Scalar Bern-Carrasco-Johansson bootstrap

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In this paper, we study tree-level scattering amplitudes of scalar particles in the context of effective field theories. We use tools similar to the soft bootstrap to build an ansatz for cyclically ordered amplitudes and impose the Bern-Carrasco-Johansson (BCJ) relations as a constraint. We obtain a set of BCJ-satisfying amplitudes as solutions to our procedure, which can be thought of as special higher-derivative corrections to SU(N) nonlinear sigma model amplitudes satisfying BCJ relations to arbitrary multiplicity at leading order. The surprising outcome of our analysis is that BCJ conditions on higher-point amplitudes impose constraints on lower-point amplitudes, and they relate coefficients at different orders in the derivative expansion. This shows that BCJ conditions are much more restrictive than soft limit behavior, allowing only for a very small set of solutions.

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I. INTRODUCTION

In recent years, the study of tree-level scattering amplitudes in effective field theories (EFT) has been an active area of research, leading to very important insights ranging from the calculation of amplitudes in the Standard Model effective field theory (SMEFT) to the intensive study of amplitudes in EFTs using modern methods such as soft bootstrap [1–6], scattering equations [7,8] and color-kinematics duality [9–12,12,13].

A key model in these investigations is the SU(N) nonlinear sigma model (NLSM) which describes the leading-order dynamics of strong interactions at low energies. Pion scattering amplitudes vanish in the soft limit, i.e. possess an Adler zero [14], as a consequence of the shift symmetry of the Lagrangian. Higher-order corrections are organized in a derivative expansion in the context of chiral perturbation theory (χPT) [15,16]. The Wilson coefficients in the χPT Lagrangian can be determined from experiments, but the theoretical derivation from the underlying QCD requires solving the theory at all scales which is incredibly hard.

Tree-level amplitudes in QFT satisfy the important properties of *locality* and *unitarity*, i.e. all poles are located at $P^2=(p_{a_1}+\cdots+p_{a_m})^2=0$ (for massless theories) and the amplitude factorizes on these poles into a product of two subamplitudes, schematically

$$A_n \xrightarrow{P^2=0} A_{n_1}\frac{1}{P^2}A_{n-n_1+2}. \qquad (1)$$

If the factorization conditions are enough to fix the amplitude uniquely, one can use (1) as input to reconstruct the n-point amplitude from lower-point amplitudes using recursion relations [17,18]. In EFTs, this assumption is not satisfied because of the presence of contact terms. These have vanishing residues on all poles and hence are not constrained by factorization, so further properties of the amplitudes are needed to specify the model.

The word bootstrap has taken on many meanings in recent years [19]. Here we refer to the method of searching the space of all tree-level amplitudes for those that satisfy certain kinematic conditions as the bootstrap. These conditions may be consequences of some underlying symmetry of the (*a priori* unknown) Lagrangian, but here we construct the amplitude from the bottom-up, without reference to Lagrangian operators. We use a prefix to indicate the type of kinematical conditions.

For scalar EFTs, the bootstrap starts with a generic rational function of variables $s_{ij}=(p_i+p_j)^2$, with poles at locations consistent with locality. Next all possible residue equations (1) are imposed as constraints, followed by kinematic conditions to fix free parameters. A natural condition for low-energy EFTs is the soft limit constraint, i.e. for small momentum p the amplitude behaves as

$$\lim_{p\to 0} A_n = \mathcal{O}(p^\sigma), \qquad \text{for some integer } \sigma. \qquad (2)$$

Using this soft bootstrap, for $\sigma=1$ we have the standard Adler zero condition. If all coefficients are then fully

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specified, the amplitude is unique and can be reconstructed using recursion relations. This is the case for NLSM amplitudes [3]. If some coefficients are left unfixed, there are more solutions which can be explained by the existence of multiple independent Lagrangians.

In this paper, we carry out a Bern-Carrasco-Johansson (BCJ) bootstrap by replacing the soft limit condition (2) by the BCJ relations [20]

$$\sum_{i=2}^{n-1}(s_{12}+\cdots+s_{1i})A_n(2,\ldots,i,1,i+1,\ldots,n)=0, \tag{3}$$

applied as a constraint to a local ansatz (together with the standard Kleiss-Kuijf relations [21]). The BCJ relations are famously satisfied by Yang-Mills amplitudes [20], as well as NLSM amplitudes [22], and play a crucial role in the double copy [23,24]. In particular, the double copy of a pair of Yang-Mills amplitudes gives rise to gravity amplitudes, while the double copy of NLSM amplitudes produces special Galileon amplitudes [7]. Further details about the double copy and the role of the BCJ relations may be found here [10].

