Cluster alphabets from generalized worldsheets:
A geometric approach to finite types

Peng Zhao[1] and Yihong Wang[2]

[1]*Joint School of the National University of Singapore and Tianjin University,*
International Campus of Tianjin University, Fuzhou 350207, China
[2]*Laboratoire d'Annecy-le-Vieux de Physique Theorique (LAPTh),*
CNRS and Universite Savoie Mont-Blanc, Annecy 74940, France

We provide a systematic derivation of cluster alphabets of finite types. The construction is based on a geometric realization of the generalized worldsheets by gluing and folding a pair of polygons. The cross ratios of the worldsheet z variables are evolved using the Y-system equations. By a new gauge choice, we obtain a simpler set of cluster alphabets than the known ones.

I. INTRODUCTION AND SUMMARY

The search for a geometric description and a simple set of variables has guided the study of scattering amplitudes in quantum field theory and string theory. Historically, the Veneziano amplitude consistent with the Regge poles and crossing symmetry was written down first, then extended to n-point amplitudes, and the notion of a worldsheet swept out by the motion of strings emerged only later.

The generalization of the Veneziano amplitude to n mesons is expressed as [1–3]

$$I_n = \left(\prod_{(i,j)}^{n-3} \int_0^1 d \log \frac{u_{i,j}}{1 - u_{i,j}} \right) \prod_{i,j}^n u_{i,j}^{\alpha' X_{i,j}}. \tag{1}$$

Here the integral is over the $n - 3$ compatible resonances, α' is the Regge slope, and $X_{i,j}$ are functions of the Mandelstam variables $s_{i,j}$. It was soon realized that the u variables may be written as cross ratios of the Koba-Nielsen z variables [4].

$$u_{i,j} = \frac{z_{i-1} - z_j}{z_{i-1} - z_{j-1}} \frac{z_i - z_{j-1}}{z_i - z_j}. \tag{2}$$

This leads to the expression of the integration measure in the Parke-Taylor form $\prod_{i=1}^n dz_i/(z_i - z_{i+1})$ and the Koba-Nielsen factor $|z_i - z_j|^{\alpha' s_{i,j}}$ familiar in modern textbooks, with the residual gauge symmetry used to fix the positions

of three points, e.g., $z_1 \to -1, z_2 \to 0, z_n \to \infty$. It is now recognized that this integral describes the tree-level amplitude of open strings, whose worldsheet is a disc with n marked points at the boundary. The string amplitude enjoys properties such as crossing symmetry, factorization, and Regge behavior.

More recently, the factorization property of the string integral was put on the center stage to define a class of generalized string integrals associated with Dynkin diagrams [5]. The so-called cluster string integrals factorize at the poles that correspond to the boundaries of the configuration space of u variables [6,7]. For example, the A_{n-3} integral (1) factorizes into an A_{n-k-2} integral and an A_{k-2} integral at its poles. The factorization property is reflected in the geometry of the generalized associahedra [8] and the integrals are interpreted as volume forms. However, like the multimeson amplitudes, the integrals are written in terms of the u variables as a generalization of (1). It was not clear what the underlying worldsheet picture is.

A second motivation for this work comes from the structure of field-theory amplitudes. The amplitudes are expressed in terms of generalized polylogarithms. The cluster bootstrap program attempts to constrain the form of the amplitude using a set of symbol alphabets [9–11]. In a related development [12], a class of alphabets based on cluster algebras of finite type was proposed using birational maps from the kinematic variables:

$$\Phi_{A_{n-3}} = \bigcup_{3 \leq i \leq n-1} \{z_i, 1 + z_i\} \cup \bigcup_{3 \leq i < j \leq n-1} \{z_i - z_j\},$$

$$\Phi_{C_{n-1}} = \Phi_{A_{n-1}}(z_3, \ldots, z_n, z_{n+2}) \cup \bigcup_{3 \leq i \leq j \leq n} \{z_i z_j + z_{n+2}\},$$

$$\Phi_{D_n} = \Phi_{A_{n-1}}(z_3, \ldots, z_n, z_{n+2}) \cup \{z_{n+3}, 1 + z_{n+3}\} \cup \bigcup_{3 \leq i \leq n} \{z_i - z_{n+3}, z_i + z_{n+2}z_{n+3}\}$$

$$\times \cup \bigcup_{3 \leq i < j \leq n} \{-z_i + z_j + z_iz_j - z_iz_{n+2} - z_iz_{n+3} + z_{n+2}z_{n+3}\}. \tag{3}$$

It was shown that the Feynman integral for the one-loop Bhabha scattering correspond to the A_3 cluster alphabet, a certain six-dimensional hexagon integral to the D_4 cluster alphabet, etc. However, the cluster alphabets were found by a clever choice of birational maps, and it was not clear how to derive them for other finite-type cluster algebras. The A-type alphabet is the set of gauge-fixed factors in the u variables (2), as string amplitudes reduce to field-theory amplitudes in the $\alpha' \to 0$ limit. It was also not clear whether the other alphabets have any geometric origin or if there is an underlying worldsheet at all.

In [13], a systematic derivation of such variables was proposed based on Y systems, and the results for D types were presented in detail. The strategy is to construct the generalized worldsheets through a "gluing" construction. We solve the u variables in terms of the worldsheet coordinates. Like in the A type, the elements of an alphabet, called letters, are the factors that appear in the u variables. As it stands, there are more ungauged letters than the number of cluster variables. Upon a choice of gauge, the alphabets are then read off from the factors. The number of letters in a cluster alphabet is shown in Table 1.

This paper aims to derive the cluster alphabets for all the finite types. Our main results are as follows:

(1) Systematic construction of the generalized worldsheet for all finite types. We provide a systematic derivation of the gluing construction of the exceptional types. We begin by reviewing the gluing construction of the D_n worldsheet. We show how the construction extends to the exceptional types, and derive an explicit cross-ratio representation of all the E_6 u coordinates. For the nonsimply laced types, we present the folding map that identifies the worldsheet coordinates. Our results may also be seen as an explicit verification of Zamolodchikov's periodicity conjecture for Y systems [14].

