Gravitational scattering up to third post-Newtonian approximation for conservative dynamics: Scalar-tensor theories

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We compute the scattering angle χ for hyperboliclike encounters in massless scalar-tensor theories up to third post-Newtonian order for the conservative part of the dynamics. To calculate the gauge-invariant scattering angle as a function of energy and orbital angular momentum, we use the approach of effective-one-body formalism as introduced in Bini and Damour [Phys. Rev. D **96**, 064021 (2017)]. We then compute the nonlocal-in-time contribution to the scattering angle by using the strategy of order reduction of nonlocal dynamics introduced for small-eccentricity orbits.

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I. INTRODUCTION

The observation of gravitational wave signals, with the first observation by the LIGO-Virgo Collaboration in 2015 [1], opened a new era to probe the dynamics of the strong-field gravity regime. The third generation of detectors [2–4], along with the next generation of telescopes, such as the Einstein telescope [5] and the Cosmic Explorer [6], will be crucial for probing the strong-field dynamics of gravity by constraining the parameters of the alternative theories of gravity.

Among the theories alternative to Einstein's general relativity (GR), the simplest theory is the addition of the massless scalar field to GR known as the scalar-tensor (ST) theory. The ST theories have been extensively studied and tested [7–12]. Besides arising naturally in the UV complete theories of gravity, the addition of the scalar field is also equivalent to $f(R)$ theories of gravity [13]. The two-body problem for ST theories has been extensively studied within the post-Newtonian (PN) approximation for both the dynamics and waveform generation in Refs. [14–21].

The detection of the gravitational wave signals relies on a large bank of (semi)analytical accurate waveform templates to match filter against the data observed in the detectors. Therefore, the two-body PN dynamics in ST theories have been mapped within the effective-one-body (EOB) formalism [22–26] to incorporate the corrections due to massless scalar-tensor theories in the EOB approach based waveform models [27–29]. These results were obtained for the elliptic motions of the compact binaries.

The EOB description of the unbound, scattering states of the binary systems was introduced in Ref. [30]. Recently, the approach was used to compute the scattering angle within the PN approximation in GR [31]. The main aim of this paper is to compute the scattering angle in ST theories up to 3PN order using the EOB Hamiltonian in ST theories.

The paper is organized as follows. In Sec. II, we give a brief reminder of ST theories and the EOB formalism in ST theories. Then, in Sec. III we derive the scattering angle for the local part of the dynamics in ST theories at the 3PN order, and in Sec. IV we derive the scattering angle for the nonlocal part of the dynamics at 3PN order using the order-reduction approach. Finally, in Sec. V we sum the local and nonlocal contributions at 3PN in a large-j expansion.

II. BRIEF SCALAR-TENSOR THEORY AND EOB REMINDER

We consider monoscalar massless ST theories described by the minimal coupling of the scalar field to the metric in the Einstein frame, and its action reads

$$S = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi) + S_m[\Psi, \mathcal{A}(\varphi)^2 g_{\mu\nu}], \tag{2.1}$$

where $g_{\mu\nu}$ is the Einstein metric, R is the Ricci scalar, φ is the scalar field, Ψ collectively denotes the matter fields, $g \equiv \det(g_{\mu\nu})$, and G is the bare Newton's constant. Here, we adopt the conventions and notations of Damour and Esposito-Farese (DEF, hereafter) [7,9]. From now on,

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we use geometric units with $c = G = 1$ unless specified explicitly.

In the Einstein frame, the field equations for ST theories are derived in [7]. The coupling of the scalar field with the matter fields gives rise to the dynamics of the scalar field where the coupling is measured by the parameter

$$\alpha(\varphi) = \frac{\partial \ln \mathcal{A}}{\partial \varphi}, \tag{2.2}$$

in the equations of motion. The scalar field is nonminimally coupled to the metric in the Jordan frame (physical frame) with the Jordan frame metric $\tilde{g}_{\mu\nu}$ defined as

$$\tilde{g}_{\mu\nu} = \mathcal{A}(\varphi)^2 g_{\mu\nu}, \tag{2.3}$$

where $\mathcal{A}(\varphi)$ is called the coupling function. The ST theory is uniquely fixed when the function $\mathcal{A}(\varphi)$ is defined and when $\mathcal{A}(\varphi) = cst$ general relativity is recovered.