Our goal is to find solutions to the BCJ relations among ordered scalar amplitudes. These relations turn out to imply the Adler zero, while also yielding higher-derivative NLSM corrections. Finally we get constraints on the couplings across varying multiplicities and orders.

The ordinary χPT construction describes pion self-interactions systematically in a derivative expansion, and hence is exact up to a particular derivative order. In this paper we address whether "BCJ perturbation theory" is consistent order-by-order and we find that it is not, because of the existence of cross-order relations. We demonstrate this both at "lower" orders, and show that it persists to higher orders and multiplicities.

Another motivation for the BCJ bootstrap is to better understand the origin of relations between coefficients in Z-theory. Abelian Z-theory is a known UV completion of NLSM amplitudes [9]. When thought of as amplitudes in an EFT, Z-amplitudes contain many relations between multiplicities and orders. Are these a consequence of BCJ or are they a special constraint of the string disk integrals that give the Z-amplitudes? We address this UV/IR origin in later sections.

Organization of the letter: We begin with a review of NLSM amplitudes and the soft bootstrap in the context of NLSM corrections. We then classify all 4-point BCJ-satisfying amplitudes and formulate the BCJ bootstrap procedure at 6-point where we proceed up to order $\mathcal{O}(p^{18})$ with some surprising results. We do limited consistency checks for eight-point amplitudes and comment on the prospects of the existence of a "BCJ Lagrangian". We comment on the connection to the Z-theory amplitude, an important solution to the BCJ constraints [9,25] that appears in the context of the CHY formula and double copy as the "stringy part" of open-string amplitudes.

II. SOFT BOOTSTRAP FOR NLSM AMPLITUDES

The tree-level amplitudes in the SU(N) nonlinear sigma model can be decomposed into flavor-ordered sectors,

$$\mathcal{A}_n^{\text{NLSM}}=\sum_{\sigma}\text{Tr}(T^{a_1}T^{a_2}\ldots T^{a_n})A_n^{\text{NLSM}}(1,2,\ldots,n), \tag{4}$$

where $A_n^{\text{NLSM}}(1,2,\ldots,n)$ is a cyclically symmetric function, T^a are the generators of the SU(N) and we sum over all permutations σ modulo cyclic ones. The only poles of A_n^{NLSM} are located at $P_{ij}^2=(p_i+\cdots+p_{j-1})^2=0$, and we use this fact as an input in the ansatz.

At four-point, there are no factorization constraints and the size of the ansatz is directly equal to the number of independent four-point amplitudes. The cyclic symmetry implies that A_4 is symmetric in $s_{12}\leftrightarrow s_{23}$, and we can write the general form for the powercounting $A_4\sim s^m$ as

$$\mathcal{O}(p^{2m})\colon\ A_4\in\{u^{m-a}(s^a+t^a)\}, \tag{5}$$

where $a=0,2,\ldots,m$, $s=(p_1+p_2)^2$, and $t=(p_2+p_3)^2$. This means that we have m independent four-point amplitudes at order $\mathcal{O}(p^{2m})$. At leading order, $A_n^{\text{NLSM}}\sim\mathcal{O}(p^2)$. Using $m=1$ in (5) gives

$$A_4=(p_1+p_3)^2=u, \tag{6}$$

the familiar four-pion NLSM amplitude. Power counting of terms in (5) can also be seen in the context of the Lagrangian in a derivative expansion,

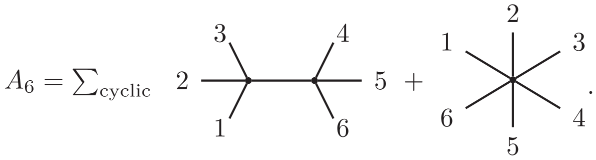
$$\mathcal{L}_{\chi\text{PT}}=\mathcal{L}_2+\mathcal{L}_4+\mathcal{L}_6+\cdots, \tag{7}$$

where the Lagrangian $\mathcal{L}_{2m}$ contains operators with $2m$ derivatives that are invariant under chiral symmetry. The usual building blocks are

$$u_\mu=i(u^\dagger\partial_\mu u-u\partial_\mu u^\dagger)\quad\text{where } u=\exp\left(\frac{i\phi^aT^a}{F\sqrt{2}}\right), \tag{8}$$

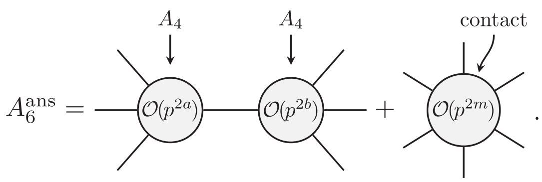
F is the pion decay constant, which we take $F\sim 1$ here for simplicity. Each Lagrangian $\mathcal{L}_{2m}$ contains multiple independent terms and construction of all such Lagrangians has been an active area of research [15,16,26,27].

