Comment on "Gravitational lensing in Weyl gravity"

Joseph Sultana[*]
Department of Mathematics, University of Malta, Msida, Malta

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In a recent paper [Phys. Rev. D **100**, 024019 (2019)] the authors calculated the bending angle of light in Schwarzschild-de Sitter (SdS) spacetime and also in the static and spherically symmetric vacuum solution of Weyl's conformal gravity, which is sometimes referred to as the Mannheim-Kazanas (MK) spacetime. To do this they used the standard Weinberg analysis which is normally used to calculate the bending angle of light in asymptotically flat spacetimes, but limited the integration to the position of the cosmological horizon in these spacetimes. In this paper we make some comments about the bending angle formulas obtained in these spacetimes. We point out that in the case of the MK spacetime this would still lead to an unphysical term in their formula for the deflection angle, which also occurred in previous light bending formulas for this spacetime based on similar analysis.

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I. INTRODUCTION

The subject of bending of light in nonasymptotically flat spacetimes, particularly the effect of the background spacetime on the bending angle has garnered a lot of interest in the last decade. This mainly started with the issue of whether the cosmological constant Λ contributes to the bending of light in the Schwarzschild-de Sitter spacetime. The fact that Λ drops out of the null geodesic equations leads one to believe [1,2] that it should not contribute to the deflection of light (apart from the dependence of the angular diameter distances on Λ in the gravitational lensing equation), but this was disproved for the first time by Rindler and Ishak [3,4] who utilized a different method for calculating the bending angle to show that this also contains a contribution from Λ. This was followed by a mixed flurry of studies [5–12] against and in favor of Rindler and Ishak's proposal and whether Λ contributes to lensing. This debate seems to have settled down with the general consensus being that the cosmological constant plays a role in gravitational lensing, but its effect is too small to be noticeable and can be ignored for practical purposes given that it will always be smaller in magnitude when compared with other lensing affects like aberration and uncertainties in cosmological distances [13,14]. Another spacetime that drew considerable interest as regards to light bending is the static and spherically symmetric vacuum solution to conformal Weyl gravity which is more commonly know as the Mannheim-Kazanas (MK) solution [15,16]. This solution which contains a linear potential term γr in its lapse function, has first been derived by Riegert [17] was later shown to predict flat rotation curves of galaxies without the need to assume the presence of the yet elusive dark matter [18–22], although recent studies [23] have criticized this claim. When calculating the deflection of light in this spacetime, Edery and Paranjape [24] (see also [25,26]) showed that the linear term γr in the metric gives rise to a negative contribution to the bending angle that increases linearly with the impact parameter, rendering the spacetime unphysical. Moreover this contribution required that the constant γ should have the opposite sign used for the prediction of flat galactic rotation curves. It turned out that this paradoxical result arises from the fact that when using the standard formula for bending of light given by Weinberg, the authors incorrectly assumed that the MK spacetime is asymptotically flat. So applying Rindler and Ishak's method to calculate the bending angle it was later shown [27,28] that the contribution of linear term in the MK metric is inversely proportional to the impact parameter and is practically insignificant for lensing purposes considering the small magnitude of constant γ obtained from the fitting of galactic rotational curves. However, the issue of the bending of light in the Weyl gravity is far from settled and recently this has been revisited several times in the literature [29–38], and so far remains inconclusive. The main disagreements arise from the different order of approximations used to derive the bending angle formula, the association of different combinations of parameters in the MK-metric with the physical mass of the lens, and the use of various alternative geometric techniques for calculating the bending angle in nonasymptotically flat spacetimes. In a recent paper Kaşikçi and Deliduman [33] obtained the angle of deflection of light in the MK-spacetime from first principles using Weinberg's analysis but limiting the integration to the position of the

[*]joseph.sultana@um.edu.mt

cosmological event horizon in this spacetime. In the next section we make a few comments about this result.

