# Disorder operators and magnetic vortices in SU(N) lattice gauge theory

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We construct the most general disorder operator for SU(N) lattice gauge theory in $(2+1)$ dimensions by using exact duality transformations. These disorder operators, defined on the plaquettes and characterized by $(\mathrm{N}-1)$ angles, are the creation and annihilation or the shift operators for the SU(N) magnetic vortices carrying $(\mathrm{N}-1)$ types of magnetic fluxes. They are dual to the SU(N) Wilson loop order operators which, on the other hand, are the creation-annihilation or shift operators for the $(\mathrm{N}-1)$ electric fluxes on their loops. The new order-disorder algebra involving SU(N) Wigner D matrices is derived and discussed. The $Z_{\mathrm{N}}(\in \mathrm{SU(N)})$ 't Hooft operator is obtained as a special limit. In this limit we also recover the standard Wilson-'t Hooft order-disorder algebra. The partition function representation and the free energies of these SU(N) magnetic vortices are discussed.

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## I. INTRODUCTION

Disorder operators, introduced originally in 1971 by Kadanoff and Ceva in the context of the two-dimensional Ising model [1], have been widely discussed and found useful in the studies of phase structures of spin models as well as Abelian and non-Abelian gauge theories [2–13]. They also play a pivotal role in differentiating the topological phases of matter [3] and in the boson-fermion transmutation through the "order $\otimes$ disorder" combinations [14]. It is generally known that the duality transformations in spin models and gauge theories naturally lead to these disorder operators as the fundamental operators describing the dual interactions. Under duality, the interacting and the noninteracting terms also interchange their roles leading to the inversion of the coupling constant in the dual interactions. The Kramers-Wannier duality in $(1+1)$ dimensional Ising spin model [2] and the Wegner duality in $(2+1)$ dimensional $Z_2$ gauge theory are the simplest examples which illustrate the above facts [3–6]. In the $(1+1)$ dimensional Ising model the disorder operators are simply the dual spin operators which describe the dual interactions with inverse coupling. They also create $Z_2$ kinks which are responsible for disordering the ground state leading to the loss of magnetization above the Curie temperature.

In Abelian and non-Abelian gauge theories the disorder operators acquire additional meaning of the dual electric potentials as the duality transformations also interchange the roles of the electric and magnetic degrees of freedom [3–6]. Again, the Wegner dualities in the simplest $Z_2$ Ising gauge theory in $(2+1)$ as well as in $(3+1)$ dimensions, clearly illustrate this additional rich feature [3–6]. More explicitly, in $(2+1)$-dimension $Z_2$ lattice gauge theory the disorder operators are the dual-spin or dual-$Z_2$ electric potential operators [4,5] which describe the interactions in the dual formulation with inverse coupling. Being conjugate to the $Z_2$ magnetic fields, they also create $Z_2$ magnetic vortices. These vortices, in turn, magnetically disorder the ground states in the confining phase [4] and are thus responsible for the confinement-deconfinement phase transition.

In general, the order (disorder) operators are related to the potentials (dual potentials) which are conjugate to electric (magnetic) fields, respectively. They can therefore be interpreted as the "translation operators" for the electric and magnetic fluxes, respectively. Moreover, the order-disorder algebra is simply the canonical commutation relations between the dual conjugate operators, i.e., between the magnetic flux and the electric potential operators [see the relations (1)–(3)]. In SU(3) lattice gauge theory or QCD the color confinement can be viewed as a consequence of magnetically disordered ground state leading to area law for the Wilson loops. Like the various cases discussed above, the magnetic disorder in QCD is produced by the magnetic vortices, which in turn are created by the SU(3) disorder operators leading to

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disordered ground state. A systematic study of these disorder operators in SU(2), SU(3), and then SU(N) lattice gauge theories, using exact duality transformations, is the subject of this work.

In 1978 Mandelstam tried to construct the SU(N) disorder operator in the continuum using the dual electric non abelian vector potentials [7,8]. In 1978 't Hooft emphasized the role of disorder operators in the context of quark confinement in SU(N) gauge theory [8]. The 't Hooft disorder operator creates magnetic fluxes which belong to the center $Z_{\mathrm{N}}$ of the gauge group SU(N). They have been extensively studied in the past analytically as well as using Monte Carlo techniques in the weak-coupling continuum limit [9–11].

It is known that the 't Hooft loop disorder operators are dual to the Wilson loop order operators in a limited sense [10,11] as they create only the center or $Z_{\mathrm{N}}$ magnetic fluxes. In this paper we construct the most general disorder operator for SU(N) lattice gauge theory in $(2+1)$ dimensions by exploiting the exact duality transformations [15,16]. These disorder operators $\Sigma_{[\vec{\theta}]}(p)$ are defined on plaquettes $p$ as

$$\Sigma^{\pm}_{[\vec{\theta}]}(p) = \exp i(\vec{\theta}(p)\cdot\vec{\mathcal{E}}_{\pm}(p)). \tag{1}$$

In (1), $\vec{\mathcal{E}}_{\pm}(p)$ are the SU(N) "electric scalar potentials" on the plaquette $p$. They are related to the SU(N) electric fields through the exact duality transformations (12) in Sec. III (also see Fig. 2). The SU(N) disorder operator $\Sigma^{\pm}_{[\vec{\theta}]}(p)$ in (1) is characterized by a set of $(\mathrm{N}-1)$ angles which are denoted by $[\vec{\theta}] \equiv (\theta_1(p), \theta_2(p), \ldots, \theta_{\mathrm{N}-1}(p))$ on each plaquette. In this work, like the Kramers-Wannier spin and Wegner gauge dualities discussed earlier, we show that the exact SU(N) duality transformations naturally lead to $\Sigma^{\pm}_{[\vec{\theta}]}(p)$ in (1). We further show that they are the creation and annihilation operators for the SU(N) magnetic vortices on the spatial plaquettes.

The Wilson loop order operators $\mathcal{W}^{[\vec{j}]}(\mathcal{C})$, on the other hand, are defined as a path-ordered product of the link holonomies along a directed loop $\mathcal{C}$,

$$\mathcal{W}^{[\vec{j}]}(\mathcal{C}) = \prod_{l\in\mathcal{C}} U^{[\vec{j}]}(l). \tag{2}$$

In (2), $U^{[\vec{j}]}(l)$ are the SU(N) link holonomies or the "magnetic vector potentials" in a general $[\vec{j}]$ representation of SU(N). Note that the SU(N) order operator $\mathcal{W}^{[\vec{j}]}(\mathcal{C})$ is characterized by a set of $(\mathrm{N}-1)$ integers on loop $\mathcal{C}$ and $[\vec{j}] \equiv (j_1, j_2, \ldots, j_{\mathrm{N}-1})$. The representation index $[\vec{j}]$ denotes the $(\mathrm{N}-1)$ eigenvalues $(j_1, j_2, \ldots, j_{\mathrm{N}-1})$ of the $(\mathrm{N}-1)$ SU(N) Casimir operators. These Casimir operators (constructed purely out of the electric field operators) acting on the SU(N) electric basis measure the net electric fluxes on the loop states created by the loop operator $\mathrm{Tr}\,(\mathcal{W}^{[\vec{j}]}(\mathcal{C}))$. In this work we also obtain the SU(N) order-disorder operators algebra:

$$\Sigma_{[\vec{\theta}]}(p)(\mathcal{W}^{[\vec{j}]}(\mathcal{C}))_{\alpha\beta}\Sigma^{-1}_{[\vec{\theta}]}(p) = \begin{cases} (D^{[\vec{j}]}(\vec{\theta})\mathcal{W}^{[\vec{j}]}(\mathcal{C}))_{\alpha\beta}, & \text{if } p \text{ inside } \mathcal{C} \\ (\mathcal{W}^{[\vec{j}]}(\mathcal{C}))_{\alpha\beta}, & \text{otherwise.} \end{cases} \tag{3}$$

In (3), $D^{[\vec{j}]}(\vec{\theta})$ denotes the SU(N)-Wigner rotation matrix in the $[\vec{j}]$ representation. If the angles $[\vec{\theta}]$ correspond to the centre element $z \in Z_{\mathrm{N}}$ with $z^{\mathrm{N}} = 1$, then using [17] $D^{[\vec{j}]}(z) = (z)^{\eta[\vec{j}]}$, where $\eta[\vec{j}](= 0, 1, 2\cdots, (\mathrm{N}-1))$ is the N-ality of the representation $[\vec{j}]$, we recover the standard 't Hooft-Wilson order-disorder algebra discussed in [8].

The plan of the paper is as follows. In Secs. II and III we summarize the Hamiltonian framework and the SU(N) duality transformations, respectively. These sections are only for setting up the notations and to explain the SU(N) duality relations which are then used directly in the following sections. The details can be found in [15,16]. The SU(N) magnetic vortex creation and annihilation or equivalently SU(N) disorder operators are discussed in Sec. IV. In order to simplify the presentation, the SU(2), SU(3), and SU(N) disorder operators are discussed one by one in the increasing order of difficulty in Secs. IV A–IV C, respectively. In the simplest SU(2) case, we construct the magnetic basis in Sec. IV A 1 using the SU(2) prepotential approach [18]. In Sec. IV A 2 we show that the SU(2) disorder operators act as SU(2) magnetic vortex creation-annihilation operators on the magnetic basis. The SU(2) order-disorder algebra is discussed in Sec. IV A 3. Some of the results in this section can also be found in [15]. We then consider the SU(3) case in detail in Sec. IV B. As expected, there are many new SU(3) features which are absent in the simple SU(2) case. In particular, we emphasize the importance of the SU(3) prepotential operators representation of the dual electric scalar potentials for constructing the SU(3) magnetic flux basis [see (55)]. In Sec. IV C we directly generalize these SU(3) results to the SU(N) case. In Sec. V, we rewrite the SU(N) disorder operator in the original Kogut-Susskind formulation. We show that they now become non-local operators and are attached with the invisible SU(N) Dirac strings. As expected, these unphysical strings can be moved around by SU(N) gauge transformations without changing their end points which specify the locations of the SU(N) gauge invariant magnetic vortices and antivortices. In Sec. VI we compute the path integral expression for the SU(N) vortex-free energy. This path integral representation should be useful for Monte Carlo simulations and to understand the role of these magnetic vortices and their condensation, if any, in the color-confinement problem. It is expected that they will condense and disorder the vacuum state for any nonzero coupling constant.

The prepotential operators create and annihilate the SU(N) electric as well as the magnetic fluxes [18]. Therefore, they provide a common platform to construct both the electric and magnetic bases in the physical loop Hilbert space of SU(N) lattice gauge theory. In these two dual bases we show that the order and disorder operators have natural action of translating the electric and magnetic fluxes respectively. These SU(N) electric and magnetic bases and the action of the order and the disorder operators on them are discussed in detail in Appendixes A and B, respectively. Appendix C shows that the SU(N) Dirac strings are unphysical.

As mentioned earlier, we work in the $(2+1)$ dimension. The notations used are as follows. The lattice sites are denoted by $(\vec{n}) \equiv (m, n)$ and the links by $l = (\vec{n}, \hat{i})$ where $\hat{i} = 1, 2$ denotes unit vectors in the two spatial directions. All the initial operators are vectors and assigned to the links $l$. All the dual operators are scalars and are defined on the plaquettes $(p)$ of the spatial two-dimensional lattice. Many times we will suppress the plaquette indices $(p)$ on the dual operators to avoid clutter.

## II. HAMILTONIAN FORMULATION

In this section, we briefly discuss SU(N) Kogut-Susskind Hamiltonian lattice gauge theory in $(2+1)$ dimensions. The Hamiltonian of SU(N) lattice gauge theory is [6,19]

$$H = \sum_{\vec{n},\hat{i}} E^2(\vec{n};\hat{i}) + K \sum_p \mathrm{Tr}(U_p + U_p^\dagger). \tag{4}$$

In Eq. (4), $E^2(\vec{n};\hat{i}) \equiv \sum_{\mathrm{a}=1}^{\mathrm{N}^2-1} (E_\pm^{\mathrm{a}}(\vec{n};\hat{i}))^2$, $U_p \equiv U(\vec{n};\hat{i}) \times U(\vec{n}+\hat{i};\vec{j})U^\dagger(\vec{n}+\hat{j};\hat{i})U^\dagger(\vec{n};\hat{j})$, and $K$ is a coupling constant. This is an electric field and magnetic-vector potential description in which each link $(\vec{n};\hat{i})$ carries an SU(N) link-flux operator $U(\vec{n};\hat{i})$. We call $U(\vec{n};\hat{i})$ the link holonomy. Their left and right link electric fields $E_\pm^{\mathrm{a}}(\vec{n};\hat{i})$ rotate the link holonomies $U(\vec{n};\hat{i})$ from the left and right, respectively or equivalently satisfy the following commutation relations:

$$\begin{aligned} [E_+^{\mathrm{a}}(\vec{n};\hat{i}), U(\vec{n};\hat{i})_{\alpha\beta}] &= -(T^{\mathrm{a}} U(\vec{n};\hat{i}))_{\alpha\beta}, \\ [E_-^{\mathrm{a}}(\vec{n}+\hat{i};\hat{i}), U(\vec{n};\hat{i})_{\alpha\beta}] &= (U(\vec{n};\hat{i})T^{\mathrm{a}})_{\alpha\beta}, \end{aligned} \tag{5}$$

where $T^{\mathrm{a}}, \mathrm{a} = 1, 2, \ldots, \mathrm{N}^2 - 1$ are the generators of fundamental representation of SU(N). These left and right electric fields are not independent and are related by the link holonomy parallel transport

$$E_-(l) = -U^\dagger(l) E_+(l) U(l), \tag{6}$$

In (6) $E_\pm(l) \equiv \sum_{\mathrm{a}=1}^{\mathrm{N}^2-1} E_\pm^{\mathrm{a}}(l) T^{\mathrm{a}}$. The commutation relations (5) and Jacobi identity imply the electric fields $E_\pm^{\mathrm{a}}(\vec{n};\hat{i})$ follow the SU(N) Lie algebra

$$\begin{aligned} [E_+^{\mathrm{a}}(\vec{n};\hat{i}), E_-^{\mathrm{b}}(\vec{n}+\hat{i};\hat{i})] &= 0, \\ [E_\pm^{\mathrm{a}}(\vec{n};\hat{i}), E_\pm^{\mathrm{b}}(\vec{n};\hat{i})] &= i f^{\mathrm{abc}} E_\pm^{\mathrm{c}}(\vec{n};\hat{i}). \end{aligned} \tag{7}$$

Also, Eq. (6) implies that their magnitudes are equal,

$$\vec{E}_+^2(n,\hat{i}) = \vec{E}_-^2(n+\hat{i};\hat{i}) \equiv \vec{E}^2(n,\hat{i}). \tag{8}$$

It is convenient to represent the independent conjugate operators on a link $l$ by $(E_+(l), U_{\alpha\beta}(l))$ or $(E_-(l), U_{\alpha\beta}(l))$ as shown in Fig. 1(a). They are the initial (before duality) electric variables representing the SU(N) electric fields $E(l)$ and their canonical conjugate magnetic vector potentials $U(l)$ on the link $l$. The SU(N) gauge transformations are

$$\begin{aligned} U(\vec{n};\hat{i}) &\to \Lambda(\vec{n}) U(\vec{n};\hat{i}) \Lambda^\dagger(\vec{n}+\hat{i}), \\ E_\pm(\vec{n};\hat{i}) &\to \Lambda(\vec{n}) E_\pm(\vec{n};\hat{i}) \Lambda^\dagger(\vec{n}). \end{aligned} \tag{9}$$