Four-point amplitudes are special because the soft behavior is automatic for any nonconstant function $F(s,t,u)$ because $s,t,u\to 0$ when any $p_i\to 0$. The first nontrivial constraints in the soft bootstrap occur at six-point. Here our ansatz consists of factorization terms that are determined by the four-point amplitudes and all possible contact terms, schematically



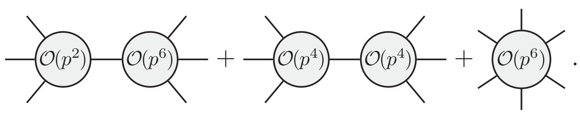
(9)

We could enlarge our ansatz to consider diagrams with more general kinematical invariants in the numerators, but using this ansatz is sufficient to guarantee correct factorization on all poles. On the other hand, soft limit behavior is not guaranteed. Additionally, there is always an ambiguity in the form of the 4-point amplitude (inserted into the vertices of the 6-point factorization diagram) as various equivalent 4-point forms do not agree on 6-point kinematics. It is convenient to choose the form written only in terms of external legs (not the internal leg P). The difference from any other form is just a contact term, which we add in the ansatz anyway.



(10)

At a fixed-derivative order $\mathcal{O}(p^{2m})$ the contact term involves $A_6 \sim s^m$, while the factorization terms contain vertices $A_4^{(a)} \sim s^a$ and $A_4^{(b)} \sim s^b$ where $a + b = m + 1$. For example at $\mathcal{O}(p^6)$ we have



(11)

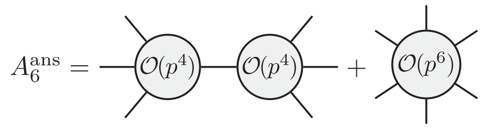
A special case is leading order NLSM i.e. $m = a = b = 1$. Here the contact contribution is uniquely fixed by soft behavior,

$$A_6^{\mathrm{NLSM}} = \left(\frac{s_{13}s_{46}}{s_{123}} + \frac{s_{26}s_{35}}{s_{345}} + \frac{s_{15}s_{24}}{s_{234}}\right) - \frac{1}{2}\sum_{\mathrm{cycl}} s_{13}. \tag{12}$$

In fact, all n-point amplitudes in the NLSM are uniquely fixed by soft recursion relations [2–4] based on the 4-point amplitude. This uniqueness is also reflected in the fact that for massless pions, the leading-order $\mathcal{O}(p^2)$ term in the χPT Lagrangian only has one term.

At general $\mathcal{O}(p^{2m})$ order, we have multiple solutions. The results at 6-point are summarized on the second line of Table II. A similar analysis has been used to count independent terms in the χPT Lagrangian [28]. To do this, one has to calculate higher-point amplitudes because at a given derivative order, some operators only contribute to eight- or higher-point amplitudes.

Note two important features of this analysis. First, constraints in the ansatz are always imposed at a fixed $\mathcal{O}(p^{2m})$ order, e.g. the cyclic sum of the second type of diagram in the $\mathcal{O}(p^6)$ ansatz (11) must satisfy the soft limit by itself. Though we still find solutions of the form



(13)

in our analysis, the allowed contact term is very special and it must come from collapsing the propagator in the factorization diagram, i.e. it must have the form

$$A_6^{\mathrm{contact}} = \sum s_{ab}s_{cd}s_{ef}, \tag{14}$$

where legs $a, b \in (1,2,3)$ and $c,d,e,f \in (4,5,6)$ (and cyclic shifts of external labels). If this were not true, the coupling constants in the $\mathcal{O}(p^4)$ Lagrangian would be linked to the $\mathcal{O}(p^6)$ Lagrangian, which is not possible when the only constraint placed is chiral symmetry. Secondly, all constraints are placed at fixed multiplicity i.e. the 6-point soft bootstrap places no constraints on 4-point coefficients.

