eq. (3.2) determines the value of $N_{f,cr}$ demarcating the lower end of the conformal window. We denote Eq. (3.2) as the linear γ critical condition, denoted as LγCC. Note that this condition is in accord with the exactly known value of $\gamma_{\bar\psi\psi,IR} = \gamma_{M,IR}$ in the present $\mathcal{N} = 1$ supersymmetric gauge theory, as is clear from the exact result (2.29).

The quadratic condition

$$\gamma_{\bar\psi\psi,IR}(2 - \gamma_{\bar\psi\psi,IR}) = 1 \tag{3.3}$$

was discussed as a critical condition for fermion condensation, and its connection with the condition (3.2) was noted in [35] [see also [60]; we are not aware of any analysis that suggests the use of a critical condition $(\gamma_{\bar\psi\psi,IR} - 1)^s$ with $s \ge 3$]. We denote Eq. (3.3) as the quadratic γ critical condition, QγCC. As is obvious from the fact that Eq. (3.3) can be rewritten as $(\gamma_{\bar\psi\psi,IR} - 1)^2 = 0$, it has a double root at $\gamma_{\bar\psi\psi,IR} = 1$. Hence, an exact solution of the quadratic equation (3.3) yields the same result as the linear condition (3.2). However, when applied in the context of series expansions such as Eq. (2.23), as calculated to finite order, the results differ from those obtained with the linear condition (3.2). This difference arises because the quadratic condition (3.3) generates higher-order terms in powers of the scheme-independent expansion variable and leads to different coefficients of lower-order terms [14,15]. As our calculations below demonstrate, this difference, in conjunction with the exponentially rapid approach of the $O(\Delta_f^p)$ series in Eq. (2.25) to the exact expression noted above, lead to the linear γCC yielding a more accurate determination of $N_{f,cr}$ for $p \ge 3$ than the quadratic γCC in this supersymmetric theory.

In a nonsupersymmetric gauge theory with N_f fermions transforming according to a single representation of the gauge group, the use of the quadratic condition (3.3) was found [14,15] to (i) show better convergence as a function of increasing order of truncation of the series (2.23) than the linear condition (3.2) and (ii) predict a larger value of $N_{f,\mathrm{cr}}$ than the linear γCC. This work in [14,15] used the general results [9,10] for $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$ to the highest order that we had calculated them, namely, $p=4$.

As noted in the introduction, an interesting question that we will investigate here is whether the quadratic γCC also converges more rapidly than the linear γCC in the above-mentioned $\mathcal{N}=1$ supersymmetric gauge theory. An additional question that we will also investigate concerns whether the values of $N_{f,\mathrm{cr}}$ obtained from the LγCC and QγCC approach the exact value $N_{f,\mathrm{cr}}=3N_c$ from above or below. Equivalently, we will determine whether the corresponding values of x_{cr} approach the exact value $x_{\mathrm{cr}}=1/2$ from above or from below. It is worthwhile to mention that a rigorous upper bound on $\gamma_{\bar\psi\psi,\mathrm{IR}}$ in a conformal field theory is that [61–63]

$$\gamma_{\bar\psi\psi,\mathrm{IR}} \le 2. \tag{3.4}$$

This is evidently less restrictive than the bound (3.1).

IV. CALCULATIONAL RESULTS

The linear γCC equation $\gamma_{\bar\psi\psi,\mathrm{IR}}-1=0$ with $\gamma_{\bar\psi\psi,\mathrm{IR}}$ calculated to order $O(\Delta_f^p)$ inclusive is Eq. (3.2). Substituting Eq. (2.32), this becomes

$$y\left(\frac{y^p-1}{y-1}\right)-1=0, \tag{4.1}$$

or, equivalently,

$$\mathrm{L}\gamma\mathrm{CC}_p:\ \left(\sum_{j=1}^{p} y^j\right)-1=0. \tag{4.2}$$