(2) New cluster alphabets. In the standard gauge choice, one may obtain the cluster alphabet for BCD, E_6, F_4, G_2 types as polynomials of degrees at most 2, 4, 5, 4, respectively. By choosing a different gauge, we produce a new, simpler set of alphabets. We obtain a linear alphabet for B type, quadratic

alphabets for CD types, and for E_6, E_7, E_8, F_4, G_2 types, polynomials of degrees at most 4, 5, 7, 4, 2, respectively.

II. THE GLUING CONSTRUCTION

A. Review on the gluing construction of the D_n worldsheet

We review the construction of the D-type worldsheet based on gluing a pair of A-type worldsheets. The basic observation is that the D_n Dynkin diagram can be written as a union of a pair of A_{n-1} Dynkin diagrams, as shown in Fig. 1. We prepare two $(n + 2)$-gons. We will call the first polygon the first sheet and the second polygon the second sheet. The vertices of the polygons can be given any labels but for convenience, we will choose them to be $(1, 2, \ldots, n + 1, n + 2)$ and $(1, 2, \ldots, n + 1, n + 3)$, respectively. We glue $n + 1$ of the common vertices together, leaving the last vertex on each polygon alone. The positions of the vertices $z_1, \ldots, z_{n+3}$ will be our worldsheet variables. We may choose a snake triangulation. Assigning a node to each diagonal and an arrow between two consecutive diagonals ordered counterclockwise around a common vertex, we see that the underlying graph precisely corresponds to a Dynkin diagram of type D_n [15].

Recall that in an A_n worldsheet, the u variables are cross ratios of their respective z variables (2). By a reparametrization $u = Y/(1 + Y)$, it is found that the cross ratios on the worldsheet satisfy a celebrated set of equations, known as Y systems:

$$Y_{i,j}Y_{i+1,j+1} = (1 + Y_{i,j+1})(1 + Y_{i+1,j}). \tag{4}$$

This can be seen as a time-evolution equation that maps $Y_{i,j} \to Y_{i+1,j+1}$ in each time step.

The Y-system equations admit a generalization to all finite types [14]:

$$Y_i(t-1)Y_i(t) = \prod_{j \to i}(1 + Y_j(t))^{-C_{i,j}}\prod_{i \to j}(1 + Y_j(t-1))^{-C_{i,j}}. \tag{5}$$

Here $C_{i,j}$ is the Cartan matrix of the root system. One may assign an orientation to the edges in the Dynkin diagram such that each node is either a source or a sink. The Zamolodchikov periodicity conjecture states that the solutions to (5) are periodic. A solution of Y systems in terms of cross ratios was used to prove the periodicity conjecture for the A type [16].

TABLE I. The dimensions of finite-type cluster algebras, which equal the number of letters in a cluster alphabet.

A_{n-3}	B_n/C_n	D_n	E_6	E_7	E_8	F_4	G_2
$n(n-3)/2$	$n(n+1)$	n^2	42	70	128	28	8
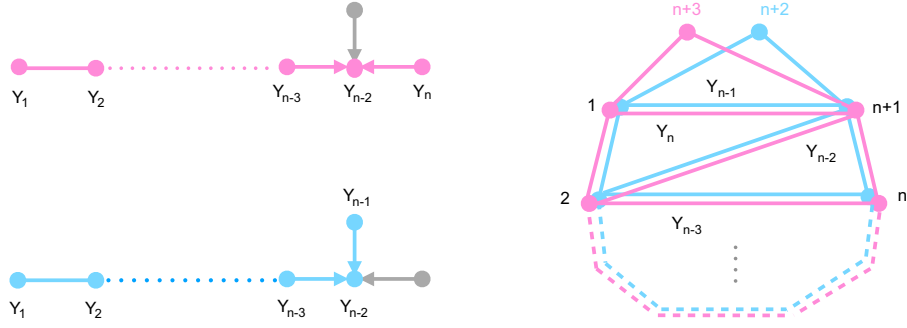
FIG. 1. The glued-polygon representation of the D_n worldsheet.

Returning to the worldsheet picture, the $n - 2$ diagonals in the triangulation are identified with the initial Y variables for each common node in the Dynkin diagram $Y_i(0)$ for $i = 1, 2, \ldots, n - 2$. The diagonals connecting z_1 with z_{n+1} in the $(1, 2, n + 1, n + 2)$ and $(1, 2, n + 1, n + 3)$ quadrilaterals provide $Y_{n-1}(0)$ and $Y_n(0)$ for the two branched nodes in the Dynkin diagram, respectively. Now we may assign cross ratios to the diagonals in the initial triangulation. The $Y_i(t)$ variables at later times are generated according to the Y-system equations. This process terminates when the Y variables return to their initial values as guaranteed by periodicity.

Because the Y-system equations are always birational transformations on the Y variables, the new Y variables will always be a rational function of the z variables. Let $z_{i,j} \coloneqq z_j - z_i$. Remarkably, once we introduce the cubic polynomials

$$w_{i,j} = z_{1,n+3} z_{i,j} z_{n+1,n+2} - z_{1,n+1} z_{i,n+3} z_{j,n+2}, \quad (6)$$

the corresponding u variables can be written as generalized cross ratios of the z, w factors:

$$u_{i,j} = \frac{z_{i,j-1} z_{i-1,j}}{z_{i,j} z_{i-1,j-1}}, \qquad u_{j,i} = \frac{w_{i,j-1} w_{i-1,j}}{w_{i,j} w_{i-1,j-1}},$$

$$u_i = \frac{z_{i,n+3} w_{i-1,i}}{z_{i-1,n+3} w_{i,i}}, \qquad \tilde{u}_i = \frac{z_{i,n+2} w_{i-1,i}}{z_{i-1,n+2} w_{i,i}}, \quad (7)$$

for $n + 1 \geq i > j > 1$. Here u_i and $\tilde{u}_i$ correspond to the two branched nodes in the Dynkin diagram. Because the D_n worldsheet is constructed from gluing a pair of A_{n-1} worldsheets, one may think of $u_{i,j}$ with $i > j$ as the cross-ratio coordinates of zs of the first sheet, and $u_{i,j}$ with $i < j$ as the cross-ratio coordinates of ws of the second sheet.