As the effective gravitational constant in ST theories depends on the scalar field, the size of the compact object and its internal gravity varies with the scalar field. Therefore, as suggested by [32], the compact, self-gravitating objects in ST theories can be considered as point particles and are described by the mass function $m_J(\phi)$ depending on the value of the scalar field in an undefined manner at the location of particles. The matter action is then given by

$$S_m = -\sum_{J=A,B} \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} m_J(\varphi), \tag{2.4}$$

where $m_J(\varphi)$ is the Einstein-frame mass of body J and λ is the affine parameter. Since $\tilde{g}_{\mu\nu} = \mathcal{A}(\varphi)^2 g_{\mu\nu}$, the Jordan-frame mass is defined as

$$\tilde{m}_J(\varphi) = \frac{m_J(\varphi)}{\mathcal{A}(\varphi)}. \tag{2.5}$$

The dimensionless body-dependent parameters that describe the scalar field effect in the Einstein frame up to the third PN order are defined using the Einstein-frame mass function $m_J(\varphi)$ following Refs. [7,9], i.e.,

$$\alpha_J = \frac{d \ln m_J(\varphi)}{d\varphi}, \tag{2.6}$$

$$\beta_J = \frac{d\alpha_J}{d\varphi}, \tag{2.7}$$

$$\beta'_J = \frac{d\beta_J}{d\varphi}, \tag{2.8}$$

$$\beta''_J = \frac{d\beta'_J}{d\varphi}. \tag{2.9}$$

These parameters are defined in the Jordan frame in Refs. [16–18] and the conversion of the Jordan-frame parameters to DEF conventions, i.e., the Einstein-frame parameters is given in Table I (see Ref. [24]).

Finally, before proceeding to the computations of the scattering angle for ST theories, we briefly review the EOB formalism proposed in [33,34] as a way to extend the validity of the PN results beyond the weak-field and slow-motion regime by resumming the PN results. The three main features of the EOB approach are the following: the description of the conservative (Hamiltonian) part of the dynamics of a two-body system, the expression for the radiation-reaction effects, and the description of GWs emitted by the coalescence of a compact binary.

The description of the conservative part of the dynamics in EOB formalism is completely described by the following effective metric up to 2PN order:

$$ds^2_{\rm eff} = -A(r)c^2 dt^2_{\rm eff} + B(r)dr^2 + r^2(d\theta^2 + \sin(\theta)^2 d\phi^2), \tag{2.10}$$

where $A(r)$ and $B(r)$ are the two EOB potentials, $t_{\rm eff}$ is the coordinate time of the effective EOB metric, and r is the radial separation in EOB coordinates. In this work, we specialize to equatorial motions, i.e., set $\theta = \pi/2$. The generalization of the EOB formalism to the 3PN order

TABLE I. Relation between the ST parameters used in the two-body Lagrangian of Ref. [17], the DEF ones and the slightly simplified notation that we are using here. The index "0" signifies a quantity evaluated at $\varphi = \varphi_0$, where φ_0 is the asymptotic constant value of the scalar field.