III. FOUR-POINT BCJ AMPLITUDES

Our goal is to use BCJ relations to constrain the ansatz instead of the amplitude's soft limit. While the Adler zero and BCJ relations can be imposed separately, every BCJ solution inherently possesses an Adler zero, as suggested in [29] (see also [22,30]) and all checks we have performed up to 6-point at $\mathcal{O}(p^{18})$ confirm this. The 4-point BCJ relation can be written as

$$sA_4(1,2,3,4) - uA_4(1,3,2,4) = 0. \tag{15}$$

Together with cyclicity this implies that at $\mathcal{O}(p^{2m})$ order we can write the canonically ordered amplitude as

$$A_4(1,2,3,4) \equiv A_4 = u \cdot F^{(2m-2)}(s,t,u), \tag{16}$$

where $F^{(2m-2)}$ is a symmetric polynomial of degree $m-1$ in s, t, u. A basis for all symmetric polynomials is

$$F_{a,b}^{(2m)} \in \{(stu)^a(s^2+t^2+u^2)^b\} \quad \text{for } 3a+2b=m. \tag{17}$$

It is easy to see that the number of such terms is $[(m+1)/2] - [(m+1)/3]$ as previously reported [31,32]. The results are summarized in Table I up to $\mathcal{O}(p^{18})$.

Note that at $\mathcal{O}(p^4)$ there are no BCJ solutions, as the equation $3a + 2b = 1$ has no integer solutions. Additionally, up to $\mathcal{O}(p^{12})$ there is only one BCJ 4-point

TABLE I. Soft and BCJ bootstrap at 4-point with the number of free coefficients that parametrize $\mathcal{O}(p^{\#})$ 4-point amplitudes compatible with soft behavior and BCJ respectively. No constraints from BCJ relations are included.

$\mathcal{O}(p^{\#})$	2	4	6	8	10	12	14	16	18
Soft amplitudes	1	2	2	3	3	4	4	5	5
BCJ amplitudes	1	0	1	1	1	1	2	1	2

amplitude at each order, but starting $\mathcal{O}(p^{14})$ there are two (or more) independent solutions. Any BCJ-satisfying 4-point amplitude can hence be expanded in a basis,

$$A_4^{\mathrm{BCJ}} = \sum_{m;a,b} \alpha_{a,b}^{(2m)} (u F_{a,b}^{(2m-2)}). \tag{18}$$

One particularly special solution to BCJ constraints is the Z-theory amplitude, the "stringy" part of the Abelian open string amplitude [9]. The derivative expansion above coincides with the α'-expansion in this case. The 4-point Z-amplitude is written as a combination of Γ-functions, the low energy expansion yields

$$\begin{aligned}\frac{1}{u}A_4^{\text{Z-theory}} &= 1 + \frac{s^2+t^2+u^2}{24\pi^2} + \frac{\zeta_3(stu)}{\pi^6} + \frac{(s^2+t^2+u^2)^2}{480\pi^4} \\ &+ \frac{(stu)(s^2+t^2+u^2)(\zeta_2\zeta_3+2\zeta_5)}{4\pi^{10}} \\ &+ \frac{51\zeta_6(s^2+t^2+u^2)^3+8(stu)^2(31\zeta_6+32\zeta_3^2)}{512\pi^{12}} \\ &+ \cdots,\end{aligned} \tag{19}$$

where we related α' and the pion decay constant to align it with the expansion in χPT. Each term individually satisfies the 4-point BCJ relations and the full Z-theory amplitude is one particular combination of them. In particular, starting $\mathcal{O}(p^{14})$, there are contributions that are compatible with 4-point BCJ but do not appear in the Z-theory amplitude. So far the coefficients in (19) are just arbitrary coefficients in front of terms which individually satisfy BCJ relations.

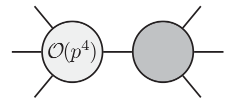
IV. BCJ AS A BOOTSTRAP CONSTRAINT

We now proceed to discuss 6-point amplitudes. In the spirit of the soft bootstrap, we fix the derivative order $\mathcal{O}(p^{2m})$ and write a local ansatz. The factorization diagrams consist of products of 4-point BCJ amplitudes of a given order with unfixed coefficients, while the contact term is given by a general polynomial ansatz. We then impose the 6-point BCJ relation

$$\begin{aligned}&s_{12}A_6(123456) + (s_{12}+s_{23})A_6(132456) \\ &\quad - (s_{25}+s_{26})A_6(134256) - s_{26}A_6(134526) = 0,\end{aligned} \tag{20}$$

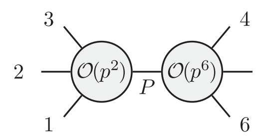
which constrains coefficients in both the factorization terms and the contact term. Constructing the ansatz in this way guarantees the satisfaction of (20) on all factorization channels but not for generic kinematics.