II. LIGHT BENDING IN WEYL GRAVITY

The static and spherically symmetric vacuum solution in conformal Weyl gravity describing the geometry outside a spherical body is given, up to a conformal factor, by the metric [15,17]

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where

$$B(r) = 1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - \frac{\Lambda}{3}r^2, \quad (2)$$

and β, γ, and Λ are integration constants. Special cases of this solution include the Schwarzschild metric ($\gamma = \Lambda = 0$) and the Schwarzschild-de Sitter ($\gamma = 0$) metric; the latter requiring the presence of a cosmological constant in general relativity. The constant γ, whose magnitude and nature remain uncertain, has dimensions of acceleration, and so the solution provides a characteristic, constant Rindler-like acceleration without the need to introduce one at the Lagrangian (such as in the relativistic implementation of MOND with TeVeS [39]). The fitting of galactic rotational curves suggests [15] that $\gamma \simeq 1/R_H$, where R_H is the Hubble length, so that the effects of this acceleration are comparable to those due to the Newtonian potential term (with $\beta\gamma \ll 1$) $2\beta/r \equiv r_s/r$ (r_s is the Schwarzschild radius), on length scales given by

$$r_s/r^2 \simeq \gamma \simeq 1/R_H \quad \text{or} \quad r \simeq (r_s R_H)^{1/2}. \quad (3)$$

For example in the case of a galaxy of mass $M \simeq 10^{11} M_\odot$ with $r_s \simeq 10^{16}$ cm and $R_H \simeq 10^{28}$ cm, this scale is $r \sim 10^{22}$ cm, i.e., roughly the size of the galaxy. Moreover Eq. (3) does not describe a particular length scale but a continuum of sizes at which the contribution from the linear term becomes significant, so that objects along this sequence do not include only galaxies but, at larger scales also galaxy clusters and at lower scales globular clusters, which were also found to require the presence of dark matter in order to account for the observed dynamics [40].

The MK solution can also be expressed in the form

$$B(r) = \sqrt{1-6m\gamma} - \frac{2m}{r} + \gamma r - \frac{\Lambda}{3}r^2, \quad (4)$$

by using the reparametrization $\beta = \frac{1-\sqrt{1-6\gamma m}}{3\gamma}$. The null geodesic equation for the metric in Eq. (1) is given by [41]

$$\frac{du}{d\phi} = \sqrt{\frac{1}{R^2} - B(u)u^2}, \quad (5)$$

where $u \equiv 1/r$ and $R \equiv L/E$ is the impact parameter; E and L being the constants of motion representing the total energy and angular momentum, respectively. In an asymptotically flat spacetime, in which the observer is assumed to be located at infinity ($r = \infty$; $u = 0$) the coordinate angle difference to the point of closest approach $r = r_0$ (or $u = u_0$) where $\frac{du}{d\phi}|_{u=u_0} = 0$ is given by

$$\Delta\phi = \phi(u_0) - \phi(0) = \int_{u_0}^{0} \frac{du}{\sqrt{\frac{1}{R^2} - B(u)u^2}}, \quad (6)$$

such that if we assume that the source is also located at infinity, the total bending angle will be

$$\Delta\phi = 2|\phi(u_0) - \phi(0)| - \pi. \quad (7)$$

Since the metric in Eq. (1) is not asymptotically flat the authors in Ref. [33] extended the integration in (6) to the position of the cosmological event horizon $r = r_h$ (or $u = u_h$) instead of $u = 0$. This position is obtained by finding the largest root of $B(r) = 0$. The distance of closest approach $r = r_0$ (or $u = u_0$) is obtained by finding the largest root of $dr/d\phi = 0$ (or equivalently $du/d\phi = 0$). Doing this, the authors used an asymptotic expansion of the elliptic integral of the first kind in (6) to obtain the following approximate expression for the bending angle of light for the MK spacetime:

$$\begin{aligned}\Delta\phi = {} & m_0\left(4 - 2\sqrt{\frac{\Lambda_0}{3}} - 2\frac{\Lambda_0}{3}\right) - 2\sqrt{\frac{\Lambda_0}{3}} + \gamma_0\sqrt{\frac{\Lambda_0}{3}} \\ & + m_0^2\left(\frac{15\pi}{4} - 4 - 3\sqrt{\frac{\Lambda_0}{3}} - 2\frac{\Lambda_0}{3}\right) + m_0\gamma_0\left(2 + \frac{\Lambda_0}{3}\right) \\ & + m_0^2\gamma_0\left(\frac{15\pi}{4} - 4 - \frac{3}{2}\sqrt{\frac{\Lambda_0}{3}}\right) + \cdots, \end{aligned} \quad (8)$$

where the dimensionless parameters are defined by $m_0 = m/r_0$, $\gamma_0 = \gamma r_0$, and $\Lambda_0 = \Lambda r_0^2$. In the case of the Schwarzschild-de Sitter metric, where $\gamma = 0$ such that $B(r) = 1–2m/r - \Lambda r^2/3$, the above expression for the bending angle written in terms of the parameters m and Λ and the distance of closest approach r_0 (where $1/r_0 \sim 1/R + m/R^2$) reduces to

$$\begin{aligned}\Delta\phi = {} & -2\sqrt{\frac{\Lambda}{3}}r_0 + \frac{m}{r_0}\left(4 - 2\sqrt{\frac{\Lambda}{3}}r_0 - 2\frac{\Lambda}{3}r_0^2\right) \\ & + \frac{m^2}{r_0^2}\left(\frac{15}{4}\pi - 4\right) - \frac{m^2}{r_0^2}\left(3\sqrt{\frac{\Lambda}{3}}r_0 + 2\frac{\Lambda}{3}r_0^2\right) + \cdots. \end{aligned} \quad (9)$$

This can be compared with the corresponding formula for the bending angle obtained by Rindler and Ishak in Ref. [4] which is expressed in terms of the impact parameter R instead of the distance of closest approach r_0, namely

$$\Delta\phi = \frac{4m}{R} + \frac{15\pi}{4}\frac{m^2}{R^2} + \frac{305}{12}\frac{m^3}{R^3} - \frac{\Lambda R^3}{6m}. \qquad (10)$$

The first term in both expressions is the Schwarzschild contribution, representing the majority of the bending, and apart from the difference between R and r_0 this takes the same form in both cases as expected. The first order contribution from the cosmological constant in both cases is negative implying that Λ diminishes the bending (see also Refs. [8,42]). However this term is different in the two cases, and again this is expected considering the different methods used to calculate the bending angle. As already mentioned above, Kaşikçi and Deliduman based their calculation on Weinberg's method given by Eq. (6) by restricting the upper limit of integration to the position of the cosmological event horizon which in the case of the SdS spacetime is given by $u_h = 1/r_h \sim \sqrt{\Lambda/3}$. On the other hand Rindler and Ishak's calculation is based on the computation of the local angle ψ between the photon orbit direction $\mathbf{d}$ and the radial direction $\boldsymbol{\delta}$ ($\phi = \text{constant}$) as shown in Fig. 1, such that

$$\tan\psi = \frac{B(r)^{1/2} r}{|A(r,\phi)|}, \qquad (11)$$

where $B(r)$ is the metric coefficient in Eq. (2) with $\gamma = 0$ and $A(r,\phi) = \frac{dr}{d\phi}$. At any point along the trajectory the one sided bending angle is $\epsilon = \psi - \phi$. In their calculation Rindler and Ishak assumed that the observer, lens and source are collinear such that the total one-sided bending angle is obtained when $\phi = 0$ which occurs at $r_{\phi=0} = R^2/2m$ such that $\epsilon = \psi_0$. Then twice ψ_0 gives the total bending angle given in Eq. (10). In their analysis Rindler and Ishak assumed that the null trajectory intersects the optic axis (the line through the coaligned source, lens and observer) at $\phi = 0$ within the cosmological horizon i.e., $R^2/2m < \sqrt{\Lambda/3}$, otherwise it would connect causally unconnected regions of spacetime. In fact, this was the reason why Kaşikçi and Deliduman assumed that the entire deflection of the null trajectory is achieved within the cosmological horizon.

