

Physical relevance of the fiducial cell in loop quantum cosmology

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A common way to avoid divergent integrals in homogeneous spatially noncompact gravitational systems, as e.g. cosmology, is to introduce a *fiducial cell* by cutting off the spatial slice at a finite region V_o . This is usually considered as an auxiliary integral regulator to be removed after performing computations by sending $V_o \rightarrow \infty$. In this paper, we analyze the dependence of the classical and quantum theory of homogeneous, isotropic, and spatially flat cosmology on this fiducial cell. We show that each fixed V_o regularization leads to a different canonically independent theory. At the classical level, the dynamics of observables is not affected by the regularization choice on-shell. For the quantum theory, however, this leads to a family of regulator dependent quantum representations labeled by V_o , and the limit $V_o \rightarrow \infty$ becomes then more subtle. First, we construct a novel isomorphism between different V_o -regularizations, which allows us to identify states in the different V_o -labeled Hilbert spaces to ensure equivalent dynamics for any value of the regulator. The $V_o \rightarrow \infty$ limit would then correspond to choosing a state for which the volume assigned to the fiducial cell becomes infinite as appropriate in the late-time regime. As the second main result of our analysis, quantum fluctuations of observables smeared over subregions $V \subset V_o$, unlike those smeared over the full V_o , explicitly depend on the size of the fiducial cell through the ratio V/V_o interpreted as the (inverse) number of subcells V homogeneously patched together into V_o . Physically relevant fluctuations for a finite region, as e.g. in the early-time regime, which would be unreasonably suppressed in a naïve $V_o \rightarrow \infty$ limit, become appreciable at small volumes. Our results suggest that the fiducial cell is not playing the role of a mere regularization but is physically relevant at the quantum level and complements previous statements in the literature.

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I. INTRODUCTION

Symmetry reduced models as cosmological or black hole spacetimes play an important role in the context of (canonical) quantum gravity approaches as e.g. loop quantum gravity (LQG) [1–3]. On the one hand, these models have a large amount of symmetries and are thus simple enough to allow for explicit quantum gravity computations. On the other hand, they describe physically interesting systems containing gravitational singularities, whose understanding and ultimate fate in a theory surpassing general relativity are among the key questions that any quantum gravity theory aims to address (see e.g. [4,5] and references therein). Last but not least, the study of symmetry-reduced systems and their quantum aspects may open the way to potentially test possible phenomenological predictions of quantum gravity models [see e.g. [6–9] for a small sample of quantum cosmological models in the framework of loop quantum cosmology (LQC)].

However, a main conceptual and technical challenge lies in the fact that homogeneous gravitational systems such as cosmology with noncompact spatial topology lead to diverging actions and Hamiltonians. In the LQC literature [6,10–16], it is thus common to introduce a *fiducial cell* to restrict the considered volume to a finite patch, labeled by the coordinate volume V_o . The physical interpretation of such a fiducial cell has, however, been debated since then. One point of view [6,7,10,17] is to consider the fiducial cell as an infrared regulator and of purely auxiliary nature. Physical results should not depend on the choice of the fiducial cell, and the regulator can then be removed by sending $V_o \rightarrow \infty$ after Poisson brackets are evaluated to get back a noncompact space. Another viewpoint based on effective quantum field theory (effective QFT) [16,18–21] consists instead in interpreting it as the physical scale of homogeneity, which should evolve along with the evolution of the universe, thus being subject to an infrared renormalization. In particular, close to the classical singularity, this scale is actually microscopic [16,20] due to the Belinskii-Khalatnikov-Lifshitz (BKL) scenario [22]. From this point of view, the limit $V_o \rightarrow \infty$, if possible, should be related to

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a renormalization group flow, and the size of the fiducial cell acquires physical relevance as a renormalization scale. The question about how to interpret this fiducial structure has in turn profound consequences on the interpretation and validity of the *effective approximation*, where the quantum physics is approximated by a smooth spacetime obeying effective modified dynamics. The common belief in the literature is that the effective dynamics is applicable and quantum fluctuations are negligible even close to the resolved classical singularity [23–27] (see, however, [28,29] for a recent review of further issues originating in effective models due to general covariance).

In this paper, we contribute to this discussion by analyzing the role of the fiducial cell in detail. We first recall the classical framework of cosmology and the necessity to introduce a fiducial cell at the off-shell level (Sec. II). Physically relevant observables and their dependence on the regularization are then discussed. The classical part closes by arguing that indeed the classical physics, which means the dynamics of observables, is not altered by the choice of fiducial cell and the regulator can be removed by sending it to infinity at the on-shell level. However, as the canonical structure depends on V_o , each cosmological theory regularized with a different fiducial cell is actually a canonically inequivalent theory.

This becomes relevant when moving to the quantum level in the second part of the paper (Sec. III), where there is not a single quantum theory but rather a family of LQC quantum representations each corresponding to a different region V_o . The V_o -rescaling symmetry of classical dynamics thus might or might not be broken at the quantum level. Our main focus then lies on investigating how the quantum representations of differently regularized classical theories can be related. More specifically, we ask the following: (1) How do we phrase the question about changing V_o in the quantum theory so that it comes to be as close as possible to the situation at the classical level? In other words, is it possible to implement a change of V_o , hence of the quantum theory, while keeping quantum dynamics the same? (2) What would then be the consequences for other quantum features such as uncertainty relations and fluctuations? In Sec. III A, we answer (1) in the affirmative by constructing a mapping between states in the different V_o -regularized Hilbert spaces with the same dynamics. In Sec. III B we then study the consequences for expectation values, higher statistical moments, and quantum fluctuations of both nonsmeared and smeared operators. In particular, the mapping between the different V_o -labeled Hilbert spaces allows us to study how these quantities scale in the different quantum representations associated with different fiducial cell sizes. We find that moments and fluctuations of nonsmeared elementary operators depend on the value of V_o , while those of the volume operator smeared over the entire fiducial cell do not, compatible with previous work. However, we further show that quantum

fluctuations and uncertainty relations of canonically conjugate observables smeared over a subregion $V \subset V_o$ of the fiducial cell *do* depend explicitly on the size of the fiducial cell through the ratio V/V_o interpreted as the (inverse) number of subcells V homogeneously patched into V_o . Given then a small region V that one is interested in tracking within a homogeneous model, its size can be operationally thought of as e.g. given by the scale resolved by a detector. Quantum fluctuations over the probe region V become relevant in the early-time universe where also the size of the region V_o over which homogeneity is imposed becomes small. In the late-time universe, where homogeneity can be imposed over large scales, quantum fluctuations are instead suppressed the more subcells are patched together into a large region V_o .

Some concluding remarks and future directions are reported in Sec. IV. More details will be worked out in a companion paper [30], where a systematic symmetry reduction at the classical level and a full theory interpretation of the fiducial cell as the scale of the homogeneous truncation in field modes is also provided.

II. CLASSICAL COSMOLOGY AND THE FIDUCIAL CELL

We restrict ourselves to the simplest cosmological setting, that is, homogeneous, isotropic, and spatially flat cosmology. The metric

$$\begin{aligned} ds^2 &= -N(t)^2 dt^2 + q_{ab} dx^a dx^b \\ &= -N(t)^2 dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2) \end{aligned} \quad (2.1)$$

is thus used as starting ansatz (see e.g. [15]). In a classical treatment, it is possible to insert this ansatz into the Einstein equations coupled to a matter source and solve the differential equations for the scale factor $a(t)$, while the lapse $N(t)$ is a pure gauge choice and determines the interpretation of the coordinate t .