The nonexistence of 4-point $\mathcal{O}(p^4)$ BCJ amplitudes means that the corresponding factorization diagrams,



for any gray vertex, (21)

are not in the ansatz, simplifying the analysis. For example, at $\mathcal{O}(p^6)$ there is only one factorization diagram consisting of the $\mathcal{O}(p^2)$ and $\mathcal{O}(p^6)$ vertices along with the $\mathcal{O}(p^6)$ contact term, i.e. (11) with the middle factorization diagram missing. One of the terms is

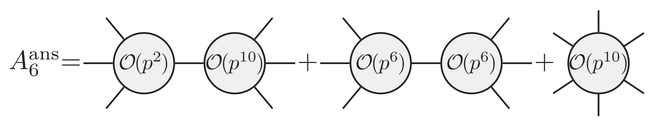


$$= A_4^{(2)}(123)\tfrac{1}{P^2}A_4^{(6)}(456), \tag{22}$$

where we defined

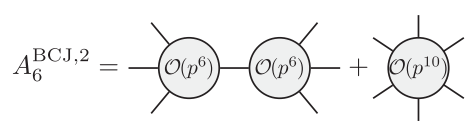
$$\begin{aligned}A_4^{(2)}(123) &\equiv A_4^{(2)}(1,2,3,P) = s_{13}, \\ A_4^{(6)}(456) &\equiv A_4^{(6)}(4,5,6,-P) = \alpha_{0,1}^{(6)} s_{46}(s_{45}^2+s_{46}^2+s_{56}^2),\end{aligned}$$

where α's are defined in (18) and we set $\alpha_{0,0}^{(2)}=1$. Imposing the BCJ condition (20) we fix the $\mathcal{O}(p^6)$ contact term uniquely and get the BCJ satisfying 6-point amplitude, see [9] for explicit formula. The same procedure works at $\mathcal{O}(p^8)$ level. At $\mathcal{O}(p^{10})$, there is a new feature—we get multiple factorization diagrams and some of them do not have an $\mathcal{O}(p^2)$ vertex,



(23)

Imposing the BCJ condition we get two solutions: the first solution consists of the first factorization diagram and a contact term [i.e. the same form as (10)] and again corresponds to some special term in the χPT expansion. The second solution is different,



(24)

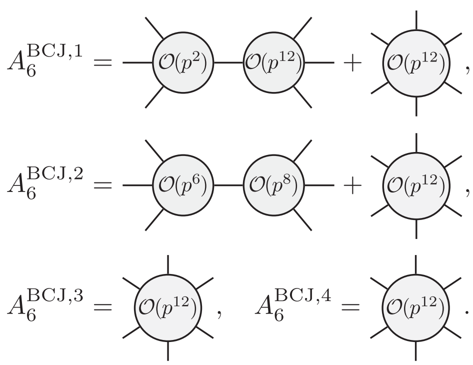
where the s_{123} factorization diagram is now equal to

$$(\alpha_{0,1}^{(6)})^2 \frac{s_{13}s_{46}(s_{12}^2+s_{13}^2+s_{23}^2)(s_{45}^2+s_{46}^2+s_{56}^2)}{s_{123}}. \tag{25}$$

This is similar to the soft bootstrap at order $\mathcal{O}(p^6)$ where one factorization diagram had two $\mathcal{O}(p^4)$ vertices. In that case, the contact term had a special form (14) and came from a collapsing propagator. Here the analysis reveals that in order to satisfy the BCJ relation in (24), we need a genuine contact term not of the special form, and we get a unique solution. This means that BCJ relates coefficients in the ansatz for the $\mathcal{O}(p^{10})$ contact term in (23) to $(\alpha_{0,1}^{(6)})^2$. The Lagrangian that could generate such amplitudes is written schematically as

$$\mathcal{L} = c_1(\partial^6\phi^4) + c_2(\partial^{10}\phi^6) + \cdots, \tag{26}$$

where BCJ now relates coefficients c_2 and c_1^2. While this never happens in the context of χPT where the Lagrangian coefficients at different $\mathcal{O}(p^{2m})$ orders can not be related by soft physics, this phenomenon occurs in the study of enhanced soft limits leading to the Dirac-Born-Infeld and special Galileon theories [4]. Here such conditions arise when we impose the BCJ relations. At the next order $\mathcal{O}(p^{12})$ we identify four solutions to the BCJ conditions,