The generators of gauge transformation at site $\vec{n}$ are Gauss operators defined by

$$\mathcal{G}^{\mathrm{a}}(\vec{n}) = \sum_{i=1}^{2} (E_+^{\mathrm{a}}(\vec{n};\hat{i}) + E_-^{\mathrm{a}}(\vec{n};\hat{i})). \tag{10}$$

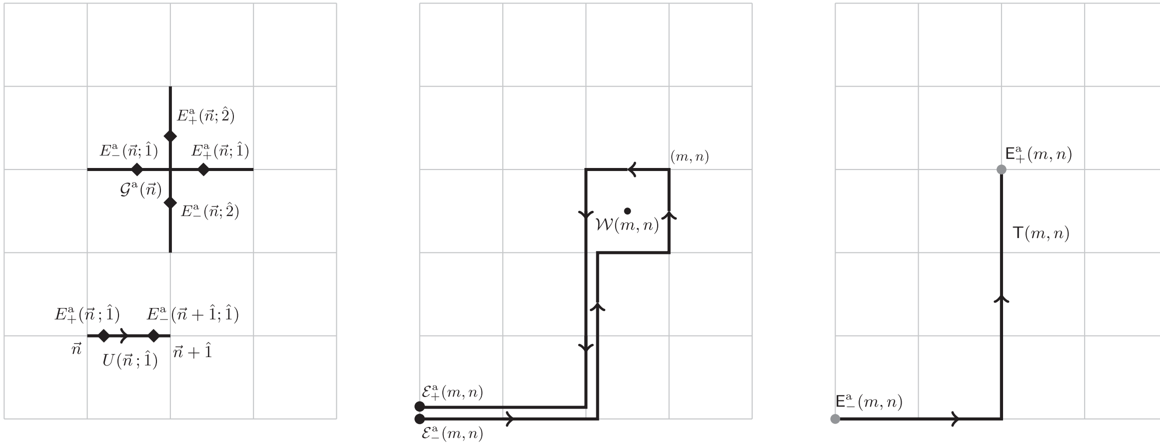
In our earlier work [15], using canonical transformations in $(2+1)$ dimensions, we solved the Gauss-law constraints (10),

$$\mathcal{G}^{\mathrm{a}}(\vec{n}) = 0, \quad \forall\, \vec{n} \neq (0,0), \tag{11}$$

to write down the SU(N) Kogut-Susskind Hamiltonian as a dual SU(N) spin model. We summarize the essential results required for the present work in the next section.

## III. DUALITY AND LOOPS

In our previous work [15], we obtained exact duality transformations through a series of canonical transformations over the entire lattice in $(2+1)$ dimensions. The dual model is written in terms of the mutually independent plaquette loops [see Fig. 1(b)] or scalar magnetic flux operators $\mathcal{W}(p)$ and their conjugate electric scalar potential $\vec{\mathcal{E}}(p)$ operators satisfying (15). The advantage of iterative canonical transformation is that the canonical commutation relations are preserved at every stage [15] leading to the exact canonical magnetic description at the end. Note that the dual operators are defined on the plaquettes or dual lattice sites while the initial Kogut-Susskind operators, discussed in the previous section, are defined on the lattice links. Such dual magnetic description has been useful in the past to study compact U(1) and SU(N) lattice gauge




(a) (b) (c)

FIG. 1. (a) Kogut-Sussking link formulation. Link operator $U(\vec{n};\hat{i})$ and its left (right) $E^{\mathrm{a}}_{+}(\vec{n};\hat{i})$ ($E^{\mathrm{a}}_{-}(\vec{n}+\hat{i};\hat{i})$) electric field. Gauss law operator at site $\vec{n}$, $\mathcal{G}^{\mathrm{a}}(\vec{n}) = \sum_{i=1}^{2}[E^{\mathrm{a}}_{+}(\vec{n};\hat{i}) + E^{\mathrm{a}}_{-}(\vec{n};\hat{i})]$ is also shown, (b) Dual physical plaquette holonomy $\mathcal{W}(\vec{n})$ and its left (right) $\mathcal{E}_{+}(\vec{n})$ ($\mathcal{E}_{-}(\vec{n})$) electric field, (c) Unphysical string holonomy $\mathsf{T}(m,n)$ and its left $\mathsf{E}_{+}(\vec{n})(=\mathcal{G}(\vec{n}))$ and right $\mathsf{E}_{-}(\vec{n})$ electric field. respectively. The convention chosen for the loop (string) electric fields is that $\mathcal{E}^{a}_{-}(\vec{n})(\mathsf{E}^{a}_{-}(\vec{n}))$ and $\mathcal{E}^{a}_{+}(\vec{n})(\mathsf{E}^{a}_{+}(\vec{n}))$ are located at the initial, end points of the flux loop (string). These string holonomies decouple from the physical Hilbert space.

theories in $(2+1)$ and $(3+1)$ dimension [15,16,20]. The dual SU(N) physical and unphysical operators [15] are summarized in the following two subsections, respectively.

### A. Magnetic flux operators $(\mathcal{W}_{\alpha\beta}(p), \mathcal{E}^{\mathrm{a}}_{+}(p))$

They are the physical magnetic operators which solve the SU(N) Gauss law constraints and define the physical Hilbert space $\mathcal{H}^{\mathrm{phys}}$. They represent the scalar SU(N) magnetic fluxes $(\mathcal{W}(p))$ on plaquette $p$ and their conjugate electric scalar potentials $\mathcal{E}_{\pm}(p)$. The SU(N) duality relations are

$$\mathcal{W}(m,n) = \mathsf{T}(m-1,n-1)U_p(m,n)\mathsf{T}^{\dagger}(m-1,n-1),$$
$$\mathcal{E}_{+}(m,n) = \sum_{n'=n}^{\infty} S^{\dagger}(m,n;n')E_{-}(m,n';\hat{1})S(m,n;n'). \quad (12)$$

The parallel transport operators $\mathsf{T}(m,n)$ and $S(m,n;n')$ in (12) are required as the dual-plaquette conjugate operators $(\mathcal{W}(m,n),\mathcal{E}(m,n))$ see only the gauge transformations $\Lambda(0,0)$ at the origin [see (18)] while the original Kogut-Susskind conjugate pairs $(U(m,n,\hat{i}), E^{a}(m,n,\hat{i}))$ have the standard gauge transformations by $\Lambda(m,n)$ given in (9). These parallel transports are given by [see Figs. 1(c) and 2]

$$\mathsf{T}(m,n) = \prod_{m'=0}^{m} U(m',0;\hat{1}) \prod_{n'=0}^{n} U(m,n';\hat{2}), \quad (13a)$$

$$S(m,n;n') \equiv \mathsf{T}(m-1,n)U(m-1,n;\hat{1}) \times \prod_{h=n}^{n'} U(m,h;\hat{2}). \quad (13b)$$

Like in the Kogut-Susskind approach, the right electric potentials are defined by

$$\mathcal{E}_{-}(p) = -\mathcal{W}^{\dagger}(p)\mathcal{E}_{+}(p)\mathcal{W}(p), \quad (14)$$

Note that $\mathcal{E}^{\mathrm{a}}_{-}(p)$ are attached to the initial end of plaquette flux line $\mathcal{W}(p)$ as shown in Fig. 1(b). The dual-operator commutation relations are [15]

$$[\mathcal{E}^{\mathrm{a}}_{+}(p), \mathcal{W}_{\alpha\beta}(p)] = -(T^{\mathrm{a}}\mathcal{W}(p))_{\alpha\beta},$$
$$[\mathcal{E}^{\mathrm{a}}_{-}(p), \mathcal{W}_{\alpha\beta}(p)] = (\mathcal{W}(p)T^{\mathrm{a}})_{\alpha\beta}. \quad (15)$$

The above commutation relations imply that $\mathcal{E}^{\mathrm{a}}_{+}(p)$ $(\mathcal{E}^{\mathrm{a}}_{-}(p))$ rotate $\mathcal{W}_{\alpha\beta}(p)$ from left (right) and therefore are the left (right) electric scalar potentials. They are mutually independent and satisfy SU(N) algebra,

$$[\mathcal{E}^{\mathrm{a}}_{+}(p), \mathcal{E}^{\mathrm{b}}_{-}(p)] = 0,$$
$$[\mathcal{E}^{\mathrm{a}}_{\pm}(p), \mathcal{E}^{\mathrm{b}}_{\pm}(p)] = if^{\mathrm{abc}}\mathcal{E}^{\mathrm{c}}_{\pm}(p). \quad (16)$$

Also, the relation (14) implies that their magnitudes are equal,

$$\vec{\mathcal{E}}_{+}{}^{2}(p) = \vec{\mathcal{E}}_{-}{}^{2}(p) \equiv \vec{\mathcal{E}}^{2}(p). \quad (17)$$

In the first two equations above we have defined $\vec{\mathcal{E}}_{\pm}{}^{2}(p) \equiv \sum_{\mathrm{a}=1}^{\mathrm{N}^2-1} \mathcal{E}^{\mathrm{a}}_{\pm}(p)\mathcal{E}^{\mathrm{a}}_{\pm}(p)$. The relations (14)–(17) in this (dual) magnetic formulation are exactly analogous to the initial relations (5)–(8), respectively in the original Kogut-Susskind electric formulation. The dual spin or magnetic flux operators transform as SU(N) adjoint matter field at the origin

$$\mathcal{W}(m,n) \to \Lambda(0,0)\mathcal{W}(m,n)\Lambda^\dagger(0,0),$$
$$\mathcal{E}_\pm(m,n) \to \Lambda(0,0)\mathcal{E}_\pm(m,n)\Lambda^\dagger(0,0). \tag{18}$$

The canonical transformations (12) can also be easily inverted to give the Kogut-Susskind electic fields in terms of the dual-electric scalar potentials [15]. These inverse relations will not be discussed as they are not relevant for the present work.

### B. String operators $(\mathsf{E}^{\rm a}_-(\vec{n}), \mathsf{T}(\vec{n}))$

String operators are unphysical operators and represent SU(N) gauge degrees of freedom at every lattice site away from the origin. They are shown in Fig. 1(c),

$$\mathsf{T}(m,n) = \prod_{m'=0}^{m} U(m',0;\hat{1}) \prod_{n'=0}^{n} U(m,n';\hat{2}),$$
$$\mathsf{E}^{\rm a}_+(m,n) = \mathcal{G}^{\rm a}(m,n) \simeq 0. \tag{19}$$

Thus, all string operators $\mathsf{T}(m,n)$ become cyclic as their conjugate electric fields $\mathsf{E}^a_+(m,n)$ turns out to be the Gauss-law operator $\mathcal{G}^a(m,n)$ [15]. Therefore they vanish on the physical Hilbert space $\mathcal{H}^p$ where the SU(N) Gauss laws are satisfied. The string operators, being unphysical, will not be relevant in this work and will not be considered henceforth.

## IV. DISORDER OPERATORS

As mentioned earlier, the order and disorder operators in SU(N) lattice theory are simply the shift or the creation-annihilation operators for the gauge invariant electric and magnetic fluxes respectively. Note that the Wilson loop operators $\mathcal{W}^{[\vec{j}]}(\mathcal{C})$, constructed in terms of the magnetic vector potentials $U(l)$ in (2), shift their conjugate electric fluxes along the loop $\mathcal{C}$. In this section, we construct the gauge-invariant disorder operators which are dual to the Wilson loop operators $\mathcal{W}^{[\vec{j}]}(\mathcal{C})$ and shift the magnetic fluxes instead. For the sake of simplicity, we first consider SU(2) case and then generalize it to SU(3) and finally to the SU(N) gauge group. All the algebraic details for the SU(N) electric and magnetic basis are given in Appendixes A and B, respectively.

### A. SU(2) disorder operator

The SU(2) magnetic plaquette flux operator is

$$\mathcal{W}^{[j=\frac{1}{2}]}(p) \equiv \exp\frac{i}{2}(\hat{n}(p)\cdot\vec{\sigma}\omega(p)). \tag{20}$$

In (20) $\hat{n}(p) = (\hat{n}^1(p), \hat{n}^2(p), \hat{n}^3(p))$ is the unit vector on every plaquette $p$ and $\vec{\sigma}(\equiv\sigma_1,\sigma_2,\sigma_3)$ are the 3 Pauli matrices. In the angle-axis representation:

$$\mathcal{W}^{[j=\frac{1}{2}]}(p) \equiv \cos\left(\frac{\omega(p)}{2}\right)\sigma_0 + i(\hat{n}(p)\cdot\vec{\sigma})\sin\left(\frac{\omega(p)}{2}\right);$$
$$\hat{n}(p)\cdot\hat{n}(p) = 1, \quad \forall\,(p). \tag{21}$$

In (21), $\sigma_0$ is $2\times 2$ unit matrix. Note that the relations (21) implies $\hat{n}^a(p) = \frac{1}{2i\sin(\frac{\omega(p)}{2})}\mathrm{Tr}(\sigma^a\mathcal{W}(p))$ and $\cos(\frac{\omega(p)}{2}) = \frac{1}{2}Tr\mathcal{W}$. Under global gauge transformation $\Lambda \equiv \Lambda(0,0)$ in (18), $(\omega,\hat{n})$ transform as

$$\omega(p) \to \omega(p),$$
$$n(p) \equiv \sum_{\rm a=1}^{3}\hat{n}^{\rm a}(p)\sigma^{\rm a} \to \Lambda(0,0)n(p)\Lambda^\dagger(0,0). \tag{22}$$

Thus the rotation angle $\omega(p)$ is invariant and the axis $\hat{n}^a(p)$ transforms as a vector.

We now define two unitary operators:

$$\Sigma^+_\theta(p) \equiv \exp i(\hat{n}(p)\cdot\mathcal{E}_+(p)\theta),$$
$$\Sigma^-_\theta(p) \equiv \exp i(\hat{n}(p)\cdot\mathcal{E}_-(p)\theta), \tag{23}$$

which are located on a plaquette $p$. They both are gauge invariant because $\mathcal{E}^{\rm a}_\pm(p)$ and $\hat{n}(p)$ gauge transform like vectors as shown in (18) and (22). In other words, $[\mathcal{G}^{\rm a},\Sigma^\pm_\theta(p)] = 0$, where $\mathcal{G}^{\rm a}$ is defined in (10). As the left and right electric scalar potentials are related through (14), $\Sigma^\pm_\theta(p)$ are not mutually independent and satisfy

$$\Sigma^+_\theta(p)\Sigma^-_\theta(p) = \Sigma^-_\theta(p)\Sigma^+_\theta(p) = \mathcal{I}. \tag{24}$$

In (24), $\mathcal{I}$ denotes the unit operator in the physical Hilbert space $\mathcal{H}^p$. The identities (24) can be easily obtained by using $\mathcal{E}_-(p) = -R^{\rm ab}(\hat{n},\omega)\mathcal{E}_+(p)$ and $R^{\rm ab}(\hat{n},\omega)\hat{n}^{\rm b} = \hat{n}^{\rm a}$, where $R^{\rm ab}(\hat{n},\omega) = \frac{1}{2}\mathrm{Tr}(\sigma^{\rm a}\mathcal{W}\sigma^{\rm b}\mathcal{W}^\dagger)$ (see [21]).