In view of quantization, it is, however, necessary to understand the off-shell structures such as action, Hamiltonian, and Poisson structure first. For this purpose, we fix the matter content of the universe to be a massless real scalar field ϕ , which will play the role of physical clock later on. Inserting the ansatz (2.1) and $\phi(t, x) = \phi(t)$ into the Einstein-Hilbert action coupled to a massless scalar field yields

$$\begin{aligned} S &= S_{\text{EH}} + S_M = \int dt \mathcal{L} + \text{boundary terms}, \\ \mathcal{L} &= \int_{\Sigma} d^3x \mathcal{L} = \int_{\Sigma} d^3x \left(-\frac{3}{\kappa} \frac{a \dot{a}^2}{N} + \frac{a^3 \dot{\phi}^2}{2N} \right), \end{aligned} \quad (2.2)$$

where $\kappa = 8\pi G$ ($c = 1$), dots refer to derivatives with respect to the t -coordinate, $S_{\text{EH}} = \frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \sqrt{g} R$ is the Einstein-Hilbert action, and $S_M = -\frac{1}{2} \int_{\mathcal{M}} d^4x \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ is the

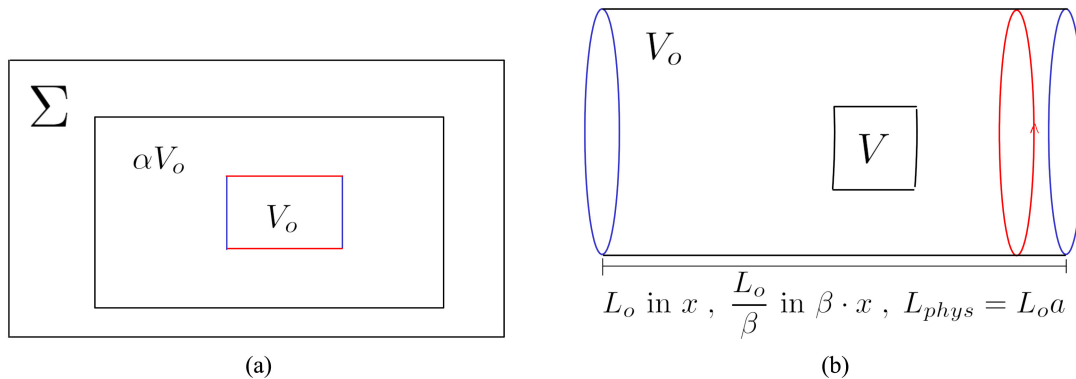


FIG. 1. (a) A compact fiducial cell $V_o \subset \Sigma$ of the noncompact spatial slice Σ is chosen. It is equivalently possible to choose another, larger fiducial cell $V_o \subset \alpha V_o \subset \Sigma$, which is a coordinate independent choice of a larger subregion. (b) To avoid boundary conditions the fiducial cell is equipped with a 3-torus topology, by identifying the blue lines and closing the red circle. The edge length of the fiducial cell V_o and the circumference of the red circle are L_o in the x -chart. This is coordinate dependent, and it reads as L_o/β in the βx -chart. The physical size is coordinate independent, but depends on initial conditions of a . On the spatial slice, we can consider finite volumes V , independent on the choice of V_o , as long as $V \subset V_o$.

matter action. The four-dimensional integral was split into a timelike integral and an integral over a spatial slice Σ . An immediate problem appears in the spatial integral, that is, $\int_{\Sigma} d^3x \rightarrow \infty$ as all fields are independent of the spatial point and thus do not satisfy proper falloff conditions. The usual remedy provided in the literature (see e.g. [6,10–16]) is to restrict the spatial integral to a finite domain $V_o \subset \Sigma$ by introducing a so-called *fiducial cell*, the latter being at this level a purely auxiliary topological construction to regularize the divergent integral. The region V_o is described by the coordinates $x, y, z \in [0, L_o]$ so that its coordinate volume¹ is $\int_{V_o} d^3x = L_o^3 =: V_o$. An important question concerns now boundary conditions at ∂V_o , which arise when Σ is assumed to be noncompact. To avoid this problem, we topologically compactify V_o by equipping it with a 3-torus topology $V_o \simeq_{\text{top}} \mathbb{T}^3$ so that $\partial V_o = \emptyset$ (cf. Fig. 1).

As this construction is purely auxiliary at first sight, it is usually expected to not enter any physical results in the end. However, it should be kept in mind that already at this stage two main approximations have been done: First, the boundaries are neglected here; i.e. we impose periodic boundary conditions, which amounts to ignoring all the interactions of the region V_o with the surrounding environment given by its complementary region in the spatial slice. Second, all other regions $\Sigma \setminus V_o$ are ignored and V_o is simply extended by homogeneity. This excludes the possibility of changes in the spatial geometry from one fiducial cell to another, thus ignoring inhomogeneities with

wavelengths larger than L_o . The latter can be safely neglected at large volumes, far from the bounce or classical singularity. However, it remains to be examined how important these modes are to the total dynamics at small volumes. In fact, changing the size of V_o makes these approximations more or less restrictive. In a follow-up paper [30], it will be presented how to do a systematic symmetry reduction at the Hamiltonian level by implementing spatial homogeneity via second-class constraints and using Dirac brackets. In this framework, the fiducial cell can be interpreted as the scale on which the spacetime is actually homogeneous, which is thus a physical requirement. Further, as we assume $x, y, z \in [0, L_o]^3$ and the torus topology here, this is the scale on which periodic boundary conditions are imposed. Using the metric (2.1), this amounts to say that the spacetime is homogeneous and satisfies periodic boundary conditions on a length scale given by

$$L_{\text{phys}} = \int_0^{L_o} dx \sqrt{g_{xx}} = L_o a(t). \quad (2.3)$$

Its actual value thus depends on the solution and the initial conditions for $a(t)$. The length L_o then enters directly the physical quantity L_{phys} . Let us emphasize here now an important difference between the coordinate size L_o and the region $V_o \subset \Sigma$. While a coordinate transformation $x \mapsto \beta x$ changes the value $L_o \mapsto L_o/\beta$, L_{phys} is not changed as $a \mapsto \beta a$ [6]. The geometry of V_o as described by a given solution $a(t)$ of the Einstein equations is not changed by this transformation, as this only amounts to describe it by means of different coordinates. However, it is also possible to simply make the scale of homogeneity/periodicity larger by mapping $L_o \mapsto \alpha L_o$, which is not a coordinate transformation, but really physically rescales V_o as a subset of Σ (see Fig. 1), when referring to the same

¹More precisely, the fiducial volume is computed with respect to a fiducial metric \hat{q}_{ab} whose coordinate axes are associated with the local triads along the edges of the cell, say $\hat{q}_{ab} dx^a dx^b = dx_1^2 + dx_2^2 + dx_3^2$ (see e.g. [6] for details). Moreover, to simplify the notation, we use the same notation V_o for both the set $V_o \subset \Sigma$ and its coordinate volume $V_o = L_o^3$. Their distinction should be clear from the context.

fixed solution.² In this case, we find $L_{\text{phys}} \mapsto \alpha L_{\text{phys}}$, which affects the physical length (2.3). From now on, we keep the coordinates fixed and only consider transformations of the second kind, where the actual physical scale of periodicity, i.e. the physical volume of V_o , is changed.

