Relating β'_* and γ'_{Q*} in the $\mathcal{N}=1$ SQCD conformal window

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In this paper we show that β'_*, the β-function slopes in the electric and magnetic theories, are equal at the corresponding infrared fixed points. This follows from the scaling of the correlators of the trace of the energy momentum tensors. The slopes β'_* determine the scaling dimensions. Our paper can be considered as a commentary to D. Anselmi, M. T. Grisaru, and A. Johansen [Nucl. Phys. **B491**, 221 (1997)]; it proposes an improved derivation not based on a rather contrived construction by D. Kutasov [Phys. Lett. B **351**, 230 (1995)], D. Kutasov and A. Schwimmer [Phys. Lett. B **354**, 315 (1995)], and D. Kutasov, A. Schwimmer, and N. Seiberg, [Nucl. Phys. **B459**, 455 (1996)]. As a byproduct we note that γ'_{Q^*}—the slopes of the matter superfield anomalous dimension—vanish at both edges of the conformal window where one of the dual theories is strongly coupled. Finally, we determine the two-coupling magnetic fixed point at weak coupling correcting the result of I. I. Kogan, M. A. Shifman, and A. I. Vainshtein [Phys. Rev. D **53**, 4526 (1996); Phys. Rev. D**59**, 109903(E) (1999)].

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I. INTRODUCTION

Yang-Mills theories with $\mathcal{N}=1$ supersymmetry produce a wide variety of exact results. One of the most important is the Seiberg duality (for reviews see Refs. [1–4]) which states that in deep infrared an SU(N_c) gauge theory of N_f flavors is dual to a SU(N_D) ($N_D \equiv N_f - N_c$) gauge theory with N_f flavors and N_f^2 color-singlet mesons. As is usual with dualities, when the original (electric) theory is weakly coupled then the dual theory, referred to as magnetic, is strongly coupled, and vice versa.

In the conformal window (CW) which lies in the interval,

$$\frac{3}{2}N_c \le N_f \le 3N_c, \tag{1}$$

both theories are asymptotically free and conformal in the IR due to an IR fixed point. We will limit ourselves to the window (1) assuming that $N_{f,c} \gg 1$ with N_f/N_c fixed. The lower and upper boundaries are referred to as the CW edges. Above the upper edge the electric theory becomes free while at $N_f < \frac{3}{2}N_c$ the same transition happens in the magnetic theory. The edges can be obtained either from the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) beta functions [5,6] or from the unitarity bound [7]. Recently, an alternative interpretation has been given in terms of a smooth matching to the chirally broken phase with pion physics [8].

In this paper we focus on the slopes of the β functions in both dual theories in the deep IR (labeled by the subscript $_*$),

$$\beta'_* = \frac{\partial}{\partial\alpha}\beta\Big|_* . \tag{2}$$

Since the above slope is related to the scaling dimension of a physically observable operator, the trace of the energy-momentum tensor (TEMT), which has a geometrical meaning, the slopes in the electric and magnetic theories must coincide,

$$\beta'_*|_{\rm el} = \beta'_*|_{\rm mag}, \tag{3}$$

for each given value of N_f from the CW. For a the definition of $\beta'_*|_{\rm mag}$ see Sec. IV.

Equation (3) was originally obtained in [9] by analyzing the Konishi currents in both dual theories on the basis of the Kutasov construction [10].[1] Our goal in this paper is to

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[1]That Eq. (3) could be true is supported indirectly by the fact that in perturbation theory the sign of $\beta'|_{\rm el}$ changes at higher orders in the expansion [11].

bypass the Kutasov construction which is not needed for the derivation of (3). Also, we derive a previously unknown relation between γ'_{Q^*} and β'_* in the electric theory and revisit numerical calculations near the CW edges.

The organization of the paper is as follows. In Sec. II we collect some basic elements of the $\mathcal{N}=1$ theories which are dual in the CW. Section III is devoted to the study of the two-point function of the TEMTs which allows us to establish the anomalous dimension of the x dependence of the two-point function in question In Sec. IV we address the analogous two-point function in the magnetic theory. In the latter, in addition to the gauge interaction, a Yukawa interaction is present too, proportional to a meson and two quark superfields. Therefore, instead of the single β function of the electric theory, in the magnetic theory we have to deal with two β functions. We define the notion of β'_* in the magnetic theory and determine this quantity. In Sec. V we derive a new relation between γ'_{Q^*} and β'_*. In Sec. VI we calculate β's and other necessary parameters at weak coupling near the edges of the CW. We also determine a stable IR fixed point which corrects the result of [12]. Finally, the Appendix is devoted to the derivation of the TEMT in the supersymmetric formalism and the R and Konishi current correlators.

II. PRELIMINARIES AND NOTATION

In this section we outline the formalism to be used below and introduce our notation. The latter follows the second book in [2] (which, in turn, is very close to that of Wess and Bagger [13]).