This LγCC$_p$ condition is a polynomial equation of degree p in the variable y, or equivalently in the variable $x=1-y$. We denote the (physical) solution of the LγCC equation (4.2), expressed in terms of the variable x, as $x_{\mathrm{cr},L,p}$. For $1\le p\le 3$, we give the analytic solutions below, with floating-point values displayed to the indicated number of significant figures:

$$x_{\mathrm{cr},L,1}=0, \tag{4.3}$$

$$x_{\mathrm{cr},L,2}=\frac{3-\sqrt{5}}{2}=0.38197, \tag{4.4}$$

and

$$\begin{aligned} x_{\mathrm{cr},L,3}&=\frac{1}{3}[-(17+3\sqrt{33})^{1/3}+2(17+3\sqrt{33})^{-1/3}+4]\\ &=0.456311. \end{aligned} \tag{4.5}$$

Although the LγCC$_p$ condition (4.2) has p formal solutions, in each case, there is no ambiguity concerning which of these is the physical solution. For example, for $p=2$, the other solution, namely, $x=(1/2)(3+\sqrt{5})=2.618$ is outside the conformal-window range, $1/2\le x<1$; for $p=3$, the other two solutions form an unphysical complex-conjugate pair, and so forth for higher p.

The quadratic γCC condition (3.3) with $\gamma_{\bar\psi\psi,\mathrm{IR}}$ calculated to $O(\Delta_f^p)$ is $(\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}-1)^2=0$. If one takes the square root of this equation to begin with, one simply recovers the linear γCC equation. If, instead, one evaluates terms at $O(\Delta_f^p)$ resulting from the quadratic expression, then one obtains the equation

$$S-1=0, \tag{4.6}$$

where the sum S has the form

$$S=\sum_{j=1}^{p}\lambda_j\Delta_f^j, \tag{4.7}$$

where the coefficients λ_j will be discussed shortly. Given an input for $\gamma_{\bar\psi\psi,\mathrm{IR}}$ calculated to $O(\Delta_f^p)$, the quadratic γCC generates terms up to $O(\Delta_f^{2p})$; however, for self-consistency, one performs the corresponding truncation of terms to $O(\Delta_f^p)$, since this is the accuracy of the input expressions for $\gamma_{M,\mathrm{IR},\Delta_f^p}$. For the coefficients λ_j we calculate that

$$\lambda_1=2\kappa_1, \tag{4.8}$$

$$\lambda_2=2\kappa_2-\kappa_1^2, \tag{4.9}$$

$$\lambda_3=2(\kappa_3-\kappa_1\kappa_2), \tag{4.10}$$

$$\lambda_4=2\kappa_4-2\kappa_1\kappa_3-\kappa_2^2, \tag{4.11}$$

$$\lambda_5=2(\kappa_5-\kappa_1\kappa_4-\kappa_2\kappa_3), \tag{4.12}$$

and so forth for higher j. In general, we find that λ_j contains a term $2\kappa_j$ and then (a) if j is odd, a sum of terms of the form $-2\kappa_r\kappa_{j-r}$, where $1\le r\le (j-1)/2$, and (b) if j is even, a sum of terms of the form $-2\kappa_r\kappa_{j-r}$ with $1\le r\le (j/2)-1$, together with a term $-\kappa_{j/2}^2$. Substituting the expression $\kappa_j=1/(N_{f,u})^j$ from Eq. (2.31), we find

$$\lambda_j=\frac{3-j}{(N_{f,u})^j} \tag{4.13}$$

and hence

$$S=\sum_{j=1}^{p}(3-j)y^j. \tag{4.14}$$

Calculating this sum in closed form, we obtain

$$S = \frac{y}{(1-y)^2}[2 - 3y + (p-2)y^p + (3-p)y^{p+1}]. \tag{4.15}$$

The numerator of the expression on the right-hand side of Eq. (4.15) contains a factor of $(1-y)^2$ which cancels the factor in the denominator, so that the result is a polynomial in y, as is obvious from its definition, Eq. (4.7), or from Eq. (4.14). The resultant quadratic γCC condition, evaluated to $O(\Delta_f^p)$, is

$$\mathrm{Q}\gamma\mathrm{CC}_p\colon\ S - 1 = \frac{1}{(1-y)^2}[-1 + 4y - 4y^2 + (p-2)y^{p+1} + (3-p)y^{p+2}] = 0. \tag{4.16}$$