In the following, we analyze the classical theory and its dependence on this fiducial cell and the topological compactification. A Legendre transform of the Lagrangian (2.2), restricted to V_o , leads to the Hamiltonian (see e.g. [6,15])

$$H_T = V_o \cdot N\mathcal{H}, \quad \mathcal{H} = -\frac{\kappa}{12} \frac{p_a^2}{a} + \frac{p_\phi^2}{2a^3} \approx 0, \quad (2.4)$$

where the momenta are defined as $p_a = \partial\mathcal{L}/\partial\dot{a}$, $p_\phi = \partial\mathcal{L}/\partial\dot{\phi}$ with $\mathcal{L} = \int_{V_o} d^3x \mathcal{L} = V_o \mathcal{L}$, and satisfy the Poisson brackets

$$\{a, p_a\} = \frac{1}{V_o}, \quad \{\phi, p_\phi\} = \frac{1}{V_o}. \quad (2.5)$$

For the purposes of quantization later, it is useful to perform the canonical transformation to the variables $v = a^3$ and $b = -p_a/(3a^2)$, leading to

$$H_T = V_o \cdot N\mathcal{H}, \quad \mathcal{H} = -\frac{3\kappa}{4} v b^2 + \frac{p_\phi^2}{2v} \approx 0, \quad (2.6a)$$

$$\{b, v\} = \frac{1}{V_o}, \quad \{\phi, p_\phi\} = \frac{1}{V_o}. \quad (2.6b)$$

The above V_o factors can be understood from a field theory viewpoint where all the quantities involved in the above equations are given by the integrals over the spatial slice of the corresponding densities. More precisely, a systematic study of the reduction procedure [30] reveals that the canonical brackets (2.6b) can actually be understood as the Dirac brackets resulting from implementing homogeneity over the region V_o via second-class constraints for the full theory variables.

The interpretation of these variables becomes clear once smeared physical quantities are constructed. In the full theory, the volume of a given region $V \subset V_o$ is given by

$$\text{vol}(V) = \int_V d^3x \sqrt{q} = \int_V d^3x v = V \cdot v. \quad (2.7)$$

Similarly, the variable b can be interpreted as the rate at which (any) volume changes as³

²Note that this is a kinematical interpretation. Taking a closer look at the Einstein equations leads to the well-known fact that they are diffeomorphism invariant. Consequently, the metric with a/α has the same initial conditions for $L_{\text{phys}} = \alpha L_o a/\alpha$ and leads to equivalent physics.

³Comparing with [31], this variable has a full theory counterpart corresponding to $-\frac{1}{\text{vol}(V)} \int_V d^3x q_{ab} P^{ab}$, with P^{ab} the conjugate momentum to the spatial metric q_{ab} which can be written in terms of the extrinsic curvature.

$$\frac{1}{N\text{vol}(V)} \frac{d\text{vol}(V)}{dt} = \frac{1}{N\text{vol}(V)} \{\text{vol}(V), H_T\} = \frac{3\kappa}{2} b, \quad (2.8)$$

which is independent of the particular volume V . Similarly, the observable $p_\phi(V) = \int_V d^3x p_\phi = V p_\phi$ gives the total matter energy of the region $V \subset V_o$. These are observables which have counterparts in the full theory and are independent of the choice of spatial coordinates. They reduce to the above simple expressions after imposing homogeneity and isotropy. In the full theory they are independent for each possible choice of $V \subset V_o$, but after imposing homogeneity and isotropy, they are all related by a linear factor. For instance, we have

$$\text{vol}(V) = \frac{V}{V_o} \text{vol}(V_o), \quad (2.9)$$

where V/V_o is simply a topological ratio (i.e. independent of the metric) determining how many times V fits into V_o . The quantities $\text{vol}(V)$ and $p_\phi(V)$ are extensive as $\text{vol}(V_1) + \text{vol}(V_2) = \text{vol}(V_1 \cup V_2)$ for any disjoint regions V_1 and V_2 ($V_1 \cap V_2 = \emptyset$). On the other hand, b and ϕ are intensive and can be interpreted as averaged values of local quantities over the compact region V_o . More details can be found in the follow-up paper [30].

Having this set up, we can study how the classical theory depends on the fiducial cell. To this aim, let us consider a *physical rescaling* $V_o \mapsto \alpha V_o$ (cf. Fig. 1), i.e. $\text{vol}(V_o) = \int_{V_o} d^3x v \mapsto \alpha \text{vol}(V_o)$. Again, this is independent of coordinates and fixes the size of the fiducial torus V_o . It is easy to see that the Hamiltonian (2.6a) is extensive and thus scales as

$$H_T \mapsto \alpha H_T. \quad (2.10)$$

Moreover, we have the following scaling behaviors:

$$\text{vol}(V) \mapsto \text{vol}(V), \quad \text{vol}(V_o) \mapsto \alpha \text{vol}(V_o), \quad (2.11a)$$

$$p_\phi(V) \mapsto p_\phi(V), \quad p_\phi(V_o) \mapsto \alpha p_\phi(V_o), \quad (2.11b)$$

$$b \mapsto b, \quad \phi \mapsto \phi, \quad (2.11c)$$

from which we see that $\text{vol}(V)$, $p_\phi(V)$, b , and ϕ are independent of the size of the fiducial cell V_o . Keeping this in mind, it becomes evident that the Poisson structure itself has to be dependent on V_o . Indeed, we have [cf. Eqs. (2.6b) and (2.7)]

$$\begin{aligned} \{b, \text{vol}(V)\} &= \frac{V}{V_o} \longmapsto \frac{1}{\alpha} \{b, \text{vol}(V)\}, \\ \{\phi, p_\phi(V)\} &= \frac{V}{V_o} \longmapsto \frac{1}{\alpha} \{\phi, p_\phi(V)\}, \end{aligned} \quad (2.12)$$

and, since the quantities entering the arguments of the Poisson bracket are independent of V_o , this scaling behavior has to be a property of the bracket itself, namely

$$\{\cdot, \cdot\} \mapsto \frac{1}{\alpha} \{\cdot, \cdot\}, \quad (2.13)$$

where the arguments are also scaled. Summarizing, changing the size of the scale of homogeneity (or in other words the periodic boundary conditions)

- (i) is not a canonical transformation as the Poisson bracket and Hamiltonian are not preserved (the Hamiltonian rescales linearly, while the Poisson bracket rescales inversely linearly);
- (ii) does not affect physical observables that do not explicitly refer to the fiducial cell.

Therefore, the choice of V_o and its compactification affects the canonical theory. Specifically, there are infinitely many canonically inequivalent cosmological theories all labeled by the value of $V_o = \int_{V_o} d^3x$ and the choice of the fiducial cell $V_o \subset \Sigma$. Still, the theories labeled by different values of the fiducial cell, say $V_o^{(1)}$ and $V_o^{(2)}$, can be related to each other in a well-defined manner. In particular, it is possible to show that the dynamics of classical observables is independent of the specific value of V_o . Indeed, let $H_T^{(1)}$, $H_T^{(2)}$, and $\{\cdot, \cdot\}_{(1)}$, $\{\cdot, \cdot\}_{(2)}$ be the Hamiltonians and Poisson brackets of the two theories with fiducial cells $V_o^{(1)}$ and $V_o^{(2)}$, respectively. For any observable \mathcal{O} , the time evolution is determined by

$$\begin{aligned} \dot{\mathcal{O}} &= \{\mathcal{O}, H_T^{(1)}\}_{(1)} \\ &= \frac{V_o^{(2)}}{V_o^{(1)}} \left\{ \mathcal{O}, \frac{V_o^{(1)}}{V_o^{(2)}} H_T^{(2)} \right\}_{(2)} \\ &= \{\mathcal{O}, H_T^{(2)}\}_{(2)}, \end{aligned} \quad (2.14)$$

which does not depend on the chosen value of V_o . Explicitly, this can be seen as the Poisson bracket depends on $1/V_o$, while the Hamiltonian is linear in V_o [cf. Eqs. (2.6)].

This result is not surprising, neither new [6,10,12], as we could have worked with the Einstein equations from the very beginning, which does not refer to V_o -dependent quantities such as an action, Hamiltonian or Poisson brackets. The Einstein equations are local and as such do not depend on the global topology⁴ so that no reference to the fiducial cell V_o is needed. In particular, it is possible to track the evolution of a finite volume $\text{vol}(V)$ and perform the limit $V_o \rightarrow \infty$ after evaluating the Poisson brackets.