#### *1. SU(2) prepotential operators*

It is extremely convenient to use the prepotential [15,18] representation for the dual electric potential on the plaquette loops to construct the electric loop (Appendix A) as well the magnetic loop (Appendix B) basis. This simplification is illustrated in Fig. 3. A further advantage is that this simple procedure can be directly generalized to all SU(N). We write the SU(2) dual plaquette loop electric potentials on any plaquette $p$ satisfying (16) as [22]

$$\mathcal{E}^{\rm a}_+(p) \equiv a^\dagger(p)\frac{\sigma^{\rm a}}{2}a(p),$$
$$\mathcal{E}^{\rm a}_-(p) \equiv -b(p)\frac{\sigma^{\rm a}}{2}b^\dagger(p). \tag{25}$$

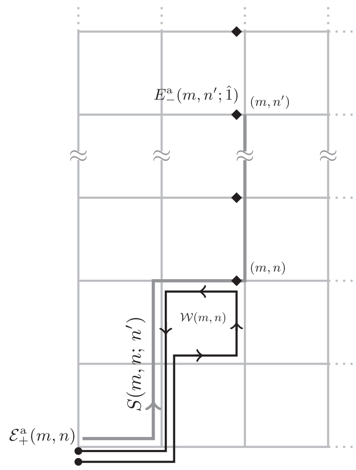



FIG. 2. Graphical representation of canonical relation (12). We have used diamond symbol to represent Kogut-Susskind electric fields $E^{\rm a}_{-}(m,n';\hat{1})$ and black dot to represent new plaquette electric fields $\mathcal{E}^{\rm a}_{+}(m,n)$. The thick gray line represents parallel transport $S(m,n;n')$ defined in Eq. (13b).

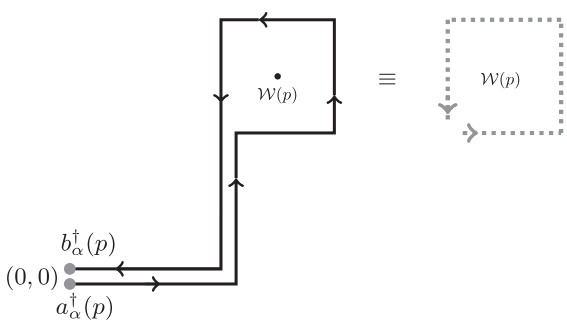



FIG. 3. SU(2) prepotential operators in the dual formulation. The two ends of the plaquette flux operator $\mathcal{W}(p)$ are associated with two doublets of the harmonic oscillators at the origin (0,0) [15,18]. Under gauge transformations at the origin, $(a^{\dagger}_{\alpha}(p), b^{\dagger}_{\beta}(p))$ transform as SU(2) doublets. The dotted plaqette on the right-hand side is a compact way to represent the plaquette holonomy $\mathcal{W}(p)$.

In (25), $a^{\dagger}_{\alpha}(p)$ and $b^{\dagger}_{\alpha}(p)$ are the two mutually commuting SU(2) doublets of harmonic-oscillator creation operators on every plaquette loop. The standard commutation relations are

$$[a_{\alpha}(p), a^{\dagger}_{\beta}(p')] = \delta_{pp'}\delta_{\alpha\beta},$$
$$[b_{\alpha}(p), b^{\dagger}_{\beta}(p')] = \delta_{pp'}\delta_{\alpha\beta}. \quad (26)$$

Using (26), it is easy to check that the representation (25) satisfies (16). The constraints (17) imply that

$$N(p) \equiv a^{\dagger}(p)\cdot a(p) = b^{\dagger}(p)\cdot b(p). \quad (27)$$

The plaquette holonomy in this representation is [18]

$$\mathcal{W}_{\alpha\beta}(p) = F(N)[b^{\dagger}_{\alpha}(p)a^{\dagger}_{\beta}(p) + \tilde{b}_{\alpha}\tilde{a}_{\beta}]F(N). \quad (28)$$

In (28), $F(N) \equiv \frac{1}{\sqrt{(N(p)+1)}}$ is the normalization factor and $\tilde{x}_{\alpha} \equiv \epsilon_{\alpha\beta}x_{\beta}$. The harmonic oscillator representation (25) implies that $a^{\dagger}_{\alpha}$ and $b^{\dagger}_{\alpha}$ transform like doublets from the right and antidoublets from the left, respectively on every plaquette (p),

$$[\mathcal{E}^{\rm a}_{+}(p), a^{\dagger}_{\alpha}(p)] = \left(a^{\dagger}(p)\frac{\sigma^{\rm a}}{2}\right)_{\alpha},$$
$$[\mathcal{E}^{\rm a}_{-}(p), b^{\dagger}_{\alpha}(p)] = -\left(\frac{\sigma^{\rm a}}{2}b^{\dagger}(p)\right)_{\alpha}. \quad (29)$$

The strong coupling vacuum on every plaquette in the dual formulation $|0\rangle_p$ satisfies:

$$\mathcal{E}^{\rm a}_{\pm}(p)|0\rangle_p = 0, \quad \forall\, p. \quad (30)$$

This is equivalent to demanding

$$a_{\alpha}(p)|0\rangle_p = 0, \qquad b_{\alpha}(p)|0\rangle_p = 0. \quad (31)$$

The relations (29) and (31) are useful to study the action of SU(2) disorder operators on the magnetic basis discussed below. Note that under SU(2) gauge transformations (18) with $\Lambda(0,0)$ at the origin [see Fig. 1(b)] these oscillators transform doublets:

$$a^{\dagger}_{\alpha}(p) \to a^{\dagger}_{\beta}(p)\Lambda_{\beta\alpha}(0,0), \quad \forall\, p,$$
$$b^{\dagger}_{\alpha}(p) \to \Lambda^{\dagger}_{\alpha\beta}(0,0)b^{\dagger}_{\beta}(p) \quad \forall\, p. \quad (32)$$

These relations are useful to construct the gauge-invariant operators in the magnetic basis constructed in the next section.

### *2. SU(2) magnetic basis*

The physical meaning of the operators $\Sigma^{\pm}_{\theta}(p)$ is simple. The non-Abelian electric scalar potentials $\mathcal{E}^{\rm a}_{\pm}(p)$ are conjugate to the magnetic flux operators $\mathcal{W}^{[j=\frac{1}{2}]}_{\alpha\beta}(p)$. They satisfy the canonical commutation relations (15). Therefore, the gauge-invariant vortex operator $\Sigma^{\pm}_{\theta}(p)$ acting on the magnetic basis of a plaquette changes the magnetic flux on it continuously as a function of $\theta$ in (43). To see this explicitly, we first construct the SU(2) magnetic basis. We note that

$$[\mathcal{W}_{\alpha\beta}(p), \mathcal{W}_{\gamma\delta}(p')] = 0, \quad \forall\, p, p'.$$

Therefore, we can diagonalize all four operators $(\mathcal{W}_{11}(p), \mathcal{W}_{12}(p), \mathcal{W}_{21}(p), \mathcal{W}_{22}(p))$ simultaneously on every plaquette. The common eigenstates $|Z(p)\rangle \equiv |z_1(p), z_2(p)\rangle$ satisfy

$$\mathcal{W}_{\alpha\beta}(p)|Z(p)\rangle = Z_{\alpha\beta}(p)|Z(p)\rangle, \qquad \alpha,\beta = 1,2. \quad (33)$$

In (33) the SU(2) matrix on the plaquette $p$ is

$$Z = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}, \quad |z_1|^2 + |z_2|^2 = 1. \tag{34}$$

The SU(2) $Z$ matrices can also be written in the SU(2) angle-axis representation

$$Z = e^{i\omega \hat{n}^a \frac{\sigma^a}{2}}. \tag{35}$$

The two SU(2) representations (34) and (35) are related by

$$z_1 = \cos\left(\frac{\omega}{2}\right) + i\hat{n}^3 \sin\left(\frac{\omega}{2}\right), \qquad z_2 = (\hat{n}^2 + i\hat{n}^1)\sin\left(\frac{\omega}{2}\right).$$

We now construct $|Z(p)\rangle$ and show that on this basis the vortex operator $\Sigma_\theta^\pm(p)$ act as the shift operators for the plaquette magnetic fluxes. The magnetic eigenstates $|Z(p)\rangle$ can be explicitly constructed in terms of SU(2) prepotential operators [18] (see Appendix B),

$$|Z(p)\rangle = \sum_{j(p)=0}^{\infty} \sqrt{d(j(p))} \frac{(a^\dagger(p)Z(p)b^\dagger(p))^{2j(p)}}{(2j(p))!} |0\rangle_p. \tag{36}$$

In (36) $d(j) \equiv (2j+1)$ is the dimension of the $j$ representation and $(a^\dagger Z b^\dagger) \equiv \sum_{\alpha,\beta=1}^{2} (a^\dagger_\alpha Z_{\alpha\beta} b^\dagger_\beta)$. From now onwards we will ignore the plaquette index $p$ on all the operators and the states as they are all defined on the lattice plaquettes. The magnetic eigenstates (36) have simple SU(2) gauge-transformation properties

$$|Z\rangle \to |\Lambda Z \Lambda^\dagger\rangle, \qquad \Lambda \equiv \Lambda(0,0). \tag{37}$$

The transformations (37) are clear from (32) and (36). In the angle-axis representation (35) the gauge transformations (37) take the simpler form

$$\omega(p) \to \omega(p), \quad \hat{n}(p) \to \Lambda \hat{n}(p) \Lambda^\dagger, \quad \Lambda \equiv \Lambda(0,0). \tag{38}$$

Thus, $\omega(p)$, $\forall\, p$ are gauge-invariant angles and $\hat{n}(p)\ \forall\, p$ transform globally like SU(2) adjoint vectors. The eigenvalues of the plaquette magnetic-field operators in the Hamiltonian (4) are

$$\mathrm{Tr}(\mathcal{W}^{[j=\frac{1}{2}]})|Z(\omega,\hat{n})\rangle = 2\cos\left(\frac{\omega}{2}\right)|Z(\omega,\hat{n})\rangle. \tag{39}$$

Now we evaluate the action of disorder operator using the prepotential relations,

$$\Sigma_\theta^+ a^\dagger_\alpha \Sigma_\theta^- = (a^\dagger e^{\frac{i}{2}\theta\hat{n}^a\sigma^a})_\alpha,$$
$$\Sigma_\theta^- b^\dagger_\alpha \Sigma_\theta^+ = (e^{-\frac{i}{2}\theta\hat{n}^a\sigma^a} b^\dagger)_\alpha. \tag{40}$$

The relations (40) can be easily established using (23) and the prepotential representation of $\mathcal{E}_\pm(p)$ in (25),

$$\Sigma_\theta^+|Z(\omega,\hat{n})\rangle = |e^{\frac{i}{2}\theta\hat{n}^a\sigma^a}Z(\omega,\hat{n})\rangle = |Z(\omega+\theta,\hat{n})\rangle,$$
$$\Sigma_\theta^-|Z(\omega,\hat{n})\rangle = |Z(\omega,\hat{n})e^{-\frac{i}{2}\theta\hat{n}^a\sigma^a}\rangle = |Z(\omega-\theta,\hat{n})\rangle. \tag{41}$$

Thus, the SU(2) plaquette disorder operator $\Sigma_\theta^\pm$ translates the plaquette magnetic fluxes. This is precisely dual to the action of the Wilson loop operators which translate the SU(2)-loop electric fluxes as shown in Appendix A [see Eq. (A10) and Fig. 6].

### 3. SU(2) order-disorder algebra

The dual-canonical commutation relations (15) involving magnetic-plaquette flux operators $\mathcal{W}(p)$ and their conjugate-electric scalar potential $\mathcal{E}(p)$ immediately lead to the SU(2) order-disorder algebra:

$$\Sigma_\theta^+(p)\mathcal{W}^{[j=\frac{1}{2}]}_{\alpha\beta}(p)\Sigma_\theta^-(p) = D^{[j=\frac{1}{2}]}_{\alpha\gamma}(\hat{n},\theta)\mathcal{W}^{[j=\frac{1}{2}]}_{\gamma\beta}(p),$$
$$\Sigma_\theta^-(p)\mathcal{W}^{[j=\frac{1}{2}]}_{\alpha\beta}(p)\Sigma_\theta^+(p) = \mathcal{W}^{[j=\frac{1}{2}]}_{\alpha\gamma}(p)D^{[j=\frac{1}{2}]}_{\gamma\beta}(\hat{n},\theta). \tag{42}$$

In (42) the Wigner matrix $D^{[j=\frac{1}{2}]} \equiv e^{i\hat{n}^a\cdot\sigma^a\frac{\theta}{2}}$ is the rotation matrix in $j=\frac{1}{2}$ representation around the magnetic axis $\hat{n}(p)$ defined through the plaquette loops $\mathcal{W}(p)$. In any higher-$[j]$ representation, we can write

$$\mathcal{W}^{[j]}_{\alpha\beta} = \mathcal{W}^{[j=1/2]}_{\{\alpha_1\beta_1}\mathcal{W}^{[j=1/2]}_{\alpha_2\beta_2}\cdots\mathcal{W}^{[j=1/2]}_{\alpha_{2j}\beta_{2j}\}},$$

where all the $\alpha$ (and therefore $\beta$) indices are completely symmetrized. Inserting the disorder operators $(\Sigma)$ and their inverses $(\Sigma^\dagger)$ in the middle, we get the SU(2) order-disorder algebra relation in $j$ representation,

$$\Sigma_\theta^+(p)\mathcal{W}^{[j]}_{\alpha\beta}(p)\Sigma_\theta^-(p) = D^{[j]}_{\alpha\gamma}(\hat{n},\theta)\mathcal{W}^{[j]}_{\gamma\beta}(p),$$
$$\Sigma_\theta^-(p)\mathcal{W}^{[j]}_{\alpha\beta}(p)\Sigma_\theta^+(p) = \mathcal{W}^{[j]}_{\alpha\gamma}(p)D^{[j]}_{\gamma\beta}(\hat{n},\theta). \tag{43}$$

In the special case when the rotations are restricted to the center $Z_2$ of the SU(2) group then $\theta = 0$ or $2\pi$ in (43) and we recover the 't Hooft Wilson order-disorder algebra with $D^{[j]}_{\alpha\beta}(\theta=2\pi) = (-1)^{2j}\delta_{\alpha\beta}$,

$$\Sigma^\pm_{\theta=2\pi}\mathcal{W}^{[j]}_{\alpha\beta} = (-1)^{2j}\mathcal{W}^{[j]}_{\alpha\beta}\Sigma^\pm_{\theta=2\pi}. \tag{44}$$

In (44), $(-1)^{2j}$ is the N-ality of the j representation. We thus recover the standard Wilson-'t Hooft loop $Z_2$ algebra [8–11] for SU(2) at $\theta = 2\pi$. The operator $\Sigma_{2\pi} \equiv \Sigma^+_{2\pi} = \Sigma^-_{2\pi}$ is the SU(2) 't Hooft operator.