LB [17]	DEF [7,9]	This paper
m_1	$m_A^0/\mathcal{A}_0$	$m_A^0/\mathcal{A}_0 \equiv \tilde{m}_A^0$
m_2	$m_B^0/\mathcal{A}_0$	$m_B^0/\mathcal{A}_0 \equiv \tilde{m}_B^0$
α	$\frac{1+\alpha_A^0\alpha_B^0}{1+\alpha_0^2}$	α_{AB}
$\tilde{G}\alpha$	$(1+\alpha_A^0\alpha_B^0)\mathcal{A}_0^2 \equiv G_{AB}\mathcal{A}_0^2$	$G_{AB}\mathcal{A}_0^2 \equiv \tilde{G}_{AB}$
$\bar{\gamma}$	$-2\frac{\alpha_A^0\alpha_B^0}{1+\alpha_A^0\alpha_B^0} \equiv \bar{\gamma}_{AB}$	$\bar{\gamma}_{AB}$
$\bar{\beta}_1$	$\frac{1}{2}\frac{(\beta_A\alpha_B^2)_0}{(1+\alpha_A^0\alpha_B^0)^2} \equiv \bar{\beta}^A_{BB}$	$\bar{\beta}_A$
$\bar{\beta}_2$	$\frac{1}{2}\frac{(\beta_B\alpha_A^2)_0}{(1+\alpha_A^0\alpha_B^0)^2} \equiv \bar{\beta}^B_{AA}$	$\bar{\beta}_B$
$\bar{\delta}_1$	$\frac{(\alpha_A^0)^2}{(1+\alpha_A^0\alpha_B^0)^2}$	δ_A
$\bar{\delta}_2$	$\frac{(\alpha_B^0)^2}{(1+\alpha_A^0\alpha_B^0)^2}$	δ_B
$\bar{\chi}_1$	$-\frac{1}{4}\frac{(\beta'_A\alpha_B^3)_0}{(1+\alpha_A^0\alpha_B^0)^3} \equiv -\frac{1}{4}\epsilon^A_{BBB}$	$-\frac{1}{4}\epsilon_A$
$\bar{\chi}_2$	$-\frac{1}{4}\frac{(\beta'_B\alpha_A^3)_0}{(1+\alpha_A^0\alpha_B^0)^3} \equiv -\frac{1}{4}\epsilon^B_{AAA}$	$-\frac{1}{4}\epsilon_B$
$\bar{\beta}_1\bar{\beta}_2/\bar{\gamma}$	$-\frac{1}{8}\frac{\beta_A^0\alpha_A^0\beta_B^0\alpha_B^0}{(1+\alpha_A^0\alpha_B^0)^3} \equiv -\frac{1}{8}\zeta_{ABAB}$	$-\frac{1}{8}\zeta$
$\bar{\kappa}_1$	$\frac{(\alpha_B^4\beta''_A)_0}{8(1+\alpha_A^0\alpha_B^0)^4}$	$\bar{\kappa}_A$
$\bar{\kappa}_2$	$\frac{(\alpha_A^4\beta''_B)_0}{8(1+\alpha_A^0\alpha_B^0)^4}$	$\bar{\kappa}_B$

introduces nongeodesic contributions (Q_e) to the effective dynamics (see Ref. [35]). The structure of the nongeodesic term Q_e at 3PN is

$$Q_e = \frac{1}{r^2}(q_1 p^4 + q_2 p^2 p_r^2 + q_3 p_r^4), \tag{2.11}$$

where $p^2 = p_r^2 + \frac{p_\phi^2}{r^2}$, p_r is the radial momentum, and p_ϕ is the angular momentum in EOB coordinates. Here, we use the gauge freedom introduced in Ref. [35], known as the Damour-Jaranowski-Schäfer (DJS) gauge, to set $q_1 = q_2 = 0$ so that Q_e only depends on the radial momentum.

The relation between the real and EOB Hamiltonian is then (see Ref. [33] for GR and Ref. [30] for ST theories)

$$\hat{H}_{\rm real}(\mathbf{r},\mathbf{p}) \equiv \frac{H_{\rm real}}{\mu} = \frac{1}{\nu}\sqrt{1+2\nu(\hat{H}_{\rm eff}-1)}, \tag{2.12}$$

where M $(=m_A^0 + m_B^0)$ is the total mass of the system, $\mu(=\frac{m_A^0 m_B^0}{M})$ is the reduced mass, and $\nu = \mu/M$ is the symmetric mass ratio. The reduced-mass effective Hamiltonian ($\hat{H}_{\rm eff}$) is given by [35]

$$\hat{H}_{\rm eff} = \frac{H_{\rm eff}}{\mu} = \sqrt{A(\hat{r})\left(1+\frac{\hat{p}_r^2}{B(\hat{r})}+\frac{\hat{p}_\phi^2}{\hat{r}^2}+\hat{Q}_e\right)}, \tag{2.13}$$

where $\hat{p}_r$ and $\hat{p}_\phi$ are the dimensionless radial and angular momenta, $\hat{r}$ is the dimensionless radial separation, and $\hat{Q}_e(\equiv Q_e/\mu^2)$ is the dimensionless nongeodesic contribution. The dimensionless variables are defined as

$$\hat{p}_r = \frac{p_r}{\mu}; \quad j = \hat{p}_\phi = \frac{p_\phi}{G_{AB}M\mu}; \quad \hat{r} = \frac{r}{G_{AB}M}. \tag{2.14}$$

Hereafter, the superscript *hat* will be used to denote the dimensionless variables.