⁴Indirectly they do for a field theory due to boundary conditions, but these are already imposed by choosing a homogeneous metric ansatz.

As a classical theory only predicts dynamics, which is local, V_o is really fiducial at the classical level, and any reference to it can be removed afterwards by performing this limit (as required in e.g. [6]), at least in principle, as long as the homogeneous approximations remains trustworthy.⁵ However, in the next section, we will see how this situation changes in the case of a quantum theory.

III. IMPLICATIONS FOR QUANTUM COSMOLOGY

A. Quantization of the symmetry-reduced theory

Let us proceed now to quantize the family of classical theories labeled by V_o using LQG techniques. Note that the quantum theory can only be well defined for a finite V_o as otherwise the Poisson bracket, and thus the commutator, would be trivial. Consequently, it is unavoidable to construct a theory at finite V_o and then examine the scaling behavior of the relevant quantities. As the classical Poisson bracket depends on the fiducial cell volume V_o [see Eqs. (2.6b) and (2.13)] but the commutator cannot have this scaling property (it is just the composition of linear operators), each value of V_o requires a separate quantum representation [6,12], i.e. different Hilbert spaces and operators.

As a first step, the system is deparametrized with the scalar field used as a clock for the quantum dynamics. The corresponding true Hamiltonian generating the evolution in ϕ -time is obtained by solving the Hamiltonian constraint Eq. (2.6a) for the canonical momentum $p_\phi(V_o)$ conjugate to ϕ . This leads to

$$p_\phi(V_o) = \sqrt{\frac{3\kappa}{2}} V_o v b =: H_{\text{true}}, \quad (3.1)$$

where we have only chosen the positive sign of the square root which is equivalent to consider only positive frequency modes. This is a suitable simplification as the aim here is only to study the dependence on V_o . The considerations in the following straightforwardly extend to the case in which both positive and negative frequency modes are considered.

The next step consists of representing the operators \hat{H}_{true} , \hat{v} , and a regularization of b on the LQC-Hilbert space \mathcal{H}_{LQC} given by [6,10,12,15]

⁵Some care is in fact still needed even at the classical level when one takes into account the fact that, after all, the homogeneous mini-superspace theory results from a mode truncation of dynamical fields in the full theory. From this perspective, the homogeneous description can be trusted as long as such a truncation provides us with a good approximation. In particular, as discussed in the follow-up paper [30], the fiducial cell sets the scale of a wavelength cutoff for the field modes, and the error committed in the truncation of the inhomogeneous modes does depend on (inverse powers of) V_o .

$$\mathcal{H}_{\text{LQC}} = L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}}), \quad |\psi\rangle = \sum_{\nu \in \mathbb{R}} \psi(\nu) |\nu\rangle; \quad (3.2a)$$

$$\langle \nu | \nu' \rangle = \delta_{\nu, \nu'}, \quad \langle b | \nu \rangle = e^{i\lambda b \nu}, \quad \psi(\nu) = \langle \nu | \psi \rangle, \quad (3.2b)$$

where $|\nu\rangle$ and $|b\rangle$ are, respectively, eigenstates of $\hat{\nu}$ and of $\widehat{e^{-i\lambda b}}$ (see below). Here, \mathbb{R}_{Bohr} denotes the Bohr compactification of the real line [6,12,32], a compact Abelian group which roughly speaking corresponds to the real line equipped with discrete topology. Note the discrete sum in (3.2a) and the Kronecker-delta rather than a Dirac-delta normalization in (3.2b). This is where the main difference between the weakly discontinuous LQC polymer representation, motivated by the discreteness of spatial geometry at a fundamental scale, and the *inequivalent* Schrödinger representation can be traced back [6,12,33]. As a consequence, only $\widehat{e^{-i\lambda b}}$ and not \hat{b} comes to be a well-defined operator on the LQC Hilbert space in analogy with the use of holonomies in LQG.⁶

The core issue is now the dependence of the quantum theory on V_o . Different choices of V_o correspond to different Hilbert spaces, although they are isomorphic and the difference is not explicitly visible. It becomes visible in the operator representations. In fact, the V_o -dependence of the Poisson bracket between ν and $e^{-i\mu b}$ necessarily implies that the representations of the corresponding quantum operators will depend on it as well. Specifically, the action of the elementary operators can be generically written as

$$\hat{\nu}|\nu\rangle = \frac{\eta^\gamma}{V_o^\gamma} \nu |\nu\rangle, \quad \widehat{e^{-i\lambda \mu b}}|\nu\rangle = \left| \nu - \frac{\lambda \mu}{\eta^\gamma V_o^\delta} \right\rangle, \quad (3.3)$$

with arbitrary powers γ, δ such that $\gamma + \delta = 1$, $\eta = \kappa^{3/2}$, and units $\hbar = 1$, $[\kappa] = \text{length}^2$, $[\nu] = [\hat{\nu}] = [\mu] = 1$, $[b] = [\lambda^{-1}] = \text{length}^{-3}$. $\mu \in \mathbb{R}$ is an arbitrary dimensionless number, and λ is the so-called polymerization scale. The powers of γ and δ are just a notation we introduced to reflect the freedom in incorporating the V_o factors in the representation of the quantum operators to ensure the Poisson brackets to be correctly represented as commutation relations. The V_o -factors in (3.3) are in fact unavoidable for the commutator to match the correspondence principle with the classical Poisson bracket, i.e.

$$[\widehat{e^{-i\lambda \mu b}}, \hat{\nu}]|\nu\rangle = \frac{\lambda \mu}{V_o} \widehat{e^{-i\lambda \mu b}}|\nu\rangle, \quad \forall |\nu\rangle, \quad (3.4)$$

so that [cf. (2.6b)]

$$[\widehat{e^{-i\lambda \mu b}}, \hat{\nu}] = i \{ \widehat{e^{-i\lambda \mu b}}, \hat{\nu} \}. \quad (3.5)$$

⁶The classical variable b is in fact directly related to the symmetry-reduced homogeneous and isotropic Ashtekar connection [6,14].

At the quantum level, the scaling property of the Poisson bracket [cf. Eq. (2.13)] is thus shifted into the quantization map for the representations of the operators associated with phase space quantities. Consequently, although the classical phase space function ν is independent of V_o , its quantum representation does depend on it and, for any two distinct values $V_o^{(1)}$ and $V_o^{(2)}$, we have

$$\hat{\nu}|_{V_o^{(1)}} = \left(\frac{V_o^{(2)}}{V_o^{(1)}} \right)^\gamma \hat{\nu}|_{V_o^{(2)}}, \quad (3.6)$$

and

$$\widehat{e^{-i\lambda \mu^{(1)} b}}|_{V_o^{(1)}} = \widehat{e^{-i\lambda \mu^{(2)} b}}|_{V_o^{(2)}}, \quad \mu^{(1)} = \left(\frac{V_o^{(2)}}{V_o^{(1)}} \right)^\delta \mu^{(2)}.$$