Since S is a polynomial in y, it follows that $S-1$ is also, and hence the expression in square brackets in Eq. (4.16) contains a factor of $(1-y)^2$, which cancels with the $(1-y)^2$ in the denominator. We denote the (physical) solution of the QγCC equation (4.16), expressed in terms of the variable x, as $x_{\mathrm{cr},Q,p}$. As is clear from Eq. (4.14), if $p \neq 3$, then the QγCC$_p$ condition is a polynomial equation of degree p in the variable y, or equivalently in the variable x, while if $p = 3$, then the coefficient of the highest-power term vanishes, so the resultant equation is of degree 2 in y. Indeed, with this cancellation, the QγCC$_3$ equation is identical to the QγCC$_2$ equation. As was the case with the LγCC$_p$ condition, although for $p \geq 2$, there are several solutions, there is no ambiguity concerning which is the physical solution; for example, for $p = 2$, the other solution is $x = 2 + \sqrt{2} = 3.414$, which is outside the conformal-window range of x. The analytic solutions to the lowest cases are

$$x_{\mathrm{cr},Q,1} = \frac{1}{2} \tag{4.17}$$

and

$$x_{\mathrm{cr},Q,2} = 2 - \sqrt{2} = 0.58579. \tag{4.18}$$

It happens that the lowest-order result $x_{\mathrm{cr},Q,1}$ is exact, but this is not generic; for $p \geq 2$, the QγCC$_p$ equation yields a value of $x_{\mathrm{cr},Q,p} > 1/2$.

In Table II, we list the results of the calculations with the linear γCC with the input value of $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$ for $1 \leq p \leq 10$, yielding the LγCC$_p$ condition. Table II includes

(1) the value of $x_{\mathrm{cr},L,p}$;

(2) the ratio of $x_{\mathrm{cr},L,p}$ to the exact value $x_{\mathrm{cr}} = 1/2$, denoted as

$$r_{\mathrm{cr},L,p} \equiv \frac{x_{\mathrm{cr},L,p}}{x_{\mathrm{cr}}} = 2x_{\mathrm{cr},L,p}; \tag{4.19}$$

(3) the fractional difference with respect to the exact value,

$$\mathrm{Diff}_{\mathrm{cr},L,p} \equiv 1 - \frac{x_{\mathrm{cr},L,p}}{x_{\mathrm{cr}}} = 1 - 2x_{\mathrm{cr},L,p}; \tag{4.20}$$

(4) the fractional difference with respect to the next lower-order value,

$$\mathrm{Diff}_{\mathrm{cr},L,p,p-1} \equiv 1 - \frac{x_{\mathrm{cr},L,p-1}}{x_{\mathrm{cr},L,p}}. \tag{4.21}$$

In Table III, we list the results of the calculations with the quadratic γCC with the input value of $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$ for $1 \leq p \leq 10$, yielding the QγCC$_p$ condition. This table includes

(1) the value of $x_{\mathrm{cr},Q,p}$;

TABLE II. In this table, the columns list (1) the value p specifying the order $O(\Delta_f^p)$ to which the linear (L) criticality condition LγCC is evaluated, yielding the LγCC$_p$ condition (4.2); (2) the value of $N_{f,\mathrm{cr}}/N_c$ calculated from this LγCC$_p$ condition, denoted $x_{\mathrm{cr},L,p}$; (3) the ratio $r_{\mathrm{cr},L,p}$ in Eq. (4.19); (4) the fractional difference with respect to the exact value, $\mathrm{Diff}_{\mathrm{cr},L,p}$ in Eq. (4.20); and (5) the fractional difference with respect to the next lower-order value, $\mathrm{Diff}_{\mathrm{cr},L,p,p-1}$ in Eq. (4.21). The abbreviation NA means "not applicable".

p	$x_{\mathrm{cr},L,p}$	$r_{\mathrm{cr},L,p}$	$\mathrm{Diff}_{\mathrm{cr},L,p}$	$\mathrm{Diff}_{\mathrm{cr},L,p,p-1}$
1	0	0	1	NA
2	0.38197	0.76393	0.23607	1
3	0.45631	0.91262	0.087378	0.16293
4	0.48121	0.96242	0.037580	0.051742
5	0.49134	0.98268	0.017321	0.020616
6	0.49586	0.99172	0.82765×10^{-2}	0.91197×10^{-2}
7	0.49798	0.99597	0.40342×10^{-2}	0.42596×10^{-2}
8	0.49901	0.99801	1.98836×10^{-3}	2.0498×10^{-3}
9	0.49951	0.99901	0.98624×10^{-3}	1.0031×10^{-3}
10	0.49975	0.99951	0.49092×10^{-3}	0.49556×10^{-3}