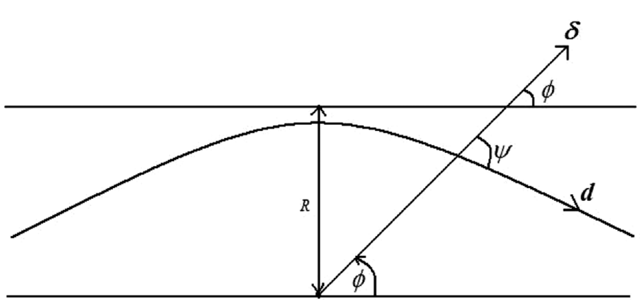

FIG. 1. The plane graph showing the deflected light trajectory with the one-sided bending angle given by $\epsilon = \psi - \phi$ (adapted from [3]).

Considering the relatively small magnitude of the cosmological constant, one would expect that the effects of the first-order terms in Λ in Eqs. (9) and (10) are much smaller than the first-order Schwarzschild contribution $\Delta\phi_{\text{sch}} \sim 4m/R$ such that the presence of the cosmological terms in the bending angle formula do not effectively contribute to gravitational lensing in a practical way [13,14]. However obtaining the value of these cosmological contributions for galaxies or cluster of galaxies (as was done in Table 1 of Ref. [4]), one finds that this is not the case. So for example if we take the case of the galaxy cluster Abell 2744 [43,44] (see Table 1 in Ref. [4]) with Einstein radius $R_E = 96.4$ Kpc, $m = 1.07 \times 10^{13} M_\odot h^{-1}$, and cosmological constant $\Lambda = 1.1056 \times 10^{-52}\ \text{m}^{-2}$ (obtained using $H_0 = 67.66 \pm 0.42\ \text{km}\,\text{s}^{-1}/\text{Mpc}$, $\Omega_\Lambda = 0.6889 \pm 0.0056$ [45]), we find that

$$\frac{4m}{R} = 5.510 \times 10^{-5}; \quad \frac{15\pi}{4}\frac{m^2}{R^2} = 2.235 \times 10^{-9};$$
$$\frac{\Lambda R^3}{6m} = 1.184 \times 10^{-5}; \quad 2\sqrt{\frac{\Lambda}{3}} r_0 = 3.612 \times 10^{-5}. \qquad (12)$$

From the above numerical values it can be seen that the first-order contributions from the cosmological constant to the bending angle in Eqs. (9) and (10) are greater than the second order term m^2/R^2 and are indeed of the same order of magnitude as $\Delta\phi_{\text{sch}}$. This in no way contradicts the conclusions reached in Refs. [13,14] about the insignificant contribution from the cosmological constant in practical gravitational lensing, considering that the SdS solution is not a realistic model of a gravitational lens embedded in a cosmological background. In a later paper Ishak *et al.* [4] applied the same method that was used earlier to derive (9) to obtain an improved formula for the light bending angle for the case of a SdS vacuole matched to a Friedmann-Robertson-Walker (FRW) background, where the source and observer are assumed to be inside the SdS vacuole such that the deflection of the light trajectory happens entirely within the vacuole. In this case it was also assumed that the source, lens and observer are coaligned and the derived bending angle is given by

$$\Delta\phi = \frac{4m}{R} + \frac{15\pi}{4}\frac{m^2}{R^2} + \frac{305}{12}\frac{m^3}{R^3} - \frac{\Lambda R r_b}{3}, \qquad (13)$$

where now the contribution from the cosmological term is expressed in terms of the radial coordinate r_b at the boundary of the vacuole. This is obtained by applying the appropriate matching conditions at $r = r_b$ which yield

$$r_{b(\mathrm{SdS})} = a(t) r_{b(\mathrm{FRW})} \tag{14}$$

and

$$m_{\mathrm{SdS}} = \frac{4\pi}{3} r^3_{b(\mathrm{SdS})} \rho_{\mathrm{m}}, \tag{15}$$