### B. SU(3) disorder operator

In this section, we construct the disorder operator for SU(3) lattice gauge theory before going to SU(N) gauge group. As in the previous SU(2) case, they are the SU(3) magnetic vortex creation-annihilation operators and are expected to magnetically disorder the weak coupling ground state [11,12]. The SU(3) plaquette magnetic flux operators can be written as

$$\mathcal{W}^{[\mathrm{p=1q=1}]}(p) = \exp i(\hat{n}(p)\cdot\vec{\lambda}\omega(p)). \tag{45}$$

In (45) $\hat{n}(p) = (\hat{n}^1(p), \hat{n}^2(p), \ldots, \hat{n}^8(p))$ is the unit vector on every plaquette $p$ and $\lambda^{\mathrm{a}}(\mathrm{a}=1,\ldots,8)$ are the 8 Gell-Mann matrices. We can also use the angle-axis representation [23] to write:

$$\mathcal{W}^{[\mathrm{p=1q=1}]}(p) \equiv A\mathcal{I} + B\vec{n}\cdot\vec{\lambda} + C\vec{n}\star\vec{n}\cdot\vec{\lambda}. \tag{46}$$

In (46) $(\vec{n}\star\vec{n})^{\mathrm{a}} \equiv d^{\mathrm{abc}}\vec{n}^{\mathrm{b}}\vec{n}^{\mathrm{c}}$ defines the second independent vector with the help of the SU(3) symmetric tensors $d^{\mathrm{abc}}$. Instead of following the standard polar decomposition (46), it is more convenient for us to construct the two independent SU(3) axes operators as [24]

$$\vec{n}^{\mathrm{a}}_{[1]}(p) = \mathrm{Tr}\lambda^{\mathrm{a}}(\mathcal{W}^{[1,1]}(p) + \mathcal{W}^{\dagger[1,1]}(p)), \tag{47a}$$

$$\vec{n}^{\mathrm{a}}_{[2]}(p) = \sqrt{3}d^{\mathrm{abc}}\vec{n}^{\mathrm{b}}_{[1]}(p)\vec{n}^{\mathrm{c}}_{[1]}(p). \tag{47b}$$

Note that $\vec{n}^{\mathrm{a}}_{[1]}(p), \vec{n}^{\mathrm{a}}_{[2]}(p)$ are real. Under SU(3) gauge transformations (18) the above two operators transform as:

$$\begin{aligned}\vec{n}^{\mathrm{a}}_{[1]}(p) &\to R^{\mathrm{ab}}(\Lambda)\vec{n}^{\mathrm{b}}_{[1]}(p)\\ \vec{n}^{\mathrm{a}}_{[2]}(p) &\to R^{\mathrm{ab}}(\Lambda)\vec{n}^{\mathrm{b}}_{[2]}(p).\end{aligned} \tag{48}$$

In (48) $R^{\mathrm{ab}}(\Lambda) = \frac{1}{2}\mathrm{Tr}(\lambda^{\mathrm{a}}\Lambda\lambda^{\mathrm{b}}\Lambda^{\dagger})$ and $\Lambda \equiv \Lambda(0,0)$. These two axes are linearly independent. It can be shown that in SU(3) case there exist only two independent axes as the third axis defined using another $d^{\mathrm{abc}}$ is the first axis $\vec{n}_{[1]}$ [25]:

$$\begin{aligned}&f^{\mathrm{abc}}\vec{n}^{\mathrm{b}}_{[2]}(p)\vec{n}^{\mathrm{c}}_{[1]}(p) = 0,\\ &d^{\mathrm{abc}}\vec{n}^{\mathrm{b}}_{[2]}(p)\vec{n}^{\mathrm{c}}_{[1]}(p) = \frac{1}{\sqrt{3}}(\vec{n}^{\mathrm{b}}_{[1]}(p)\vec{n}^{\mathrm{b}}_{[1]}(p))\vec{n}^{\mathrm{a}}_{[1]}(p).\end{aligned}$$

Now we define the SU(3) disorder operators which translate these two gauge invariant magnetic fluxes:

$$\begin{aligned}\Sigma^{+}_{\theta_1,\theta_2}(p) &\equiv \exp i\left\{\left(\sum_{h=1}^{2}\theta_h(p)\hat{n}^{\mathrm{a}}_{[h]}\right)\mathcal{E}^{\mathrm{a}}_{+}(p)\right\},\\ \Sigma^{-}_{\theta_1,\theta_2}(p) &\equiv \exp i\left\{\left(\sum_{h=1}^{2}\theta_h(p)\hat{n}^{\mathrm{a}}_{[h]}\right)\mathcal{E}^{\mathrm{a}}_{-}(p)\right\}.\end{aligned} \tag{49}$$

In (49) $(\theta_1,\theta_2) \equiv (\theta_1(p),\theta_2(p))$ are the external angular parameters characterizing the SU(3) disorder operator. Like in the SU(2) case, the two operators in (49) are unitary and Hermitian conjugate of each other

$$\Sigma^{+}_{\theta_1,\theta_2}(p)\Sigma^{-}_{\theta_1,\theta_2}(p) = \mathcal{I} = \Sigma^{-}_{\theta_1,\theta_2}(p)\Sigma^{+}_{\theta_1,\theta_2}(p). \tag{50}$$

Like SU(2) case this can also be proved using the properties of the SU(3) $\lambda$ matrices.

#### *1. SU(3) prepotential operators*

The SU(3) prepotential operators on plaquettes are defines as

$$\begin{aligned}\mathcal{E}^{\mathrm{a}}_{+} &\equiv \sum_{h=1}^{2}a^{\dagger}[h]\frac{\lambda^{\mathrm{a}}}{2}a[h],\\ \mathcal{E}^{\mathrm{a}}_{-} &\equiv -\sum_{h=1}^{2}b[h]\frac{\lambda^{\mathrm{a}}}{2}b^{\dagger}[h].\end{aligned} \tag{51}$$

In (51), $(a^{\dagger}_{\alpha}[h], a_{\alpha}[h])$ and $(b^{\dagger}_{\alpha}[h], b_{\alpha}[h])$ where $\alpha = 1,2,3$; $h = 1,\ 2$ are the mutually independent SU(3) triplets of harmonic oscillator creation-annihilation operators on every plaquette [26]. They are attached to the initial and the end points of the plaquette loops [see Fig. 1(b)]. The summation over $[h] = 1,2$ is over the rank of the group. As all operators are defined on plaquettes, we suppress the plaquette index "$p$" throughout this section. The harmonic oscillator commutation relations and (51) imply that $a^{\dagger}_{\alpha}[h]$ and $b^{\dagger}_{\alpha}[h]$ transform like triplets from right and anti-triplets from left respectively on every plaquette ($p$): transformations:

$$\begin{aligned}[\mathcal{E}^{\mathrm{a}}_{+}, a^{\dagger}_{\alpha}[h]] &= \left(a^{\dagger}[h]\frac{\lambda^{\mathrm{a}}}{2}\right)_{\alpha}, \quad h=1,2,\\ [\mathcal{E}^{\mathrm{a}}_{-}, b^{\dagger}_{\alpha}[h]] &= -\left(\frac{\lambda^{\mathrm{a}}}{2}b^{\dagger}[h]\right)_{\alpha}, \quad h=1,2.\end{aligned} \tag{52}$$

Like in the SU(2) case (32), the SU(3) gauge transformations (18) with $\Lambda(0,0)$ at the orgin [see Fig. 1(b)] the SU(3) oscillators on every plaquette transform as SU(3) triplets,

$$\begin{aligned}a^{\dagger}_{\alpha}[h] &\to a^{\dagger}_{\beta}[h]\Lambda_{\beta\alpha}(0,0), \quad h=1,2,\\ b^{\dagger}_{\alpha}[h] &\to \Lambda^{\dagger}_{\alpha\beta}(0,0)b^{\dagger}_{\beta}[h], \quad h=1,2.\end{aligned} \tag{53}$$

These relations are again useful for the gauge covariant parametrization of the SU(3) magnetic basis in the angle-axis representation and is discussed in the next section. The SU(3) strong coupling vacuum in the dual description $|0\rangle$ satisfies

$$a_{\alpha}[h]|0\rangle_p = 0, \qquad b_{\alpha}[h]|0\rangle_p = 0, \quad h=1,2. \tag{54}$$

This strong coupling vacuum state $|0\rangle_p \equiv |0\rangle$ is used to construct the SU(3) magnetic basis in the next section.

### 2. SU(3) magnetic basis

We now show that $\Sigma^{\pm}_{\theta_1,\theta_2}$ operating on the SU(3) plaquette magnetic basis act like a translation operators for the two gauge-invariant magnetic fields. As shown in Appendix B, the SU(3) magnetic basis can be written in terms of SU(3) prepotentials [18] as

$$|Z\rangle = \sum_{\mathrm{p,q}=0}^{\infty} \sqrt{d(\mathrm{p,q})} \frac{(a^\dagger[1]Zb^\dagger[1])^{\mathrm{p}}}{\mathrm{p}!} \frac{(a^\dagger[2]Zb^\dagger[2])^{\mathrm{q}}}{\mathrm{q}!} |0\rangle. \quad (55)$$

In the above equation, the plaquette index has been suppressed and

$$d(\mathrm{p,q}) = \frac{1}{2}(\mathrm{p}+1)(\mathrm{q}+1)(\mathrm{p}+\mathrm{q}+2),$$

is the dimension of the $[\mathrm{p,q}]$ representation of SU(3) [27], $Z_{\alpha\beta}$ are the elements of SU(3) matrix and correspond to the eigenvalues of $\mathcal{W}^{[\mathrm{p}=1,\mathrm{q}=1]}_{\alpha\beta}(p)$, and we have ignored plaquette index $p$ in (55). In the axis-angle representation $Z$ can be written as [28]

$$Z(p) = Z(\omega_1, \omega_2) = \exp i(\omega_1 \hat{n}^{\mathrm{a}}_{[2]} + \omega_2 \hat{n}^{\mathrm{a}}_{[2]}) \frac{\lambda^{\mathrm{a}}}{2}. \quad (56)$$

In (56) we have labeled the SU(3) group manifold by $Z(\omega_1,\omega_2) \equiv Z(\hat{n}_{[1]}, \hat{n}_{[2]}; \omega_1, \omega_2)$. The two axes $(\hat{n}_{[1]}, \hat{n}_{[2]})$ are suppressed for the notational simplicity. Under SU(3) gauge transformations at the origin (18),

$$|Z\rangle \to |\Lambda Z \Lambda^\dagger\rangle, \qquad \Lambda \equiv \Lambda(0,0). \quad (57)$$

We have used (53) and the defining Eq. (56) to obtain the above covariant transformations. The gauge transformations (57) show that

$$\omega_h \to \omega_h, \qquad \hat{n}_{[h]} \to \Lambda \hat{n}_{[h]} \Lambda^\dagger h = 1,2. \quad (58)$$

Thus $(\omega_1, \omega_2)$ are the gauge-invariant angles and the two axes $\hat{n}_{[h]} \equiv \sum_{\mathrm{a}=1}^{8} \hat{n}^{\mathrm{a}}_{[h]} \lambda^{\mathrm{a}}$ transform like the adjoint vectors on every plaquette.

In order to evaluate the action of the disorder operator on this magnetic basis we first write down the following equations, which can be easily established using the commutation relations in (52),

$$\Sigma^{+}_{\theta_1,\theta_2} a^\dagger_\alpha[h] \Sigma^{+\dagger}_{\theta_1,\theta_2} = \left(a^\dagger[h] e^{i(\theta_1 \hat{n}^{\mathrm{a}}_{[2]} + \theta_2 \hat{n}^{\mathrm{a}}_{[2]})\frac{\lambda^{\mathrm{a}}}{2}}\right)_\alpha, \quad (59\mathrm{a})$$

$$\Sigma^{-}_{\theta_1,\theta_2} b^\dagger_\alpha[h] \Sigma^{-\dagger}_{\theta_1,\theta_2} = \left(e^{-i(\theta_1 \hat{n}^{\mathrm{a}}_{[1]} + \theta_2 \hat{n}^{\mathrm{a}}_{[2]})\frac{\lambda^{\mathrm{a}}}{2}} b^\dagger[h]\right)_\alpha. \quad (59\mathrm{b})$$

Using the above equations we can easily prove that

$$\begin{aligned}
\Sigma^{+}_{\theta_1,\theta_2} |Z(\omega_1,\omega_2)\rangle &= |e^{(i(\theta_1 \hat{n}^{\mathrm{a}}_{[1]} + \theta_2 \hat{n}^{\mathrm{a}}_{[2]})\frac{\lambda^{\mathrm{a}}}{2})} Z(\omega_1,\omega_2)\rangle \\
&= |Z(\omega_1+\theta_1, \omega_2+\theta_2)\rangle, \\
\Sigma^{-}_{\theta_1,\theta_2} |Z(\omega_1,\omega_2)\rangle &= |Z(\omega_1,\omega_2) e^{(-i(\theta_1 \hat{n}^{\mathrm{a}}_{[1]} + \theta_2 \hat{n}^{\mathrm{a}}_{[2]})\frac{\lambda^{\mathrm{a}}}{2})}\rangle \\
&= |Z(\omega_1-\theta_1, \omega_2-\theta_2)\rangle,
\end{aligned}$$

or

$$\Sigma^{\pm}_{\theta_1,\theta_2} |Z(\omega_1,\omega_2)\rangle = |Z(\omega_1 \pm \theta_1, \omega_2 \pm \theta_2)\rangle. \quad (60)$$

Therefore, the disorder operator in (49) translates the two gauge-invariant angles. We can thus interpret them as the creation-annihilation operators for the SU(3) magnetic vortices.

### 3. SU(3) order-disorder algebra

The SU(3) order-disorder algebra is

$$\begin{aligned}
\Sigma^{+}_{[\vec{\theta}]}(p) \mathcal{W}^{[\mathrm{p,q}]}_{\alpha\beta}(p) \Sigma^{+\dagger}_{[\vec{\theta}]}(p) &= D^{[\mathrm{p,q}]}_{\alpha\gamma}(\vec{\theta}) \mathcal{W}^{[\mathrm{p,q}]}_{\gamma\beta}(p), \\
\Sigma^{-}_{[\vec{\theta}]}(p) \mathcal{W}^{[\mathrm{p,q}]}_{\alpha\beta}(p) \Sigma^{-\dagger}_{[\vec{\theta}]}(p) &= \mathcal{W}^{[\mathrm{p,q}]}_{\alpha\gamma}(p) D^{[\mathrm{p,q}]}_{\gamma\beta}(\vec{\theta}). \quad (61)
\end{aligned}$$

In (61), $D^{[\mathrm{p,q}]}(\theta_1,\theta_2) \equiv \exp\left(i(\theta_1 \hat{n}^{\mathrm{a}}_{[1]} + \theta_2 \hat{n}^{\mathrm{a}}_{[2]})\frac{\lambda^{\mathrm{a}}}{2}\right)$ is SU(3) Wigner D-matrix in the $[\mathrm{p,q}]$ representation. Similar to the SU(2) case, we have used the dual-canonical commutation relations (15) to obtain the SU(3) order-disorder algebra in (61).