We shall denote the collection of polynomial factors that appear in the u variables as an "ungauged alphabet." The ungauged D_n alphabet is

$$\bigcup_{1 \leq i \leq n+1} \{z_{i,n+3}\} \cup \bigcup_{1 \leq i < j \leq n+2} \{z_{i,j}\} \cup \bigcup_{2 \leq i < j \leq n} \{w_{i,j}\}. \quad (8)$$

There are $n^2 + n + 3$ independent variables. Upon gauge fixing $z_1 \to -1$, $z_2 \to 0$, $z_{n+1} \to \infty$, $n + 3$ variables corresponding to $z_{1,2}$ and $z_{i,n+1}$ for all $i \neq n + 1$ are removed, and we obtain the n^2 letters (3).

While the u variables are written nicely as generalized cross ratios, the interpretation of the w variables remains mysterious. Here we provide a new, determinant representation:

$$w_{i,j} = \det \begin{pmatrix} 1 & 1 & 1 \\ z_i + z_{n+1} & z_1 + z_j & z_{n+2} + z_{n+3} \\ z_i z_{n+1} & z_1 z_j & z_{n+2} z_{n+3} \end{pmatrix}. \quad (9)$$

It is symmetric on the pairs of indices $(i, n + 1)$, $(1, j)$, $(n + 2, n + 3)$, but is antisymmetric when the pairs are exchanged, much like the symmetries of a Riemann tensor.

B. The construction of E_n worldsheets

Consider an E_n-type Dynkin diagram, where $n = 6, 7, 8$. It may be written as a union of A_{n-1} and A_{n-2} diagrams. We prepare a $(n + 2)$-gon and a $(n + 1)$-gon and glue n of the common vertices, leaving one vertex on the first polygon and two vertices on the second polygon free. We work out the E_6 example explicitly, as shown in Fig. 2. The initial set of variables are
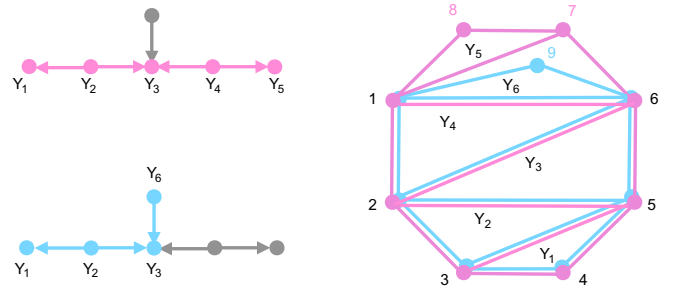
FIG. 2. The glued-polygon representation of the E_6 worldsheet.

$$Y_i(0) = \{Y_{3,5}, Y_{2,5}, Y_{2,6}, Y_{6,1}, Y_{7,1}, \tilde{Y}_{6,1}\},$$

$$= \left\{ \frac{z_{2,5}z_{3,4}}{z_{2,3}z_{4,5}}, \frac{z_{1,5}z_{2,4}}{z_{1,2}z_{4,5}}, \frac{z_{1,6}z_{2,5}}{z_{1,2}z_{5,6}}, \frac{z_{1,5}z_{6,8}}{z_{1,8}z_{5,6}}, \frac{z_{1,6}z_{7,8}}{z_{1,8}z_{6,7}}, \frac{z_{1,5}z_{6,9}}{z_{1,9}z_{5,6}} \right\}. \tag{10}$$

The Y variables are written in terms of the z variables as

$$Y_{i,j} = \frac{z_{i-1,j}z_{i,j-1}}{z_{i-1,i}z_{j-1,j}}. \tag{11}$$

Note that on the first sheet, the vertex that comes before z_1 is z_8; on the second sheet, the vertex that comes before z_1 is z_9.

We evolve the Y-system equations (5) as before, generating all the $Y_i(t)$ at later times. Among the nonlinear

factors appearing in this parametrization of the E_6 Y system, there are 12 cubic polynomials of the form

$$w_{i,j}^E = z_{1,n+1}z_{i,j}z_{n,n+3} - z_{1,n}z_{i,n+1}z_{j,n+3},$$

$$\tilde{w}_{i,j}^E = z_{1,n+2}z_{i,j}z_{n,n+3} - z_{1,n}z_{i,n+2}z_{j,n+3}. \tag{12}$$

Note that $w_{i,j}^E$ for E_6 is slightly different from $w_{i,j}$ for D_n due to a difference in the labels. There are also four quartic polynomials of the form

$$w_{i,j,k} = z_{1,i}z_{j,n+2}z_{k,n+1}z_{n,n+3} - z_{1,n+3}z_{n+2,i}z_{n+1,j}z_{n,k}, \tag{13}$$

and a sextic polynomial

$$w_{i,j,k,l} = z_{1,n+3}z_{1,n}z_{i,n+2}z_{j,n+1}z_{k,n+3}z_{l,n} - z_{1,n+3}z_{1,n}z_{n,n+3}z_{i,n+2}z_{j,k}z_{l,n+1}$$

$$+ z_{1,i}z_{1,n}z_{n,n+3}z_{j,n+3}z_{k,n+2}z_{l,n+1} + z_{1,n+2}z_{1,n+3}z_{n,n+1}z_{n,n+3}z_{i,l}z_{j,k}. \tag{14}$$

The indices are taken to lie in $2 \leq i < j < k < l \leq 5$ so the last polynomial is simply $w_{2,3,4,5}$. When some of the indices are allowed to coincide, the sextic polynomial factorizes into a product of the lower-order polynomials, e.g.,

$$w_{i,i,j,k} = w_{i,k}^E \tilde{w}_{i,j}^E, \qquad w_{i,j,j,k} = z_{1,n}z_{j,n+3}w_{i,j,k}. \tag{15}$$

