Universal scaling dimensions for highly irrelevant operators in the local potential approximation

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We study d-dimensional scalar field theory in the local potential approximation of the functional renormalization group. Sturm-Liouville methods allow the eigenoperator equation to be cast as a Schrödinger-type equation. Combining solutions in the large field limit with the Wentzel-Kramers-Brillouin approximation, we solve analytically for the scaling dimension d_n of high dimension potential-type operators $\mathcal{O}_n(\varphi)$ around a nontrivial fixed point. We find that $d_n = n(d - d_\varphi)$ to leading order in n as $n \to \infty$, where $d_\varphi = \frac{1}{2}(d - 2 + \eta)$ is the scaling dimension of the field φ and determine the power-law growth of the subleading correction. For $O(N)$ invariant scalar field theory, the scaling dimension is just double this, for all fixed $N \geq 0$ and additionally for $N = -2, -4, \ldots$. These results are universal, independent of the choice of cutoff function which we keep general throughout, subject only to some weak constraints.

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I. INTRODUCTION

The functional renormalization group (FRG) is one of the most widely used approaches to study quantum field theories in nonperturbative regimes, as evidenced by an extensive literature (see, for instance, the reviews [1–6]). Various realizations of the FRG exist [7–18], but the most prevalent version [12–18] focuses on the flow of an appropriately defined Legendre effective action Γ_Λ (also referred to as the effective average action), with respect to an infrared cutoff scale Λ. This flow equation is given by

$$\frac{\partial}{\partial \Lambda}\Gamma_\Lambda = -\frac{1}{2}\mathrm{Tr}\left[\frac{1}{\Delta_\Lambda}\frac{\partial \Delta_\Lambda}{\partial \Lambda}\left(1 + \Delta_\Lambda \Gamma_\Lambda^{(2)}\right)^{-1}\right]. \qquad (1.1)$$

Here, Tr stands for a spacetime trace and $\Gamma_\Lambda^{(2)}$ is the Hessian matrix with respect to the fields. The propagator $\Delta_\Lambda(q) = C_\Lambda(q)/q^2$ is modified by the inclusion of a multiplicative infrared cutoff function $C_\Lambda(q) = C(q^2/\Lambda^2)$, which is nonnegative, monotonically increasing, and satisfies $C(0) = 0$ and $C(\infty) = 1$.

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In practical applications, some form of approximation becomes necessary. One frequently employed approximation is the local potential approximation (LPA) [19–28], which simplifies the flow equations by disregarding the momentum dependence of the effective action, except for a local potential term V_Λ. For a scalar field φ in d Euclidean dimensions, the effective action then takes the form

$$\Gamma_\Lambda = \int d^d x \left(\frac{1}{2}(\partial_\mu \varphi)^2 + V_\Lambda(\varphi)\right). \qquad (1.2)$$

While an exact analytical solution to this truncated FRG formulation is still not possible in general, the LPA enables numerical treatments that provide valuable insights into the system's behavior. It allows for numerical estimates of various physical quantities, including critical exponents and the scaling equation of state [1–6,23,29–31]. Moreover, the LPA serves as the initial step in a systematic derivative expansion [1,23,29–31], which facilitates a more comprehensive exploration of the system's properties [1–6,30].

Nevertheless, it is important to acknowledge the limitations of the LPA and more generally the derivative expansion. Since such truncations do not correspond to a controlled expansion in some small parameter, the errors incurred can be expected to be of the same order in general as the quantities being computed.[1] Furthermore, quantities that should be universal, and thus independent of the specific form of the

[1]See however Refs. [32–35].

cutoff, are not (for example, for the critical exponent ν at the Wilson-Fisher fixed point in $d = 3$ dimensions, the LPA yields $\nu = 0.689$ with a sharp cutoff [23] while for a power-law cutoff one obtains $\nu = 0.660$ [29]).

It has long been understood that an exception to this is the general form of a nontrivial fixed potential $V(\varphi)$ in the large field regime [1,23,29,31], which follows from asymptotic analysis,

$$V(\varphi) = A|\varphi|^{d/d_\varphi} + \cdots \quad \text{as} \quad \varphi \to \pm\infty, \tag{1.3}$$

where the ellipses stand for subleading terms (see later). The leading term coincides with the scaling equation of state precisely at the fixed point. It is a simple consequence of dimensional analysis on using the scaling dimension $d_\varphi = \frac{1}{2}(d - 2 + \eta)$ for the field φ at the fixed point, η being its anomalous dimension. However, asymptotic analysis does not fix the amplitude A or the anomalous dimension η, which have to be found by other means, for example, by numerical solution of truncated fixed point equations.

In this paper, we will show that within LPA, asymptotic analysis combined with Sturm-Liouville (SL) and Wentzel-Kramers-Brillouin (WKB) analysis[2] also allows one to determine asymptotically the scaling dimension d_n of the highly irrelevant ($d_n \gg 1$) eigenoperators $\mathcal{O}_n = \mathcal{O}_n(\varphi)$ of potential type (those containing no spacetime derivatives). Ordering them by increasing scaling dimension, we will show that $d_n = n(d - d_\varphi)$ to leading order in n. In the case of $O(N)$ invariant scalar field theory with fixed $N \geq 0$ the dimension d_n is doubled to $d_n = 2n(d - d_\varphi)$. The scaling dimension is thus independent of N. It agrees with the result for the single scalar field since these eigenoperators are functions of $\varphi^2 = \varphi^a\varphi^a$ and thus pick out only the even eigenoperators (those symmetric under $\varphi \leftrightarrow -\varphi$) in the $N = 1$ case. We also show that the scaling dimension is $d_n = 2n(d - d_\varphi)$ whenever $N = -2k$, where k is a nonnegative integer.

Once again these results are independent of the choice of cutoff and thus universal. Indeed, in this paper, we will keep the cutoff function completely general throughout, subject only to some weak technical constraints that we derive later. Note that, like the fixed point equation of state (1.3), the d_n take the same form, independent of the choice of fixed point, provided only that $d_\varphi > 0$ and that the fixed point potential is nonvanishing. We also show that the next to leading correction to d_n behaves as a power of n. The power is universal although the coefficient of the subleading correction is not.

Actually this approach was first employed to determine the scaling dimension of highly irrelevant eigenoperators in an $f(R)$ approximation [38,39] to the asymptotic safety scenario [40–42] in quantum gravity. The $f(R)$ approximation serves as a close analog to the LPA in this context [43–45]. However, while the resulting scaling dimensions d_n exhibit a simple nearly universal form for large values of n, they nevertheless retained strong dependence on the choice of cutoff. This issue can be traced back [39] to the so-called single-metric (or background field) approximation [40], where the identification of the quantum metric with the background metric is made in order to close the equations. The present paper thus completes the circle by demonstrating that, indeed, without such an approximation, the results become truly universal. Additionally, it showcases the power of these methods in a simpler context.

The paper is organized as follows. We first analyze the functional renormalization group equations for a single scalar field in the LPA. From the eigenoperator equation we write the resulting SL equation in Schrödinger form and thus, by taking the large field limit, deduce the asymptotic form of the renormalization group eigenvalues in the WKB limit. Section III extends the analysis to $O(N)$ scalar field theory using the same approach. Finally in Sec. IV we conclude and discuss the results, placing them in a wider context.