## C. SU(N) disorder operator

We now use the SU(N) dual electric scalar potentials $\mathcal{E}(p)$ in (12) to define the SU(N) disorder operator

$$\Sigma^{\pm}_{[\theta_1\theta_2\cdots\theta_{\mathrm{N}-1}]}(p) = \exp i\{\vec{\theta}(p)\cdot\vec{\mathcal{E}}_{\pm}(p)\}. \quad (62)$$

In (62) $\vec{\theta}(p) \equiv [\theta_1(p), \theta_2(p), \ldots, \theta_{\mathrm{N}-1}(p)]$ are the $(\mathrm{N}-1)$ external angular parameters characterizing the SU(N) disorder operator on the plaquette $(p)$ and

$$\vec{\theta}^a(p) \equiv \sum_{h=1}^{(\mathrm{N}-1)} \theta_h(p) \hat{n}^a_{[h]}(p). \quad (63)$$

The invariance (18) demands that the operator $\vec{\theta}(p)$ in (62) is the most general vector operator constructed out of the magnetic-flux operator $\mathcal{W}_{\alpha\beta}(p)$. In other words, they depend on the $(\mathrm{N}-1)$ directions of the SU(N) magnetic fields. In the SU(2) and SU(3) cases in the previous sections we have already constructed one and two independent axes, respectively, using the plaquette magnetic-flux operators. In the same way we now iteratively define the $(\mathrm{N}-1)$

linearly independent "SU(N) magnetic axes" using the SU(N) symmetric structure constants $d^{\rm abc}$ as follows:

$$\vec{n}^{\rm a}_{[h+1]}(p) \equiv d^{\rm abc}\vec{n}^{b}_{[h]}(p)\vec{n}^{c}_{[1]}(p) h = 1, 2, \ldots, {\rm N}-2. \quad (64)$$

The first magnetic axis is defined as $\vec{n}^{\rm a}_{[1]}(p) \equiv {\rm Tr}(\Lambda^{\rm a}(\mathcal{W}+\mathcal{W}^\dagger))$ where $\Lambda^{\rm a}({\rm a}=1,2,\ldots,({\rm N}^2-1))$ are the SU(N) fundamental representation matrices. The iterative procedure ends as [29] $\vec{n}^{\rm a}_{[{\rm N}]} \equiv d^{\rm abc}\vec{n}^{\rm b}_{[{\rm N}-1]}(p)\vec{n}^{\rm c}_{[1]}(p) = \vec{n}^{\rm a}_{[1]}(p)$. The $({\rm N}-1)$ SU(N) magnetic-field operators $\vec{n}^{\rm a}_{[h]}; h = 1, 2, \ldots, ({\rm N}-1)$ are Hermitian as the symmetric structure constants $d^{\rm abc}$ are always real. Under the gauge transformation (18), these axes transform as vectors

$$\vec{n}^{\rm a}_{[h]}(p) \to R^{\rm ab}(\Lambda)\vec{n}^{\rm b}_{[h]}(p), \quad h = 1, 2, \ldots, \quad ({\rm N}-1). \quad (65)$$

The disorder operator is invariant under the gauge transformations (18) as $\vec{\theta}(p)$ and the dual electric potentials $\vec{\mathcal{E}}(p)$ both transform as vectors. As in the case of SU(2) [see (24)] and SU(3) [see (50)], $\Sigma^{+}_{[\vec{\theta}]}(p)$ and $\Sigma^{-}_{[\vec{\theta}]}(p)$ are not independent and satisfy

$$\Sigma^{+}_{[\vec{\theta}]}(p)\Sigma^{-}_{[\vec{\theta}]}(p) = \mathcal{I} = \Sigma^{-}_{[\vec{\theta}]}(p)\Sigma^{+}_{[\vec{\theta}]}(p). \quad (66)$$

Here $\mathcal{I}$ is unity operator in the physical Hilbert space. The relations (66) follow from the parallel transport relating the two electric scalar potentials: $\mathcal{E}^{\rm a}_{-}(p) = -R^{\rm ab}(\mathcal{W}(p))\mathcal{E}^{\rm b}_{+}(p)$ and $\hat{n}^{\rm a}_{[h]}(p) = -R^{\rm ab}(\mathcal{W}(p))\hat{n}^{\rm b}_{[h]}(p); h = 1, 2 \cdots, ({\rm N}-1)$. We now briefly discuss the SU(N) prepotential operators to be used in the Sec. IV C 2 for the construction of the SU(N) magnetic basis.

### 1. SU(N) prepotential operators

The SU(N) dual-electric scalar potentials $\mathcal{E}^{a}(p)$ can be written in terms of the $({\rm N}-1)$ N-plets of harmonic oscillators at each of the two ends of the plaquette $p$. We define

$$\mathcal{E}^{\rm a}_{+}(p) = \sum_{h=1}^{({\rm N}-1)} \underbrace{\left[\sum_{\alpha,\beta=1}^{\rm N} a^\dagger_\alpha[h]\left(\frac{\Lambda^{\rm a}}{2}\right)_{\alpha\beta} a_\beta[h]\right]}_{\equiv \mathcal{E}^{a}_{+}[h]},$$
$$\mathcal{E}^{\rm a}_{-}(p) = \sum_{h=1}^{({\rm N}-1)} \underbrace{\left[\sum_{\alpha,\beta=1}^{\rm N} b_\alpha[h]\left(-\frac{\Lambda^{\rm a}}{2}\right)_{\alpha\beta} b^\dagger_\beta[h]\right]}_{\equiv \mathcal{E}^{a}_{-}[h]}. \quad (67)$$

In (67), we have introduced prepotential N-plets $(a_\alpha[h], a^\dagger_\alpha[h])$ and $(b_\alpha[h], b^\dagger_\alpha[h])$ for each of the $({\rm N}-1)$ fundamental representations of SU(N). They are denoted by $h = 1, 2, \ldots, ({\rm N}-1)$ and we have suppressed the additional plaquette index on the right-hand side of (67) for convenience. The $\frac{\Lambda^a}{2}$ are the $({\rm N}^2-1)$ SU(N) matrices in the fundamental representation. The harmonic-oscillator commutation relations of the SU(N) prepotentials imply

$$[\mathcal{E}^{\rm a}_{+}[h], a^\dagger_\alpha[h']] = \delta_{h,h'}\frac{1}{2}a^\dagger_\beta[h]\Lambda^{\rm a}_{\beta\alpha},$$
$$[\mathcal{E}^{\rm a}_{+}[h], b^\dagger_\alpha[h']] = -\delta_{h,h'}\frac{1}{2}\Lambda^{\rm a}_{\alpha\beta}b^\dagger_\beta[h]. \quad (68)$$

We also note that under SU(N) gauge transformations (18) with $\Lambda \equiv \Lambda(0,0)$ [see Fig. 1(b)] these oscillators transform as

$$a^\dagger_\alpha[h] \to a^\dagger_\beta[h]\Lambda_{\beta\alpha}, \quad \forall\, h = 1, 2, \ldots, \quad ({\rm N}-1),$$
$$b^\dagger_\alpha[h] \to \Lambda^\dagger_{\alpha\beta}b^\dagger_\beta[h], \quad \forall\, h = 1, 2, \ldots, \quad ({\rm N}-1). \quad (69)$$

Like in SU(2) and SU(3) cases, the relations (68) and (69) will be useful in constructing the SU(N) magnetic basis in the next section.

### 2. SU(N) magnetic basis

In this section, we construct the SU(N) magnetic basis for all SU(N) and show that the disorder operators on a magnetic basis act as shift operators for the $N-1$ magnetic fields. The SU(N) magnetic basis has been constructed in Appendix B and is given by

$$|Z\rangle = \sum_{[\vec{j}]=0}^{\infty}\sqrt{d(\vec{j})}\prod_{h=1}^{{\rm N}-1}\frac{1}{j_h!}(a^\dagger[h]Zb^\dagger[h])^{2j_h}|0\rangle. \quad (70)$$

In (70) $\sqrt{d(\vec{j})}$ is the dimension of the SU(N) $[\vec{j}](\equiv(j_1, j_2, \ldots, j_{{\rm N}-1})$ representation. The SU(N) strong-coupling vacuum $|0\rangle$ in the dual description on every plaquette satisfies

$$a_\alpha[h]|0\rangle = 0, \quad b_\alpha[h]|0\rangle = 0, \quad h = 1, 2, \ldots, \quad ({\rm N}-1). \quad (71)$$

Like in SU(2) and SU(3) cases we parametrize the SU(N) matrix $Z \equiv Z(p)$ in (70) on every plaquette $p$ in the angle-axis representation as

$$Z = Z(\omega_1, \omega_2, \ldots, \omega_{{\rm N}-1}) = \exp i\left(\omega_h\hat{n}^{\rm a}_{[h]}\frac{\Lambda^{\rm a}}{2}\right). \quad (72)$$

In (72) the $({\rm N}-1)$ linearly independent unit vectors are defined as

$$\vec{n}^{\rm a}_{[r+1]}(p) \equiv d^{\rm abc}\hat{n}^{\rm b}_{[r]}(p)\hat{n}^{\rm c}_{[1]}(p), \quad r = 1, 2, \ldots, {\rm N}-2. \quad (73)$$

We have again suppressed the $({\rm N}-1)$ axes $\hat{n}^{a}_{[h]}$ in $Z(\omega_1, \omega_2, \ldots, \omega_{{\rm N}-1})$ for the notational simplicity.

In order to evaluate the action of disorder operators on the magnetic basis (70), we use (68) to obtain,

$$\Sigma^{+}_{[\vec{\theta}]} a^{\dagger}_{\alpha}[h] \Sigma^{+\dagger}_{[\vec{\theta}]} = \left( a^{\dagger}[h] e^{i(\theta_h \hat{n}^{\mathrm{a}}_{[h]})\frac{\Lambda^{\mathrm{a}}}{2}} \right)_{\alpha}, \tag{74}$$

$$\Sigma^{-}_{[\vec{\theta}]} b^{\dagger}_{\alpha}[h] \Sigma^{-\dagger}_{[\vec{\theta}]} = \left( e^{-i(\theta_h \hat{n}^{\mathrm{a}}_{[h]})\frac{\Lambda^{\mathrm{a}}}{2}} b^{\dagger}[h] \right)_{\alpha}. \tag{75}$$

Therefore, the action of disorder operators on the magnetic basis is given by

$$\Sigma^{+}_{[\vec{\theta}]} |Z\rangle = |e^{i(\theta_h \hat{n}^{\mathrm{a}}_{[h]})\frac{\Lambda^{\mathrm{a}}}{2}} Z\rangle, \tag{76}$$

$$\Sigma^{-}_{[\vec{\theta}]} |Z\rangle = |Z e^{i(\theta_h \hat{n}^{\mathrm{a}}_{[h]})\frac{\Lambda^{\mathrm{a}}}{2}}\rangle. \tag{77}$$

We now use axis-angle representation (72) to get

$$\begin{aligned} \Sigma^{+}_{[\vec{\theta}]} |Z(\omega_h)\rangle &= |e^{i(\theta_h \hat{n}^{\mathrm{a}}_{[h]})\frac{\Lambda^{\mathrm{a}}}{2}} e^{i(\omega_h \hat{n}^{\mathrm{a}}_{[h]})\frac{\Lambda^{\mathrm{a}}}{2}}\rangle = |Z(\omega_h + \theta_h)\rangle, \\ \Sigma^{-}_{[\vec{\theta}]} |Z(\omega_h)\rangle &= |e^{i(\omega_h \hat{n}^{\mathrm{a}}_{[h]})\frac{\Lambda^{\mathrm{a}}}{2}} e^{-i(\theta_h \hat{n}^{\mathrm{a}}_{[h]})\frac{\Lambda^{\mathrm{a}}}{2}}\rangle = |Z(\omega_h - \theta_h)\rangle. \end{aligned} \tag{78}$$

Therefore, the disorder operator on a plaquette $p$ translates the $\mathrm{N}-1$ gauge invariant angles defining the SU(N) magnetic fluxes.

#### 3. SU(N) order-disorder algebra

Using the canonical commutation relations in the dual description (15) we get Similarly, the SU(N) order-disorder algebra is

$$\begin{aligned} \Sigma^{+}_{[\vec{\theta}]}(p) \mathcal{W}^{[\vec{j}]}_{\alpha\beta}(p) \Sigma^{-}_{[\vec{\theta}]}(p) &= D^{[\vec{j}]}_{\alpha\gamma}([\vec{\theta}]) \mathcal{W}^{[\vec{j}]}_{\gamma\beta}(p), \\ \Sigma^{-}_{[\vec{\theta}]}(p) \mathcal{W}^{[\vec{j}]}_{\alpha\beta}(p) \Sigma^{+}_{[\vec{\theta}]}(p) &= \mathcal{W}^{[\vec{j}]}_{\alpha\gamma}(p) D^{[\vec{j}]}_{\gamma\beta}([\vec{\theta}]). \end{aligned} \tag{79}$$

In (79) the Wigner matrix $D^{[\vec{j}]}([\vec{\theta}])$ represent the SU(N) rotations around the magnetic axes $\hat{n}_{[h]}$ by $\theta_h$ with $h = 1, 2, \ldots, (\mathrm{N}-1)$.

#### 4. Reduction to 't Hooft algebra

In the special case when the rotations are in the center of SU(N) with $Z \in Z_{\mathrm{N}}$ and $Z^{\mathrm{N}} = 1$, we get

$$D^{[\vec{j}]}(Z) = (z)^{\eta[\vec{j}]} \mathcal{I}, \qquad z^{\mathrm{N}} = 1, \tag{80}$$

where $\mathcal{I}$ is the unit matrix and $\eta[\vec{j}]$ is the N-ality of the $[\vec{j}]$ representation. The SU(N) center elements in (80) are

$$z = e^{\frac{2\pi i m}{\mathrm{N}}}, \quad m = 0, 1, \ldots, \quad (\mathrm{N}-1). \tag{81}$$

We thus get the 't Hooft Wilson order-disorder algebra [8–11].

$$\begin{aligned} \Sigma^{+}_{[Z_{\mathrm{N}}]}(p) \mathcal{W}^{[\vec{j}]}_{\alpha\beta}(p) \Sigma^{-}_{[Z_{\mathrm{N}}]}(p) &= (z)^{\eta[\vec{j}]} \mathcal{W}^{[\vec{j}]}_{\alpha\beta}(p), \\ \Sigma^{-}_{[Z_{\mathrm{N}}]}(p) \mathcal{W}^{[\vec{j}]}_{\alpha\beta}(p) \Sigma^{+}_{[Z_{\mathrm{N}}]}(p) &= (z)^{\eta[\vec{j}]} \mathcal{W}^{[\vec{j}]}_{\alpha\beta}(p). \end{aligned} \tag{82}$$

### V. SU(N) DIRAC STRINGS

The disorder operators defined in the previous section can also be written in terms of the Kogut-Susskind link holonomies and their electric fields using the exact duality transformations (12). As expected, these disorder operators $\Sigma(p)$ are highly nonlocal operators in the original description but their physical action is completely local. This leads to the invisible SU(N) Dirac strings discussed in this section. Using the exact duality relations we write

$$\begin{aligned} &\Sigma^{+}_{[\theta_1\theta_2\cdots\theta_{\mathrm{N}-1}]}(m,n) \\ &= \exp\left\{ i\vec{\theta}^{\mathrm{a}}(m,n) \cdot \sum_{n'=n+1}^{\infty} R^{\mathrm{ab}}(S(m,n;n')) E^{\mathrm{b}}_{-}(m,n';\hat{1}) \right\}. \end{aligned} \tag{83}$$