TABLE III. In this table, the columns list (1) the value p specifying the order $O(\Delta_f^p)$ to which the quadratic criticality condition QγCC is evaluated, yielding the QγCC$_p$ condition (4.16); (2) the value of $N_{f,\mathrm{cr}}/N_c$ calculated from this QγCC$_p$ condition, denoted $x_{\mathrm{cr},Q,p}$; (3) the ratio $r_{\mathrm{cr},Q,p}$ in Eq. (4.19); (4) the fractional difference with respect to the exact value, $\mathrm{Diff}_{\mathrm{cr},Q,p}$ in Eq. (4.20); and (5) the fractional difference with respect to the next lower-order value, $\mathrm{Diff}_{\mathrm{cr},Q,p,p-1}$ in Eq. (4.21). Other notation is as in Table II.

p	$x_{\mathrm{cr},Q,p}$	$r_{\mathrm{cr},Q,p}$	$\mathrm{Diff}_{\mathrm{cr},Q,p}$	$\mathrm{Diff}_{\mathrm{cr},Q,p,p-1}$
1	0.5	1	0	NA
2	0.58579	1.17157	−0.17157	0.14644
3	0.58579	1.17157	−0.17157	0
4	0.57421	1.14843	−0.14843	−0.020155
5	0.56145	1.12289	−0.12289	−0.022738
6	0.54982	1.09964	−0.09964	−0.021147
7	0.53985	1.07969	−0.079692	−0.018475
8	0.53153	1.06305	−0.063050	−0.0156545
9	0.52470	1.04940	−0.049404	−0.013004
10	0.51918	1.03836	−0.038361	−0.010635

(2) the ratio of $x_{\mathrm{cr},Q,p}$ to the exact value $x_{\mathrm{cr}} = 1/2$, denoted as

$$r_{\mathrm{cr},Q,p} \equiv \frac{x_{\mathrm{cr},Q,p}}{x_{\mathrm{cr}}} = 2x_{\mathrm{cr},Q,p}; \qquad (4.22)$$

(3) the fractional difference with respect to the exact value,

$$\mathrm{Diff}_{\mathrm{cr},Q,p} \equiv 1 - \frac{x_{\mathrm{cr},Q,p}}{x_{\mathrm{cr}}} = 1 - 2x_{\mathrm{cr},Q,p}; \qquad (4.23)$$

(4) the fractional difference with respect to the next lower-order value,

$$\mathrm{Diff}_{\mathrm{cr},Q,p,p-1} \equiv 1 - \frac{x_{\mathrm{cr},Q,p-1}}{x_{\mathrm{cr},Q,p}}. \qquad (4.24)$$

We see that in this theory, (i) for a given order $O(\Delta_f^p)$ with $p \geq 3$, the linear γCC yields a value of $x_{\mathrm{cr},L,p}$ that is closer to the exact value $x_{\mathrm{cr}} = 1/2$ than the value $x_{\mathrm{cr},Q,p}$ obtained from the quadratic γCC, so that the linear γCC yields an estimate of x_{cr} that approaches the exact value more rapidly than the estimate from the quadratic γCC. This is our main result. This result can be understood as a consequence of the exponentially rapid approach of the $O(\Delta_f^p)$ series in Eq. (2.25) to the exact expression (2.29) for $\gamma_{\bar\psi\psi,\mathrm{IR}}$ that enters in the linear γCC, together with the fact that the quadratic γCC, when expanded out, introduces different coefficients $\lambda_j \neq \kappa_j$ in an expansion in powers of Δ_f in Eq. (4.7). Furthermore, while the linear γCC yields a value of $x_{\mathrm{cr},L,p}$ that approaches the exact value from below, the quadratic γCC at order $p \geq 2$ yields a value of $x_{\mathrm{cr},Q,p}$ that approaches the exact value from above. These findings are evident in Tables II and III. We have checked that these properties also hold at higher truncation order beyond the highest order, $p = 10$, shown in these tables.