The electric theory contains SU(N_c) gauge bosons and the following matter sector; $2N_f$ chiral superfields in the fundamental representation, namely, N_f fundamentals Q_k and N_f antiuindamentals $\tilde{Q}^k$, $k=1,2,\ldots,N_f$. The Lagrangian has the form

$$\mathcal{L}=\left(\frac{1}{4g^2}\int d^2\theta W^{\alpha a}W^a_\alpha+\text{H.c.}\right)+\sum_{\text{all flavors}}\left\{\int(d^2\theta d^2\bar{\theta}\bar{Q}^{\bar{f}}e^V Q_f+\int d^2\theta d^2\bar{\theta}\bar{\tilde{Q}}^{\bar{f}}e^{-V}\tilde{Q}_f)\right\}. \tag{4}$$

The above Lagrangian is written in the ultraviolet; as we descend down to the IR, $1/g^2$ is replaced by the running constant $1/g(\mu)^2$ and the matter-field Z factor $[Z_Q(\mu)]$ appears in front of the second term in (4).[2]

In the dual magnetic theory the dual quark superfields are denoted as q_f and $\tilde{q}^f$ and the dual color is

$$N_D\equiv N_f-N_c. \tag{5}$$

[2]We omit the subscript "el" where there is no option of confusion.

In addition, in the magnetic theory one has to introduce a color-singlet matter field represented by the matrix M^i_j and the superpotential

$$\mathcal{W}=\mathrm{f}\,M^i_j q_i\tilde{q}^j, \tag{6}$$

where f is the Yukawa coupling which can be chosen to be real. The corresponding kinetic term is normalized canonically, $\mathrm{Tr}(\bar{M}M)$.

The β function and the matter anomalous dimension are defined as

$$\beta(\alpha)=\frac{\partial\alpha(\mu)}{\partial L},\qquad \alpha=\frac{g^2}{4\pi},\qquad \gamma_Q=-\frac{d\log Z_Q}{dL},\qquad L=\log\mu. \tag{7}$$

In the dual magnetic theory we will introduce g_D^2 and $\alpha_D=\frac{g_D^2}{4\pi}$.

Finally we will need the expression for the NSVZ beta function [5,6],

$$\beta=-\frac{\alpha^2}{2\pi}[3N_c-N_f(1-\gamma_Q)]\left(1-\frac{N_c\alpha}{2\pi}\right)^{-1}. \tag{8}$$

At one loop

$$\gamma_Q=-\frac{\alpha}{\pi}\frac{N_c^2-1}{2N_c}\to-\frac{\alpha}{2\pi}N_c. \tag{9}$$

We will also need the expression for the hypercurrent divergence which includes all three geometric anomalies. In the operator form it can be read off from Eq. (A1) provided we omit the first line which is needed only in the magnetic theory. Then we conclude that

$$(\theta^\rho_\rho)_{\rm el}=\mathcal{C}_G\beta(\alpha)\mathcal{O}_G, \tag{10}$$

where

$$\mathcal{O}_G=-2\mathrm{Re}[W^{\alpha a}W^a_\alpha]_{\theta^2\mathrm{el}}=[G^a_{\mu\nu}G^{\mu\nu\,a}-2D^2-4i\bar{\lambda}^a_{\dot{\alpha}}\mathcal{D}^{\dot{\alpha}\alpha}\lambda^a_\alpha]_{\rm el}, \tag{11}$$

and

$$\mathcal{C}_G=(16\pi\alpha^2)^{-1}. \tag{12}$$

Here and in what follows we will refer to $\mathcal{O}_G$ in the right-hand side of (11) as G^2; hence, the corresponding notation.

III. $\langle\theta^\rho_\rho(x)\theta^\alpha_\alpha(0)\rangle$ IN THE ELECTRIC THEORY

For a generic local operator $O(x)$ the two-point function $\langle O(x)O(0)\rangle$ in the conformal limit takes the form $(x^2)^{-\Delta_O}$

where Δ_O is the sum of the normal (engineering) and anomalous dimensions of the operator O,

$$\Delta_O = d_O + \gamma_O. \tag{13}$$

The engineering dimension of TEMT is obviously $d_{G^2} = 4$. We will focus on the anomalous dimension.

The energy-momentum tensor $\theta_{\alpha\beta}$ is symmetric and conserved and has a geometric nature. This tensor itself does not change under the variation of μ, as we descend from the UV to the IR. Hence, its trace is not renormalized either. The particular expression for θ^ρ_ρ depends on how we normalize the gluon-field stress tensor, but the final result for the x-scaling dependence near the conformal point is unambiguously determined by the theory and is physical,

$$\langle \mathcal{O}_G(x)\mathcal{O}_G(0)\rangle \propto \frac{1}{(x^2)^{\Delta_{G^2}}}, \qquad \Delta_{G^2} = d_{G^2} + \gamma_{G^2}, \tag{14}$$

where d_{G^2} is the engineering dimension of the operator $\mathcal{O}_G$ and γ_{G^2} is the anomalous dimension of $\mathcal{O}_G$.