[2] See, e.g., Ref. [36] for textbook discussion of SL methods and Ref. [37] for WKB methods.

II. FLOW EQUATIONS IN LPA

The LPA amounts to setting the field φ in the Hessian to a spacetime constant, thus dropping from a derivative expansion all terms that do not take the form of a correction to the potential. The flow equation for $V_\Lambda(\varphi)$ then takes the form

$$\left(\partial_t + d_\varphi \varphi \frac{\partial}{\partial\varphi} - d\right) V_\Lambda(\varphi) = -\frac{1}{2}\int \frac{d^d q}{(2\pi)^d} \frac{\dot{\Delta}}{\Delta} \frac{1}{1 + \Delta V''_\Lambda(\varphi)}, \tag{2.1}$$

where $\partial_t = -\Lambda\partial_\Lambda$, t being the renormalization group "time" which, following [7], we have chosen to flow toward the IR. Here the momentum, potential, and field are already scaled by the appropriate power of Λ to make them dimensionless. Then $\Delta = C(q^2)/q^2$ no longer depends on Λ. The same is true of $\partial_t \Delta_\Lambda$, which after scaling we write as $\dot{\Delta}$, where

$$\dot{\Delta} = 2C'(q^2). \tag{2.2}$$

Since $C(q^2)$ is monotonically increasing, we have that $\dot{\Delta} > 0$.

The scaling dimension of the field is $d_\varphi = \frac{1}{2}(d - 2 + \eta)$, where η is the anomalous dimension. Since η arises from the renormalization group running of the field and is typically inferred from corrections to the kinetic term, one would naturally conclude that it vanishes in LPA [2,7,19–26]. Nevertheless, as noticed in Refs. [27,28], this assumption is not necessary. The flow equation (2.1) is still a mathematically consistent equation with $\eta \neq 0$. However, since we cannot determine η directly from (2.1), its value needs to be input from elsewhere (either from experiment or other theoretical studies). We will follow this strategy, in the expectation that it improves the accuracy of our final estimates for d_n.

Let us recall that the flow equation (2.1) is an implementation of the Wilsonian renormalization group (RG) [1,7]. Lowering the cutoff Λ implements the Kadanoff blocking [46], while rescaling the cutoff back to the original size is equivalently implemented by "measuring" all quantities in units of Λ, i.e., by making them dimensionless using the appropriate power of Λ [1] as we have done above. Then at a critical point corresponding to a continuous phase transition, the solutions $V_\Lambda(\varphi)$ remain finite but the distinguishing feature is that they become independent of Λ (see, e.g., [1]).

Thus, at such a fixed point (FP) $V_\Lambda(\varphi) = V(\varphi)$, and η, have no renormalization group time dependence. The eigenoperator equation follows from linearizing about a FP,

$$V_\Lambda(\varphi) = V(\varphi) + \varepsilon v(\varphi) \mathrm{e}^{\lambda t}, \tag{2.3}$$

ε being infinitesimal. Here λ is the RG eigenvalue. It is the scaling dimension of the corresponding coupling and is positive (negative) for relevant (irrelevant) operators. The scaling dimension of the operator $v(\varphi)$ itself is then $d - \lambda$. We write the eigenoperator equation in the same form as Refs. [38,39,45],

$$-a_2(\varphi) v''(\varphi) + a_1(\varphi) v'(\varphi) + a_0(\varphi) v(\varphi) = (d - \lambda) v(\varphi), \tag{2.4}$$

where the φ-dependent coefficients multiplying the eigenoperators are given by

$$a_0(\varphi) = 0, \tag{2.5}$$

$$a_1(\varphi) = d_\varphi \varphi, \tag{2.6}$$

$$a_2(\varphi) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{\dot{\Delta}}{(1 + \Delta V'')^2} > 0, \tag{2.7}$$

and we have noted that a_2 is positive. We can now repeat the analysis carried out in [38,39,45] to solve for λ in the case of high dimension eigenoperators.

A. Asymptotic solutions

For large φ, the rhs of (2.1) can be neglected. Thus, at a fixed point, the equation reduces to a first order ordinary differential equation (ODE) which is easily solved. It gives the first term (1.3) in an asymptotic series solution [29],

$$V(\varphi) = A|\varphi|^m + O(|\varphi|^{2-m}) \quad \text{as} \quad \varphi \to \pm\infty, \tag{2.8}$$

where for convenience we introduce

$$m = d/d_\varphi, \tag{2.9}$$

and A is a real constant (that is determined by solving for the full FP solution). The subleading terms arise from iterating the leading order contribution to next order.

Of course there is always the trivial $V(\varphi) \equiv 0$ fixed point solution, corresponding to the Gaussian fixed point. We will not be interested in that (the scaling dimensions in that case are exactly known and reviewed in the discussion in Sec. IV). Instead we focus on nontrivial FP solutions for which $A \neq 0$. In principle, A could be different in the two limits $\varphi \to \pm\infty$, although in practice the fixed point potentials (2.8) are symmetric. Anyway, we will see that A drops out of the analysis in a few further steps.

It is helpful for the following to note that $m > 3$, since this inequality ensures that the m-dependent asymptotic solutions we are about to derive are valid. To see that $m > 3$, first note that if η is neglected (typically $\eta \ll 1$, see, e.g., [47]), m is a decreasing function of d for all $d > 2$. In practice, nontrivial FP solutions only exist for $2 \leq d < 4$ (see, e.g., [23]). In the limit $d \to 4^-$, $\eta \to 0$ (by the ϵ expansion [47]), and thus $m \to 4$. Therefore, if we can neglect η, we see that m is bounded below by $m \geq 4$. In practice, one finds that the values of η increase as d is lowered, but even in $d = 2$ dimensions they are not large enough to destroy this bound. In $d = 2$ dimensions, the asymptotic solution (2.8) corresponds to that of a unitary minimal model [48,49]. The one with the largest anomalous dimension is that of the Ising model universality class which has $\eta = 1/4$, thus in $d = 2$ dimensions we have in fact $m \geq 8$ for all the unitary minimal models. In this way, we see that we are safe to bound $m > 3$ in practice.

Note that the solution (2.8) has a single free parameter even though the FP equation is a (nonlinear) second order ODE. The second parameter, if it exists, can be deduced by linearizing around (2.8), writing $V(\varphi) \mapsto V(\varphi) + \delta V(\varphi)$, and solving the flow equation (2.1) at the FP this time for δV. Since δV satisfies a *linear* second order ODE and one solution is already known, namely $\delta V = \partial_A V(\varphi)$, it is easy to find the solution that corresponds at the linearized level to the missing parameter [23,29]. However, one then discovers that these "missing" linearized solutions are rapidly growing exponentials. Such a linearized perturbation is not valid asymptotically since for diverging φ it is much larger than the solution (2.8) we perturbed around. Hence, the FP asymptotic solutions only have the one free parameter, A.