This allows us to write the Y variables, or equivalently the u variables, of E_6 compactly as generalized cross ratios[1]:

$$u_i(t) = \begin{pmatrix} \frac{z_{2,5}z_{3,4}}{z_{2,4}z_{3,5}} & \frac{z_{3,6}z_{4,5}}{z_{3,5}z_{4,6}} & \frac{w_{2,2,2,4}z_{5,6}}{w_{2,2,2,5}z_{4,6}} & \frac{w_{2,3,3,3}w_{2,3,3,5}}{w_{2,2,3,5}w_{3,3,3,3}} & \frac{z_{4,8}w_{3,4,4,4}}{z_{3,8}w_{4,4,4,4}} & \frac{z_{4,7}w_{4,4,4,5}}{z_{5,7}w_{4,4,4,4}} & \frac{z_{5,8}z_{6,7}}{z_{5,7}z_{6,8}} \\ \frac{z_{1,5}z_{2,4}}{z_{1,4}z_{2,5}} & \frac{z_{2,6}z_{3,5}}{z_{2,5}z_{3,6}} & \frac{z_{4,6}w_{2,2,3}}{z_{3,6}w_{2,2,4}} & \frac{w_{2,2,2,5}w_{2,3,3,4}}{w_{2,2,2,4}w_{2,3,3,5}} & \frac{w_{2,3,4,5}w_{3,3,3,5}}{w_{2,3,3,5}w_{3,3,4,5}} & \frac{w_{3,4,4,5}w_{4,4,4,4}}{w_{3,4,4,4}w_{4,4,4,5}} & \frac{w_{3,5,5,5}w_{4,5,5,5}}{w_{3,4,5,5}w_{5,5,5,5}} \\ \frac{z_{1,6}z_{2,5}}{z_{1,5}z_{2,6}} & \frac{z_{3,6}w_{2,2,2,2}}{z_{2,6}w_{2,2,2,3}} & \frac{w_{2,2,2,4}w_{2,3,3,3}}{w_{2,2,2,3}w_{2,3,3,4}} & \frac{w_{2,3,3,5}w_{2,3,4,4}}{w_{2,3,3,4}w_{2,3,4,5}} & \frac{w_{2,4,4,5}w_{3,3,4,5}}{w_{2,3,4,5}w_{3,4,4,5}} & \frac{w_{3,5,5,5}w_{4,4,4,5}}{w_{3,4,4,5}w_{4,5,5,5}} & \frac{z_{1,4}w_{5,5,5,5}}{z_{1,5}w_{4,5,5,5}} \\ \frac{z_{1,5}z_{6,8}}{z_{1,6}z_{5,8}} & \frac{z_{1,7}z_{2,6}}{z_{1,6}z_{2,7}} & \frac{w_{2,2,2,3}w_{2,2,2,4}}{w_{2,2,2,2}w_{2,2,3,4}} & \frac{w_{2,3,3,4}w_{3,3,3,3}}{w_{2,3,3,3}w_{3,3,3,4}} & \frac{w_{2,3,4,5}w_{3,4,4,4}}{w_{2,4,4,5}w_{3,3,4,4}} & \frac{w_{2,5,5,5}w_{3,4,4,5}}{w_{2,4,4,5}w_{3,5,5,5}} & \frac{z_{1,3}w_{4,5,5,5}}{z_{1,4}w_{3,5,5,5}} \\ \frac{z_{1,6}z_{7,8}}{z_{1,7}z_{6,8}} & \frac{z_{1,8}z_{2,7}}{z_{1,7}z_{2,8}} & \frac{z_{3,8}w_{2,3,3,3}}{z_{2,8}w_{3,3,3,3}} & \frac{z_{3,7}w_{3,3,3,4}}{z_{4,7}w_{3,3,3,3}} & \frac{w_{2,4,4,5}w_{4,4,4,5}}{w_{2,4,5,5}w_{4,4,4,4}} & \frac{z_{1,2}w_{3,5,5,5}}{z_{1,3}w_{2,5,5,5}} & \frac{z_{1,4}z_{2,3}}{z_{1,3}z_{2,4}} \\ \frac{z_{1,5}z_{6,9}}{z_{1,6}z_{5,9}} & \frac{z_{1,9}z_{2,6}}{z_{1,6}z_{2,9}} & \frac{z_{2,7}w_{2,2,2,3}}{z_{3,7}w_{2,2,2,2}} & \frac{w_{2,3,3,4}w_{3,3,3,4}}{w_{2,3,4,4}w_{3,3,3,3}} & \frac{w_{2,3,4,5}w_{3,4,4,4}}{w_{2,4,4,4}w_{3,3,4,5}} & \frac{w_{3,4,4,4}w_{3,4,4,5}}{w_{3,3,4,5}w_{4,4,4,4}} & \frac{z_{5,8}w_{4,5,5,5}}{z_{4,8}w_{5,5,5,5}} \end{pmatrix}. \tag{16}$$

Unlike the D_n case, the cross ratios involving $w_{i,j,k,l}$ are not unique and can be transformed using the identities

$$\frac{w_{i,i,j+1,k}w_{i,i,j,k+1}}{w_{i,i,j,k}w_{i,i,j+1,k+1}} = \frac{w_{i+1,j,k,k}w_{i,j+1,k,k}}{w_{i,j,k,k}w_{i+1,j+1,k,k}} = 1. \tag{17}$$