The three EOB potentials (A, B, Q_e) up to 3PN in the DJS gauge choice of Ref. [35] formally read

$$A(\hat{r}) = 1 - \frac{2}{\hat{r}} + \frac{a_2(\nu)}{\hat{r}^2} + \frac{a_3(\nu)}{\hat{r}^3} + \frac{a_4(\nu)}{\hat{r}^4}, \tag{2.15}$$

$$B(\hat{r}) = 1 + \frac{b_1(\nu)}{\hat{r}} + \frac{b_2(\nu)}{\hat{r}^2} + \frac{b_3(\nu)}{\hat{r}^3}, \tag{2.16}$$

$$\hat{Q}_e(\hat{r}) = q_3(\nu)\frac{\hat{p}_r^4}{\hat{r}^2}, \tag{2.17}$$

where the ν-dependent coefficients a_i, b_i, and q_i take into account both GR and ST corrections which are separated as

$$a_i = a_i^{\rm GR} + a_{i,\rm ST}, \tag{2.18}$$

$$b_i = b_i^{\rm GR} + b_{i,\rm ST}, \tag{2.19}$$

$$q_3 = q_3^{\rm GR} + q_{3,\rm ST}, \tag{2.20}$$

where $a_i^{\rm GR}$, $b_i^{\rm GR}$, $q_i^{\rm GR}$, $a_i^{\rm ST}$, $b_i^{\rm ST}$, and $q_i^{\rm ST}$ are ν-dependent coefficients. The GR coefficients are fully known analytically up to 4PN order and are analytically known at 6PN except for some unknown coefficients proportional to ν^2 [33,35–38]. As for the ST theories, the nonlocal-in-time contributions start at the 3PN order [17,18], and the 3PN coefficients can be decomposed as

$$a_{4,\rm ST} = a_{4,\rm ST}^{\rm I} + a_{4,\rm ST}^{\rm II}, \tag{2.21}$$

$$b_{3,\rm ST} = b_{3,\rm ST}^{\rm I} + b_{3,\rm ST}^{\rm II}, \tag{2.22}$$

$$q_{3,\rm ST} = q_{3,\rm ST}^{\rm I} + q_{3,\rm ST}^{\rm II}, \tag{2.23}$$

where the superscripts I and II denote the local and nonlocal contributions, respectively. These coefficients can be further decomposed as

$$a_{4,\rm ST}^{\rm I} = a_{4,\rm ST}^{\rm loc} + a_{4,\rm ST}^{\rm log}\ln(u), \tag{2.24}$$

$$b_{3,\rm ST}^{\rm I} = b_{3,\rm ST}^{\rm loc} + b_{3\rm ST}^{\rm log}\ln(u), \tag{2.25}$$

$$q_{3,\rm ST}^{\rm I} = q_{3,\rm ST}^{\rm loc} + q_{3,\rm ST}^{\rm log}\ln(u). \tag{2.26}$$

These corrections to the EOB potentials in the ST theories up to 3PN order have been derived in [22–26] based on the real two-body Lagrangian up to 3PN order given in Refs. [14–18].

III. SCALAR-TENSOR SCATTERING ANGLE: LOCAL CONTRIBUTIONS

In this section, we derive the contribution to the scattering angle for encounters of two nonspinning bodies for the local part of the conservative dynamics up to third PN order in ST theories. As the nonlocal-in-time (tail) effects start only at the 3PN level in ST theories, the scattering angle up to 3PN can be separated as a sum of functions,

$$\chi = \chi_{\rm loc} + \chi_{\rm nonloc}, \tag{3.1}$$

where $\chi_{\rm loc}$ and $\chi_{\rm nonloc}$ are, respectively, local and nonlocal contributions to the scattering angle.

The convenient way to compute the scattering angle is to use the Hamilton-Jacobi approach.[1] For ST theories as GR,

[1]In the Hamiltonian-Jacobi formalism, the solution of the equation of motion proceeds as follows. First, using the Hamiltonian we form the Hamiltonian-Jacobi equation,

$$\frac{\partial S}{\partial t} + H\left(q;\frac{\partial S}{\partial q};t\right) = 0, \tag{3.2}$$

and find the complete integral of motion $S(q,\alpha,t)$, where α are the arbitrary constants. Then by differentiating $S(q,\alpha,t)$ with respect to α and equating this equation with a constant, we find the solutions of the equations of motion, i.e., the coordinates q. (For details refer to Sec. [47] of Ref. [39].)