Substituting (2.8) into (2.7), we see that asymptotically $a_2(\varphi)$ scales as follows:

$$a_2(\varphi) = F|\varphi|^{2(2-m)} + O(|\varphi|^{3(2-m)}) \quad \text{as} \quad \varphi \to \pm\infty, \tag{2.10}$$

where F is positive and cutoff dependent,

$$F = \frac{1}{2(m(m-1)A)^2}\int \frac{d^dq}{(2\pi)^d}\frac{\dot{\Delta}}{\Delta^2} = -\frac{1}{(m(m-1)A)^2}\int \frac{d^dq}{(2\pi)^d}q^4\frac{\partial}{\partial q^2}C^{-1}(q^2). \quad (2.11)$$

We will assume that the integral converges. This imposes some weak constraints on the cutoff profile. From (2.11), we see that we require $C(q^2)$ to vanish slower than q^{d+2} as $q \to 0$, and $C \to 1$ faster than $1/q^{d+2}$ as $q \to \infty$. This is true, for example, for the popular form of additive (i.e., mass-type) cutoff [13] [which was the one used in the analogous $f(R)$ analyses in Refs. [38,39]],

$$r(q^2) = \frac{q^2}{\exp(aq^{2b}) - 1}, \qquad a > 0, \quad b \geq 1, \quad (2.12)$$

provided also we set $b < \frac{1}{2}(d+2)$, the relation to $C(q^2)$ being $q^2C^{-1}(q^2) = q^2 + r(q^2)$.

Given that $a_2(\varphi)$ vanishes asymptotically, it is tempting to neglect the a_2 term in (2.4). We will shortly justify this. By neglecting the a_2 term, the ODE becomes linear first order giving a unique solution up to normalization. Thus we deduce that the eigenoperators asymptotically scale as a power of the field,

$$v(\varphi) \propto |\varphi|^{\frac{d-\lambda}{d_\varphi}} + \cdots, \quad (2.13)$$

where the ellipses stand for subleading corrections.

The neglect of the a_2 is justified as follows. The missing solution is one that grows exponentially [again, so that $a_2(\varphi)v''(\varphi)$ cannot be neglected]. Since the ODE is linear, these are allowed solutions to (2.4), but they are ruled out because, on treating such perturbations at the nonperturbative level, it can be shown that they do not evolve multiplicatively in the RG no matter how close one starts to the FP [1,30,38,39,50,51], i.e., the RG time dependence never takes the form in Eq. (2.3). [Such perturbations do not then have a well-defined scaling dimension, and in fact it can be shown that as soon as Λ is lowered, they can be expanded as a convergent sum over the power-law solutions (2.13). For more details, see Refs. [1,30,38,39,50,51].]

Now, the asymptotic solution (2.13) imposes two boundary conditions (one for each limit $\varphi \to \pm\infty$) on the second order ODE (2.4), but since the ODE is linear this overconstrains the equation,[3] which thus leads to quantization of the RG eigenvalue λ. We index the solutions as $v_n(\varphi)$, ordering them so that λ_n decreases as n increases. We can now perform a SL transformation and deduce the asymptotic dependence of the eigenvalues λ_n on n, as $n \to \infty$.

[3]We can see this, for example, by imposing a normalization condition on v.

B. SL analysis

We can rewrite the eigenvalue equation (2.4) in a SL form by multiplying it with the SL weight function

$$w(\varphi) = \frac{1}{a_2(\varphi)}\exp\left\{-\int_0^\varphi d\varphi' \frac{a_1(\varphi')}{a_2(\varphi')}d\varphi'\right\}, \quad (2.14)$$

which is always positive due to the positivity of a_2. Then the eigenvalue equation becomes

$$-(a_2(\varphi)w(\varphi)v'(\varphi))' = (d-\lambda)w(\varphi)v(\varphi). \quad (2.15)$$

The SL operator on the left, $L = -\frac{d}{d\varphi}\left(a_2 w \frac{d}{d\varphi}\cdot\right)$, is self-adjoint when acting on the space spanned by the eigenoperators, i.e., it satisfies

$$\int_{-\infty}^{\infty} d\varphi\, u_1(\varphi)Lu_2(\varphi) = \int_{-\infty}^{\infty} d\varphi\, u_2(\varphi)Lu_1(\varphi), \quad (2.16)$$

when the u_i are linear combinations of the eigenoperators. This is so because the boundary terms at infinity, generated by integration by parts, vanish in this case. This follows because, from (2.13), the u_i diverge at worst as a power of φ, while $w(\varphi) \to 0$ exponentially fast as $\varphi \to \pm\infty$.

Thus, from SL analysis [36], we know that the eigenvalues λ_n are real, discrete, with a most positive (relevant) eigenvalue, and an infinite tower of ever more negative (more irrelevant) eigenvalues, $\lambda_n \to -\infty$ as $n \to \infty$ [30]. Let us define a "coordinate" x,

$$x = \int_0^\varphi \frac{1}{\sqrt{a_2(\varphi')}}d\varphi' \quad (2.17)$$

(always taking the positive root in fractional powers). Defining the wave function as

$$\psi(x) = a_2^{1/4}(\varphi)w^{1/2}(\varphi)v(\varphi) \quad (2.18)$$

enables us to recast (2.15) as

$$-\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = (d-\lambda)\psi(x). \quad (2.19)$$

This is a one-dimensional time-independent Schrödinger equation for a particle of mass $m = 1/2$, with energy $E = d - \lambda$, i.e., just the eigenoperator scaling dimension, and with potential [38,39,45]

$$U(x) = \frac{a_1^2}{4a_2} - \frac{a_1'}{2} + a_2'\left(\frac{a_1}{2a_2} + \frac{3a_2'}{16a_2}\right) - \frac{a_2''}{4}, \quad (2.20)$$

where the terms on the right-hand side are functions of φ.

From the limiting behavior of $a_2(\varphi)$, (2.10), we see that asymptotically the coordinate x scales as

$$x = \int_0^{\varphi} \left(\frac{|\varphi'|^{m-2}}{\sqrt{F}} + O(1) \right) d\varphi' = \pm \frac{|\varphi|^{m-1}}{(m-1)\sqrt{F}} + O(|\varphi|) \quad \text{as} \quad \varphi \to \pm\infty, \tag{2.21}$$

so in particular when $\varphi \to \pm\infty$ we have $x \to \pm\infty$. On the right-hand side of (2.20), the first term dominates at leading order (LO) and next-to-leading order (NLO). Since, asymptotically,

$$\frac{a_1^2(\varphi)}{4a_2(\varphi)} = \frac{d_\varphi^2}{4F} |\varphi|^{2m-2} + O(|\varphi|^m), \tag{2.22}$$

we thus find that

$$U(x) = \frac{1}{4}(d - d_\varphi)^2 x^2 + O(|x|^{1+\frac{1}{m-1}}) \quad \text{as} \quad x \to \pm\infty. \tag{2.23}$$