Finally, standard LQC ordering and regularization prescriptions for the Hamiltonian [34] lead to

$$\begin{aligned} \hat{H}_{\text{true}} = & \sqrt{\frac{3\kappa}{2}} V_o \sqrt{|\hat{\nu}|} \left(\frac{\widehat{\sin(\lambda b)}}{2\lambda} \text{sign}(\hat{\nu}) \right. \\ & \left. + \text{sign}(\hat{\nu}) \frac{\widehat{\sin(\lambda b)}}{2\lambda} \right) \sqrt{|\hat{\nu}|}, \end{aligned} \quad (3.7)$$

whose action, after defining

$$\nu = \frac{\lambda}{\eta^\gamma V_o^\delta} n, \quad n \in \mathbb{R}, \quad \theta = \frac{\lambda}{\eta^\gamma V_o^\delta}, \quad (3.8)$$

can be recast into the following finite difference equation:

$$\begin{aligned} \hat{H}_{\text{true}}\psi(\nu) = & \langle \nu | \hat{H}_{\text{true}} \psi \rangle \\ = & \frac{i}{4} \sqrt{\frac{3\kappa}{2}} \left(s_+(n) \sqrt{|n||n+1|} \psi(\theta \cdot (n+1)) \right. \\ & \left. - s_-(n) \sqrt{|n||n-1|} \psi(\theta \cdot (n-1)) \right), \end{aligned} \quad (3.9)$$

with $s_\pm(n) = \text{sign}(n \pm 1) + \text{sign}(n)$. As usual the zero volume state $\nu = 0$ is annihilated and positive and negative branches are preserved. The quantity θ entering Eq. (3.9) can be thought of as an “effective” polymerization scale, which is explicitly V_o -dependent [cf. Eq. (3.8)]. Moreover, we note that the linear scaling behavior of the classical Hamiltonian [cf. Eq. (2.10)] is again changed by the V_o -quantum representation and is rather undetermined. However, to have a well-defined dynamics given by the Schrödinger equation generated by \hat{H}_{true}

$$i \frac{\partial}{\partial \phi} |\psi\rangle = \hat{H}_{\text{true}} |\psi\rangle, \quad |\psi; \phi\rangle = e^{i\phi \hat{H}_{\text{true}}} |\psi\rangle, \quad (3.10)$$

such a scaling behavior has to be determined. In this respect, it is possible to identify states in the different

V_o -quantum theories which have the same evolution behavior. Let us emphasize that, in principle, it might be possible to allow V_o dependent dynamics and forget about the following identification. However, as the classical dynamics is independent of V_o , it seems reasonable to demand the same for quantum dynamics.

For this construction, we make use of the fact that there exists [34,35] an analytic expression for a function $\Psi_E: \mathbb{R} \rightarrow \mathbb{C}$ satisfying

$$-\frac{i}{2} \sqrt{\frac{3\kappa}{2}} \cdot \left(s_+(n) \sqrt{|n||n+1|} \Psi_E(n+1) - s_-(n) \sqrt{|n||n-1|} \Psi_E(n-1) \right) = E \Psi_E(n). \quad (3.11)$$

Consequently, the states $\psi_E(\nu) = \psi_E(\theta \cdot n) = \Psi_E(n)$ are eigenstates of the Hamiltonian (3.9) with eigenvalue E . As shown in [34], the operator (3.7) is essentially self-adjoint with an absolutely continuous, nondegenerate spectrum equal to the real line. The explicit expression of the corresponding eigenfunctions can be found in [34,35] but is not relevant for the considerations that follow. Consider now two quantum representations resulting from quantizing the classical theories associated with two distinct values of V_o , say $V_o^{(1)}$ and $V_o^{(2)}$, respectively. In both representations one could find the eigenstates of the corresponding Hamiltonian according to the same identification (3.11) (V_o and thus θ are different). It follows that in both quantum representations one finds the same spectrum for \hat{H}_{true} and, recalling Eq. (3.8), the eigenstates can be related by

$$\psi_E^{(1)}(\nu) = \psi_E^{(2)} \left(\left(\frac{V_o^{(1)}}{V_o^{(2)}} \right)^\delta \nu \right), \quad (3.12)$$

where the superscripts (1) and (2) denote the eigenstates in the corresponding V_o -quantum theory. Let us stress that *a priori* the carrier Hilbert spaces for the different quantum representations are *different*, and it is only due to the simplicity of the model under consideration that they can easily be identified as above. As anticipated above, it is *a priori* not necessary to identify the states in the different quantum theories as in (3.12) and, depending on the aims and/or the physical situation to be described, one could in principle find different arguments leading to a different identification. Here, the relation (3.12) is to ensure the dynamics to be the same in the two quantum theories, similar to the classical case. To see this, we note that the above identification of eigenstates allows one to define a well-defined transformation behavior of the Hamiltonian operator according to

$$\begin{aligned} \hat{H}_{\text{true}}|_{V_o^{(1)}} &= \int dE E \langle \psi_E^{(1)} | \cdot \rangle \psi_E^{(1)} \mapsto \hat{H}_{\text{true}}|_{V_o^{(2)}} \\ &= \int dE E \langle \psi_E^{(2)} | \cdot \rangle \psi_E^{(2)}, \end{aligned} \quad (3.13)$$

from which it is evident that two states $\psi^{(1)} \in \mathcal{H}_{\text{LQC}}^{(1)}$ and $\psi^{(2)} \in \mathcal{H}_{\text{LQC}}^{(2)}$ have the exact same ϕ -time evolution as long as $\langle \psi_E^{(1)} | \psi^{(1)} \rangle_{(1)} = \langle \psi_E^{(2)} | \psi^{(2)} \rangle_{(2)}$ for all E . This in turn induces, under the demanding of dynamics to remain V_o -independent, the following dynamics-preserving isomorphism between the two different quantum theories

$$\mathcal{I}: \mathcal{H}_{\text{LQC}}^{(1)} \rightarrow \mathcal{H}_{\text{LQC}}^{(2)}$$

by

$$\begin{aligned} \psi^{(1)} &\mapsto \psi^{(2)} = \mathcal{I}(\psi^{(1)}) \\ \psi^{(2)}(\nu) &= \psi^{(1)} \left(\left(\frac{V_o^{(2)}}{V_o^{(1)}} \right)^\delta \nu \right). \end{aligned} \quad (3.14)$$

Therefore, it is possible to make the whole families of quantum theories labeled by the values of V_o dynamically equivalent, as it was the case for the classical theory. However, a quantum theory is more than just dynamics, but also includes quantum uncertainty relations and fluctuations. Their dependence on V_o is discussed in the next subsection. In this respect, let us notice that the above isomorphism implementing a notion of fiducial cell rescaling at the quantum level is in fact the sought mapping e.g. in [26] and, as we shall see in the next subsection, can be used for a detailed analysis of expectation values and higher moments [see Eq. (3.17) below]. The present results can in particular be used to study the scaling properties of semiclassical states as e.g. those considered in [26,36,37]. We refer the interested reader to the companion paper [30] for details.

B. Uncertainty relations and quantum fluctuations

Having determined the mapping (3.14) between the Hilbert spaces associated with different V_o -valued quantum representations, we can now study whether and how expectation values and higher statistical moments change from one V_o value to the other. Note that all the expectation values in what follows are ϕ -time dependent and, since the isomorphism preserves time evolution, the following statements hold in a fully dynamical sense. However, to ease the notation, we shall suppress the explicit ϕ -time dependence and write $\psi(\phi) = \psi$, $|\psi; \phi\rangle = |\psi\rangle$.