(27)

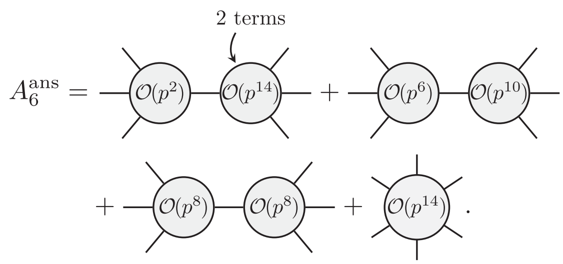
The first two solutions correspond to a factorization diagram, each completed by a particular $\mathcal{O}(p^{12})$ contact term, while the last two solutions are pure contact terms. Of course, any linear combination of BCJ amplitudes is a BCJ-satisfying amplitude. Our results are summarized in Table II, where the number of soft/BCJ amplitudes gives a total number of solutions, while the number of contact terms is a subset of that.

TABLE II. Soft and BCJ bootstrap at 6-point: The table shows the number of free coefficients that parametrize $\mathcal{O}(p^\#)$ 6-point amplitudes compatible with soft behavior and BCJ respectively, this includes products of 4-point coefficients that appear in pole terms and independent contact terms (whose numbers are separately indicated). The counting takes into account all relations that stem from 6-point BCJ.

$\mathcal{O}(p^\#)$	2	4	6	8	10	12	14	16	18
Soft amplitudes	1	2	10	29	83	207	461	945	1819
- from that contacts	0	0	5	22	70	191	434	915	1772
BCJ amplitudes	1	0	1	1	2	4	7	16	36
- from that contacts	0	0	0	0	0	2	4	13	31

V. FOUR-POINT RELATIONS

In principle, relations between coefficients at various $\mathcal{O}(p^{2m})$ come from UV physics and should not be accessible by any IR conditions—certainly not soft limits. At order $\mathcal{O}(p^{14})$, we find first evidence of such relations arising from the BCJ bootstrap. Here something surprising happens; not all factorization diagrams with BCJ-satisfying 4-point amplitudes can be completed into BCJ-satisfying 6-point amplitudes by the addition of contact terms. The ansatz for the amplitude takes the form,



(28)

Note that there are two different BCJ-satisfying $\mathcal{O}(p^{14})$ 4-point amplitudes, so the first factorization diagram represents two terms. Schematically, the ansatz is

$$A_6^{\mathrm{ans}} = \alpha_{2,0}^{(14)}(\ldots) + \alpha_{0,3}^{(14)}(\ldots) + \alpha_{0,1}^{(6)}\alpha_{0,2}^{(10)}(\ldots) + (\alpha_{1,0}^{(8)})^2(\ldots) + \sum_k \alpha_k^{\mathrm{ct}}(\ldots), \tag{29}$$

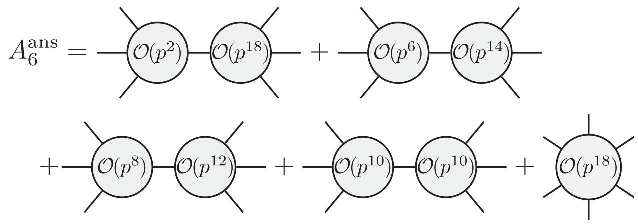
where $(\ldots)$ stands for some kinematical expressions. Note that in this case there are four BCJ-satisfying contact terms which always appear as solutions to the BCJ conditions, even if we turn off the factorization terms. Modulo them, we find three BCJ-satisfying solutions. Naively one would expect four if all the 4-point BCJ-satisfying amplitudes get uplifted to 6-point. Instead, the 4-point coefficients must satisfy a constraint,

$$\alpha_{2,0}^{(14)} - \frac{8}{3}\alpha_{0,3}^{(14)} - \frac{8}{3}\alpha_{0,1}^{(6)}\alpha_{0,2}^{(10)} - \frac{1}{2}(\alpha_{1,0}^{(8)})^2 = 0, \tag{30}$$

which allows us to solve for one of the parameters in terms of the others. It is interesting to see how the Z-theory amplitude satisfies this relation. The $\mathcal{O}(p^{14})$ coefficients have ζ_6 and ζ_3^2 parts while $\alpha_{0,2}^{(10)} \sim \zeta_4$, $\alpha_{1,0}^{(8)} \sim \zeta_3$ and $\alpha_{0,1}^{(6)} \sim \zeta_2$. The equality thus splits into an equation for the π^6 and ζ_3^2 coefficients.