To LO, this is the potential of a simple harmonic oscillator of the form $\frac{1}{2} m\omega^2 x^2$, where

$$\omega = d - d_\varphi = \frac{1}{2}(d + 2 - \eta). \tag{2.24}$$

C. WKB analysis

We can now use WKB analysis to compute the asymptotic form of the energy levels, also known as operator scaling dimensions, E_n, at large n. This follows from solving the equality

$$\int_{-x_n}^{x_n} dx \sqrt{E_n - U(x)} = \left(n + \frac{1}{2} \right) \pi, \tag{2.25}$$

for the total phase of the wave oscillations described by $\psi(x)$, in the limit of large E_n [37]. Here x_n are the classical turning points, i.e., such that $E_n = U(\pm x_n)$. Now, the above integral is dominated by the regions close to the turning points, where we can substitute the asymptotic form (2.23). Including the subleading correction proportional to some constant γ (that depends on the cutoff profile) the integral is

$$\begin{aligned} &\frac{\omega}{2} \int_{-x_n}^{x_n} dx \sqrt{x_n^2 + \gamma x_n^{1+\frac{1}{m-1}} - x^2 - \gamma |x|^{1+\frac{1}{m-1}}} \\ &= \frac{\omega}{2} x_n^2 \int_{-1}^{1} dy \sqrt{1 - y^2 + \gamma x_n^{\frac{1}{m-1}-1} (1 - |y|^{1+\frac{1}{m-1}})}. \end{aligned} \tag{2.26}$$

Since the x_n are also large we can now evaluate the right-hand side and thus from (2.25) we get the asymptotic relation between x_n and n,

$$\frac{\omega\pi}{4} x_n^2 + O\left(x_n^{1+\frac{1}{m-1}} \right) = n\pi. \tag{2.27}$$

Hence, using (2.23), (2.24), and (2.27), the scaling dimension of the eigenoperators takes the form

$$\begin{aligned} d_n = E_n = d - \lambda_n = U(x_n) &= n\omega + O\left(n^{\frac{m}{2(m-1)}} \right) \\ &= n(d - d_\varphi) + O\left(n^{\frac{m}{2(m-1)}} \right) \quad \text{as} \quad n \to \infty. \end{aligned} \tag{2.28}$$

The subleading correction to the critical exponents contains information about the cutoff via the constant γ introduced in (2.26). However, at leading order the result is independent of the cutoff and is hence universal.

III. $O(N)$ SCALAR FIELD THEORY

Now let us apply the same treatment to N scalar fields φ^a ($a = 1, \ldots, N$) with an $O(N)$ invariant potential $V_\Lambda(\varphi^2) = V_\Lambda(\rho)$, in the LPA. We use the shorthand $\rho = \varphi^a \varphi^a = \varphi^2$. The flow equation (2.1) becomes [31,52]

$$\left(\partial_t - d + 2d_\varphi \rho \frac{\partial}{\partial\rho} \right) V_\Lambda(\rho) = -\frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{\dot{\Delta}}{\Delta} (M^{-1})^{aa}, \tag{3.1}$$

where the matrix M is given by

$$\begin{aligned} M^{ab} &= \delta^{ab} + \Delta \frac{\partial^2 V_\Lambda(\rho)}{\partial\varphi^a \partial\varphi^b} \\ &= \delta^{ab} + 2\Delta[\delta^{ab} V'_\Lambda(\rho) + 2\varphi^a \varphi^b V''_\Lambda(\rho)]. \end{aligned} \tag{3.2}$$

Inverting and tracing yields

$$(M^{-1})^{aa} = \frac{N-1}{1 + 2\Delta V'_\Lambda(\rho)} + \frac{1}{1 + 2\Delta V'_\Lambda(\rho) + 4\Delta\rho V''_\Lambda(\rho)}. \tag{3.3}$$

In the limit of large ρ, the right-hand side of the flow equation (3.1) can be neglected at leading order. This implies that a FP solution $V_\Lambda(\rho) = V(\rho)$ takes the following asymptotic form:

$$V(\rho) = A\rho^{\frac{m}{2}} + O(\rho^{1-\frac{m}{2}}) \quad \text{as} \quad \rho \to \infty, \tag{3.4}$$

where as before the subleading term has been calculated by iterating the leading contribution to next order.

The RG eigenvalue equation follows by linearizing (3.1) around the fixed point solution,

$$V_\Lambda(\rho) = V(\rho) + \varepsilon v(\rho) e^{\lambda t}, \tag{3.5}$$

giving an equation for $v(\rho)$ with the same structure as (2.4), i.e.,

$$-a_2(\rho) v'' + a_1(\rho) v' + a_0(\rho) v = (d - \lambda) v, \tag{3.6}$$

the same value for $a_0(\rho) = 0$, but different expressions for $a_1(\rho)$,

$$a_1(\rho) = 2d_\varphi\rho - \int \frac{d^dq}{(2\pi)^d}\dot{\Delta}\left[\frac{1}{(1+2\Delta V' + 4\Delta\rho V'')^2} + \frac{N-1}{(1+2\Delta V')^2}\right], \tag{3.7}$$

and $a_2(\rho)$, which however is again always positive,

$$a_2(\rho) = \int \frac{d^dq}{(2\pi)^d}\frac{2\dot{\Delta}\rho}{(1+2\Delta V' + 4\Delta\rho V'')^2}. \tag{3.8}$$

Using the asymptotic fixed point solution (3.4) (and assuming $A \neq 0$) we get that asymptotically a_2 scales as follows:

$$a_2(\rho) = 4F\rho^{3-m} + O\left(\rho^{4-\frac{3m}{2}}\right) \quad \text{as} \quad \rho \to \infty, \tag{3.9}$$

where F was already defined in (2.11). By similar arguments as before, we see that $m > 3$ in practice, so this implies $a_2(\rho) \to 0$. We also find that a_1 scales as follows:

$$a_1(\rho) = 2d_\varphi\rho + O(\rho^{2-m}) \quad \text{as} \quad \rho \to \infty. \tag{3.10}$$

If we substitute $\rho = \varphi^2$ into the above asymptotic expansions, they differ from the large φ behavior (2.6) of $a_1(\varphi)$ and (2.10) of $a_2(\varphi)$. However, they reproduce the previous results once we transform the ODE (3.6) by changing variables $\rho = \varphi^2$. Thus, by the same arguments as before, cf. (2.13), we also know that for $\rho \to \infty$, we must have

$$v(\rho) \propto \rho^{\frac{d-\lambda}{2d_\varphi}} + \cdots. \tag{3.11}$$

However, this now imposes only one boundary condition on the linear ODE (3.6) since ρ is restricted to be nonnegative. On the other hand we see from (3.8) that $a_2(0) = 0$, so the ODE has a so-called fixed singularity at $\rho = 0$. In order to ensure that $v(\rho)$ remains nonsingular at this point, an additional boundary condition is then required,

$$a_1(0)v'(0) = (d-\lambda)v(0). \tag{3.12}$$

Now we again have two boundary conditions, overconstraining the equation and leading to quantization of the RG eigenvalue λ.