Without loss of generality we can choose $\mathcal{O}_G$ normalized in accordance with Eq. (4). The TEMT is given in (10)–(12) where the β function in the electric theory is presented in (8). Differentiating both sides over $L = \log\mu$ we arrive at

$$0 = \mathcal{O}_G \frac{\partial}{\partial L}\left(\frac{\beta}{\alpha^2}\right) + \frac{\beta}{\alpha^2}\frac{\partial \mathcal{O}_G}{\partial L}, \tag{15}$$

which implies

$$\beta(\alpha)\mathcal{O}_G \frac{\partial}{\partial\alpha}\left(\frac{\beta}{\alpha^2}\right) - \gamma_{G^2}\,\mathcal{O}_G\frac{\beta}{\alpha^2} = 0. \tag{16}$$

Taking into account that $\beta(\alpha_*) = 0$ while $\alpha_* \neq 0$ we find the well-known result

$$(\gamma_{G^2})_* = \frac{\partial\beta}{\partial\alpha}\bigg|_* \equiv \beta'_*. \tag{17}$$

Finally we can present the two-point function of the TEMTs at the points x and 0. Note that the x dependence is fully determined by the two-point function (14); therefore,

$$\begin{aligned}\langle\theta^\rho_\rho(x)\theta^\rho_\rho(0)\rangle &\propto [(16\pi\alpha^2)^{-1}\beta(\alpha)|_\mu]^2\langle\mathcal{O}_G(x)\mathcal{O}_G(0)\rangle_\mu \\ &\propto [(16\pi\alpha^2)^{-1}\beta(\alpha)|_\mu]^2\frac{1}{(x^2)^4}\frac{1}{(x^2\mu^2)^{\beta'_*}}.\end{aligned} \tag{18}$$

The μ dependence in Eq. (18) enters explicitly through $\mu^{-2\beta'_*}$, and implicitly, through the prefactor. At the conformal point $\mu \to 0$ the scale dependence of the prefactor

$$P = (16\pi\,\alpha^2)^{-1}\beta(\alpha)|_\mu$$

is

$$P \to \left(\frac{\mu}{\Lambda}\right)^{\beta'_*}, \tag{19}$$

where Λ is a μ independent scale parameter which can be seen as the analog of $\Lambda_{\rm QCD}$ that determines the logarithmic running in the chirally broken phase. As a result, we arrive at the following final result for the correlator at hand,

$$\langle\theta^\rho_\rho(x)\theta^\alpha_\alpha(0)\rangle \propto \frac{1}{(x^2)^4}\frac{1}{(x^2\Lambda^2)^{\beta'_*}}. \tag{20}$$

This is our main result in this section. For completeness, and since it follows rather directly, we derive the R and Konishi current correlators in Appendix B.

At first sight, Eq. (20) might seem confusing. Indeed, as is well-known in the conformal theory the TEMT scaling dimension is four since the dilatation current is conserved. In the conformal limit the β function exactly vanishes and so does the two-point function (up to contact terms) under consideration. This is consistent with the TEMT vanishing on "physical states."

Our analysis is carried out in the vicinity of the conformal point where the β function can be approximated by the first nontrivial term of its expansion; namely,

$$\beta = \beta'(\alpha_*)(\alpha - \alpha_*), \qquad \beta(\alpha_*) = 0,$$

with $\alpha - \alpha_*$ small. At the very end we take the limit $\alpha - \alpha_* \to 0$ where possible. Our strategy should be viewed as a perturbation theory around the conformal point with the correlator expanded in terms of β as can be seen from Eq. (18).

IV. $\langle\theta^\rho_\rho(x)\theta^\alpha_\alpha(0)\rangle$ IN THE MAGNETIC THEORY

In the magnetic theory the number of colors N_D is given in (5). In other words, $\mathrm{SU}_{\rm gauge}(N_c) \to \mathrm{SU}_{\rm gauge}(N_f - N_c)$, with the same CW

$$\frac{3}{2}N_D \leq N_f \leq 3N_D. \tag{21}$$

The matter fields of the magnetic theory q_i and $\tilde{q}^j$ belong to the (anti)fundamental representations of $\mathrm{SU}(N_D)$. In addition one must add a color-singlet matrix field M^i_j, depending on the flavor indices, and the superpotential shown in Eq. (6) where f is the "second" holomorphic coupling constant of the theory. The Lagrangian takes the form

$$\begin{aligned}\mathcal{L} = &\left(\frac{1}{4g_D^2(\mu)}\int d^2\theta W^{a\alpha}W^a_\alpha + \text{H.c.}\right) \\ &+ \sum_{\text{all flavors}} Z_q(\mu)\int d^2\theta d^2\bar\theta \tilde{q}^{\bar f} e^V q_f \\ &+ \int d^2\theta d^2\bar\theta [Z_M(\mu)\mathrm{Tr}\,(\bar{M}M)] \\ &+ \left(\int d^2\theta\mathcal{W}(\tilde{q},q,M) + \text{H.c.}\right),\end{aligned} \tag{22}$$