Contrasting these results with those in the corresponding nonsupersymmetric gauge theory, one must first recall that the value of $N_{f,\mathrm{cr}}$ (depending on the gauge group G and the fermion representation $\mathcal{R}$) is not known exactly, so that one cannot make a precise comparison with it. However, one can, at least, determine the fractional changes in the values of the solutions for $x_{\mathrm{cr},L,p}$ and $x_{\mathrm{cr},Q,p}$ as functions of the order $O(\Delta_f^p)$ to which one has calculated $\gamma_{\bar\psi\psi,\mathrm{IR}}$. At an IRFP in a nonsupersymmetric gauge theory with fermions in one representation, the maximum order to which the scheme-independent calculations have been performed is $p = 4$, with results given in our Refs. [9,10]. It was found in [14,15] (and confirmed in [16]), using these results for $\gamma_{\bar\psi\psi,\Delta_f^p}$ from [9,10], that the quadratic γCC converges more rapidly than the linear γCC. Thus, for $p \geq 3$, the relative accuracies and convergence rates of the linear versus the quadratic γCC that we find for this $\mathcal{N} = 1$ supersymmetric theory are opposite to the behavior that was found in the nonsupersymmetric theory. Moreover, in the nonsupersymmetric gauge theory, the linear and quadratic γCC conditions yield estimates of $N_{f,\mathrm{cr}}$ that increase as a function of the truncation order, p [14,15]. This is also true for the values of $N_{f,\mathrm{cr}}$ and thus $x_{\mathrm{cr},L,p}$ obtained from the LγCC$_p$ equation in the supersymmetric gauge theory studied here; i.e., $x_{\mathrm{cr},L,p}$ approaches the exact value $x_{\mathrm{cr}} = 1/2$ from below. In contrast, in this supersymmetric theory, for $p \geq 2$ the value of $x_{\mathrm{cr},Q,p}$ calculated from the QγCC$_p$ equation approaches the exact value of x_{cr} from above.

V. CONCLUSIONS

In conclusion, in this paper we have performed a comparison of the linear and quadratic critical conditions $\gamma_{\bar\psi\psi,\mathrm{IR}} = 1$ and $\gamma_{\bar\psi\psi,\mathrm{IR}}(2 - \gamma_{\bar\psi\psi,\mathrm{IR}}) = 1$, where $\gamma_{\bar\psi\psi,\mathrm{IR}}$ is the anomalous dimension of the fermion bilinear $\bar\psi\psi$ at an infrared fixed point in the conformal window in an $\mathcal{N} = 1$ supersymmetric gauge theory with N_f pairs of chiral superfields Φ_i and $\tilde\Phi_i$ transforming according to the $\mathcal{R}$ and $\bar{\mathcal{R}}$ representations of the gauge group G, respectively. This theory has the appeal that both $\gamma_{\bar\psi\psi,\mathrm{IR}}$ and the value $N_{f,\mathrm{cr}}$ at the lower boundary of the conformal window are known exactly. We find that, as a function of the order $O(\Delta_f^p)$ to which one uses the truncated calculation of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ as input, for $p \geq 3$, the linear critical condition yields an estimate of $x_{\mathrm{cr}} = N_{f,\mathrm{cr}}/N_{f,u}$ that is more accurate than the quadratic critical condition. This behavior is opposite to what was found for nonsupersymmetric gauge theories. It should be emphasized that the use of both the linear and quadratic critical conditions with finite-order inputs for $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$ are approximate perturbative methods. Thus, differences between predictions for the lower end of the conformal window obtained with these methods provide one measure of the importance of higher-order terms in the inputs, $\gamma_{\bar{\psi}\psi,\mathrm{IR},\Delta_f^{p'}}$ with $p' > p$. Studies that elucidate the properties of IR-conformal gauge theories and, in particular, the location of the lower boundary of the conformal window in these theories are of continuing interest, both for basic quantum field theory and for possible phenomenological applications. The comparative analysis reported herein provides some further insight into predictions from different critical conditions for the lower boundary of the conformal window.

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