At $\mathcal{O}(p^{16})$, the situation is similar to the $\mathcal{O}(p^{12})$ case. Applying the BCJ relations does not place any constraints on the 4-point coefficients. In other words, each factorization diagram individually leads to a BCJ-satisfying 6-point amplitude after the addition of appropriate contact terms. This seems to be related to the fact that there is only one $\mathcal{O}(p^{16})$ 4-point BCJ amplitude.

At $\mathcal{O}(p^{18})$ order, the ansatz is



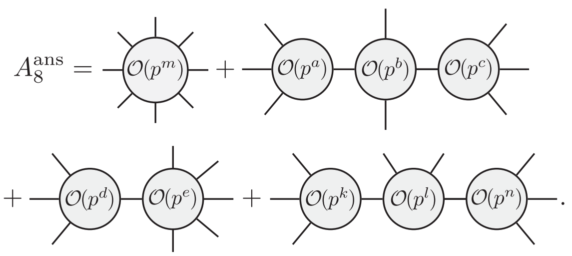
(31)

Imposing the BCJ conditions, we find 31 contact terms (out of 2132 in the ansatz) satisfy our conditions, but not all factorization terms can be completed individually into BCJ-satisfying 6-point amplitudes. This is because we get one constraint on 4-point coupling constants,

$$\alpha_{2,1}^{(18)} - 8\alpha_{0,4}^{(18)} + \alpha_{0,1}^{(6)}\alpha_{2,0}^{(14)} - 8\alpha_{0,1}^{(6)}\alpha_{0,3}^{(14)} - \alpha_{1,0}^{(8)}\alpha_{1,1}^{(12)} - 4(\alpha_{0,2}^{(10)})^2 = 0, \tag{32}$$

which has to be satisfied to get BCJ amplitudes. Hence in total at $\mathcal{O}(p^{18})$ we have 36 BCJ amplitudes, where we naively would have expected 37. The Z-theory 4-point amplitude again satisfies this relation in a nontrivial way, now dividing into three groups of terms proportional to ζ_8, $\zeta_3\zeta_5$ and $\zeta_2\zeta_3^2$.

It is important to check if there are any new constraints coming from 8-point amplitudes. Generating these within the ansatz presents a challenge due to the extensive combinatorial complexity of the contact terms—we were able to check up to $\mathcal{O}(p^{10})$. The general form of the ansatz for the 8-point amplitude is



(33)

The power counting conditions are

$$m = a + b + c - 4 = d + e - 2 = k + l + n - 4. \tag{34}$$

We refer to the two types of diagrams as *double-* and *single-factorization* diagrams. Note that again we use BCJ-satisfying 4-point amplitudes in the factorization diagrams in the ansatz. For the 6-point vertex of the single-factorization diagram and for the 8-point contact term, we use a generic ansatz. In principle, on finding a solution, one should cross-check the solution for the 6-point vertex (appearing in the eight-point amplitude) with the solution for the BCJ-satisfying 6-point contact term, to find possible 6-point relations. Here, we do not do this and instead focus only on generating further conditions on the 4-point BCJ amplitudes.

At $\mathcal{O}(p^6)$ and $\mathcal{O}(p^8)$ orders, nothing interesting happens; we find one BCJ 8-point amplitude at each order and no constraints on $\alpha_{0,1}^{(6)}$ and $\alpha_{1,0}^{(8)}$. The result takes the general form of (33), with one insertion of $\mathcal{O}(p^6)$ [resp. $\mathcal{O}(p^8)$] vertex and rest $\mathcal{O}(p^2)$ vertices. As there is no quadratic dependence on higher-derivative terms, we could in principle expand both $\mathcal{O}(p^6)$ and $\mathcal{O}(p^8)$ BCJ solutions in the basis of terms in χPT.

At $\mathcal{O}(p^{10})$ in the double-factorization terms in (33), we have either two $\mathcal{O}(p^6)$ vertices or one $\mathcal{O}(p^{10})$ vertex (the rest are $\mathcal{O}(p^2)$ vertices). The kinematical ansatz is

$$A_8^{\rm ans} = \alpha_{0,2}^{(10)}(\dots) + (\alpha_{0,1}^{(6)})^2(\dots) + \sum_k \alpha_k^{\rm sf,ct}(\dots), \tag{35}$$

where we grouped all single-factorization (sf) and contact (ct) terms in the last term, as we focus on the 4-point couplings here. Naively, we would expect two solutions which contain double-factorization terms if constants $\alpha_{0,2}^{(10)}$ and $\alpha_{0,1}^{(6)}$ are truly independent. Imposing the BCJ relations we find 21 solutions, but only one contains double factorization diagrams. Hence we get the following constraint on 4-point couplings,