In (83) we have used (12)

$$\mathcal{E}^{a}(m,n) = \sum_{b} R^{\mathrm{ab}}(S(m,n;n')) E^{\mathrm{b}}_{-}(m,n';\hat{1}),$$

where the parallel transports $S(m,n;n')$ and the vectors $\vec{\theta}^{a}(m,n)(\equiv\vec{\theta}^{a}(p))$ are defined in (13b) and (63) respectively. Thus, the local SU(N) disorder operator (62) in the dual description becomes a nonlocal operator (83) when rewritten in terms of the original Kogut-Susskind link operators. As is clear from (83), it now rotates all the horizontal Kogut-Susskind link operators $U(m-1,n';\hat{1}), n' \geq n$. We can define the axis of rotation associated with each rotated link as

$$\vec{\Theta}^{\mathrm{a}}(m, n' > n) = R^{\mathrm{ab}}(\mathsf{S}(m,n;n')) \vec{\theta}^{\mathrm{b}}(m,n), \tag{84}$$

which can also be recast in an iterative relation

$$\vec{\Theta}^{\mathrm{a}}(m, n'+1) = R^{\mathrm{ab}}(U(m,n';\hat{2})) \vec{\Theta}^{\mathrm{b}}(m,n'). \tag{85}$$

We can similarly obtain $\Sigma^{-}_{[\vec{\theta}]}$ by using (12), (14), and (62). Now we have

$$\Sigma^{+}_{[\vec{\theta}]}(m,n) = \exp\left\{ i \sum_{n'=n+1}^{\infty} \vec{\Theta}^{\mathrm{a}}(m,n') \cdot E^{\mathrm{a}}_{-}(m,n';\hat{1}) \right\}. \tag{86}$$

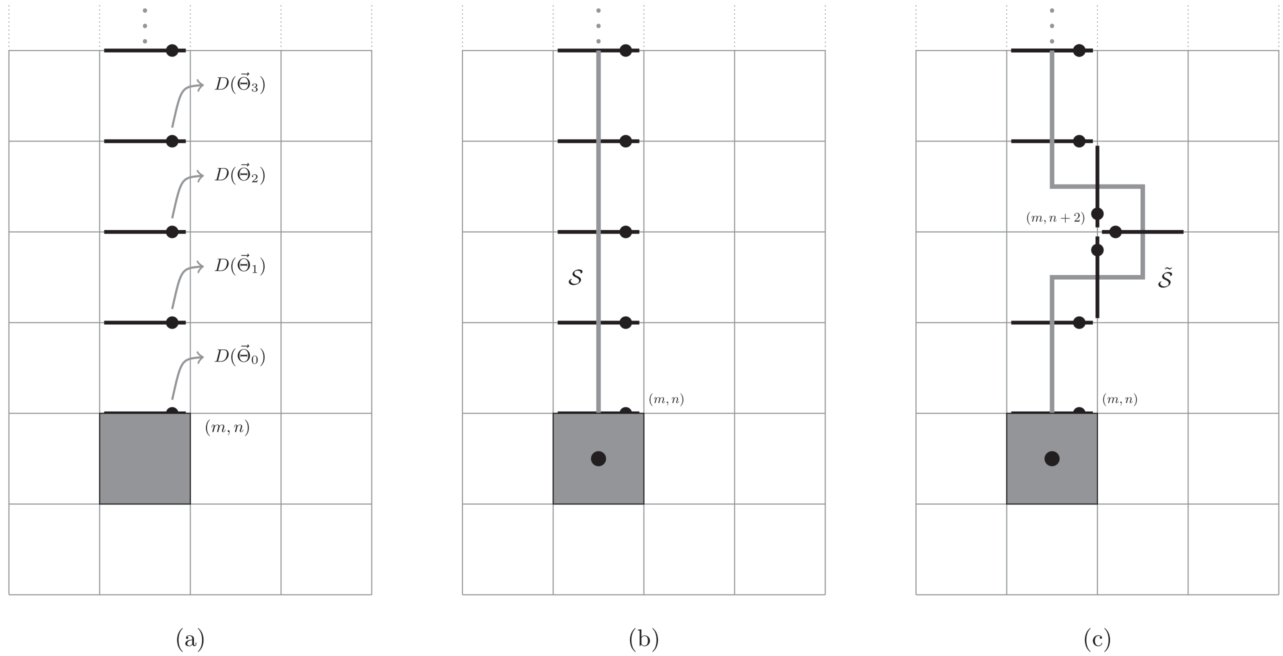



FIG. 4. (a) The disorder operator $\Sigma^{+}_{[\vec{\theta}]}(m,n)$, defined in Eq. (83) rotates all horizontal links $U(m-1,n';\hat{1})$, $\forall\ n' \geq n$ around an axis $\vec{\Theta}(m,n')$ (for $n'=n,n+1,n+2,\ldots$ they are denoted by $\vec{\Theta}_0,\vec{\Theta}_1,\vec{\Theta}_2,\ldots$), (b) Invisible SU(N) Dirac string $\mathcal{S}$. The rotated links $\mathsf{l}\in\mathcal{S}$ are the dark horizontal links, (c) Shape of Dirac string can be deformed without affecting the endpoint or the location of the magnetic vortex. The SU(N) gauge transformations at site $(m,n+2)$ changes the shape of the Dirac string from $\mathcal{S}$ to $\tilde{\mathcal{S}}$.

They rotate the links

$$\Sigma^{+}_{[\vec{\theta}]}(m,n)U_{\alpha\beta}(m,n';\hat{1})\Sigma^{+\dagger}_{[\vec{\theta}]}(m,n) = U_{\alpha\gamma}(m,n';\hat{1})D_{\gamma\beta}(\vec{\Theta}(m,n')), \quad \forall\, n' \geq n. \tag{87}$$

These rotations of the horizontal link holonomies are shown in Fig. 4(a). The rotational axes of these link holonomies are related through the parallel transport equations (85) which, in turn, are obtained by the exact duality transformations (12). These special relations ensure that they create magnetic flux only on the plaquette located at the end point $(m,n)$ keeping all the other plaquette fluxes unaffected (see Appendix C). Therefore, this local action by the nonlocal operator (86) creates an invisible non-Abelian Dirac string $\mathcal{S}$ originating from the corresponding plaquette [see Fig. 4(b)]. In Appendix C is shown that using gauge transformations these Dirac strings can be deformed arbitrarily except their gauge invariant endpoints.

## VI. PATH INTEGRAL REPRESENTATION

In this section, we construct the path integral representation of the SU(N) disorder operators so that their behavior can also be studied using Monte Carlo simulations in future studies. Such construction for the $Z_2$ 't Hooft disorder operator in pure SU(2) lattice gauge theory can be found in [9,11]. The ground-state wave functional depends on the links in the two-dimensional surface $\Sigma$ at time $t=0$ [9],

$$\Psi_0(U) \equiv \langle U|\psi(0)\rangle = \int \prod_{l>0} dU(l) e^{\beta\sum_{p>0}\mathrm{Tr}(U_p+U_p^{\dagger})}. \tag{88}$$

In (88) the integration is done over all links $l>0$ which are the links at time $t>0$. Similarly the plaquettes involved in the summation are in the upper half lattice at $t>0$. Thus the ground state $\Psi_0(U)$ depends only on the spatial links at $t=0$. The expectation values of any functional $F[U(l)]$ in the ground state $|\psi(0)\rangle$ is defined as

$$\langle F[U(l)]\rangle = \langle\psi(0)|F[U(l)]|\psi(0)\rangle.$$

The path integral representation is

$$\langle F[U(l)]\rangle = \frac{1}{Z(\beta)}\int d\mu(U)F[U(l)]e^{\beta\mathrm{tr}(U_p+U_p^{\dagger})}, \tag{89}$$

where $d\mu(U) \equiv \prod_l dU(l)$ and $l$, $p$ now denote all the links and plaquettes in the three-dimensional lattice and $\beta = \frac{2\mathrm{N}}{g^2}$. The partition function $Z(\beta)$ is given by

$$Z(\beta) = \int \prod_l dU(l) e^{\beta\sum_p(\mathrm{tr}(U_p+U_p^{\dagger}))}. \tag{90}$$

The action of $\Sigma^{+}_{[\vec{\theta}]}(m,n)$ rotates all the links crossing the Dirac string by the appropriate SU(N) Wigner-$D$ matrices as shown in Fig. 4(a). Therefore the expectation value

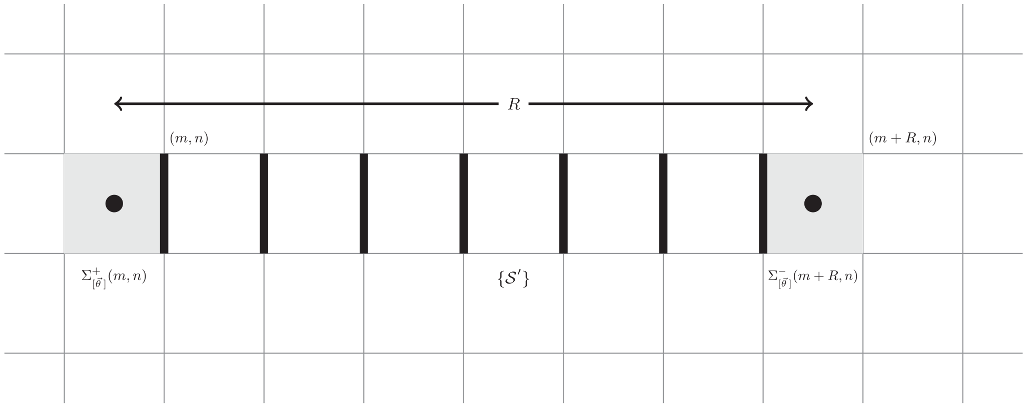



FIG. 5. Action of the disorder operators $\Sigma^+_{[\vec{\theta}]}(m,n)\Sigma^-_{[\vec{\theta}]}(m+R,n)$ creating SU(N) vortex-antivortex at a distance $R$ apart. The SU(N) transformations rotate the dark vertical links denoted by $\mathsf{l}' \in \mathcal{S}'$ in (93). This set of vertical dark links $\mathsf{l}'$ is denoted by $\{\mathcal{S}'\}$.

$\Sigma^+_{[\vec{\theta}]}(m,n)$ or the free energy of the SU(N) magnetic vortex can be defined as

$$\langle \Sigma^+_{[\vec{\theta}]}(m,n)\rangle = \langle e^{-\beta\sum_{\mathsf{l}\in\mathcal{S}}[\mathrm{tr}(D(\vec{\theta})U_p + U_p^\dagger D^\dagger(\vec{\theta})) - \mathrm{tr}(U_p+U_p^\dagger)]}\rangle$$
$$\equiv e^{-\beta F_{\mathrm{mag}}(\vec{\theta})}. \quad (91)$$

In (91) the summation sign includes only those plaquettes which protrude from the links $\mathsf{l}\in\{\mathcal{S}\}$ [see Fig. 4(b)] in the +ve time direction and $F_{\mathrm{mag}}(\vec{\theta})$ denotes the free energy of the magnetic vortex. Note that the path integral representation for the SU(N) vortex (91) is analogous to the path integral representations for the defects in the 2D Ising model [1] and $Z_N$ vortices in SU(N) gauge theory [8] obtained by Kadanoff and 't Hooft, respectively. We can also define SU(N) electric free energy of the vortex as the SU(N) Fourier transform

$$e^{-\beta F_{elec}(\vec{j})} \equiv \int d\theta_1 \int d\theta_2 \cdots \int d\theta_{\mathrm{N}-1}\chi_{[\vec{j}]}(\vec{\theta})e^{-\beta F_{\mathrm{mag}}(\vec{\theta})}. \quad (92)$$

In (92), $\chi_{[\vec{j}]}(\vec{\theta})$ is the SU(N) character in the $[\vec{j}] = (j_1, j_2, \ldots, j_{\mathrm{N}-1})$ representation of SU(N).

The Monte Carlo simulation of $\langle\Sigma^+_{[\vec{\theta}]}(m,n)\rangle$ in (91) is problematic because of the presence of the infinite Dirac string attached to a vortex contradicts the periodic boundary conditions imposed on a finite lattice. On the other hand one can easily compute the vortex-antivortex correlation functions as shown in Fig. 5,

$$\langle \Sigma^+_{[\vec{\theta}]}(m,n)\Sigma^-_{[\vec{\theta}]}(m+R,n)\rangle$$
$$\equiv e^{-\beta F(\vec{\theta},R)}$$
$$= \langle e^{-\beta\sum_{\mathsf{l}'\in\mathcal{S}'}[\mathrm{tr}(D(\vec{\theta})U_p + U_p^\dagger D^\dagger(\vec{\theta})) - \mathrm{tr}(U_p+U_p^\dagger)]}\rangle. \quad (93)$$

In (93) $\mathcal{S}'$ denotes the set of dark links $\mathsf{l}'$ in Fig. 5 and the summation sign includes only those plaquettes which protrude from the links $\mathsf{l}'\in\{\mathcal{S}'\}$ in the +ve time direction. It will be interesting to study the above free energies and hence the role of SU(N) vortices in the ground state and their magnetic disorder in the large $R$ limit using Monte Carlo simulations near the continuum $\beta\to\infty$.

## VII. SUMMARY AND DISCUSSION

In this work we have constructed the most general disorder operators for SU(N) lattice gauge theory in $(2+1)$ dimensions in the Hamiltonian formulation. Being exactly dual to the Wilson loop operators, these operators create and annihilate $(\mathrm{N}-1)$ types of SU(N) magnetic fluxes. The SU(N) order-disorder algebra is simply the canonical commutation relations in the dual formulation, i.e., the commutation relations between the electric scalar potentials and their conjugate magnetic fluxes.

In the strong coupling limit the disorder and order operators satisfy,

$$\langle 0|\Sigma^{\pm}_{[\vec{\theta}]}|0\rangle \longrightarrow g^2\to\infty 1, \quad \forall\,[\vec{\theta}], \quad (94a)$$

$$\langle 0|\mathrm{Tr}(\mathcal{W}^{[\vec{j}]}_{\mathcal{C}})|0\rangle \longrightarrow g^2\to\infty 0, \quad \forall\,[\vec{j}]. \quad (94b)$$

In the first limit equation we used the nonlocal expression for $\Sigma^{\pm}_{[\vec{\theta}]}$ in (83). The strong coupling limits in (94a) and (94b) show a complete magnetic disorder at least in the strong coupling ground state $|0\rangle$. The study of $\langle\Sigma^{\pm}(\theta)\rangle$ and the vacuum correlation functions of $\langle\Sigma^{\pm}_{\vec{\theta}}(p)\Sigma^{\mp}_{\vec{\theta}}(p')\rangle$, as $|p-p'|\to\infty$ for different $[\vec{\theta}]$ in the weak-coupling continuum limit is required to further probe the relevance of these magnetic disorder operators in the problem of color confinement. These studies across the finite temperature confinement-deconfinement transition will also be useful to understand the magnetic disorder in confining vacuum. We further note that the SU(N) disorder operators are meaningful even in the presence of dynamical matter fields in any SU(N) representation. These canonical transformation techniques can also be generalized to obtain the SU(N) disorder operator in $(3+1)$ dimensions where the dual electric potentials are also the dual gauge fields on the dual links. Thus, like Wilson loop operators $\mathcal{W}_{[\vec{j}]}(\mathcal{C})$, the disorder operator $\Sigma_{[\vec{\theta}]}(\mathcal{C}')$ will also be defined on the closed curves $\mathcal{C}'$ on the dual lattice. The work in these directions is in progress.