In the standard gauge choice ($z_1 = -1$, $z_2 = 0$, $z_6 = \infty$), the E_6 alphabet consists of 42 letters that are polynomials with a degree of at most 4:

$$\Phi_{E_6} = \Phi_{A_5}(z_3, z_4, z_5, z_7, z_8) \cup \{z_9, 1 + z_9\} \cup \bigcup_{3 \leq i \leq 5} \{z_{i,9}, z_i + z_7 z_9, z_i + z_8 z_9\} \cup$$

$$\times \bigcup_{3 \leq i < j \leq 5} \{-z_i + z_j + z_i z_j - z_i z_7 - z_i z_9 + z_7 z_9, -z_i + z_j + z_i z_j - z_i z_8 - z_i z_9 + z_8 z_9,$$

$$\times z_i z_j - z_i z_7 + z_i z_8 - z_j z_8 + z_i z_8 z_9 - z_7 z_8 z_9\} \cup$$

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[1]The u variables for E_6 can alternatively be realized by the Grassmannian cluster algebra G(4,7) [17].

$$\times \{-z_3z_4 + z_3z_7 + z_4z_5 - z_4z_7 + z_4z_8 - z_5z_8 + z_3z_4z_5 - z_3z_4z_7 - z_3z_4z_9 - z_3z_5z_8 + z_3z_7z_8$$
$$+ z_3z_7z_9 + z_4z_8z_9 - z_7z_8z_9,$$
$$- z_3z_5 + z_4z_5 + z_3z_4z_5 - z_3z_4z_7 + z_3z_4z_8 - z_3z_5z_8 - z_3z_5z_9 - z_3z_8z_9 + z_4z_7z_9 + z_5z_8z_9$$
$$+ z_3z_4z_8z_9 - z_3z_7z_8z_9 - z_3z_8z_9^2 + z_7z_8z_9^2\}. \tag{18}$$

In Sec. V, we shall derive a simpler alphabet by a different gauge choice.

III. THE BOUNDARY STRUCTURE OF CLUSTER CONFIGURATION SPACES

Recall that one of the main features of the A_{n-3} worldsheet is that each diagonal divides an n-gon into an $(n - k + 1)$-gon and a $(k + 1)$-gon. The string amplitude factorizes at each pole (boundary of the u space) as

$$A_{n-k-2} \times A_{k-2} \subset \partial A_{n-3}. \tag{19}$$

We can make similar statements for the other types with the picture of glued polygons. Each diagonal in the initial triangulation corresponds to a node on the Dynkin diagram and slices the polygon into two parts. The D_5 example is shown in Fig. 3.

$$\partial D_n = n \left(\sum_{i=1}^{n-2} A_{i-1} \times D_{n-i} + 2A_{n-1} \right). \tag{20}$$

The multiplicity n is determined by the periodicity n of the D_n Y system.

The boundaries of the E_n u space can be obtained similarly, as shown in Fig. 4 for E_6.

$$\partial E_6 = 7(A_1 \times A_2 \times A_2 + 2A_1 \times A_4 + A_5 + 2D_5). \tag{21}$$

There is an overall factor because the E_6 system has period 7. Note that for E_n types, there are more boundaries than diagonals available. We will only identify the possible types of boundaries using the initial cluster, and the remaining boundaries will be obtained by evolving the Y-system equations. All the $u_i(t)$ at the same ith node correspond to the same type of boundary. For example, the 128 boundaries of the E_8 worldsheet are

$$\partial E_8 = 16(A_2 \times D_5 + A_1 \times E_6 + A_1 \times A_2 \times A_4$$
$$+ A_3 \times A_4 + A_1 \times A_6 + A_7 + D_7 + E_7). \tag{22}$$

IV. NONSIMPLY LACED TYPES FROM FOLDING

The worldsheet parametrization for the nonsimply laced types can be achieved by a process known as folding. The folding map on the z parameters is derived from the
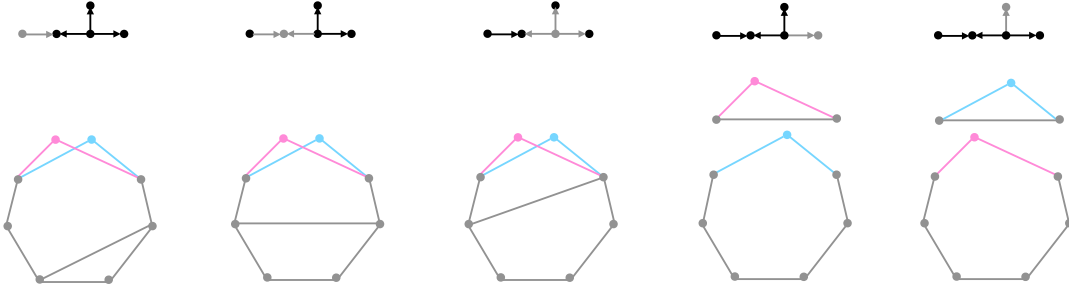