As for the operator \hat{v} , recalling the action Eq. (3.3) together with the definitions Eq. (3.8), we find

$$\begin{aligned}
 \langle \hat{v} |_{V_o^{(1)}} \rangle_{\psi^{(1)}} &:= \langle \psi^{(1)} | \hat{v} |_{V_o^{(1)}} | \psi^{(1)} \rangle = \sum_{\nu \in \mathbb{R}} \psi^{(1)*}(\nu) \frac{\eta^\gamma}{(V_o^{(1)})^\gamma} \nu \psi^{(1)}(\nu) \\
 &= \sum_{\nu \in \mathbb{R}} \psi^{(2)*} \left(\left(\frac{V_o^{(1)}}{V_o^{(2)}} \right)^\delta \nu \right) \frac{\eta^\gamma}{(V_o^{(1)})^\gamma} \nu \psi^{(2)} \left(\left(\frac{V_o^{(1)}}{V_o^{(2)}} \right)^\delta \nu \right) \\
 &= \frac{V_o^{(2)}}{V_o^{(1)}} \sum_{\nu' \in \left(\frac{V_o^{(1)}}{V_o^{(2)}} \right)^\delta \mathbb{R}} \psi^{(2)*}(\nu') \frac{\eta^\gamma}{(V_o^{(2)})^\gamma} \nu' \psi^{(2)}(\nu') \\
 &= \frac{V_o^{(2)}}{V_o^{(1)}} \langle \hat{v} |_{V_o^{(2)}} \rangle_{\psi^{(2)}}, \tag{3.15}
 \end{aligned}$$

where $\nu' = (V_o^{(1)}/V_o^{(2)})^\delta \nu$ in the second to last line. By construction of (3.14), we further have that

$$\langle \hat{H}_{\text{true}} |_{V_o^{(1)}} \rangle_{\psi^{(1)}} = \langle \hat{H}_{\text{true}} |_{V_o^{(2)}} \rangle_{\psi^{(2)}}, \tag{3.16}$$

and it is straightforward to verify the following relations:

$$\langle \hat{v} |_{V_o^{(1)}}^n \rangle_{\psi^{(1)}} = \left(\frac{V_o^{(2)}}{V_o^{(1)}} \right)^n \langle \hat{v} |_{V_o^{(2)}}^n \rangle_{\psi^{(2)}}, \tag{3.17a}$$

$$\langle e^{-i\lambda \hat{v}} |_{V_o^{(1)}}^n \rangle_{\psi^{(1)}} = \langle e^{-i\lambda \hat{v}} |_{V_o^{(2)}}^n \rangle_{\psi^{(2)}}, \tag{3.17b}$$

$$\Delta_{\psi^{(1)}} \hat{v} |_{V_o^{(1)}} = \frac{V_o^{(2)}}{V_o^{(1)}} \Delta_{\psi^{(2)}} \hat{v} |_{V_o^{(2)}}, \tag{3.17c}$$

$$\Delta_{\psi^{(1)}} e^{-i\lambda \hat{v}} |_{V_o^{(1)}} = \Delta_{\psi^{(2)}} e^{-i\lambda \hat{v}} |_{V_o^{(2)}}, \tag{3.17d}$$

for the higher moments and variances of the operators in Eq. (3.3). With this transformation behavior at our disposal, we can then investigate possible physical effects on the observables

$$\text{vol}(\widehat{V}_o) = V_o \hat{v}, \quad \text{vol}(\widehat{V}) = V \hat{v} = \frac{V}{V_o} \text{vol}(\widehat{V}_o), \tag{3.18}$$

corresponding to the integrated volume of the fiducial cell $V_o \subset \Sigma$ or any region $V \subseteq V_o$ (see Fig. 1). According to the results in Eqs. (3.17), we have

$$\langle \text{vol}(\widehat{V}_o^{(1)}) |_{V_o^{(1)}} \rangle_{\psi^{(1)}} = \langle \text{vol}(\widehat{V}_o^{(2)}) |_{V_o^{(2)}} \rangle_{\psi^{(2)}} \tag{3.19}$$

i.e. in contrast to the classical theory [cf. Eq. (2.11a)], the expectation value of the operator for the integrated volume of the cell is independent of the value of its coordinate volume. Note that changing V_o actively changes the subset in the spatial slice Σ and the region on which periodic boundary conditions are imposed.

Similarly, using the relations (3.5), (3.17), and (3.19), the uncertainty relations read as⁷

$$\begin{aligned}
 \frac{1}{2} |\langle \cos(\lambda b) |_{V_o^{(1)}} \rangle_{\psi^{(1)}}| &\leq \Delta_{\psi^{(1)}} \text{vol}(\widehat{V}_o^{(1)}) |_{V_o^{(1)}} \Delta_{\psi^{(1)}} \frac{\sin(\lambda b)}{\lambda} \Big|_{V_o^{(1)}} \\
 &= \Delta_{\psi^{(2)}} \text{vol}(\widehat{V}_o^{(2)}) |_{V_o^{(2)}} \Delta_{\psi^{(2)}} \frac{\sin(\lambda b)}{\lambda} \Big|_{V_o^{(2)}} \\
 &\geq \frac{1}{2} |\langle \cos(\lambda b) |_{V_o^{(2)}} \rangle_{\psi^{(2)}}|. \tag{3.20}
 \end{aligned}$$

Therefore, the expectation values and fluctuations of the elementary operators depend on the value of V_o as per Eqs. (3.17), while those for the observable measuring the size of the full fiducial cell are independent of the specific value of V_o . Independently of which subset $V_o \subset \Sigma$ is chosen, the expectation values and uncertainty relations involving the volume operator smeared over the entire V_o are only state dependent. Moreover, if the state $\psi^{(1)}$ saturates the uncertainty relation, then $\psi^{(2)}$ does too. Similar is true for the energy density

$$\rho_\psi(\phi) := \frac{\langle p_\phi(\widehat{V}_o) \rangle_\psi^2}{\langle \text{vol}(\widehat{V}_o) \rangle_\psi^2} = \frac{\langle \hat{H}_{\text{true}} \rangle_\psi^2}{\langle \text{vol}(\widehat{V}_o) \rangle_\psi^2}, \tag{3.21}$$

which due to Eqs. (2.10) and (3.19) is independent of the V_o -representation. Further, as the isomorphism Eq. (3.14) preserves dynamics, the energy $1/2\lambda^2$ at the bounce is independent, too. This works also for a subregion $V \subset V_o$ as both quantities at the numerator and denominator of (3.21) are extensive so that the multiplicative factors would cancel yielding an intensive ratio.

It is then insightful to evaluate the expectation value of the total volume for an eigenstate of \hat{v} , i.e. $\psi = \delta_{\nu, \nu_o}$, which reads as

$$\langle \text{vol}(\widehat{V}_o) \rangle_{\delta_{\nu, \nu_o}} = V_o \frac{\eta^\gamma}{V_o^\gamma} \nu_o^{\gamma+\delta-1} V_o^\delta \eta^\gamma \nu_o^{(3.8)} = \lambda n_o. \tag{3.22}$$

Consequently, for any choice of V_o , it is only possible to assign to this region the volume λn_o , where $n_o \in \mathbb{N}$ after imposing the Hamiltonian constraint (and restricting to positive volumes). Compatibly with (3.19), this is obviously independent of V_o ; i.e. the (topological) scale on which homogeneity and periodicity are imposed. This is plausible as $V_o^{(1)}$ or $V_o^{(2)}$ are purely topological constructions, and there is no reference to any geometry. The geometry in turn is then specified by the choice of state ψ , which is in a sense the quantum equivalent of specifying

⁷To ease the comparison with existing literature as e.g. [23,24], we focus here on the operator corresponding to the simplest regularization for the conjugate momentum b by combination of point holonomies $e^{\pm i\lambda b}$ yielding the sin function.

initial conditions for the classical (coordinate-independent) observable $\text{vol}(V_o)$. As there is no additional reference field in the system, there is no way to distinguish $V_o^{(1)}$ from $V_o^{(2)}$.