the Hamiltonian $H_{\rm eff}$ (and $H_{\rm real}$) are invariant under time translations and space rotations, and their associated constants of motion are, respectively, the energy and the angular momentum of the binary system. The EOB action by separating the EOB coordinates takes the form

$$S(t_{\rm eff}, r, \phi; \mathcal{E}_{\rm eff}, p_\phi) = -\mathcal{E}_{\rm eff} t_{\rm eff} + p_\phi \phi + \int^r dr p_r(r, \mathcal{E}_{\rm eff}, p_\phi), \quad (3.3)$$

where $\mathcal{E}_{\rm eff}$ is the energy in EOB formalism and $t_{\rm eff}$ is the EOB metric coordinate time. Using the Hamiltonian-Jacobi formalism, the solution of the equation of motion of the orbit (ϕ) is obtained from (Sec. [47] of Ref. [39]),

$$\frac{\partial S}{\partial p_\phi} = \phi_0 = \text{const}, \quad (3.4)$$

and the scattering angle related to the orbital phase is then [31]

$$\frac{1}{2}(\chi(\bar{E}, j) + \pi) = -\int_0^{u_{(\max)}} \frac{1}{u^2}\frac{\partial}{\partial j}\hat{p}_r(\bar{E}, j, u), \quad (3.5)$$

where $u = 1/\hat{r}$, $u_{(\max)} = 1/r_{\min}$ is the distance of the closest approach of two bodies, and $\bar{E}$ is the dimensionless energy variable defined as [31]

$$\bar{E} \equiv \frac{1}{2}(\hat{\mathcal{E}}_{\rm eff}^2 - 1) \equiv \frac{1}{2}p_\infty^2, \quad (3.6)$$

where $\hat{\mathcal{E}}_{\rm eff} = \mathcal{E}_{\rm eff}/\mu$ is the dimensional EOB energy and $p_\infty^2 \equiv 2\bar{E}$ is an energy variable. (The notation p_∞ is introduced instead of the notation v_∞ of Ref. [31] to be consistent with the current literature [see Eq. (10.10) [40]].)

A. PN-expanded $\chi_{\rm loc}$ for scalar-tensor theories

Let us now first compute the radial momentum $\hat{p}_r$ as a function of $u = 1/\hat{r}$, orbital angular momentum, and energy, that would then be used to compute the explicit integral of Eq. (3.5). This is obtained by iteratively solving in $\hat{p}_r^2$ the EOB energy conservation law, $\hat{\mathcal{E}}_{\rm eff}^2 = \hat{H}_{\rm eff}^2$ [see Eq. (2.13)],

$$\hat{\mathcal{E}}_{\rm eff}^2 = A(u)\left(1 + \frac{\hat{p}_r^2}{B(u)} + j^2u^2 + q_3(\nu)\hat{p}_r^4u^2\right), \quad (3.7)$$

which yields

$$\hat{p}_r^2(\bar{E}, j, u) = [\hat{p}_r^2]^0 + [\hat{p}_r^2]^1\eta^2 + [\hat{p}_r^2]^2\eta^4 + [\hat{p}_r^2]^3\eta^6 + \mathcal{O}(\eta^8) \quad (3.8)$$

with $\eta \sim 1/c$ as a PN-order marker. The Newtonian order contribution to $\hat{p}_r^2$ is

$$[\hat{p}_r^2]^0 = 2\bar{E} - j^2u^2 + 2u. \quad (3.9)$$

The explicit expressions of $\hat{p}_r^2$ and hence $\hat{p}_r$ up to 3PN order in ST theory are given in Supplemental Material [41].

This kind of formal PN expansion of $\hat{p}_r$, and hence the expansion of the integrand of Eq. (3.5) along with the PN expansion of the upper limit function $u_{(\max)}$, generates a sequence of divergent integral on the limit $[0, u_{(\max)}]$. However, it was shown in [42] that the correct value of a PN expanded integral such as that of Eq. (3.5) is obtained by first using the Newtonian limit of $u_{(\max)}$ as the upper limit of the integral, PN-expanding only the integrand, and taking the Hadamard partie finie (Pf) of the divergent integrals generated. The upper limit of the integral, $u_{(\max)}$, is the positive root closest to zero corresponding to the circular motion of Eq. (3.7), i.e.,

$$\hat{\mathcal{E}}_{\rm eff}^2 = A(u)(1 + j^2u^2), \quad (3.10)$$

and at the Newtonian level it reads

$$u_{(\max)} = \frac{1 + \sqrt{1 + 2\bar{E}j^2}}{j^2} + \mathcal{O}\left(\frac{1}{c^2}\right). \quad (3.11)$$