A. SL analysis

The last step is to perform the SL analysis, which also differs because of the $\rho = 0$ boundary. For small ρ we have

$$a_2(\rho) = 2G\rho + O(\rho^2) \quad \text{and} \quad a_1(\rho) = -GN + O(\rho), \tag{3.13}$$

where we have set

$$G = \int \frac{d^dq}{(2\pi)^d}\frac{\dot{\Delta}}{[1+2\Delta V'(0)]^2}. \tag{3.14}$$

Note that G is of course positive. [By Taylor expanding (3.1) one sees that its convergence is guaranteed for any such solution to the flow equation.] The SL weight function now takes the form

$$w(\rho) = \frac{1}{a_2(\rho)}\exp\left\{-\int_{\rho_0}^{\rho} d\rho' \frac{a_1(\rho')}{a_2(\rho')}\right\}, \tag{3.15}$$

where by (3.13) a nonzero lower limit, $\rho_0 > 0$, is required to avoid the integral diverging (when $N \neq 0$).

Using $w(\rho)$ we can now cast (3.6) in SL form (2.15). However, for the SL operator to be self-adjoint, we need the boundary contributions that appear on integration by parts to vanish. This is still true for large fields since as $\rho \to \infty$, the eigenoperators diverge at worst as a power, while from (3.9) we have $a_2(\rho) \to 0$, and thus $w(\rho) \to 0$ exponentially fast. At the $\rho = 0$ boundary we require[4]

$$\lim_{\rho\to 0} a_2(\rho)w(\rho)(v_i(\rho)v_j'(\rho) - v_j(\rho)v_i'(\rho)) = 0, \tag{3.16}$$

for any two eigenfunctions $v_i(\rho)$ and $v_j(\rho)$. This is true for all $N > 0$ since by (3.13) and (3.15) we see that for small ρ,

$$a_2(\rho)w(\rho) \propto \rho^{N/2}[1 + O(\rho)]. \tag{3.17}$$

We have thus determined that the SL operator is self-adjoint for all $N > 0$.

Actually, $N = 0$ is also interesting since it corresponds to the universality class of fluctuating long polymers [47]. In this case, the above analysis shows that $a_2(0)w(0) > 0$, which would appear to imply that (3.16) is no longer satisfied. However, from (3.13) we see that $a_1(0) = 0$ now and thus, from (3.12), either $\lambda_i = d$ or $v_i(0) = 0$ [31]. The first possibility corresponds to the uninteresting solution $v(\rho) \equiv 1$, i.e., the unit operator, which we discard. All the other eigenoperators must thus satisfy $v_i(0) = 0$, and so (3.16) is satisfied in this reduced space. Therefore, with this one proviso, the SL operator is actually self-adjoint for all $N \geq 0$.

For general $N < 0$, the SL operator fails to be self-adjoint, and thus SL analysis is no longer applicable. However, for $N = -2k$, where k is a non-negative integer, something special happens. The first $k+1$ eigenoperators with the lowest scaling dimension turn out to have exactly soluble scaling dimensions, in fact coinciding with the Gaussian ones [53–55]. (The case $N = 0$ above is the first example, the lowest dimension operator being the unit operator with scaling dimension zero.) Again, the SL

[4]Using (3.12) and (3.13), this can be reduced to $\lim_{\rho\to 0} a_2(\rho)w(\rho)(\lambda_i - \lambda_j)v_i(\rho)v_j(\rho) = 0$ (when $N \neq 0$).

operator is self-adjoint in the remainder of the space. For example, for $N = -2$, one knows from Ref. [31] that the remaining eigenoperators satisfy $v_i(0) = v_i'(0) = 0$, and thus $v_i(\rho) \propto \rho^2$ for small ρ, while for $N = -4$ boundary conditions force the remaining eigenoperators to satisfy $v_i(\rho) \propto \rho^3$ for small ρ. From that analysis it is clear that in general, at $N = -2k$, we have that the remaining operators satisfy

$$v_i(\rho) \propto \rho^{k+1} \quad \text{as} \quad \rho \to 0. \tag{3.18}$$

Combining these observations with (3.16) and (3.17), we see that the SL operator is indeed self-adjoint in the reduced space defined by excluding the first $k+1$ operators.

The SL equation can now be recast in the same way as before, using (2.17) for x and (2.18) for $\psi(x)$ (except for the obvious replacement of φ by ρ). The resulting Schrödinger equation is then precisely as before, viz. (2.19), and the potential $U(x)$ also takes precisely the same form in terms of the a_i, viz. (2.20). However, the $\rho = 0$ boundary turns into an $x = 0$ boundary since, by (3.13) and (2.17), we have

$$x = \sqrt{2\rho/G} + O\left(\rho^{\frac{3}{2}}\right) \quad \text{as} \quad \rho \to 0. \tag{3.19}$$

Thus, using a_2 from (3.13) and $a_2 w$ from (3.17), we see that

$$\psi(x) \propto x^{\frac{N-1}{2}} v(x) \tag{3.20}$$

for small x. Hence for all $N > 1$, $\psi(x)$ vanishes as $x \to 0$. On taking into account the behavior (3.18) we see that in the reduced space, $\psi(x)$ also vanishes for the special cases $N = -2k$. In this limit, the leading contributions to the potential come from the first, third, and fourth terms in (2.20), and thus we find

$$U(x) = \frac{(N-1)(N-3)}{4x^2} + O(1) \quad \text{as} \quad x \to 0. \tag{3.21}$$

The cases $N = 1$, 3 are exceptional since this leading behavior then vanishes, while the range $1 < N < 3$ will need a separate treatment because the potential is then unbounded from below.

At the other end of x's range, we find that

$$\begin{aligned} x &= \int_0^\rho d\rho' \left(\frac{(\rho')^{\frac{1}{2}(m-3)}}{2\sqrt{F}} + O\left(\rho'^{-\frac{1}{2}}\right) \right) \\ &= \frac{\rho^{\frac{1}{2}(m-1)}}{(m-1)\sqrt{F}} + O\left(\rho^{\frac{1}{2}}\right) \quad \text{as} \quad \rho \to \infty. \end{aligned} \tag{3.22}$$

Identifying $\rho = \varphi^2$, this is the same formula (2.21) as before. The potential $U(x)$ is again dominated by the first term in (2.20), both at LO and NLO. Substituting the asymptotic expressions (3.10) and (3.9) for a_1 and a_2, we find exactly the same formula (2.23) for the large x behavior of $U(x)$. In particular, the leading term is again that of a simple harmonic oscillator with angular frequency $\omega = d - d_\varphi$.

B. WKB analysis

For the cases $N > 3$, $0 < N < 1$, and $N = -2m$, we can now proceed with the WKB analysis in the usual way. In this case, we have for the total phase of the wave function,

$$\int_{x_n^-}^{x_n^+} dx \sqrt{E_n - U(x)} = \left(n + \frac{1}{2}\right)\pi, \tag{3.23}$$

where x_n^- and x_n^+ are the classical turning points, i.e., $E_n = d - \lambda_n = U(x_n^-) = U(x_n^+)$. In contrast to the previous case, the potential is not symmetric and there is no simple relation between x_n^- and x_n^+.