where $a(t)$ is the scale factor in the FRW metric, and ρ_{m} is the density of the universe at the moment when light passes by the lens which is located at the centre of the SdS vacuole. From these equations one notes that although the size of the hole $r_{b(\mathrm{FRW})}$ is constant in comoving coordinates, the physical size of the hole $r_{b(\mathrm{SdS})}$ increases in static coordinates due to the expansion of the universe. So now for the Abell 2744 galaxy cluster mentioned above $\Lambda R r_b/3 = 1.603 \times 10^{-8}$, which is still larger than the second order term $\mathrm{m}^2/\mathrm{R}^2$, but it's now significantly smaller than the Schwarzschild term $\Delta\phi_{\mathrm{sch}}$. In a way this result is expected considering that the effect of the cosmological background on the bending of light depends on the position of the source and observer with respect to the lens. So for this particular example one finds that the three coordinate radii satisfy the inequality $r_b < r_{\phi=0} < r_h$. Therefore although the exact de-Sitter spacetime, being conformally flat does not contribute to bending of null trajectories, the SdS being asymptotically conformally flat would still cause a deflection of light even far away from the lens itself. One should also mention the fact that the SdS vacuole matched to FRW spacetime considered by Ishak *et al.* [4] is still far from being a realistic model of gravitational lensing in a cosmological background. In a recent paper Hu *et al.* [14] (see also Ref. [13]) improved this model by assuming that the source and observer are within the FRW background and are therefore comoving with the expansion of the universe. They have also considered the effect of the change in the size of the SdS vacuole as light propagates through it. They found that in this case the contribution from the cosmological constant is even smaller than that obtained by Ishak *et al.* [4] and can be almost entirely attributed to the Λ dependence of the angular diameter distances used in the lensing equation $\Delta\phi = D_S\theta_E/D_{LS}$, where D_S, D_{LS} are the angular diameter distances of the source from the observer and the source from the lens respectively and θ_E is the Einstein angle related to the Einstein radius by $R = \theta_E D_L$; D_L being the angular diameter distance of the lens from the observer.

Turning back to the MK spacetime given by (2) or (4) one can easily check that unlike the SdS vacuole embedded in an FRW background, this spacetime provides a smooth transition from a Schwarzschild-like metric near its source to a general FRW metric in the background, and so it may provide a more realistic example of a lens in a cosmological background. So for large r when the β terms in (2) can be ignored, the resulting spacetime is conformally related to the FRW metric having an arbitrary scale factor $a(t)$ and spatial curvature $\kappa = -\Lambda/3 - \gamma^2/4$. This can be seen by applying the coordinate transformation [15]

$$\rho = \frac{4r}{2(1+\gamma r - \frac{\Lambda r^2}{3})^{1/2} + \gamma r + 2} \quad \text{and} \quad \tau = \int a(t)dt, \tag{16}$$

such that the line element in (1) with $\beta = 0$ reduces to

$$ds^2 = \frac{1}{a^2(\tau)} \frac{[1-\rho^2(\gamma^2/16+\Lambda/12)]^2}{[(1-\gamma\rho/4)^2 + \Lambda\rho^2/12]^2} \times \left(-d\tau^2 + \frac{a^2(\tau)}{[1-\rho^2(\gamma^2/16+\Lambda/12)]^2}(d\rho^2+\rho^2 d\Omega)\right). \tag{17}$$

So choosing the static coordinates in (1), one is again faced with the same issue mentioned above for SdS spacetime about the position of the source and observer with respect to the lensing object, considering that the final result will depend on these positions. In Ref. [27] the Rindler-Ishak method is used to obtain the bending angle for the MK-metric given in (2) assuming a coaligned source, lens and observer and using the formula in (11) evaluated at $\phi = 0$ corresponding to the position $r_{\phi=0} = 2R^2/(2\beta(2-3\beta\gamma) - \gamma R^2)$, which in terms of the parameter m introduced in (4) can be written as $r_{\phi=0} = 2R^2/(4m - R^2\gamma)$. The resulting bending angle to first order in γ and Λ is given by

$$\Delta\phi = \frac{4\beta}{R} - \frac{2\beta^2\gamma}{R} - \frac{\Lambda R^3}{6\beta}. \tag{18}$$