## ACKNOWLEDGMENTS

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## APPENDIX A: ELECTRIC LOOP BASIS

It is easy to construct the loop basis in terms of the dual-electric scalar potentials on the plaquette loops (see Fig. 3). In the prepotential representation

$$\mathcal{E}^{\rm a}_{-}(p) \equiv a^\dagger(p)\frac{\sigma^{\rm a}}{2}a(p), \quad \mathcal{E}^{\rm a}_{+}(p) \equiv -b(p)\frac{\sigma^{\rm a}}{2}b^\dagger(p). \quad \text{(A1)}$$

Using the facts that the left and the right electric fields are independent, $[\mathcal{E}^{\rm a}_{+}(p), \mathcal{E}^{\rm b}_{-}(p)] = 0$, and their magnitudes are equal, $\sum_{\rm a=1}^{3} \mathcal{E}^{\rm a}_{+}\mathcal{E}^{\rm a}_{+} = \sum_{\rm a=1}^{3} \mathcal{E}^{\rm a}_{-}\mathcal{E}^{\rm a}_{-} \equiv \mathcal{E}^2$, we define the first set of a complete set of commuting operators on every plaquette $p$ as: $[\mathcal{E}^2, \mathcal{E}^{\rm a=3}_{+}, \mathcal{E}^{\rm a=3}_{-}]$. The SU(2) electric-loop decoupled basis on every plaquette $p$ is

$$\begin{aligned} |jm_+m_-\rangle &\equiv |jm_+\rangle \otimes |jm_-\rangle \\ &= \phi^j_{m_+}(a^\dagger_1, a^\dagger_2)|0\rangle_a \otimes \phi^j_{m_-}(b^\dagger_1, b^\dagger_2)|0\rangle_b, \end{aligned} \quad \text{(A2)}$$

where we have defined

$$\phi^j_m(a^\dagger_1, a^\dagger_2) = \frac{(a^\dagger_1)^{j+m}}{\sqrt{(j+m)!}}\frac{(a^\dagger_2)^{j-m}}{\sqrt{(j-m)!}}. \quad \text{(A3)}$$

Under SU(2) gauge transformations at the origin $\Lambda = \Lambda(0,0)$

$$\begin{aligned} \phi^j_{m_+}(a^\dagger_1, a^\dagger_2) &\to D^j_{m_+m'_+}(\Lambda)\phi^j_{m'_+}(a^\dagger_1, a^\dagger_2), \\ \phi^j_{m_-}(b^\dagger_1, b^\dagger_2) &\to D^j_{m_-m'_-}(\Lambda^\dagger)\phi^j_{m'_-}(b^\dagger_1, b^\dagger_2). \end{aligned} \quad \text{(A4)}$$

The electric flux states transform as

$$|jm_+m_-\rangle = \sum_{m'_+,m'_-} D^j_{m_+m'_+}(\Lambda)D^j_{m_-m'_-}(\Lambda^\dagger)|jm'_+m'_-\rangle. \quad \text{(A5)}$$

At this stage, it is convenient to work with the coupled basis instead of the decoupled basis (A5). We define the complete set of commuting operators (CSCO) on every plaquette as

$$\mathcal{E}^2 = \mathcal{E}^2_+ = \mathcal{E}^2_-, \quad L^{\rm a} \equiv \mathcal{E}^{\rm a}_+ + \mathcal{E}^{\rm a}_-, \quad L^{\rm a=3} \equiv \mathcal{E}^{\rm a=3}_- + \mathcal{E}^{\rm a=3}_+. \quad \text{(A6)}$$

The loop coupled basis on every lattice plaquette $|nlm\rangle$ can be constructed as

$$|nlm\rangle = N_{nlm}(k_+)^{n-l-1}(L_-)^{l-m}(a^\dagger_1)^l(b^\dagger_2)^l|0,0\rangle. \quad \text{(A7)}$$

In (A7) $k_+ \equiv a^\dagger \cdot b^\dagger \equiv \sum_{\alpha=1}^{2} a^\dagger_\alpha b^\dagger_\alpha$ and

$$N_{nlm} \equiv \sqrt{\frac{n(l+m)!}{(l-m)!(l!)^2(n-l-1)!(m+l)!}}.$$

The corresponding eigenvalue equations are

$$\begin{aligned} \mathcal{E}^2|nlm\rangle_p &= \left(\frac{n^2-1}{4}\right)\Big|nlm\rangle_p, \\ \vec{L}^2|nlm\rangle_p &= l(l+1)|nlm\rangle_p, \\ L^{\rm a=3}|nlm\rangle_p &= m|nlm\rangle_p. \end{aligned} \quad \text{(A8)}$$

In above $l = 0, 1, 2, \ldots, n-1$ and $m = -l, -l+1, \ldots, l$. Under gauge transformations at the origin $\Lambda = \Lambda(0,0)$, these states have much simpler transformation property

$$|nlm\rangle = \sum_{\bar{m}} D^l_{m\bar{m}}(\Lambda)|nl\bar{m}\rangle. \quad \text{(A9)}$$

In other words the principal $(n)$ and the angular momentum $(l)$ quantum numbers remain invariant.

### 1. The Wilson loops as translation operators

In the loop basis (A8), the plaquette operators, $\mathcal{W}(p)$ which are unit size Wilson loop order operator acts as a translation operator for the electric flux $n$. Using (28) we get

$${\rm Tr}\mathcal{W}(p)|nlm\rangle = A_p|n-1lm\rangle + B_p|n+1lm\rangle. \quad \text{(A10)}$$

Here we have ignored the plaquette index p on all the three quantum numbers and

$$A_p = \frac{\sqrt{(n-l-1)(n+l)}}{(n-1)}, \quad B_p = \frac{\sqrt{(n+l-1)(n-l)}}{(n+1)}.$$

The above translative action of the fundamental plaquette loop operator $\mathcal{W} \equiv \mathcal{W}(p)$ is valid on each plaquette $p$ and we have suppresses the plaquette index $p$ on both sides of (A10). The action (A10) is illustrated in Fig. 6. An arbitrary

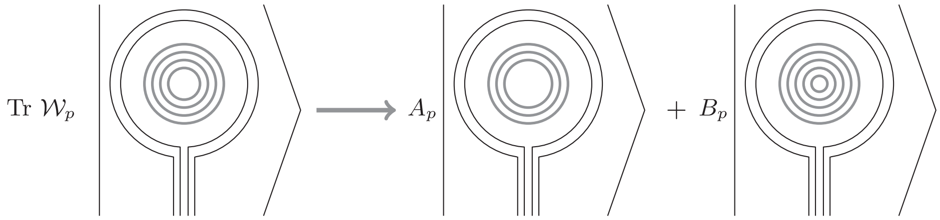


FIG. 6. The action of the Wilson loop on the loop state $|n = 4, l = 2, m\rangle$ described in the coupled basis. The circles in the three figures represent the SU(2) electric flux circulating in a loop within the plaquette and $2l$ is the number of open flux lines. The action of Tr $(\mathcal{W})$ simply translates $n$ to $n \pm 1$ in (A10).

Wilson loop operator $\mathcal{W}(\mathcal{C})$ can be written in terms of the fundamental plaquette Wilson loop operators as

$$\mathcal{W}(\mathcal{C}) = \prod_{p \in \mathcal{C}} \mathcal{W}(p) = \mathcal{W}(p_1)\mathcal{W}(p_2)\cdots\mathcal{W}(p_{n_\mathcal{C}}). \quad \text{(A11)}$$

The above product over plaquettes is taken from the bottom right corner as shown in Fig. 7. More explicitly, the curve $\mathcal{C}$ is obtained by traversing the $n_\mathcal{C}$ plaquettes in the order $p_1 \to p_2 \cdots \to p_{n_\mathcal{C}}$ as shown in Fig. 7. These $n_\mathcal{C}$ plaquette paths are shown in Fig. 3. Therefore, the end effect of $\mathcal{W}(\mathcal{C})$ is to translate the electric fluxes of all plaquette loops inside the closed curve $\mathcal{C}$,

$$\begin{aligned}\mathcal{W}(\mathcal{C}) \prod_{p \in \mathcal{C}} |nlm\rangle_p \\ = \prod_{p \in \mathcal{C}} (A_p|n-1lm\rangle_p + B_p|n+1lm\rangle_p). \quad \text{(A12)}\end{aligned}$$

Note that in the SU(N) case the Wilson loop operators will shift all the $(N-1)$ eigenvalues of the Casimir $\mathcal{E}^2[h]$ $(h = 1, 2, \ldots, (N-1))$ on the plaquettes $p \in \mathcal{C}$ by $\pm 1$.

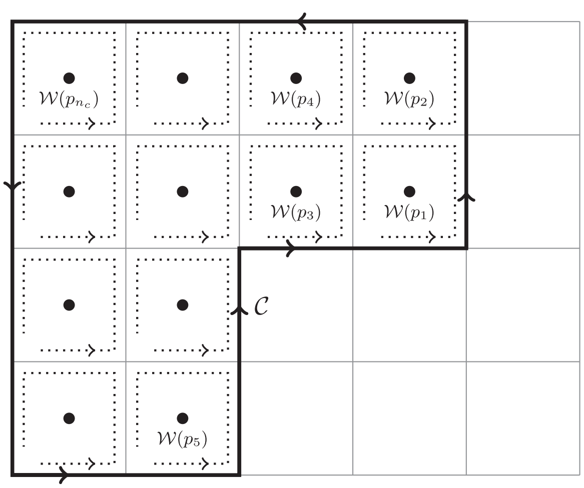


FIG. 7. The Wilson loop $\mathcal{W}(C)$ of any shape and size can be written as an ordered product of all plaquette operators $\mathcal{W}(p)$ inside $\mathcal{C}$ as in (A11). These dotted plaquettes inside $\mathcal{C}$ are illustrated in Fig. 3.

## APPENDIX B: MAGNETIC LOOP BASIS

We now construct the magnetic basis for plaquette flux operators and show that the disorder operator has natural translative action on them. The group manifold for SU(2) group is $S^3$. We define it on every plaquette $p$ through complex doublets $\vec{z}(p) \equiv (z_1(p), z_2(p))$ that satisfy the constraint $|z_1(p)|^2 + |z_2(p)|^2 = 1, \ \forall \ p$. A configuration on $S^3$ is

$$Z(p) = \begin{bmatrix} z_1(p) & z_2(p) \\ -z_2^*(p) & z_1^*(p) \end{bmatrix}. \quad \text{(B1)}$$

We write eigenvalue equations for the magnetic flux operators as

$$\mathcal{W}_{\alpha\beta}(p)|Z(p)\rangle = Z_{\alpha\beta}(p)|Z(p)\rangle. \quad \text{(B2)}$$

Here $Z_{\alpha\beta}(p)$ are the matrix elements of the matrix $Z(p)$ in (B1). These states form a complete orthonormal basis on $S^3$

$$\begin{aligned}\int_{S^3} d\mu(\vec{z})|Z(p)\rangle\langle Z(p)| = 1, \\ \langle Z(p)\rangle Z'(p) = \delta(Z(p) - Z'(p)). \quad \text{(B3)}\end{aligned}$$

The SU(2) group manifold integrations is defined as $\int_{SU(2)} d\mu(\vec{z}) \equiv \frac{1}{16\pi^2} \int d^2z_1 d^2z_2 \delta(z_1^* z_1 + z_2^* z_2 - 1)$.

The magnetic eigenvectors $|Z(p)\rangle$ can be expanded in the complete orthonormal electric basis as

$$\begin{aligned}|Z(p)\rangle &\equiv |z_1(p), z_2(p)\rangle \\ &= \sum_{j=0}^{\infty} \sqrt{(2j+1)} \sum_{m_+, m_-} D^j_{m_+m_-}(Z(p))|jm_+m_-\rangle. \quad \text{(B4)}\end{aligned}$$

The construction of magnetic states can be easily checked by directly applying $\mathcal{W}$ on both sides above equations and realizing that $\mathcal{W}$ acts on the electric field basis as the raising and lowering operators for $(j, m_+, m_-)$ and using the recurrence relations for the $D$-functions connecting $D^j_{m_+m_-}$ to $D^{j\pm\frac{1}{2}}_{m_+\pm\frac{1}{2}m_-\pm\frac{1}{2}}$. For SU(N), N ≥ 3, this approach gets extremely complicated as it requires the recurrence relations for the SU(N) Winger $D$-functions. We will first write down these states in terms of SU(2) prepotentials where they take a much simpler form and then verify the eigenvalues equations (B2). Now use Eq. (A2)

$$|Z(p)\rangle = \sum_{j=0}^{\infty}\sqrt{(2j+1)} \times \sum_{m_+,m_-} D^j_{m_+m_-}(Z(p))\phi^j_{m_+}(a_1^\dagger,a_2^\dagger)\phi^j_{m_-}(b_1^\dagger,b_2^\dagger)|0\rangle. \tag{B5}$$

We call $\phi^j_m(x_1,x_2)$ the SU(2) structure functions. These SU(2) structure functions have the following orthonormal properties:

$$\int_{SU(2)} d\mu(\vec{z})\phi^{j*}_m(z_1,z_2)\phi^j_{m'}(z_1,z_2) = \frac{\delta_{m,m'}}{(2j+1)!}, \quad \sum_m \phi^{j*}_m(z_1,z_2)\phi^j_m(w_1,w_2) = \frac{(z_1^* w_1 + z_2^* w_2)^{2j}}{(2j)!}. \tag{B6}$$

Further, we can also write SU(2) Wigner D-function in terms of these structure functions as follows:

$$D^j_{m,n}(z_1,z_2) = d_j\int_{SU(2)} d^2w_1 d^2w_2 \phi^{j*}_m(w_1,w_2)\phi^j_n(z_1^w,z_2^w), \tag{B7}$$

where

$$\begin{bmatrix} z_1^w \\ z_2^w \end{bmatrix} \equiv \begin{bmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \qquad d_j \equiv (2j+1).$$

Using properties of structure functions and Wigner $D$ functions,

$$\sum_{m_-=-j}^{j} D^j_{m_+m_-}(z_1,z_2)\phi^j_{m_-}(w_1,w_2) = \phi^j_{m_+}(w_1^z,w_2^z)$$

in (B5), we get

$$|Z(p)\rangle = \sum_{j=0}^{\infty}\sqrt{(2j+1)} \times \sum_{m_+}\phi^j_{m_+}(a_1^\dagger,a_2^\dagger)\phi^j_{m_+}(b_1^{z\dagger},b_2^{z\dagger})|0\rangle, \tag{B8}$$

where

$$\begin{bmatrix} b_1^{z\dagger} \\ b_2^{z\dagger} \end{bmatrix} \equiv \begin{bmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{bmatrix}\begin{bmatrix} b_1^\dagger \\ b_2^\dagger \end{bmatrix}.$$

Now we can sum the remaining magnetic index to get

$$|Z(p)\rangle = \sum_{j=0}^{\infty}\sqrt{d_j}\frac{(a^\dagger Z(p) b^\dagger)^{2j}}{(2j)!}|0,0\rangle, \tag{B9}$$

where $d_j = (2j+1)$ is the dimension of $[j]$ representation. The eigenvalues equation (B2) holds at each point of the group manifold. We first prove it for $Z = I$ where $I$ is the identity element of SU(2) group. First, we prove that

$$\mathcal{W}^{j=1/2}_{\alpha\beta}|I\rangle = \delta_{\alpha\beta}|I\rangle, \tag{B10}$$

where

$$|I\rangle = \sum_{j=0}^{\infty}\frac{(2j+1)^{1/2}}{(2j)!}(a^\dagger\cdot b^\dagger)^{2j}|0,0\rangle. \tag{B11}$$