In the large n limit, the contribution from the right-hand boundary gives half of what we obtained before. To see this in detail, let x_0^+ be some fixed finite value but sufficiently large to trust the asymptotic form (2.23) of the potential, then the contribution from the right-hand boundary is

$$\begin{aligned} &\int_{x_0^+}^{x_n^+} dx \sqrt{E_n - U(x)} \\ &= \frac{\omega}{2}(x_n^+)^2 \int_{x_0^+/x_n^+}^1 dy \sqrt{1 - y^2 + \gamma (x_n^+)^{\frac{1}{m-1}-1}(1 - |y|^{1+\frac{1}{m-1}})}. \end{aligned} \tag{3.24}$$

Taking into account the multiplying factor of $(x_n^+)^2$ we see that the lower limit x_0^+/x_n^+ of the integral can be set to zero, since the correction is of order $O(x_n^+)$ which is smaller than that given by the γ correction. Thus, we get half the integral in (2.26) (with x_n replaced by x_n^+) giving half the left-hand side of (2.27),

$$\int_{x_0^+}^{x_n^+} dx \sqrt{E_n - U(x)} = \frac{\omega\pi}{8}(x_n^+)^2 + O\left((x_n^+)^{1+\frac{1}{m-1}}\right). \tag{3.25}$$

Using the asymptotic form of the potential, we see that the leading term can be written as $\pi E_n/(2\omega)$. In the large n limit, the left-hand boundary makes a contribution that can be neglected in comparison. To see this let x_0^- be some fixed finite value but sufficiently small to use (3.21). Then the contribution from the left-hand boundary is

$$\begin{aligned} &\int_{x_n^-}^{x_0^-} dx \sqrt{E_n - U(x)} \\ &= \frac{1}{2}\sqrt{(N-1)(N-3)} \int_1^{x_0^-/x_n^-} dy \left(\frac{\sqrt{y^2 - 1}}{y} + O(x_n^-) \right). \end{aligned} \tag{3.26}$$

Since x_n^- is vanishing for large E_n, we see that this integral is $O(1/x_n^-)$ or, using again the relation (3.21), $O(E_n^{1/2})$. That only leaves the portion of the integral that goes from x_0^- to x_0^+, but since these boundaries are fixed and finite, we see that this part also grows as $\sqrt{E_n}$ and thus it too can be neglected in comparison to (3.25).

Therefore, asymptotically, the integral in (3.23) is given by (3.25). Inverting the relation to find $(x_n^+)^2$ asymptotically in terms of n, we thus find

$$d_n = E_n = d - \lambda_n = U(x_n^+) = 2n\omega + O\left(n^{\frac{m}{2(m-1)}}\right)$$
$$= 2n(d - d_\varphi) + O\left(n^{\frac{m}{2(m-1)}}\right) \quad \text{as} \quad n \to \infty, \tag{3.27}$$

i.e., precisely double the value we found for a single component field in (2.28) and independent of N.

We see that technically this arises because the WKB integral is precisely half as large in the $O(N)$ case, the leading contribution coming from the x_n^+ boundary only. Recall that, at $N = 1, 3$, the leading behavior (3.21) of $U(x)$ is no longer applicable. Since the potential is now finite as $x \to 0$, it is clear from the above analysis that the left-hand boundary continues to contribute at most $O(E_n^{1/2}) \sim \sqrt{n}$ and so can be neglected. Thus we see that (3.27) applies also to these exceptional cases. Thus also for $N = 1$ we find twice the previous scaling dimension as a function of large index n. This is in agreement with that single field result, however, because these eigenoperators are a function of φ^2 only. Hence for a single component field, the current n indexes only the even eigenoperators (those symmetric under $\varphi \leftrightarrow -\varphi$).

Finally, let us show that our result (3.27) is also applicable to the range $1 < N < 3$. Although in this case, from (3.21), the potential $U(x) \to -\infty$ as $x \to 0$, we know from (3.20) that the solutions we need have $\psi(x)$ vanishing there. These solutions are consistent with the Schrödinger equation (2.19) because for small x we have, by (3.20), a diverging second derivative,

$$-\frac{d^2\psi(x)}{dx^2} \propto -\frac{(N-1)(N-3)}{4x^2}\psi(x), \tag{3.28}$$

which is precisely the right behavior to cancel the divergence in the Schrödinger equation coming from the $U(x)\psi(x)$ term. Meanwhile the $v(x)$ term in (3.20) is well behaved in terms of oscillations at small x, behaving similar to the above cases. Therefore, we are only neglecting a subleading contribution to the total phase, if we work instead with a modified WKB integral where we replace the lower limit in (3.23) with some finite value x_0^-. By the above analysis we then recover (3.27) again. In this way we have shown that the result (3.27) is actually applicable for all $N \geq 0$ and to the special cases $N = -2k$ (where k is a non-negative integer).

IV. SUMMARY AND DISCUSSION

We have used SL theory and WKB methods to derive the scaling dimension d_n of highly irrelevant operators $\mathcal{O}_n$ around a nontrivial fixed point for scalar field theory, in the local potential approximation. The scaling dimensions d_n are ordered so that they increase with increasing index n. The d_n are derived following the methods developed in [38]. They are given to leading order in n, together with the power-law dependence on n of the next-to-leading order. The results apply to all the nontrivial (multi)critical fixed points in $2 < d < 4$, for single component scalar field theory and for $O(N)$ invariant scalar field theory, and also to the unitary minimal models in $d = 2$ dimensions. The d_n are universal, independent of the choice of fixed point (except through the anomalous dimension η), and independent of the cutoff choice which we have left general throughout, apart from the weak technical constraints discussed below Eq. (2.11). In particular, these constraints allow for the popular smooth cutoff choice (2.12). The crucial property leading to universality is that the results depend only on asymptotic solutions at large field, which can be derived analytically, and are also universal in the same sense. Although nonuniversal cutoff-dependent terms, in particular (2.11) and (3.14), enter into the calculation at intermediate stages, they drop out in the final stages. For a single component real scalar field, d_n is given in (2.28). For $O(N)$ scalar field theory, the d_n are just twice this, cf. (3.27), independent of N. This is in agreement with the single field result because here n indexes the eigenoperators that are a function of φ^2 only.

The first steps in deriving these results is to recast the eigenoperator equation in SL form and then establish that the SL operator is self-adjoint in the space spanned by the eigenoperators. For a single component scalar field this follows after demonstrating that the SL weight decays exponentially for large field, since the eigenoperators grow at most as a power of the field. For the $O(N)$ case the analysis is more subtle because the relevant space is now the positive real line (parametrized by $\rho = \varphi^2 \geq 0$) and thus the SL operator is self-adjoint only if the boundary terms at $\rho = 0$ also vanish. By analytically determining the small ρ dependence of the relevant quantities we see that the SL operator is self-adjoint when $N > 0$. For $N \leq 0$, the SL operator is not self-adjoint and the analysis does not apply. Presumably in these cases one would find that the scaling dimensions d_n are no longer real. However, for a sequence of special cases $N = -2k$, where k is a non-negative integer, the SL operator is self-adjoint on a reduced space spanned by all eigenoperators apart from the first $k + 1$. The analysis can then proceed on this reduced space. As we already noted, while most of these special cases are presumably only of theoretical interest, the $N = 0$ case describes the statistical physics of long polymers.