The integrals of Eq. (3.5) for ST theories can all be explicitly computed using the standard techniques (see Ref. [43]) except for one logarithmic integral arising at 3PN order. To simplify the expressions of the scattering angle, we introduce an auxiliary variable[2]

$$\alpha = \frac{1}{\sqrt{2\bar{E}j^2}} \quad (3.12)$$

and a function

$$B(\alpha) = \arctan(\alpha) + \frac{\pi}{2}. \quad (3.13)$$

The scattering angle for local contribution $\frac{1}{2}\chi_{\rm loc}$ up to 3PN order can be decomposed as a sum of contributions from each PN order, i.e.,

$$\frac{1}{2}\chi_{\rm loc}^{\rm (N)} = B(\alpha) - \frac{\pi}{2}, \quad (3.14)$$

$$\frac{1}{2}\chi_{\rm loc}^{\rm (1PN)} = \frac{1}{j^2}[C^{\rm B}_{(1PN)}B(\alpha) + C^0_{\rm (1PN)}], \quad (3.15)$$

$$\frac{1}{2}\chi_{\rm loc}^{\rm (2PN)} = \frac{1}{j^4}[C^{\rm B}_{\rm (2PN)}B(\alpha) + C^0_{\rm (2PN)}], \quad (3.16)$$

$$\frac{1}{2}\chi_{\rm loc}^{\rm (3PN)} = \frac{1}{j^6}[C^{\rm B}_{\rm (3PN)}B(\alpha) + C^0_{\rm (3PN)}] + I_\chi, \quad (3.17)$$

[2]Note that the parameter α defined here is different from the ST parameters α_J, $\alpha(\varphi)$, and α_0.