where the superpotential $\mathcal{W}$ is defined in (6). The M superfield is in the magnetic representation; therefore its dimension in the UV is one, and the coupling f is dimensionless. The trace in the second line of (22) runs over flavors. The Lagrangian (22) explicitly exhibits the effect of the renormalization group (RG) flow. At the classical level it is scale and conformally invariant. The μ dependence breaks the scale symmetry and gives rise to two β functions, β_D for the gauge coupling and β_{f} for the super-Yukawa coupling. The latter appears only due to the $Z_{q,M}$ factors when we apply the equations of motion. Indeed, in passing to the canonically normalized matter kinetic terms we obtain

$$\frac{\mathrm{f}^2}{4\pi} \to \frac{\mathrm{f}(\mu)^2}{4\pi} = \frac{f^2}{4\pi}[Z_q(\mu)^2 Z_M(\mu)]^{-1}. \tag{23}$$

In what follows we will use the notation

$$a_{\mathrm{f}} \equiv \frac{\mathrm{f}^2}{4\pi}. \tag{24}$$

Then, the expressions for the magnetic β functions are as follows:

$$\begin{aligned}\beta_{\mathrm{f}}(\alpha_D, a_{\mathrm{f}}) &\equiv \frac{\partial}{\partial L} a_{\mathrm{f}} = a_{\mathrm{f}}[\gamma_M(\alpha_D, a_{\mathrm{f}}) + 2\gamma_q(\alpha_D, a_{\mathrm{f}})],\\ \beta_D(\alpha_D, a_{\mathrm{f}}) &= -\frac{\alpha_D^2}{2\pi}[3N_D - N_f(1-\gamma_q(\alpha_D, \mathrm{f}))]\\ &\quad \times \left(1 - \frac{N_D \alpha_D}{2\pi}\right)^{-1},\end{aligned} \tag{25}$$

where β_D is the β function (8) with $N_c \to N_D$ and $\gamma_Q \to \gamma_q$ and the one for β_{f} is obtained by differentiating (23). The latter holds in principle up to nonperturbative corrections since the nonrenormalization theorem of the superpotential is perturbative in nature. However, at weak coupling, nonperturbative corrections are exhausted by instantons but they are absent since the R-symmetry implies that the zero modes do not match for $N_f > N_c + 1$. Moreover, it has been argued that the absence of renormalon ambiguities inside the conformal window implies the absence of non-perturbative corrections [14]. We therefore conjecture that the expression for β_{f} is formally correct for $N_f > N_c + 1$.

At the critical values of the coupling constants the sum of anomalous dimensions vanishes,

$$\gamma_{M*} + 2\gamma_{q*} = 0.$$

Explicit leading-order expressions are given in Sec. VI.

In what follow we aim to show that (20) holds equally for the magnetic theory when suitably adapted. This requires a bit more work since in the magnetic theory we have two couplings. The TEMT in the magnetic theory, given in (A2), reads

$$T^{\rho}{}_{\rho} = \mathcal{C}_G \beta_D(\alpha_D, a_{\mathrm{f}})\mathcal{O}_G + \mathcal{C}_{\mathcal{W}}\beta_{\mathrm{f}}(\alpha_D, a_{\mathrm{f}})\mathcal{O}_{\mathcal{W}}, \tag{26}$$

where $\mathcal{O}_G$ and $\mathcal{O}_{\mathcal{W}}$ corresponds to the gluon and superpotential part respectively, see Eqs. (A3)–(A5) in the Appendix A. We may linearize both β functions around the IR fixed point. If we define the coupling vector

$$\delta\underline{\alpha} \equiv \begin{pmatrix} \alpha_D - \alpha_{D_*} \\ a_{\mathrm{f}} - a_{\mathrm{f}_*} \end{pmatrix}, \tag{27}$$

the linearized β function can be written as

$$\frac{\partial}{\partial L}\delta\underline{\alpha} = B_* \delta\underline{\alpha} + \mathcal{O}((\delta\underline{\alpha})^2), \tag{28}$$

where B_* is the gradient matrix

$$B_* = \begin{pmatrix} \partial_{\alpha_D}\beta_D & \partial_{a_{\mathrm{f}}}\beta_D \\ \partial_{\alpha_D}\beta_f & \partial_{a_{\mathrm{f}}}\beta_f \end{pmatrix}_*, \tag{29}$$

evaluated at the IR fixed point. The derivatives in the matrix B are just numbers independent of the running α_D and a_{f} which can depend, however, on the numerical values of α_{D_*} and a_{f_*} (cf. Sec. VI).