The next step is to cast the SL equation in the form of a one-dimensional time-independent Schrödinger equation with energy levels $E_n = d_n$ and potential $U(x)$. For the single component field this potential is symmetric, and in order to determine the energy levels E_n asymptotically at large n, using the WKB approximation, we need only the behavior of $U(x)$ at large x. The latter follows from our asymptotic analysis. For $O(N)$ scalar field theory, the space is the positive real line $x \geq 0$, and thus for WKB analysis we need also the behavior of the potential $U(x)$ at small x. Here we find that the range $1 \leq N \leq 3$ requires a separate treatment because the leading term in $U(x)$ turns negative leading to a potential unbounded from below. Nevertheless, we are able to treat this case and the end result for d_n, (3.27), is the same, thus applying universally to all $N \geq 0$ and the $N = -2k$ special cases.

Although these results are universal, they are still derived within the LPA, which is an uncontrolled model approximation. One might reasonably hope, however, that the fact that these results are universal, in the sense of being independent of the detailed choice of cutoff, is an indication that they are nevertheless close to the truth. On the other hand, the LPA [22] of the Polchinski flow equation [9] is in fact completely cutoff independent, although this property arises rather trivially. It is actually equivalent under a Legendre transformation [56] to the flow equation (2.1) for the Legendre effective action in LPA, as we study here, but only for a special (but actually popular) choice of additive cutoff known as the optimized cutoff [57]. However, the optimized cutoff does not satisfy our technical constraints given below (2.11) so our analysis is invalid for this case; nor in fact does a sharp cutoff [14,20,23,58] or power-law cutoff [29] satisfy the technical constraints. What this means is that these particular cutoffs fail to regularize completely the region of large fields, in the sense that a_2, defined by (2.7) or (3.8), no longer has an asymptotic expansion given simply by integrating over the asymptotic expansion of its integrand. For these three particular cutoffs, regions of momenta far from Λ alter the asymptotic expansion of a_2 so that it is no longer of the form (2.10), or (3.9), and for this reason these cutoffs are less satisfactory.

Nevertheless, following our methods, it would be straightforward to derive the asymptotic scaling dimensions d_n in LPA for any or all of these three special choices of cutoff, by using the particular form of the LPA flow equation in these cases (which are known in closed form, since the momentum integrals can be calculated analytically in these cases). The results will differ from the d_n derived here and among themselves, but their investigation would improve insight into the accuracy of the LPA in this regime. Furthermore, it would seem possible to generalize any of these special choices of cutoff to their own class of cutoffs with similar properties and thus understand the extent to which the results could still be cutoff independent, up to some appropriate constraints, in these cases, and gain a more detailed understanding of why the d_n differ.

Unfortunately our d_n do not seem to match in a useful way to existing results in the literature. The LPA restricts us to eigenoperators that contain no spacetime derivatives, and thus our index n counts only over these. In reality all eigenoperators (apart from the unit operator) contain spacetime derivatives, so in particular it is not clear how our index n would map into the exact sequence.

However, in some special limits the LPA is effectively exact. This is true for the Gaussian fixed point, for example, where $d_n = nd_\varphi$ (with $\eta = 0$). Our scaling dimensions d_n differ from this, but the Gaussian fixed point is specifically excluded from our analysis since our results apply only to nontrivial fixed points such that the asymptotic expansion of the fixed point potential takes the form (1.3) or (3.4) with $A \neq 0$.

The LPA also becomes effectively exact in the large N limit [52], and there the scaling dimensions are $d_n = 2n$ (with $\eta = 0$) which again differs from our result (as well as differing from the Gaussian fixed point result). Furthermore, they continue to disagree even if we now take a second limit such that both n and N are sent to infinity. However, in this case we have an example where the order of the limits matters. The $N \to \infty$ result is derived for d_n while first holding n fixed, while our result applies first for fixed N while $n \to \infty$.

The difference can be seen at the technical level. The first term on the right-hand side of the flow equation (3.1) is proportional to N. In our analysis, however, it is the denominators that dominate. On the other hand, in the large N analysis, only the first term survives, resulting in a first order ODE with no SL properties (or Schrödinger equation representation). The universal results fall out, on the one hand, in our analysis from the asymptotic behavior at large field, but on the other hand, in large N they fall out from a Taylor expansion around the minimum of the fixed point potential [52]. There seems unfortunately to be no way to bridge the gap between these two limiting regimes.

An even clearer example where the exchange of limits does not commute is provided by the special cases $N = -2k$. As we recalled in Sec. III, in these cases the first $k + 1$ eigenoperators degenerate, gaining Gaussian scaling dimensions. But our d_n apply to the highly irrelevant eigenoperators that are found in the reduced space, which excludes these first $k + 1$ operators, and hence have nontrivial scaling dimensions. However, if instead we fix on the nth eigenoperator and let $N \to -\infty$ by sending $k \to \infty$, we see that this nth eigenoperator will fall into the excluded space and thus end up with Gaussian scaling dimensions. The disagreement between the two results will then remain even if we choose next to send $n \to \infty$.