where

$$
\begin{aligned}
C^{\mathrm{B}}_{(1\mathrm{PN})} &= \frac{1}{2}\{-a_{2,\mathrm{ST}} + b_{1,\mathrm{ST}}\} + 3,\\
C^{0}_{(1\mathrm{PN})} &= \frac{1}{2\alpha(1+\alpha^2)}\{-\alpha^2 a_{2,\mathrm{ST}} + \alpha^2 b_{1,\mathrm{ST}} + b_{1,\mathrm{ST}}\} + \frac{3\alpha^2+2}{\alpha(1+\alpha^2)},\\
C^{\mathrm{B}}_{(2\mathrm{PN})} &= \frac{1}{4\alpha^2}\bigg[105\alpha^2 - 3\alpha^2 a_{2,\mathrm{ST}} b_{1,\mathrm{ST}} - 42\alpha^2 a_{2,\mathrm{ST}} + \frac{3}{2}\alpha^2 (a_{2,\mathrm{ST}})^2 - 6\alpha^2 a_{3,\mathrm{ST}} - 2a_{2,\mathrm{ST}} + 9\alpha^2 b_{1,\mathrm{ST}} + 3\alpha^2 b_{2,\mathrm{ST}} - \frac{3}{4}\alpha^2 b_{1,\mathrm{ST}}^2\\
&\quad + b_{1,\mathrm{ST}} + b_{2,\mathrm{ST}} - \frac{b_{1,\mathrm{ST}}^2}{4} + 15 + (-30\alpha^2 - 6)\nu\bigg],\\
C^{0}_{(2\mathrm{PN})} &= \frac{1}{4\alpha(1+\alpha^2)^2}\bigg[81\alpha + 190\alpha^3 + 105\alpha^5 - 3\alpha^5 a_{2,\mathrm{ST}} b_{1,\mathrm{ST}} - 5\alpha^3 a_{2,\mathrm{ST}} b_{1,\mathrm{ST}} - 2\alpha a_{2,\mathrm{ST}} b_{1,\mathrm{ST}} - 42\alpha^5 a_{2,\mathrm{ST}} + \frac{3}{2}\alpha^5 a_{2,\mathrm{ST}}^2\\
&\quad - 6\alpha^5 a_{3,\mathrm{ST}} - 72\alpha^3 a_{2,\mathrm{ST}} + \frac{5}{2}\alpha^3 a_{2,\mathrm{ST}}^2 - 10\alpha^3 a_{3,\mathrm{ST}} - 26\alpha a_{2,\mathrm{ST}} - 4\alpha a_{3,\mathrm{ST}} + 9\alpha^5 b_{1,\mathrm{ST}} + 3\alpha^5 b_{2,\mathrm{ST}} - \frac{3}{4}\alpha^5 b_{1,\mathrm{ST}}^2\\
&\quad + 16\alpha^3 b_{1,\mathrm{ST}} + 6\alpha^3 b_{2,\mathrm{ST}} - \frac{3}{2}\alpha^3 b_{1,\mathrm{ST}}^2 + 7\alpha b_{1,\mathrm{ST}} + 3\alpha b_{2,\mathrm{ST}} - \frac{3}{4}\alpha b_{1,\mathrm{ST}}^2 + (-30\alpha^5 - 56\alpha^3 - 26\alpha)\nu\bigg],\\
C^{\mathrm{B}}_{(3\mathrm{PN})} &= \frac{1}{\alpha^4}\left[\frac{9\nu^2}{8} - \frac{3\nu}{2} - \frac{3q_{3,\mathrm{ST}}}{16}\right] + \frac{1}{\alpha^2}\bigg[\frac{3}{32}a_{2,\mathrm{ST}} b_{1,\mathrm{ST}}^2 - \frac{21 a_{2,\mathrm{ST}} b_{1,\mathrm{ST}}}{8} - \frac{3a_{2,\mathrm{ST}} b_{2,\mathrm{ST}}}{8} + \frac{9\nu a_{2,\mathrm{ST}}}{4} + \frac{3a_{2,\mathrm{ST}}^2}{2} - \frac{273 a_{2,\mathrm{ST}}}{8}\\
&\quad - \frac{3a_{3,\mathrm{ST}} b_{1,\mathrm{ST}}}{8} - \frac{27 a_{3,\mathrm{ST}}}{4} - \frac{3a_{4,\mathrm{ST}}^{\mathrm{loc}}}{4} - \frac{3b_{1,\mathrm{ST}} b_{2,\mathrm{ST}}}{8} + \frac{3\nu b_{1,\mathrm{ST}}}{2} + \frac{3b_{1,\mathrm{ST}}^3}{32} - \frac{3b_{1,\mathrm{ST}}^2}{16} + \frac{45 b_{1,\mathrm{ST}}}{8} + \frac{9b_{2,\mathrm{ST}}}{4} + \frac{3b_{3,\mathrm{ST}}^{\mathrm{loc}}}{4} + \frac{45\nu^2}{4}\\
&\quad + \frac{123\pi^2\nu}{128} - 109\nu - \frac{9q_{3,\mathrm{ST}}^{\mathrm{loc}}}{8} + \frac{315}{4}\bigg] + \frac{15 a_{2,\mathrm{ST}} a_{3,\mathrm{ST}}}{4} + \frac{15}{16}a_{2,\mathrm{ST}}^2 b_{1,\mathrm{ST}} + \frac{15}{32}a_{2,\mathrm{ST}} b_{1,\mathrm{ST}}^2 - \frac{105 a_{2,\mathrm{ST}} b_{1,\mathrm{ST}}}{8} - \frac{15 a_{2,\mathrm{ST}} b_{2,\mathrm{ST}}}{8}\\