$$\alpha_{0,2}^{(10)} = \frac{6}{5}(\alpha_{0,1}^{(6)})^2. \tag{36}$$

This is again satisfied by the Z-theory amplitude (19) and corresponds to the quadratic recurrence of the Riemann zeta function at even integers. The same is true for the relevant parts of (30) and (32). Note that the Abelian Z-theory amplitude (19) that we compare to, can be obtained by a sum over orderings of bicolored Z-theory amplitudes. Recently, generalizations of such bicolored Z-theory amplitudes were proposed [13,33]. The Abelianization of results in [13,33] can be achieved by a special selection of Wilson coefficients in our analysis. Thus, the analysis presented here is more general, leading to the natural question of whether there exists a different EFT description of bicolored Z-theory.

VI. OUTLOOK: TOWARDS A BCJ LAGRANGIAN

In this letter, we study the constraints placed on the coupling constants of higher-derivative scalar amplitudes by the BCJ relations. Starting with the BCJ-satisfying 4-point amplitude, we learn that when contributing to 6-point and 8-point BCJ amplitudes, the 4-point couplings must satisfy a number of constraints. BCJ relates terms of different derivative orders, which is very unlike how the Adler zero constrains amplitudes that give rise to χPT Lagrangians $\mathcal{L}_{\chi{\rm PT}}$. Indeed the BCJ relations seem to interpolate between IR and UV physics, as they both imply the vanishing soft limit of amplitudes, as well as place constraints on couplings of various orders. We have shown that demanding the BCJ relations are satisfied at higher-point would put even more constraints on lower-point amplitudes and their couplings.

The fact that BCJ relations constrain couplings at different multiplicity is linked to the question of the existence of "BCJ Lagrangians" which give rise to BCJ satisfying amplitudes. We know one solution to this problem is the leading-order 2-derivative NLSM Lagrangian. Additionally, we now know that in a putative higher-derivative Lagrangian, the couplings of various orders must be dependent—we indeed found some of these dependencies, but more work is needed to establish if there exists a higher-derivative Lagrangian that satisfies the BCJ relations at arbitrary multiplicity. One possible solution to the problem is Z-theory, whose Lagrangian, though unknown explicitly, is expected to exist.

It is clear that such a BCJ Lagrangian is a special version of χPT Lagrangian with tuned Wilson coefficients as the Adler zero condition is included in the BCJ constraint. Since the χPT Lagrangian is known formally only up to a relatively low level $O(p^8)$ [27], we can only explore a restricted case. General Lagrangian operators involve covariant derivatives [e.g. at $O(p^{10})$ they can include $\nabla_\alpha u_\mu$ or $\nabla_\alpha \nabla_\beta u_\mu$—see [31] for details]. However, there is one class of terms we can write down in a closed form to all orders—terms without covariant derivatives. Starting at $O(p^n)$ with n pions, for generic dimension they are given by

$$\mathcal{L}^n_{\chi\mathrm{PT}} = \sum_{j=1}^{d_n} c_j \langle u_{\mu_{j_1}} \ldots u^{\mu_{j_1}} \ldots u_{\mu_{j_{n/2}}} \ldots u^{\mu_{j_{n/2}}} \rangle, \qquad (37)$$

where we sum over all symmetric chord diagrams ($d_n = 1, 2, 5, 17, 79, \ldots$). Interestingly, we find that BCJ imposes the following condition on the constants c_j:

$$\mathrm{BCJ}\colon \sum_{j=1}^{d_n} c_j = 0, \quad \text{for } n > 2. \qquad (38)$$

We have verified this conjecture up to $O(p^8)$. It might represent the structure of a necessary condition on the full higher-derivative Lagrangian and could be a hint in the search for a more general BCJ Lagrangian and even an off shell version of color-kinematics duality.

Our work reinforces that BCJ corrections automatically satisfy soft behavior constraints. It would be helpful to construct a formal proof of this observation. Additionally, it would be useful to understand what these BCJ corrections double copy to and whether there is a symmetry principle that selects specifically these corrections to special Galileon theory.

An even more ambitious goal is to see both the Adler zero and the BCJ relations arise from some underlying geometric structure in the framework of positive geometry [34] such as the amplituhedron [35,36] for planar $\mathcal{N} = 4$ SYM amplitudes or the ABHY Associahedron for biadjoint ϕ^3 amplitudes [37].

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