Next, we can diagonalize the matrix B_*, which generically has two unequal real eigenvalues $\lambda_- \leq \lambda_+$.[3] Indeed, let us introduce the matrix U diagonalizing B_*,

$$B_{\mathrm{diag}} \stackrel{\mathrm{def}}{=} \hat{B} = U^{-1} B_* U, \qquad \hat{B} = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}. \tag{30}$$

Correspondingly, the "diagonalized form" for the column $\delta\underline{\alpha}$ in (27) becomes

$$\delta\underline{\hat{\alpha}} \equiv U^{-1}\delta\underline{\alpha}. \tag{31}$$

Then we can write

$$\frac{d}{dL}\delta\underline{\alpha} = B_*\delta\underline{\alpha} = U\hat{B}\delta\underline{\hat{\alpha}}, \tag{32}$$

or, alternatively

$$\frac{d}{dL}\delta\underline{\hat{\alpha}} = \hat{B}\delta\underline{\hat{\alpha}}. \tag{33}$$

The solution of the equation above takes the form,

$$\delta\underline{\hat{\alpha}} = \begin{pmatrix} (\frac{\mu}{\Lambda_-})^{\lambda_-} \\ (\frac{\mu}{\Lambda_+})^{\lambda_+} \end{pmatrix}, \tag{34}$$

where the eigenvalues $\lambda_{\mp}$ have to be non-negative and $\Lambda_{\mp}$ are related to the choice of initial condition, cf. Fig. 1.

[3]These eigenvalues are scheme independent under analytic coupling redefinitions e.g., [15].

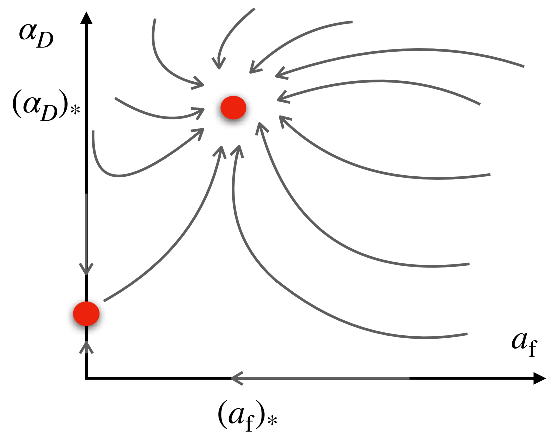

FIG. 1. Illustration of the RG flow in the magnetic theory. Red dots denote IR fixed points given in (52) for N_f just above $\frac{3}{2}N_c$ where the magnetic theory is weakly coupled.

Now let us return to Eq. (26). Introducing a row of operators $\mathcal{O}$,

$$\mathcal{O} = \{\mathcal{C}_G\mathcal{O}_G, \mathcal{C}_W\mathcal{O}_W\}, \tag{35}$$

we can rewrite (26), using the linear approximation in Eq. (32), as follows:

$$\theta^\rho{}_\rho = \mathcal{O}B_*\delta\underline{\alpha} = \hat{\mathcal{O}}\,\hat{B}\,\delta\underline{\hat{\alpha}}, \qquad \hat{\mathcal{O}} = \mathcal{O}U. \tag{36}$$

Finally, we can find the matrix of anomalous dimensions Γ,

$$\Gamma = \begin{pmatrix} \gamma_{\mathcal{O}_1} & 0 \\ 0 & \gamma_{\mathcal{O}_2} \end{pmatrix}, \tag{37}$$

for the operators $\mathcal{O}$, or to be more exact, for two linear combinations in $\hat{\mathcal{O}}$. To this end we differentiate both sides in (36) over ∂L and arrive at

$$0 = -\hat{\mathcal{O}}\Gamma\hat{B}\delta\underline{\hat{\alpha}} + \hat{\mathcal{O}}\,\hat{B}\,\hat{B}\,\delta\underline{\hat{\alpha}}, \tag{38}$$

implying, in turn, that

$$\Gamma = \hat{B}, \tag{39}$$

cf. Eq. (30). In deriving (38) we used Eq. (33). From the above we conclude that

$$\langle\hat{\mathcal{O}}_1(x)\hat{\mathcal{O}}_1(0)\rangle \propto \frac{1}{(x^2)^4}\frac{1}{(x^2\mu^2)^{\lambda_-}}, \tag{40}$$

where λ_- is the lowest eigenvalue of the matrix $\hat{B}$ and, hence, of the matrix B_*, see Eq. (29).[4] Following [9] to simplify notation we will denote

$$\lambda_- = \beta'_{\rm mag}. \tag{41}$$

Summarizing this section we conclude that

$$\langle T^\rho{}_\rho(x)T^\alpha{}_\alpha(0)\rangle_{\rm mag} \propto \frac{1}{(x^2)^{4+\beta'_*|_{\rm mag}}}. \tag{42}$$

Since the Seiberg dual correlators must coincide in the corresponding IR fixed points,

$$\langle T^\rho{}_\rho(x)T^\alpha{}_\alpha(0)\rangle_{\rm mag} \overset{\rm IR}{\longleftrightarrow} \langle T^\rho{}_\rho(x)T^\alpha{}_\alpha(0)\rangle_{\rm el}, \tag{43}$$

we confirm that $\beta'_*|_{\rm el} = \beta'_*|_{\rm mag}$ holds as stated in Eq. (3) in the introduction. This is the central result of our paper.