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[1] Tim R. Morris, Elements of the continuous renormalization group, Prog. Theor. Phys. Suppl. **131**, 395 (1998).
[2] C. Bagnuls and C. Bervillier, Exact renormalization group equations. An introductory review, Phys. Rep. **348**, 91 (2001).
[3] Juergen Berges, Nikolaos Tetradis, and Christof Wetterich, Nonperturbative renormalization flow in quantum field theory and statistical physics, Phys. Rep. **363**, 223 (2002).
[4] Andrea Pelissetto and Ettore Vicari, Critical phenomena and renormalization group theory, Phys. Rep. **368**, 549 (2002).
[5] Oliver J. Rosten, Fundamentals of the exact renormalization group, Phys. Rep. **511**, 177 (2012).
[6] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J. M. Pawlowski, M. Tissier, and N. Wschebor, The nonperturbative functional renormalization group and its applications, Phys. Rep. **910**, 1 (2021).
[7] K. G. Wilson and John B. Kogut, The renormalization group and the epsilon expansion, Phys. Rep. **12**, 75 (1974).
[8] Franz J. Wegner and Anthony Houghton, Renormalization group equation for critical phenomena, Phys. Rev. A **8**, 401 (1973).
[9] Joseph Polchinski, Renormalization and effective Lagrangians, Nucl. Phys. **B231**, 269 (1984).
[10] Jose I. Latorre and Tim R. Morris, Exact scheme independence, J. High Energy Phys. 11 (2000) 004.
[11] Stefano Arnone, Antonio Gatti, and Tim R. Morris, A proposal for a manifestly gauge invariant and universal calculus in Yang-Mills theory, Phys. Rev. D **67**, 085003 (2003).
[12] J. F. Nicoll and T. S. Chang, An exact one particle irreducible renormalization group generator for critical phenomena, Phys. Lett. A **62**, 287 (1977).
[13] Christof Wetterich, Exact evolution equation for the effective potential, Phys. Lett. B **301**, 90 (1993).
[14] Tim R. Morris, The exact renormalization group and approximate solutions, Int. J. Mod. Phys. A **09**, 2411 (1994).
[15] Steven Weinberg, Critical phenomena for field theorists, in *14th International School of Subnuclear Physics: Understanding the Fundamental Constitutents of Matter Erice, Italy, 1976* (Springer, New York, 1976), p. 1, https://doi.org/10.1007/978-1-4684-0931-4.
[16] M. Bonini, M. D'Attanasio, and G. Marchesini, Perturbative renormalization and infrared finiteness in the Wilson renormalization group: The massless scalar case, Nucl. Phys. **B409**, 441 (1993).
[17] Ulrich Ellwanger, Flow equations for N point functions and bound states, Z. Phys. C **62**, 503 (1994).
[18] D. Morgan, Quartet: Baryogenesis, bubbles of false vacuum, quantum black holes, and the renormalization group, Ph.D. thesis, University of Texas, Austin, 1991.
[19] J. F. Nicoll, T. S. Chang, and H. E. Stanley, Approximate renormalization group based on the Wegner-Houghton differential generator, Phys. Rev. Lett. **33**, 540 (1974).
[20] Anna Hasenfratz and Peter Hasenfratz, Renormalization group study of scalar field theories, Nucl. Phys. **B270**, 687 (1986).
[21] G. Felder, Renormalization group in the local potential approximation, Commun. Math. Phys. **111**, 101 (1987).
[22] Richard D. Ball, Peter E. Haagensen, I. Latorre, Jose, and Enrique Moreno, Scheme independence and the exact renormalization group, Phys. Lett. B **347**, 80 (1995).
[23] Tim R. Morris, On truncations of the exact renormalization group, Phys. Lett. B **334**, 355 (1994).
[24] Tim R. Morris, Momentum scale expansion of sharp cutoff flow equations, Nucl. Phys. **B458**, 477 (1996).
[25] Ken-Ichi Aoki, Kei-ichi Morikawa, Wataru Souma, Jun-ichi Sumi, and Haruhiko Terao, The effectiveness of the local potential approximation in the Wegner-Houghton renormalization group, Prog. Theor. Phys. **95**, 409 (1996).
[26] Jordi Comellas and Alex Travesset, O(N) models within the local potential approximation, Nucl. Phys. **B498**, 539 (1997).
[27] H. Osborn and D. E. Twigg, Reparameterisation invariance and RG equations: Extension of the local potential approximation, J. Phys. A **42**, 195401 (2009).
[28] C. Bervillier, Revisiting the local potential approximation of the exact renormalization group equation, Nucl. Phys. **B876**, 587 (2013).
[29] Tim R. Morris, Derivative expansion of the exact renormalization group, Phys. Lett. B **329**, 241 (1994).
[30] Tim R. Morris, Three-dimensional massive scalar field theory and the derivative expansion of the renormalization group, Nucl. Phys. **B495**, 477 (1997).
[31] Tim R. Morris and Michael D. Turner, Derivative expansion of the renormalization group in O(N) scalar field theory, Nucl. Phys. **B509**, 637 (1998).
[32] I. Balog, H. Chaté, B. Delamotte, M. Marohnić, and N. Wschebor, Convergence of nonperturbative approximations to the renormalization group, Phys. Rev. Lett. **123**, 240604 (2019).
[33] G. De Polsi, I. Balog, M. Tissier, and N. Wschebor, Precision calculation of critical exponents in the O(N) universality classes with the nonperturbative renormalization group, Phys. Rev. E **101**, 042113 (2020).
[34] G. De Polsi, G. Hernández-Chifflet, and N. Wschebor, Precision calculation of universal amplitude ratios in $\mathrm{O}(N)$ universality classes: Derivative expansion results at order $O(\partial 4)$, Phys. Rev. E **104**, 064101 (2021).
[35] G. De Polsi and N. Wschebor, Regulator dependence in the functional renormalization group: A quantitative explanation, Phys. Rev. E **106**, 024111 (2022).
[36] E. L. Ince, *Ordinary Differential Equations* (Dover Publications, New York, 1956).
[37] P. M. C. Morse and H. Feshbach, *Methods of Theoretical Physics*, International Series in Pure and Applied Physics (McGraw-Hill, New York, 1953).
[38] Alex Mitchell, Tim R. Morris, and Dalius Stulga, Provable properties of asymptotic safety in $f(R)$ approximation, J. High Energy Phys. 01 (**2022**) 041.
[39] Tim R. Morris and Dalius Stulga, The functional $f(R)$ approximation, in *Handbook of Quantum Gravity* (Springer, Singapore, 2023).
[40] M. Reuter, Nonperturbative evolution equation for quantum gravity, Phys. Rev. D **57**, 971 (1998).
[41] Robert Percacci, *An Introduction to Covariant Quantum Gravity and Asymptotic Safety*, 100 Years of General Relativity Vol. 3 (World Scientific, Singapore, 2017).

[42] Martin Reuter and Frank Saueressig, *Quantum Gravity and the Functional Renormalization Group: The Road Towards Asymptotic Safety* (Cambridge University Press, Cambridge, England, 2019), p. 1.
[43] Dario Benedetti and Francesco Caravelli, The local potential approximation in quantum gravity, J. High Energy Phys. 06 (2012) 017.
[44] Juergen A. Dietz and Tim R. Morris, Asymptotic safety in the f(R) approximation, J. High Energy Phys. 01 (2013) 108.
[45] Dario Benedetti, On the number of relevant operators in asymptotically safe gravity, Europhys. Lett. **102**, 20007 (2013).
[46] L. P. Kadanoff, Scaling laws for Ising models near T(c), Physics **2**, 263 (1966).
[47] Jean Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (2002), https://doi.org/10.1093/acprof:oso/9780198509233.001.0001.
[48] A. B. Zamolodchikov, Conformal symmetry and multicritical points in two-dimensional quantum field theory. (In Russian), Sov. J. Nucl. Phys. **44**, 529 (1986).
[49] Tim R. Morris, The renormalization group and two-dimensional multicritical effective scalar field theory, Phys. Lett. B **345**, 139 (1995).
[50] Tim R. Morris, On the fixed point structure of scalar fields, Phys. Rev. Lett. **77**, 1658 (1996).
[51] I. Hamzaan Bridle and Tim R. Morris, Fate of nonpolynomial interactions in scalar field theory, Phys. Rev. D **94**, 065040 (2016).
[52] Marco D'Attanasio and Tim R. Morris, Large N and the renormalization group, Phys. Lett. B **409**, 363 (1997).
[53] R. Balian and G. Toulouse, Critical exponents for transitions with $n = -2$ components of the order parameter, Phys. Rev. Lett. **30**, 544 (1973).
[54] Michael E. Fisher and Classical, n-component spin systems or fields with negative even integral n, Phys. Rev. Lett. **30**, 679 (1973).
[55] Michael E. Fisher, The renormalization group in the theory of critical behavior, Rev. Mod. Phys. **46**, 597 (1974).
[56] Tim R. Morris, Equivalence of local potential approximations, J. High Energy Phys. 07 (2005) 027.
[57] Daniel F. Litim, Optimization of the exact renormalization group, Phys. Lett. B **486**, 92 (2000).
[58] T. S. Chang J. F. Nicoll and H. E. Stanley, A differential generator for the free energy and the magnetization equation of a differential generator for the free energy and the magnetization equation of state, Phys. Lett. **57A**, 7 (1976).