FIG. 3. The factorizations of the D_5 stringy integral as seen on the worldsheet.
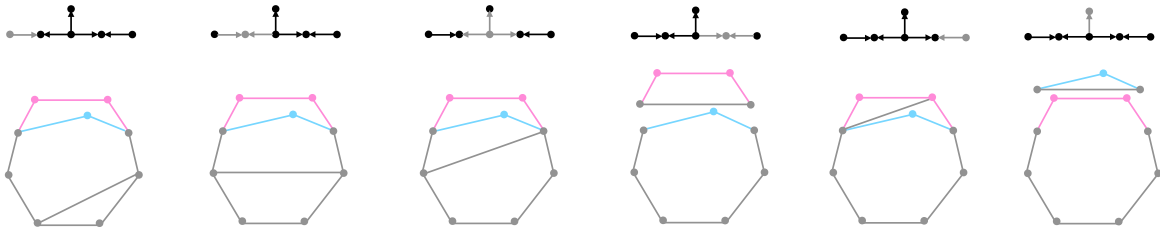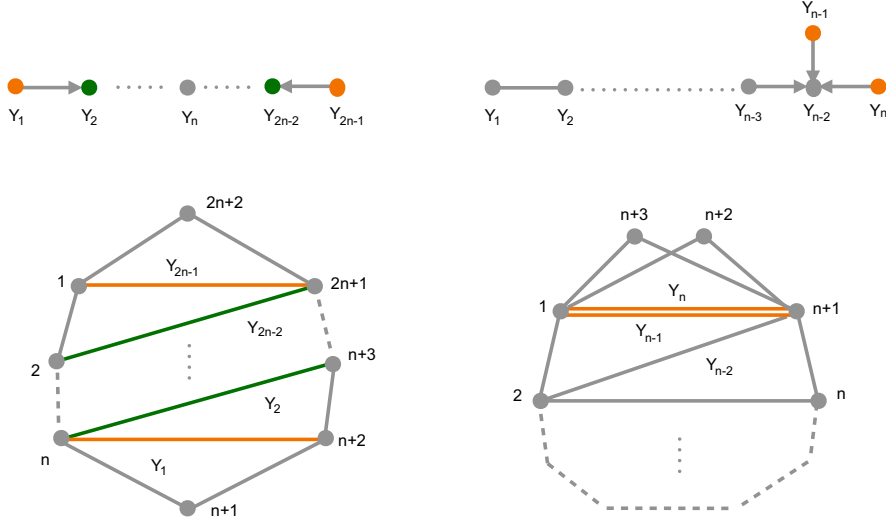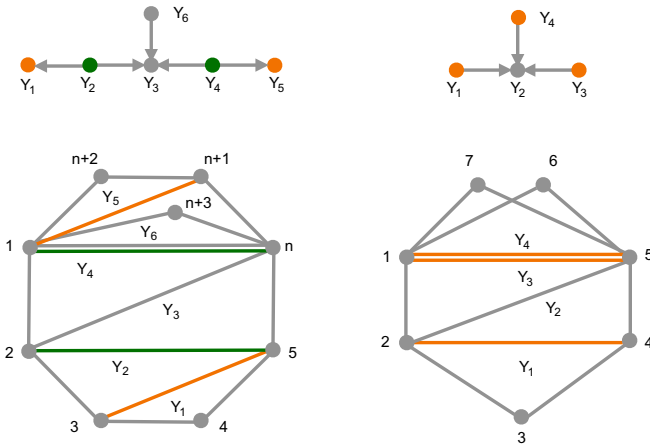

FIG. 4. The factorizations of the E_6 stringy integral as seen on the worldsheet.

FIG. 5. The folding from A_{2n-1} to C_n and D_n to B_{n-1}.

FIG. 6. The folding from E_6 to F_4 and D_4 to G_2.

standard folding of the root systems combined with the birational map in the ADE types. See Figs. 5 and 6.

$A_{2n-1} \rightarrow C_n$ *folding.* To obtain the C_n worldsheet, we fold the A_{2n-1} worldsheet by identifying the diagonals according to the roots as

$$Y_{2n-1} = Y_1, \quad Y_{2n-2} = Y_2, \quad \cdots Y_{n+1} = Y_{n-1}. \quad (23)$$

Solving the cross-ratio relations, we obtain a fractional map of the A_{2n-1} worldsheet variables in terms of the C_n worldsheet variables:

$$z_{2n+3-i} = \frac{z_{n+3}z_{1,2}z_{n+2-i,n+2} - z_1 z_{2,n+1} z_{n+2,n+3}}{z_{1,2}z_{n+2-i,n+2} - z_{2,n+1}z_{n+2,n+3}} \quad (24)$$

for $i = 1, 2, \ldots, n-1$. In the standard gauge choice $(z_1 \rightarrow -1, z_2 \rightarrow 0, z_{n+3} \rightarrow \infty)$, the folding map (24) reduces to a simple gauge-fixed map

$$z_{2n+3-i} = -\frac{z_{n+2}}{z_{n+2-i}}. \quad (25)$$

We recover the quadratic C_n alphabet from the linear A_{2n-1} alphabet by examining all the polynomial factors that appear in the cross ratio (2) under the folding map. Equivalently, one may perform the folding map directly on the alphabet (3) and read off all the factors.

$D_n \rightarrow B_{n-1}$ *folding.* We identify

$$Y_n = Y_{n-1}. \quad (26)$$

This is equivalent to $z_{n+3} = z_{n+2}$.

$E_6 \rightarrow F_4$ *folding.* To obtain the F_4 worldsheet, we fold the E_6 worldsheet by identifying the diagonals according to the roots as

$$Y_5 = Y_1, \qquad Y_4 = Y_2. \quad (27)$$

Solving the cross-ratio relations, we obtain

$$z_7 = \frac{z_6 z_{1,2} z_{3,5} - z_1 z_{2,3} z_{5,6}}{z_{1,2} z_{3,5} - z_{2,3} z_{5,6}}, \quad z_8 = \frac{z_6 z_{1,2} z_{4,5} - z_1 z_{2,4} z_{5,6}}{z_{1,2} z_{4,5} - z_{2,4} z_{5,6}}. \quad (28)$$

Upon gauge fixing as $z_1 \rightarrow -1, z_2 \rightarrow 0, z_6 \rightarrow \infty$, this reduces to a simple map

$$z_7 = -\frac{z_5}{z_3}, \qquad z_8 = -\frac{z_5}{z_4}. \quad (29)$$

$D_4 \rightarrow G_2$ *folding.* To obtain G_2, we fold the D_4 worldsheet by identifying the diagonals according to the roots as

$$Y_4 = Y_3 = Y_1. \quad (30)$$

Solving the cross-ratio relations, we obtain

$$z_6 = \frac{z_5 z_{1,2} z_{3,4} - z_1 z_{2,3} z_{4,5}}{z_{1,2} z_{3,4} - z_{2,3} z_{4,5}}. \tag{31}$$

Upon gauge fixing as $z_1 \to -1, z_2 \to 0, z_5 \to \infty$, this reduces to a simple map

$$z_6 = -\frac{z_4}{z_3}. \tag{32}$$

The alphabets of nonsimply laced types may be obtained by applying the gauge-fixed maps directly to the alphabets of simply laced types. The results were already quoted in [13] without derivation. In the next section, we shall derive a simpler set of alphabets by first applying the general folding maps and then fixing the gauge.