The situation is, however, different for a finite region $V \subset V_o^{(1)}$, $V_o^{(2)} \subset \Sigma$ as can be seen by studying the change of physical volume assigned to it and relating it to the total fiducial cell. As discussed in Sec. II, changing V_o in the classical theory is a symmetry of the dynamics and the limit $V_o \rightarrow \infty$, $\text{vol}(V_o) \rightarrow \infty$ can be taken for any finite initial conditions on v .⁸ The quantum case requires more care. To this aim, let us study the scaling behavior of the expectation values of the operator $\widehat{\text{vol}}(V)$ and the corresponding uncertainty relations. These can be readily computed to be (for a given value of V_o)

$$\langle \widehat{\text{vol}}(V) \rangle_\psi = \frac{V}{V_o} \langle \widehat{\text{vol}}(V_o) \rangle_\psi, \quad (3.23)$$

$$\Delta_\psi \widehat{\text{vol}}(V) \Delta_\psi \frac{\sin(\lambda b)}{\lambda} \geq \frac{V}{2V_o} |\langle \cos(\lambda b) \rangle_\psi|, \quad (3.24)$$

from which we have the following relations between the quantum theories corresponding to two different values $V_o^{(1)}$ and $V_o^{(2)}$ [cf. (3.18), (3.19), and (3.17)]

$$\langle \widehat{\text{vol}}(V) \rangle_{\psi^{(2)}} = \frac{V_o^{(1)}}{V_o^{(2)}} \langle \widehat{\text{vol}}(V) \rangle_{\psi^{(1)}}, \quad (3.25)$$

$$\Delta_{\psi^{(2)}} \widehat{\text{vol}}(V) \Delta_{\psi^{(2)}} \frac{\sin(\lambda b)}{\lambda} = \frac{V_o^{(1)}}{V_o^{(2)}} \Delta_{\psi^{(1)}} \widehat{\text{vol}}(V) \Delta_{\psi^{(1)}} \frac{\sin(\lambda b)}{\lambda}. \quad (3.26)$$

Thus, not only the expectation value and fluctuations of the elementary operator \hat{v} in (3.17) but also those for the volume of V depend on the value of V_o . This is to be contrasted with Eqs. (3.19) and (3.20) for the entire fiducial cell which have been the focus of previous work. It is then worth noticing that Eq. (3.24) extends previous results in the literature to the case in which one considers a subregion V of the fiducial cell V_o and, consistently, previous results are recovered for $V = V_o$ [cf. Eqs. (20) and (34) in [24]].

Let us discuss the above V_o -dependence in more detail. The quantum dynamics of the volume $\widehat{\text{vol}}(V)$ of the finite region V is not affected either by the coordinate volume V_o

or the *physical volume* assigned to the fiducial cell V_o as long as

$$\langle \widehat{\text{vol}}(V) \rangle_\psi = \frac{V}{V_o} \langle \widehat{\text{vol}}(V_o) \rangle_\psi \xrightarrow{V_o \rightarrow 0, \langle \widehat{\text{vol}}(V_o) \rangle_\psi \rightarrow \infty} \text{finite}. \quad (3.27)$$

The quantity (3.27) might be a proper cosmological observable as the double scaling limit might be performed such that $\langle \widehat{\text{vol}}(V) \rangle_\psi \sim \text{size of universe today}$. Consequently, the state ψ and the observable $\widehat{\text{vol}}(V)$ provide a physically reasonable description of the volume of the universe. Now, as the (expectation value of the) total volume of the region V_o in a given state does not depend on the value of V_o [cf. Eq. (3.19)], the quantum counterpart of a putative classical $V_o \rightarrow \infty$ limit would be to choose a state for which $\langle \widehat{\text{vol}}(V_o) \rangle_\psi \rightarrow \infty$; i.e. the region of homogeneity is enlarged geometrically. This is the case in late-time cosmology where the homogeneous approximation can be safely trusted on large scales and, for finite V and sufficiently large V_o , quantum fluctuations (3.24) are suppressed by the ratio V/V_o and remain so over different, yet large sizes of V_o [cf. (3.26)].

The situation is rather different at the small scales of the early-time universe where the volume of both V and V_o become sufficiently small so that quantum fluctuations cannot be neglected. The importance of fluctuations for finite small cells has already been emphasized e.g. in [16,20]. Evolving the system backward following the collapse of an initially large-scale homogeneous universe, structure forms and inhomogeneities build up within a comoving volume of given coordinate size V_o . For the collapse process to be still described using a homogeneous model, a smaller region over which homogeneity is imposed should be selected when inhomogeneities become appreciable so that more and more inhomogeneous modes with a wavelength larger than the fiducial cell can be neglected within that region. This would be in particular required to be eventually consistent with an asymptotic BKL scenario according to which spacetime dynamics is locally homogeneous when approaching the spacelike singularity. From the scaling property (3.26) we thus see that, in the early-time regime where $V_o^{(1)} \sim V_o^{(2)} \sim V_o$ are comparable and small, not only the quantum fluctuations for the elementary operators (3.17) but also those for the smeared volume of a small region $V \subset V_o$ are not suppressed by the V/V_o factor in (3.24). The states yielding small physical volumes are thus very quantum and the quantum description of V at small scales cannot be made effectively classical to an arbitrary precision.

Given the region V one is interested in describing within the homogeneous approximation as e.g. a small elementary cell inside V_o , and thinking of the latter as been made of multiple identical elementary cells patched

⁸Let us remind the reader that, even though the dynamics of classical observables is not affected in the $V_o \rightarrow \infty$ limit, such a limit would spoil the canonical structure of the classical theory [cf. Eq. (2.6b)]. A naïve limit which ignores the fact that different quantum representations and Hilbert spaces are identified by the different values of V_o , would in turn spoil the (off-shell) starting point for the canonical commutation relations.

together homogeneously,⁹ quantum fluctuations of the individual cells are present at small scales, and it is only when going from the early-time small scales to the late-time large scales that they are suppressed as the number V_o/V of subcells fitting into a bigger and bigger V_o grows.¹⁰ As emphasized in Sec. III A, the mapping between the different V_o -labeled quantum theories is *a priori* not unique. Therefore, one could in principle seek for a different mapping for which the resulting scaling behaviors are such that quantum fluctuations remain small when shrinking V_o . However, if such a mapping exists, the dynamics of the states will be modified under a change of the region V_o . In either case, it is clear that the size of the fiducial cell has a physical significance and is not just a regulator that can be removed at the end of the day with no effects at the quantum level.

IV. CONCLUSIONS

In this paper, we analyzed the classical and quantum relevance of the so-called *fiducial cell* in homogeneous and isotropic cosmology, that is, the compact region $V_o \subset \Sigma$ to which otherwise-divergent integrals over the noncompact spatial slice Σ are restricted. The assigned coordinate volume of the latter is V_o and, in the limit $V_o \rightarrow \infty$, the region V_o approaches again the spatial slice Σ . While the volume V_o is coordinate dependent, the quantity $\text{vol}(V_o) = \int_{V_o} d^3x a^3 = V_o a^3$ is not. Such a fiducial cell is commonly assumed to be a nonphysical auxiliary construction to be removed later on. A closer inspection initiated in the present paper and complemented by a companion paper [30], where also a systematic study of the reduction to the homogeneous theory is presented, suggests that the fiducial cell is not so fiducial after all. In agreement with some previous investigations [16,18–21] based on a different starting point motivated by analogy with effective QFT, V_o has a physical interpretation as the scale on which homogeneity is imposed or in other words the scale on which periodic boundary conditions are imposed. However, a physical length scale to this region can only be assigned after solving for the dynamical metric.

In the first part of the paper the classical theory was reviewed with emphasis on the way in which V_o enters its canonical structure. It was shown that it is possible to construct observables independent of the fiducial cell.

⁹As discussed in the companion paper [30], this is precisely the picture one gets by explicitly carrying out the homogeneous reduction of the gravitational field modes where the spatial slice is partitioned into disjoint cells and homogeneity is imposed only on a finite number of them $V_o = \bigsqcup_n V_n$.