V. RELATION BETWEEN γ'_{Q^*} AND β'_*

The β function of the electric theory (8) is essentially a relation between the matter-field anomalous dimension γ_Q and the β function itself. Since we have gained information on β'_* we may exploit this fact to deduce information on γ'_{Q*} by directly differentiating at the IR fixed point

$$\beta'_* = -\frac{\alpha_*^2}{2\pi}\frac{N_f}{1-\frac{\alpha_*}{2\pi}N_c}\gamma'_{Q*}. \tag{44}$$

Hence, β'_* and γ'_{Q^*} are proportional to each other throughout the CW with a coefficient depending on the unknown critical coupling α_*. We note that the relation (44) is generally scheme dependent and so is the fixed-point coupling α_* but β'_* and γ'_{Q^*} are scheme independent under analytic redefinitions of the coupling, cf. footnote 3. The fact that both β'_* and γ'_{Q^*} are zero simultaneously might not be completely accidental. They both describe the perturbation around the fixed point for a gauge theory with massive mater, e.g., [16]. We remind the reader that γ_Q equals minus the anomalous dimension of the mass to all orders in perturbation theory in $\mathcal{N}=1$ supersymmetry.

With (3) one obtains a strong coupling relation but unfortunately the right-hand side contains the two unknowns α_* and γ'_{Q^*} for which we cannot solve simultaneously. Nevertheless, one can deduce interesting information. Since, β'_* as a function of N_f starts at zero for $N_f = 3N_c$ and then raises and lowers towards zero again at $N_f = \frac{3}{2}N_c$, there must be at least two number of flavors, $N_f^{(w)} > N_f^{(s)}$, for which

$$\beta'_*|_{N_f^{(w)}} = \beta'_*|_{N_f^{(s)}}, \tag{45}$$

holds (the superscripts w and s stand for weakly and strongly coupled with regards to the electric coupling). As we expect the electric coupling to become continuously stronger towards the lower edge of the CW one finds

[4] In fact, tracking the μ dependence in the prefactors one would be able to see that μ in the denominator will be replaced by Λ_-, in much the same way as in passing from Eq. (18) to Eq. (20).

$$\alpha^*|_{N_f^{(w)}} < \alpha^*|_{N_f^{(s)}} \Leftrightarrow \gamma'_{Q^*}|_{N_f^{(w)}} > \gamma'_{Q^*}|_{N_f^{(s)}}, \qquad (46)$$

which shows the curious result that γ'_{Q^*} is larger when the theory is weakly coupled and vice versa.

VI. NEAR THE EDGES OF THE CONFORMAL WINDOW

With the knowledge of the β functions one can investigate them at weak coupling. This is particularly interesting in the magnetic case where there are two couplings.

As a warm-up we will first consider the electric case. The β function is given in (3) and since the electric theory is weakly coupled for N_f just below $3N_c$ we expand in the following quantity

$$\epsilon \equiv \frac{3N_c - N_f}{N_f} \ll 1, \qquad (47)$$

for which we find the critical coupling and the slope to be

$$\alpha_* = \frac{2\pi}{N_c}\epsilon, \qquad \beta'_* = 3\epsilon^2, \qquad (48)$$

upon using Eq. (9) for γ_Q.

In the magnetic case we first need to obtain the explicit form of the Yukawa β function in (25). We need the anomalous dimension at leading order in the couplings. We have computed them explicitly

$$\gamma_q = -\frac{\alpha_D}{\pi}\frac{N_D^2 - 1}{2N_D} + \frac{a_{\mathrm{f}}}{2\pi}N_f, \qquad \gamma_M = \frac{a_{\mathrm{f}}}{2\pi}N_D, \qquad (49)$$

and find agreement with the results found in Sec. 8 of [12]. The gauge coupling part is identical to the electric case with the replacement $N_c \to N_D$ and the part proportional to the Yukawa coupling a_{f} is related to the computation in the Wess-Zumino model.[5] Assembling (25) and (49) we get the explicit Yukawa β function to leading order,

$$\beta_{\mathrm{f}} = a_{\mathrm{f}}\left[\frac{a_{\mathrm{f}}}{2\pi}(N_D + 2N_f) - \frac{2\alpha_D}{\pi}\frac{N_D^2 - 1}{2N_D}\right]. \qquad (50)$$

The function β_{f} differs from Eq. (64) in [12] by a simple typo, namely, in [12] one should replace

$$\frac{\alpha_D}{\pi} \to \frac{2\alpha_D}{\pi},$$

cf. our Eq. (50). This typo seems to have propagated further in their analysis and we thereby correct the IR fixed point found in that paper. In analogy to the electric case we define

$$\epsilon_D = \frac{3N_D - N_f}{N_f} \ll 1, \qquad (51)$$

for the dual magnetic theory to find the fixed point for the β functions given in (50) with β_{f} approximated as above. Assuming an ansatz of the form $\alpha_D, a_{\mathrm{f}} \propto \epsilon_D$ we find the following two solutions:

$$\frac{N_D}{2\pi}(\alpha_D, a_{\mathrm{f}})_* = \epsilon_D \begin{cases} (1,0) & a_{\mathrm{f}} = 0 \\ (7,2) & a_{\mathrm{f}} \neq 0 \end{cases}. \qquad (52)$$

The first fixed point with no Yukawa is of the Banks-Zaks type whereas the second one with the Yukawa coupling switched on is less well-known. The Banks-Zaks fixed point is unstable as the RG flow tends to the other fixed point for $a_{\mathrm{f}} \neq 0$ (cf. Fig. 1 and [12]).