V. NEW CLUSTER ALPHABETS

In the standard gauge choice, which is usually taken to be $z_1 \to -1, z_2 \to 0, z_n \to \infty$, we recover the known cluster alphabets of types A, C, D and obtain new ones for type E and the nonsimply laced ones. However, it is reasonable to suspect that we have not found the simplest possible choice. Unlike the A_n case, not all worldsheet variables are

on an equal footing. Different gauge choices will lead to different alphabets. While there is no canonical choice of the alphabet, choosing a gauge that yields letters that are polynomials of the lowest order is preferable. We say two alphabets are equivalent if seen as a collection of hypersurfaces, they have the same topological property. That is,

(1) The number of letters equals the dimension of the cluster algebra.
(2) They give the same point count in the hypersurface complement.

A simpler D_n alphabet is obtained by the gauge choice $z_{n+3} \to -1, z_1 \to 0, z_{n+1} \to \infty$. Let

$$a_{i,j} = z_i - z_{n+2} + z_i z_j - z_i z_{n+2} \tag{33}$$

be the gauge-fixed version of $w_{i,j}$. The D_n alphabet is

$$\Phi_{D_n} = \Phi_{A_{n-1}}(z_2, \ldots, z_n) \cup \{z_{n+2}\} \cup$$
$$\times \bigcup_{2 \le i \le n} \{z_{i,n+2}\} \cup \bigcup_{2 \le i < j \le n} \{a_{i,j}\}. \tag{34}$$

A similar gauge choice $(z_9 \to -1, z_1 \to 0, z_6 \to \infty)$ removes the terms containing z_i^2 in the E_6 alphabet (18). Let

$$\tilde{a}_{i,j} = z_i - z_{n+1} + z_i z_j - z_i z_{n+1},$$
$$a_{i,j,k} = z_i z_j - z_i z_{n+1} - z_j z_{n+2} + z_{n+1} z_{n+2} + z_i z_j z_k - z_i z_j z_{n+1} - z_i z_k z_{n+2} + z_i z_{n+1} z_{n+2},$$
$$a_{i,j,k,l} = z_i z_j - z_i z_{n+1} - z_j z_{n+2} + z_{n+1} z_{n+2} + z_i z_j z_k + z_i z_j z_l - z_i z_j z_{n+1} - z_i z_j z_{n+2}$$
$$\quad - z_i z_k z_{n+1} + z_i z_k z_{n+2} - z_i z_l z_{n+2} + z_i z_{n+1} z_{n+2} - z_j z_k z_{n+2} + z_j z_{n+1} z_{n+2}$$
$$\quad + z_i z_j z_k z_l - z_i z_j z_k z_{n+1} - z_i z_j z_l z_{n+2} + z_i z_j z_{n+1} z_{n+2}. \tag{35}$$

be the gauge-fixed versions of $\tilde{w}_{i,j}^E, w_{i,j,k}, w_{i,j,k,l}$, respectively. A new E_6 alphabet may be written succinctly as

$$\Phi_{E_6} = \Phi_{A_4}(z_2, \ldots, z_5) \cup \{z_7, z_8, z_{7,8}, a_{2,3,4,5}\} \cup \bigcup_{2 \le i \le 5} \{z_{i,7}, z_{i,8}\} \cup \bigcup_{2 \le i < j \le 5} \{a_{i,j}, \tilde{a}_{i,j}\} \cup \bigcup_{2 \le i < j < k \le 5} \{a_{i,j,k}\}. \tag{36}$$

Gauge fix then fold vs fold then gauge fix. To obtain the alphabet for nonsimply laced types, we may apply the gauge-fixed folding maps [(25), (26), (29), (32)] to the alphabets of simply laced types. However, the gauge choice may not be optimal for the nonsimply laced types. Alternatively, we can also first fold the ungauged alphabets using the general maps [(24), (26), (28), (31)], and then choose the gauge that produces the nicest alphabets for nonsimply laced types.

If we fold the D_n alphabet to obtain the B_{n-1} alphabet, then some of the letters are still quadratic. If instead the ungauged D_n alphabet are first folded as $z_{n+3} = z_{n+2}$ and then gauge fixed as $z_1 \to -1, z_{n+1} \to 0, z_{n+2} \to \infty$, then we obtain a linear alphabet

$$\Phi_{B_{n-1}} = \Phi_{A_{n-1}}(z_2, \ldots, z_n) \cup \bigcup_{2 \le i < j \le n} \{1 - z_{i,j}\}. \tag{37}$$

We chose this particular gauge because it produces the simplest possible set of linear letters. Here we see the advantage of having an ungauged description of the worldsheet. It is not possible to obtain the linear B_{n-1} alphabet from folding the quadratic D_n alphabet because we have gauge fixed $z_{n+1} \to \infty$ to obtain the D_n alphabet, whereas the B_{n-1} alphabet corresponds to gauge-fixing $z_{n+2}, z_{n+3} \to \infty$. This shows that the B-type alphabet may be realized by a set of hyperplanes.

For C_n, we were not able to find a gauge choice that leads to a linear alphabet. The ungauged C_n letters obtained from

the general folding map are quadratic in $n-1$ of the variables. An interesting question would be to find a map that realizes the C_n alphabet as a hyperplane arrangement.