¹⁰This is also in line with some of the discussions about coarse graining and renormalization with coherent states in LQC [38,39] according to which, by patching together N cells, the fluctuations in the large cell grow only as \sqrt{N} , so that the relative fluctuations are vanishing in the infinite cell limit.

In particular, their dynamical evolution is also independent of it so that the on-shell physical predictions for classical dynamics are not affected by the choice of the fiducial cell. In other words, changing the size of the homogeneity region (fiducial cell) is a symmetry of classical dynamics. This is plausible as the regulator and compactification of the spatial slice are only relevant off-shell to construct the action, Lagrangian, Hamiltonian, and Poisson brackets, but not on-shell at the level of the Einstein equations. As the local Einstein equations are sufficient to describe the full classical physics, it is not surprising that the size of V_o does not affect the dynamics of these observables. Only the off-shell quantities such as the Hamiltonian and the action diverge in the limit $V_o \rightarrow \infty$ where the regulator is removed. Furthermore, studying the V_o -dependence of the canonical structure, it was argued that classically each fixed value of V_o describes a different canonical theory and the different theories cannot be related by a canonical transformation. Therefore, the regularization leads to a whole family of classical homogeneous and isotropic theories, all leading to the same classical dynamics.

The fact that V_o labels canonically inequivalent classical theories has important consequences at the quantum level. In the second part of the paper, the quantization of these classical theories was then studied in the framework of homogeneous and isotropic loop quantum cosmology. It was argued that each value of V_o corresponds to a different quantum representation characterized by different Hilbert spaces labeled by the V_o -values. By relating states and expectation values within the Hilbert spaces carrying the different V_o -representations, it was possible to study the dependence on the change of V_o of quantum operators which may also exhibit different scaling behaviors from their classical counterparts. Moreover, it was possible to find an isomorphism between states of the Hilbert spaces associated with different V_o -regularizations, which preserves the action of the Hamiltonian and thus dynamics of quantum observables. As discussed, this is not a necessity, but leads to a situation similar to the classical theory where dynamics is not dependent on the particular choice of fiducial cell and its coordinate volume. In particular, as we have already noticed, the isomorphism proposed in this work comes to be the mapping sought in previous literature [26].

The fact that the states so related have the same dynamics, however, does not mean that a change of V_o has no effect in the quantum theory where not only dynamics but also expectation values, higher moments, and quantum fluctuations are relevant. In fact, two states $\psi^{(1)}$ and $\psi^{(2)}$ related as in (3.14) can be physically distinct. For example, the scaling behaviors (3.17) of the elementary operators suggest that the point around which semiclassical states are peaked and their widths will transform under a change of V_o [30]. This might have consequences for the effective equations in that the classical trajectory over

which the state is peaked could be rather different in the two quantum theories depending on the ratio $V_o^{(2)}/V_o^{(1)}$.

A quantum theory, moreover, also includes predictions about quantum fluctuations. As we discussed with the help of the proposed mapping between the different quantum theories, these have a well-defined scaling behavior when the V_o -representation is changed, for both the smeared and nonsmeared relevant operators. It was shown that the volume of $V_o \subset \Sigma$ in the quantum theory is actually independent of its coordinate value, which is consistent with diffeomorphism invariance and the fact that there is no reference field with respect to which the volume of V_o can be measured. It was further argued that the quantum analog of removing the regulator consists in choosing a state for which $\langle \text{vol}(\widehat{V}_o) \rangle_\psi \rightarrow \infty$ and thus the physical volume assigned to the fiducial cell V_o becomes infinite. This is the case in late-time cosmology where the universe can be well approximated by the homogeneous model over large scales. Nonetheless, by looking at observables respectively smeared over the fiducial cell V_o and a subregion of it $V \subset V_o$, it was shown that, unlike the former, the quantum fluctuations of the latter explicitly depend on the size of the fiducial cell through the ratio V/V_o interpreted as the (inverse) number of subcells V homogeneously patched together into V_o . In particular, the uncertainty relations for smeared *physical observables* such as $\widehat{\text{vol}}(V)$ are affected as these would vanish in such a limit, leading to a seemingly classical theory even for a finite region V as long as homogeneity can be safely imposed over large scales. This observation is consistent and generalizes the arguments in [24], where no distinction between V and V_o was considered. In particular, this is in agreement with the observations that for large fiducial volumes the effective equations are valid [23,24,27,36,37]. However, quantum fluctuations become relevant for a finite region, as e.g. in the early time regime, so that the fiducial cell is not playing the role of a mere regularization but comes to be of physical relevance at the quantum level.

A possible interpretation of this is the fact that a classical theory is purely local, while a quantum theory is not. The latter includes the description of nonlocal fluctuations and correlations. The amount of fluctuations of a finite region $V \subset V_o$ are always measured with respect to the fluctuations of the full fiducial cell V_o . From this point of view, it is not surprising that the fluctuations for V become negligible when the volume of V_o becomes sufficiently large. It should nevertheless be emphasized that V can have still a finite volume as e.g. the size of our universe, even if $\langle \text{vol}(V_o) \rangle_\psi \rightarrow \infty$. Thus, classically the fiducial cell has the *physical* interpretation as the scale on which homogeneity and periodic boundary conditions are imposed. Classical dynamics is independent of it, at least in the large volume regime where inhomogeneities can be safely neglected and the homogeneous description provides us with a good

approximation. The situation changes at the quantum level, where the scale of homogeneity has a physical effect on the quantum fluctuations. Measuring the quantum fluctuations of $\widehat{\text{vol}}(V)$ allows one to measure in principle the size of V_o , which is then a physical observable due to quantum effects. Therefore, one needs to think carefully about the relevant quantum state which is studied and the volume that is assigned to V_o in this way. In the follow-up paper [30], we provide a more detailed analysis to understand these features from a full theory point of view by imposing homogeneity constraints systematically for the modes of classical fields, determining the resulting canonical structure of the symmetry-reduced classical theory, and studying its quantization.

As a final note, we remark that changing V_o at the quantum level also affects the relevant couplings, here the polymerization scale λ , and forces one to perform a very precise identification of states in order to be consistent with the dynamics. This looks very much like a very naïve renormalization procedure, and in future work it would be interesting to relate this to previous work in the context of coarse graining and renormalization in loop quantum cosmology [38–41]. More specifically, as discussed in detail in the follow-up paper [30], the classical homogeneous minisuperspace theory results from a twofold procedure consisting in neglecting the inhomogeneous modes with wavelength larger than the cell size and also in truncating those remaining inhomogeneous modes inside the cell so that one is left only with the zero mode. When considering then multiple cells patched together into a bigger cell, the fully homogeneous analysis discussed here demands the zero modes in different cells to be equal while setting to zero all inhomogeneous modes. This essentially amounts to replicate the physics in one cell into all the others and in turn leads to the mapping of states (3.14) between different Hilbert spaces and no renormalization of the Hamiltonian yet. The present setting can therefore be extended in the following ways. First, even neglecting interactions between the cells which in our framework should be encoded in boundary terms at the surfaces of the individual cells, the inclusion of inhomogeneous modes with a wavelength larger than a single cell size would require us to impose different gluing conditions for the field modes. Therefore, we expect the dynamics for states in the quantum theories of a single cell or many cells not to be the same anymore and, consequently, the mapping between different V_o -valued Hilbert spaces to be modified. Second, a more complete picture would then require the interactions between neighboring cells to also be included. In this respect, it would be interesting to compare the resulting analysis with previous work on perturbations around homogeneous cosmological spacetimes as initiated in [42], and systematically study the regime of validity of perturbative treatments of inhomogeneities at the quantum level. We believe that the framework presented in this paper

(and in its follow-up paper [30]) offers interesting starting points for investigating these questions which are thus left for future investigation.

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