We have suppressed the plaquette index $p$. Using prepotential representation (28) for $\mathcal{W}_{\alpha\beta}$ we get

$$\mathcal{W}_{\alpha\beta}|I\rangle = \sum_{j=0}^{\infty}\frac{(2j+1)^{1/2}}{(2j)!}\left[\frac{a_\beta b_\alpha}{(2j)^{1/2}} - \frac{\tilde{a}^\dagger_\alpha\tilde{b}^\dagger_\beta}{(2j+2)^{1/2}}\right] \times \frac{1}{\sqrt{(2j+1)}}(a^\dagger\cdot b^\dagger)^{2j}|0\rangle.$$

Now we replace $2j$ by $2j+1$ in the first term and $2j$ by $2j-1$ in the second term of above equation to get

$$\mathcal{W}_{\alpha\beta}|I\rangle = \sum_{j=0}^{\infty}\frac{(2j+1)^{1/2}}{(2j)!}\left[\frac{1}{(2j+1)^2}a_\beta b_\alpha(a^\dagger\cdot b^\dagger)^{2j+1} - \frac{(2j)}{(2j+1)}a^\dagger_\alpha b^\dagger_\beta(a^\dagger\cdot b^\dagger)^{2j-1}\right]|0\rangle$$

we evaluate first term using the prepotential commutation relations, $a_\alpha b_\beta(a^\dagger\cdot b^\dagger)^{2j+1}|0\rangle = a_\alpha b_\beta(\tilde{a}^\dagger\cdot\tilde{b}^\dagger)^{2j+1}|0\rangle = [(2j+1)^2\delta_{\alpha\beta}(a^\dagger\cdot b^\dagger)^{2j} + (2j)(2j+1)\tilde{a}^\dagger_\alpha\tilde{b}^\dagger_\beta(a^\dagger\cdot b^\dagger)^{2j-1}]|0\rangle$ and substitute in above equation to get

$$\mathcal{W}_{\alpha\beta}|I\rangle = \sum_{j=0}^{\infty}\frac{(2j+1)^{1/2}}{(2j)!}\delta_{\alpha\beta}(a^\dagger\cdot b^\dagger)^{2j+1}|0\rangle = \delta_{\alpha\beta}|I\rangle.$$

Now we can prove the eigenvalue equation (B2), by considering a transformation of oscillators $b^\dagger_\alpha \to (Z^\dagger b^\dagger)_\alpha$. Under these transformations

$$\mathcal{W}_{\alpha\beta} \to Z^\dagger_{\alpha\gamma}\mathcal{W}_{\gamma\beta}, \quad |I\rangle \to |Z\rangle$$

which yields

$$Z^\dagger_{\alpha\gamma}\mathcal{W}^{j=1/2}_{\gamma\beta}(p)|Z(p)\rangle = |Z(p)\rangle. \tag{B12}$$

As $Z^\dagger Z = I$, we get the eigenvalue equation (B2).

The conjugate electric fields act on this basis as differential operators on this plaquette holonomy basis,

$$\mathcal{E}^{\mathrm{a}}_{+}|Z\rangle = -\frac{\sigma^{\mathrm{a}}_{\alpha\beta}}{2} Z_{\gamma\beta} \frac{\partial}{\partial Z_{\alpha\gamma}} |Z\rangle, \tag{B13}$$

$$\mathcal{E}^{\mathrm{a}}_{-}|Z\rangle = \frac{\sigma^{\mathrm{a}}_{\alpha\beta}}{2} Z_{\gamma\alpha} \frac{\partial}{\partial Z_{\gamma\beta}} |Z\rangle. \tag{B14}$$

For SU(3) these magnetic states are given by

$$|Z\rangle = \sum_{\mathrm{p,q}} \sqrt{d(\mathrm{p,q})} \frac{(a^\dagger[1]Zb^\dagger[1])^{\mathrm{p}}}{\mathrm{p}!} \frac{(a^\dagger[2]Zb^\dagger[2])^{\mathrm{q}}}{\mathrm{q}!} |0\rangle, \tag{B15}$$

where $d(\mathrm{p,q}) = \frac{1}{2}(\mathrm{p}+1)(\mathrm{q}+1)(\mathrm{p}+\mathrm{q}+2)$ is the dimension of $[\mathrm{p,q}]$ representation. For the general SU(N) case, these magnetic states are given as

$$|Z\rangle = \sum_{[\vec{j}]=0}^{\infty} \sqrt{d(\vec{j})} \prod_{h=1}^{\mathrm{N}-1} \frac{1}{j_h!} (a^\dagger[h]Zb^\dagger[h])^{2j_h} |0\rangle, \tag{B16}$$

where $d(\vec{j})$ is the dimension of the $[\vec{j}]$ representation and $Z$ represents $(\mathrm{N}\times\mathrm{N})$ SU(N) matrix.

## APPENDIX C: INVISIBILITY OF DIRAC STRING

In this appendix, we explicitly show that disorder operators in (83) creates magnetic flux only on one plaquette $U_p(m,n)$ located at $(m,n)$. They leave all the other plaquettes unaffected. The disorder operator involves Kogut-Susskind electric fields $E_-(m,n'\geq n;\hat{1})$, therefore it trivially commutes with all other plaquettes which do not involve $U(m,n\geq n';\hat{1})$. The only relevant plaquette are $U_p(m,n'\geq n)$. Now we evaluate its action case by case:

(1) First we consider $n'=n$, plaquette $U^p \equiv U_p(m,n)$. For convenience we define $U_p(m,n) = U(m-1,n-1;\hat{1})U(m,n-1;\hat{2})U^\dagger(m-1,n;\hat{1}) \times U^\dagger(m-1,n-1;\hat{2}) \equiv U_1U_2U_3^\dagger U_4^\dagger$. The disorder operator $\Sigma^+_{[\vec{\theta}]}$ will only rotate link $U_3^\dagger$ around the axis $\vec{\Theta}^{\mathrm{a}}_0 = R^{\mathrm{ab}}(U_1U_2)\vec{\theta}^{\mathrm{b}}$,

$$\begin{aligned}\Sigma^+_{[\vec{\theta}]} U^p_{\alpha\beta} \Sigma^{+\dagger}_{[\vec{\theta}]} &= [U_1U_2(\Sigma^+_{[\vec{\theta}]}U_3^\dagger\Sigma^{+\dagger}_{[\vec{\theta}]})U_4^\dagger]_{\alpha\beta}\\ &= [U_1U_2D(\vec{\Theta}_0)U_3^\dagger U_4^\dagger]_{\alpha\beta}\\ &= [U_1U_2D(U_2^\dagger U_1^\dagger\vec{\theta}U_1U_2)U_3^\dagger U_4^\dagger]_{\alpha\beta}.\end{aligned}$$

Now we use a property of Wigner $D$-matrices namely $D(U_2^\dagger U_1^\dagger\vec{\theta}U_1U_2) = U_2^\dagger U_1^\dagger D(\vec{\theta})U_1U_2$ to get

$$\Sigma^+_{[\vec{\theta}]} U^p_{\alpha\beta} \Sigma^{+\dagger}_{[\vec{\theta}]} = [D(\vec{\theta})U^p]_{\alpha\beta}.$$

Therefore, the disorder operators $\Sigma^+_{[\vec{\theta}]}(m,n)$ create magnetic flux at plaquette $U_p(m,n)$.

(2) For $n'>n$, consider plaquette $U^p \equiv U_p(m,n')$ For convenience we define $U_p(m,n') = U(m-1,n'-1;\hat{1})U(m,n'-1;\hat{2})U^\dagger(m-1,n';\hat{1}) \times U^\dagger(m-1,n'-1;\hat{2}) \equiv U_1U_2U_3^\dagger U_4^\dagger$. The disorder operator $\Sigma^+_{[\vec{\theta}]}$ will rotate two horizontal links $U_1$ and $U_3^\dagger$ around the axes $\vec{\Theta}_{n'}$ and $\vec{\Theta}_{n'+1}$, respectively. Due to Eq. (85), these two axes are related as $\vec{\Theta}^{\mathrm{a}}_{n'+1} = R^{\mathrm{ab}}(U_2)\vec{\Theta}^{\mathrm{b}}_{n'}$,

$$\begin{aligned}\Sigma^+_{[\vec{\theta}]} U^p_{\alpha\beta} \Sigma^{+\dagger}_{[\vec{\theta}]} &= [(\Sigma^+_{[\vec{\theta}]}U_1\Sigma^{+\dagger}_{[\vec{\theta}]})U_2(\Sigma^+_{[\vec{\theta}]}U_3^\dagger\Sigma^{+\dagger}_{[\vec{\theta}]})U_4^\dagger]_{\alpha\beta}\\ &= [U_1D^\dagger(\vec{\Theta}_{n'})U_2D(\vec{\Theta}_{n'+1})U_3^\dagger U_4^\dagger]_{\alpha\beta}\\ &= [U_1D^\dagger(\vec{\Theta}_{n'})U_2D(U_2^\dagger\vec{\Theta}_{n'}U_2)U_3^\dagger U_4^\dagger]_{\alpha\beta}\\ &= [U_1D^\dagger(\vec{\Theta}_{n'})U_2U_2^\dagger D(\vec{\Theta}_{n'})U_2U_3^\dagger U_4^\dagger]_{\alpha\beta}\\ &= U^p_{\alpha\beta}.\end{aligned}$$

Therefore, the disorder operators $\Sigma^+_{[\vec{\theta}]}(m,n)$ leave all plaquette $U_p(m,n'>n)$ unaffected.

Thus we have shown that the disorder operator rotates a plaquette by a Wigner-$D$ matrix and hence creates a SU(N) magnetic vortex there. Now we will show that the shape of the Dirac string which is a vertical line $\mathcal{S}$ in Fig. 4 can be deformed by gauge transformations and is therefore unphysical. Deformation of Dirac string by unit plaquette at site $(m,n')$ can be achieved by replacing $E^{\mathrm{a}}_+(m,n';\hat{1})$ by $-(E^{\mathrm{a}}_-(m,n';\hat{1}) + E^{\mathrm{a}}_+(m,n';\hat{2}) + E^{\mathrm{a}}_-(m,n';\hat{2}))$ in Eq. (83). Deformed string $\tilde{\mathcal{S}}$ is shown in Fig. 4. Applying similar replacements we can change the shape of the Dirac string arbitrarily with a fixed endpoint.

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[17] This result follows from the SU(N) Young tableau in the $[\vec{j}]$ representation with total $L$ fundamental boxes. If each of them is rotated by the center element $z$ then we get $D^{[\vec{j}]}(z) = (z)^L\mathcal{I} = (z)^{\eta[\vec{j}]}\mathcal{I}$ as $z^{\mathrm{N}} = 1$. Here $\mathcal{I}$ is the identity matrix in the $[\vec{j}]$ representation of SU(N).

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[21] We have used the relation $n^{\rm a}(p)\mathcal{E}^{\rm a}_{-}(p) = -n^{\rm a}(p) \times R^{\rm ab}(\mathcal{W}^\dagger(p))\mathcal{E}^{\rm b}_{+}(p)$ and

$$\begin{aligned} & n^{\rm a}(p)R^{\rm ab}(\mathcal{W}^\dagger(p)) \\ & = {\rm Tr}(\sigma^{\rm a}\mathcal{W}(p))\frac{1}{2}{\rm Tr}(\sigma^{\rm a}\mathcal{W}^\dagger(p)\sigma^{\rm b}\mathcal{W}(p)) \\ & = \frac{1}{2}(\sigma^{\rm a}_{\alpha\beta}\sigma^{\rm a}_{\gamma\delta})\sigma^{\rm b}_{\eta\rho}\mathcal{W}_{\beta\alpha}(p)\mathcal{W}^\dagger_{\delta\eta}(p)\mathcal{W}_{\rho\gamma}(p) \\ & = \frac{1}{2}(2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta})\sigma^{\rm b}_{\eta\rho}\mathcal{W}_{\beta\alpha}(p)\mathcal{W}^\dagger_{\delta\eta}(p)\mathcal{W}_{\rho\gamma}(p) \\ & = \sigma^{\rm b}_{\eta\rho}\mathcal{W}_{\rho\eta}(p) = n^{\rm b}(p) \end{aligned}$$

[22] In defining $\mathcal{E}^{a}_{-}(p)$ we have used the fact that like $(\sigma^a)$, their transpose with a negative sign $(-\tilde{\sigma}^a)$ also satisfies the same SU(2) Lie algebra.

[23] K. S. Mallesh and N. Mukunda, The algebra and geometry of SU(3) matrices, Pramana **49**, 371 (1997).

[24] The two definitions for $\vec{n}_{[1]}(p)$ and $\vec{n}_{[2]}(p)$ in (47a) and (47b) respectively are easily generalizable to SU(N) case discussed in the next section.

[25] We have used the following two identities
(1) $(f^{\rm abe}d^{\rm dce} + f^{\rm ace}d^{\rm bde} + f^{\rm ade}d^{\rm bce}) = 0.$
(2) $(d^{\rm abe}d^{\rm dce} + d^{\rm ace}d^{\rm bde} + d^{\rm ade}d^{\rm bce}) = \frac{1}{3}(\delta^{\rm ab}\delta^{\rm cd} + \delta^{\rm ac}\delta^{\rm bd} + \delta^{\rm bc}\delta^{\rm ad}).$

[26] We are ignoring the SU(3) multiplicity problem here as the aim in this work is to construct the SU(3) magnetic eigenstates and not to worry about SU(3) multiplicities. One can trivially replace all SU(3) prepotentials by SU(3) irreducible prepotentials [18] at the end without changing any results of this section. The same strategy will be adapted in the next SU(N) section to keep the discussion simple.

[27] H. Georgi, *Lie Algebras in Particle Physics: From Isospin to Unified Theories* (Perseus Books, Reading, Massachusetts, 1999).

[28] Advantage of this representation is that it has the following property:

$$\begin{aligned} Z(\omega_1,\omega_2)Z(\theta_1,\theta_2) &= e^{(i(\omega_1\hat{n}^{\rm a}_{[1]}+\omega_2\hat{n}^{\rm a}_{[2]})\frac{\lambda^{\rm a}}{2})}e^{(i(\theta_2\hat{n}^{\rm b}_{[1]}+\theta_2\hat{n}^{\rm b}_{[2]})\frac{\lambda^{\rm b}}{2})} \\ \because\ e^Xe^Y &= e^{X+Y+\frac{1}{2}[X,Y]+\cdots}, [\lambda^{\rm a},\lambda^{\rm b}] = 2if^{\rm abc}\lambda^{\rm c}, \\ f^{\rm abc}\hat{n}^{\rm b}_{[h]}\hat{n}^{\rm c}_{[h']} &= 0, h, h' = 1, 2 \\ &= e^{(i((\omega_1+\theta_1)\hat{n}^{\rm a}_{[1]}+(\omega_1+\theta_2)\hat{n}^{\rm a}_{[2]})\frac{\lambda^{\rm a}}{2})} \\ &= Z(\omega_1+\theta_1,\omega_2+\theta_2) \end{aligned}$$

Which we will use to show the translation of two gauge invariant angles through the action of the disorder operator.

[29] We have used the property: $(d^{\rm abe}d^{\rm cde} + d^{\rm ace}d^{\rm bde} + d^{\rm ade}d^{\rm bce}) = \frac{1}{3}(\delta^{\rm ab}\delta^{\rm cd} + \delta^{\rm ac}\delta^{\rm bd} + \delta^{\rm bc}\delta^{\rm ad})$ for SU(3). It can be similarly generalized to SU(N) with $({\rm N}-1)$ d structure functions.