The first and last term in the above formula correspond to the Schwarzschild and cosmological contributions, similar to those in the SdS spacetime as given in (10). The contribution to the bending angle from the linear term in the metric is inversely proportional to the impact parameter and has a negative sign just like the cosmological term, meaning that it suppresses the amount of bending. Considering that the parameter γ is related to the gravitational source [see Eq. (18) in Ref. [46] and also Refs. [20–22]], it has been argued [30] that this term should be positive so that it enhances the bending angle, in the same way that it provides an explanation to the flat galactic rotational curves in the absence of dark matter. This issue has been settled [47,48] when it was shown that the sign of the linear term in the metric can be reversed by a simple gauge transformation, thus leading to a positive contribution from γ in the bending angle formula. Considering the asymptotic form of the MK spacetime given in (17) in which the spatial curvature of the FRW background depends on γ one can also state that the γr term in the metric has an effect on the cosmological background even when $\beta = 0$ (or $m = 0$). So in a way the presence of the linear term in the MK solution facilitates the embedding of gravitational source in a cosmological background. The derivation of the bending angle in the MK spacetime using Rindler and Ishak's method obtained elsewhere (see for example Refs. [28,30,31]) resulted in slightly different formulas for the bending angle, with the main difference arising from the order of the approximations taken, the association of different parameters (m vs β) with the geometric mass of the lens and the different points in the calculations where higher powers of $m/R, \beta/R, \gamma$, and Λ are discarded. However in all these formulas the first-order contribution from γ to the bending angle is always inversely proportional to the impact parameter R, meaning that just like the Einstein contribution $4m/R$, it diminishes with distance from the lens, which is expected if γ is really associated with the gravitational lens itself. In the bending angle formula obtained by Kaşikçi and Deliduman [33] given by (8) the first-order (and main) contribution from γ is given by the term $\gamma_0\sqrt{\Lambda_0/3} = \gamma r_0^2\sqrt{\Lambda/3}$ and therefore it increases with the distance of closest approach r_0 (which is related to the impact parameter R). This is similar but smaller in magnitude than the contribution $-\gamma r_0$ obtained by Edery and Paranjape [see Eq. (21) in Ref. [24]] which has been termed as unphysical considering that increases linearly with the closest approach distance r_0 from the lens. Although both Edery and Paranjape's and Kaşikçi and Deliduman's derivations are based on Weinberg's method, the latter derivation takes into consideration the asymptotic nonflatness of the MK spacetime and limits the integration (and so the deflection of the null trajectory) to the position of the cosmological event horizon $r = r_h$ in this spacetime, while in the former case the integration is extended to infinity, thereby yielding a larger contribution. However, one can easily check that the leading contribution from γ in (8) is significantly higher than that in (18). So, if we again take the example of Abell 2744 and the value $\gamma \sim 1/R_H = 10^{-26}\,\mathrm{m}^{-1}$, we obtain $\gamma r_0^2\sqrt{\Lambda/3} = 5.37\times 10^{-10}$ and $2\beta^2\gamma/R = 1.128\times 10^{-14}$. Still these are both insignificant for practical gravitational lensing when compared to the Einstein bending angle. It's also interesting to note that in (8) the leading γ contribution to the bending angle is coupled to the cosmological constant instead of the geometric mass of the lens as in (18) and in the other similar formulas obtained in Refs. [28,30,31]). This would lead us to rethink the exact nature of the linear term in the MK metric, i.e., whether this term is derived from the gravitational source as previously claimed or whether it is associated to the asymptotic region of the spacetime. This is still an open issue.

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