To obtain the G_2 alphabet, we now need to solve the folding equation (30) for z_2, apply the map on the ungauged D_4 alphabet, and then gauge fix as $z_1 \to -1$, $z_5 \to 0, z_6 \to \infty$. We then arrive at a new G_2 alphabet that is at most quadratic, as opposed to quartic as found previously from the standard gauge fixing [13]:

$$\Phi_{G_2} = \Phi_{B_2}(z_3, z_4) \cup \{z_3 - z_4 - z_3 z_4, z_4 - z_3^2 + z_3 z_4\}. \tag{38}$$

Note that as in the $D_n \to B_{n-1}$ folding, we cannot obtain a simplified form of the F_4 alphabet if we directly folded the E_6 alphabet as (29), where z_6 is already fixed at infinity. Instead, we fold the ungauged alphabet and then gauge fix as $z_9 \to -1, z_1 \to 0, z_5 \to \infty$. A new F_4 alphabet consists of 28 letters of polynomial degree of at most 4:

$$\Phi_{F_4} = \Phi_{A_4}(z_2, z_3, z_4, z_6) \cup \bigcup_{3 \le i \le j \le 4} \{a_{i,j}, b_{i,j}, c_{i,j}\} \cup \{z_2 z_3 - z_2 z_4 + z_2 z_6 - z_3^2 - z_3^2 z_4 + z_2 z_3 z_6,$$

$$z_2 z_3 - z_2 z_4 + z_2 z_6 - z_3 z_4 - z_3 z_4^2 + z_2 z_3 z_6, z_2^2 z_6 + z_2 z_3^2 - 2 z_2 z_3 z_4 + z_2 z_3^2 z_6 - z_3^2 z_4^2,$$

$$2 z_2 z_3 - 2 z_2 z_4 + z_2 z_6 - z_3^2 + z_2 z_3^2 + 2 z_2 z_3 z_6 - z_2 z_4^2 - 2 z_3^2 z_4 + z_2 z_3^2 z_6 - z_3^2 z_4^2,$$

$$z_2^2 z_6 - 2 z_2 z_3 z_4 + z_2 z_4^2 - 2 z_2 z_4 z_6 + z_2 z_6^2 + 2 z_3^2 z_4 - z_3^2 z_6 + z_2^2 z_6^2 - 2 z_2 z_3 z_4 z_6 + z_3^2 z_4^2\}, \tag{39}$$

where

$$a_{i,j} = z_2 + z_i z_j, \qquad b_{i,j} = z_2 z_6 - z_i z_j,$$
$$c_{i,j} = z_2 + z_6 - z_i - z_j + z_2 z_6 - z_i z_j. \tag{40}$$

ACKNOWLEDGMENTS

APPENDIX: ALPHABETS FOR E_7 AND E_8

The alphabets of E_7 and E_8 may be constructed similarly. The gauge choice is $z_{n+3} \to -1, z_1 \to 0, z_n \to \infty$. The E_7 alphabet consists of letters with a degree of at most 5. Introduce two new highest-order letters

$$a_{i,j,k,l,m} = a_{j,m} a_{i,k,l} + z_{n+2} z_{i,j} z_{k,n+1} z_{l,m},$$
$$\tilde{a}_{i,j,k,l,m} = \tilde{a}_{j,l} a_{i,k,m} - z_{n+2} z_{i,j} z_{k,l} z_{n+1,n+2}. \tag{A1}$$

The 70 letters of the E_7 alphabet are

$$\Phi_{E_7} = \Phi_{A_5}(z_2, \ldots, z_6) \cup \{z_8, z_9, z_{8,9}, a_{2,3,4,5,6}, \tilde{a}_{2,3,4,5,6}\} \cup \bigcup_{2 \le i \le 6} \{z_{i,8}, z_{i,9}\}$$

$$\times \cup \bigcup_{2 \le i < j \le 6} \{a_{i,j}, \tilde{a}_{i,j}\} \cup \bigcup_{2 \le i < j < k \le 6} \{a_{i,j,k}\} \cup \bigcup_{2 \le i < j < k < l \le 6} \{a_{i,j,k,l}\}. \tag{A2}$$

The E_8 alphabet consists of letters with a degree of at most 7. We introduce six degree-7 letters

$$c_2 = a_{2,4,6} a_{2,3,5,7} + z_2 z_{10} z_{2,3} z_{4,5} z_{6,7} z_{9,10},$$

$$c_3 = a_{3,4,6} a_{2,3,5,7} + z_{10} z_{2,3} z_{3,4} z_{5,6} z_{9,10},$$

$$c_4 = a_{2,4,7} a_{3,4,5,6} - z_{10} z_{2,3} z_{4,5} z_{9,10} a_{4,6},$$

$$c_5 = a_{2,5,6} a_{3,4,5,7} + z_{2,3} z_{5,9} z_{6,7} z_{10} \tilde{a}_{4,5},$$

$$c_6 = a_{3,5,6} a_{2,4,6,7} + z_2 z_{10} z_{3,4} z_{5,6} z_{6,7} z_{9,10},$$

$$c_7 = a_{3,5,7} a_{2,4,6,7} + z_{10} z_{2,3} z_{4,5} z_{6,7} z_{9,10}, \tag{A3}$$

and three degree-6 letters

$$b_1 = a_{3,7} a_{2,4,5,6} + z_{10} z_{2,3} z_{6,7} a_{4,5},$$
$$b_2 = \tilde{a}_{2,6} a_{3,4,5,7} + z_{10} z_{2,3} z_{6,7} \tilde{a}_{4,5},$$
$$b_3 = a_{2,3,5,6,7} z_{4,10} - z_{3,4} z_{9,10} a_{2,5,6}. \tag{A4}$$

The 128 letters of the E_8 alphabet are

$$\Phi_{E_8} = \Phi_{A_6}(z_2, \ldots, z_7) \cup \{z_9, z_{10}, z_{9,10}, b_1, b_2, b_3\} \cup \bigcup_{2 \leq i \leq 7} \{z_{i,9}, z_{i,10}, c_i\} \cup \bigcup_{2 \leq i < j \leq 7} \{a_{i,j}, \tilde{a}_{i,j}\}$$

$$\times \cup \bigcup_{2 \leq i < j < k \leq 7} \{a_{i,j,k}\} \cup \bigcup_{2 \leq i < j < k < l \leq 7} \{a_{i,j,k,l}\} \cup \bigcup_{2 \leq i < j < k < l < m \leq 7} \{a_{i,j,k,l,m}, \tilde{a}_{i,j,k,l,m}\}. \tag{A5}$$

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