In order to obtain the slope we need the eigenvalues of the B_* matrix (29), for which we find

$$\lambda_- = 21\epsilon_D^2, \qquad \lambda_+ = 14\epsilon_D, \qquad (53)$$

such that $\lambda_- < \lambda_+$ for $\epsilon_D \ll 1$. Since the slope of the β function is determined by the minimal eigenvalue we finally get the slope in terms of ϵ_D[6]

$$\beta'_*|_{\mathrm{el}} = \beta'_*|_{\mathrm{mag}} = 21\epsilon_D^2. \qquad (54)$$

It is also interesting to consider the eigenvectors is this approximation. We find the following nonorthogonal eigenvectors:

$$(\vec{v}_-)^T = \left(1, \frac{63}{2}\epsilon_D\right), \qquad (\vec{v}_+)^T = \frac{1}{\sqrt{53}}(2, -7), \qquad (55)$$

corresponding to the eigenvalues given above. We infer that for small ϵ_D the gluonic operator dominates over the Yukawa term.

VII. CONCLUSIONS

In this paper we have shown that the slopes of the β function at the IR fixed point, are equal to each other (3) in the electric and the magnetic theories of the Seiberg duality. This result was derived some time ago using the Konishi currents and the Kutasov construction [9]. We found a simpler way to obtain this result by matching the two-point function of the trace of the energy-momentum tensor in the electric and magnetic theory. By the very assumption of the

[5] In the Wess-Zumino model with superpotential $W(\Phi) = \frac{Y}{6}\Phi^3$, the Z-factor of the superfield Φ, as given in Eq. (2.7) in [17] for example, is related to $Z_q = Z_\Phi|_{Y^2 \to 2\mathrm{f}^2 N_f}$. Moreover, $Z_M = Z_q|_{N_f \to N_D, g_D \to 0}$. Our explicit computation passes this cross-check.

[6] The result in (54) can be compared to the one in [9] where they obtained $\beta'_*|_{\mathrm{mag}} = \frac{21}{4}\epsilon_D^2$, upon using the conversion $\sigma \equiv \frac{3}{2} - \frac{N_f}{N_c} = \frac{3}{4}\epsilon_D$, which differs by a factor of 4.

Seiberg duality such a geometric quantity has to match and since its scaling is governed by β'_* the result follows.

In passing we obtained a new relation between between γ'_{Q^*} and β'_* given in Eq. (44). The RG flow near the edge of the conformal window previously discussed is [12] is corrected. We obtain in addition the corresponding eigenvalues and eigenvectors of the flow in the magnetic theory. These results might be useful in that β'_* and γ'_{Q^*} are the quantities that describe perturbations around a fixed point in a gauge theory with matter.

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APPENDIX A: THE TRACE OF THE ENERGY-MOMENTUM TENSOR

In superfields, the anomalies in the TEMT and in the divergence of the R current are given by a unified formula for the hypercurrent $\mathcal{J}_{\alpha\dot{\alpha}}$ (see the second reference in [2], Sec. X.27.4) which for our choice of the superpotential (6) takes on the form

$$\partial^{\alpha\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = -\frac{i}{3}D^2\left\{\left[-\left(\frac{\gamma_M}{2}+\gamma_q\right)\mathcal{W}\right] - \frac{1}{16\pi^2}[3N_D - N_f + N_f\gamma_q]\mathrm{Tr}W^2\right\} + \mathrm{H.c.}, \tag{A1}$$

where D is the spinorial derivative which singles out the θ^2 component on the right-hand side. Equation (A1) refers to the magnetic theory. In the electric theory $\mathcal{W}=0$, so the first line disappears, $N_D \to N_c$ and $\gamma_q \to \gamma_Q$. Equation (A1) implies

$$\theta^\rho_\rho = \mathcal{C}_G\beta(\alpha_D, a_\mathrm{f})\mathcal{O}_G + \mathcal{C}_\mathcal{W}\beta_\mathrm{f}(\alpha_D, a_\mathrm{f})\mathcal{O}_\mathcal{W}, \tag{A2}$$

where

$$\mathcal{C}_G = (16\pi\alpha^2)^{-1}, \qquad \mathcal{C}_\mathcal{W} = \sqrt{\frac{4\pi}{a_\mathrm{f}}}, \tag{A3}$$

and

$$\begin{aligned}\mathcal{O}_G &= -2\mathrm{Re}[W^{\alpha a}W^a_\alpha]_{\theta^2\mathrm{mag}} \\ &= [G^a_{\mu\nu}G^{\mu\nu a} - 2D^2 - 4i\bar{\lambda}^a_{\dot{\alpha}}\mathcal{D}^{\dot{\alpha}\alpha}\lambda^a_\alpha]_\mathrm{mag},\end{aligned} \tag{A4}$$

$$\mathcal{O}_\mathcal{W} = -2\mathrm{Re}[M^i_j q_i\tilde{q}^j]_{\theta^2} = -2\mathrm{Re}[qF_M\tilde{q} + \psi\tilde{\psi}M + \mathrm{perm}]. \tag{A5}$$

Here D is the D-term of the gauge superfield and F is the F-term of the chiral superfields.

APPENDIX B: THE R CURRENT AND KONISHI CURRENT CORRELATORS

Continuing from Sec. III we can find the scaling dimension of the R and Konishi current in the electric theory without much further effort. The (unimproved) R current, enters the same supermultiplets as the energy-momentum tensor,

$$R_\mu = -\frac{1}{g^2}\lambda^a\sigma_\mu\bar{\lambda}^a + \frac{1}{3}\sum_f(\psi_f\sigma_\mu\bar{\psi}_f - 2i\phi_f\overleftrightarrow{D}_\mu\bar{\phi}_f). \tag{B1}$$

This (unimproved) current is not conserved because of the chiral anomaly. The R symmetry is anomalous,

$$\partial_\mu R^\mu = [(-24\pi\alpha^2)^{-1}\beta(\alpha)(G\tilde{G}+\cdots)]_\mu, \tag{B2}$$

where $G\tilde{G} \equiv G^a_{\mu\nu}\tilde{G}^{\mu\nu a}$ with the dual tensor $\tilde{G}^{\mu\nu a} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}G^{\alpha\beta a}$ and

$$G^a_{\mu\nu}\tilde{G}^{\mu\nu a} + \cdots \propto \mathrm{Im}W^2,$$

cf. (A4). Taking into account the fact that the anomalous dimension of $\mathrm{Im}W^2$ is the same as that of $\mathrm{Re}W^2$ we can readily calculate the two-point function

$$\langle\partial_\mu R^\mu(x)\partial_\nu R^\nu(0)\rangle \propto [(-24\pi\alpha^2)^{-1}\beta(\alpha)|_\mu]^2\frac{1}{(x^2\Lambda^2)^{\beta'_*}}, \tag{B3}$$

from the anomaly in the hypercurrent which includes both operators R_μ and $\theta_{\mu\nu}$ [2].

The result is the same as in (18), with the replacement $G^2 \to G\tilde{G}$. Then, we can drop the derivatives in (B3) to obtain

$$\langle R_\mu(x)R_\nu(0)\rangle \propto \frac{1}{(x^2)^3}\frac{1}{(x^2\Lambda^2)^{\beta'_*}}. \tag{B4}$$

This x scaling law differs from that in (20) by the engineering dimension of R_μ, namely $d_R = 3$ vs $d_\theta = 4$. The anomalous dimensions are exactly the same as they have to be since they belong in the same supermultiplet.

Finally, let us consider the two-point function of the Konishi current. There is a small nuance here which deserves to be discussed. The Konishi current is defined as

$$K_\mu = \sum_{\psi_f,\bar{\psi}^f}(-\psi_f\sigma_\mu\bar{\psi}^f - \phi_f i\overleftrightarrow{\mathcal{D}}_\mu\bar{\phi}^f). \tag{B5}$$

By the same token, the flavor-singlet Konishi current is not conserved due to the anomaly,[7]

[7]In the superfield language $\bar{D}^2\mathcal{J}_K = \frac{N_f}{2\pi^2}\mathrm{Tr}W^2$. The relation between K_μ in (B5) and $\mathcal{J}_K$ is as follows: K_μ is the $\theta\bar{\theta}$ component of $\mathcal{J}_K$.

$$\partial^\mu K_\mu = \frac{1}{48\pi^2} N_f G\tilde{G}. \tag{B6}$$

Next, we note that the operator $G\tilde{G}$ on the right-hand side resides in the same superfield W^2 as G^2. Therefore, the anomalous dimension of $G\tilde{G}$ is the same as that of G^2,

$$(\gamma_{G\tilde{G}})_* = (\gamma_{G^2})_*, \tag{B7}$$

where the latter has already been given in (17). Is there a difference compared to the cases of TEMT and R_μ?

The answer is positive. Indeed, θ^ρ_ρ has the zero-anomalous dimension. This is the reason why Eq. (20) has no sliding scale μ. The cancellation of μ is achieved thanks to the prefactor defined above Eq. (19). At the same time, K_μ has a nonvanishing anomalous dimension. Hence, as a result, the sliding scale μ is present in the correlation function

$$\langle K_\mu(x) K_\nu(0)\rangle \propto \frac{1}{(x^2)^3} \frac{1}{(x^2\mu^2)^{\beta'_*}}. \tag{B8}$$

If we compare this with Eq. (B4) we will see the sliding μ^2 instead of fixed Λ^2—this is the only difference. The IR scaling law (B8) for the Konishi current was first derived in [9].

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