&\quad + \frac{75\nu a_{2,\mathrm{ST}}}{4} - \frac{1}{16}5a_{2,\mathrm{ST}}^3 + \frac{195 a_{2,\mathrm{ST}}^2}{8} - \frac{1485 a_{2,\mathrm{ST}}}{8} - \frac{15 a_{3,\mathrm{ST}} b_{1,\mathrm{ST}}}{8} - \frac{135 a_{3,\mathrm{ST}}}{4} - \frac{15 a_{4,\mathrm{ST}}^{\mathrm{loc}}}{4} - \frac{5b_{1,\mathrm{ST}} b_{2,\mathrm{ST}}}{8} + \frac{5b_{1,\mathrm{ST}}^3}{32}\\
&\quad + \frac{175 b_{1,\mathrm{ST}}}{8} - \frac{15 b_{1,\mathrm{ST}}^2}{16} + \frac{25 b_{2,\mathrm{ST}}}{4} + \frac{5b_{3,\mathrm{ST}}^{\mathrm{loc}}}{4} + \frac{105\nu^2}{8} + \frac{615\pi^2\nu}{128} - \frac{625\nu}{2} - \frac{15 q_{3,\mathrm{ST}}^{\mathrm{loc}}}{16} + \frac{1155}{4},\\
C^{0}_{(3\mathrm{PN})} &= \frac{1}{(1+\alpha^2)^3}\bigg\{\alpha^5\bigg[\frac{15 a_{2,\mathrm{ST}} a_{3,\mathrm{ST}}}{4} + \frac{15}{16}a_{2,\mathrm{ST}}^2 b_{1,\mathrm{ST}} + \frac{15}{32}a_{2,\mathrm{ST}} b_{1,\mathrm{ST}}^2 - \frac{105 a_{2,\mathrm{ST}} b_{1,\mathrm{ST}}}{8} - \frac{15 a_{2,\mathrm{ST}} b_{2,\mathrm{ST}}}{8}\\
&\quad + \nu\left(\frac{75 a_{2,\mathrm{ST}}}{4} + \frac{615\pi^2}{128} - \frac{625}{2}\right) - \frac{1}{16}5a_{2,\mathrm{ST}}^3 + \frac{195 a_{2,\mathrm{ST}}^2}{8} - \frac{1485 a_{2,\mathrm{ST}}}{8} - \frac{15 a_{3,\mathrm{ST}} b_{1,\mathrm{ST}}}{8} - \frac{135 a_{3,\mathrm{ST}}}{4} - \frac{15 a_{4,\mathrm{ST}}^{\mathrm{loc}}}{4}\\
&\quad - \frac{5b_{1,\mathrm{ST}} b_{2,\mathrm{ST}}}{8} + \frac{5b_{1,\mathrm{ST}}^3}{32} + \frac{175 b_{1,\mathrm{ST}}}{8} - \frac{15 b_{1,\mathrm{ST}}^2}{16} + \frac{25 b_{2,\mathrm{ST}}}{4} + \frac{5b_{3,\mathrm{ST}}^{\mathrm{loc}}}{4} + \frac{105\nu^2}{8} - \frac{15 q_{3,\mathrm{ST}}^{\mathrm{loc}}}{16} + \frac{1155}{4}\bigg]\\
&\quad + \alpha^3\bigg[10 a_{2,\mathrm{ST}} a_{3,\mathrm{ST}} + \nu\left(\frac{209 a_{2,\mathrm{ST}}}{4} + \frac{3b_{1,\mathrm{ST}}}{2} + \frac{1763\pi^2}{128} - \frac{2827}{3}\right) + \frac{5}{2}a_{2,\mathrm{ST}}^2 b_{1,\mathrm{ST}} + \frac{43}{32}a_{2,\mathrm{ST}} b_{1,\mathrm{ST}}^2 - \frac{301 a_{2,\mathrm{ST}} b_{1,\mathrm{ST}}}{8}\\
&\quad - \frac{43 a_{2,\mathrm{ST}} b_{2,\mathrm{ST}}}{8} - \frac{1}{6}5a_{2,\mathrm{ST}}^3 + \frac{133 a_{2,\mathrm{ST}}^2}{2} - \frac{4233 a_{2,\mathrm{ST}}}{8} - \frac{43 a_{3,\mathrm{ST}} b_{1,\mathrm{ST}}}{8} - \frac{387 a_{3,\mathrm{ST}}}{4} - \frac{43 a_{4,\mathrm{ST}}^{\mathrm{loc}}}{4} - \frac{49 b_{1,\mathrm{ST}} b_{2,\mathrm{ST}}}{24} + \frac{49 b_{1,\mathrm{ST}}^3}{96}\\
&\quad + \frac{1535 b_{1,\mathrm{ST}}}{24} - \frac{43 b_{1,\mathrm{ST}}^2}{16} + \frac{227 b_{2,\mathrm{ST}}}{12} + \frac{49 b_{3,\mathrm{ST}}^{\mathrm{loc}}}{12} + \frac{185\nu^2}{4} - \frac{29 q_{3,\mathrm{ST}}^{\mathrm{loc}}}{8} + \frac{3395}{4}\bigg]\\
&\quad + \alpha\bigg[\frac{33 a_{2,\mathrm{ST}} a_{3,\mathrm{ST}}}{4} + \nu\left(\frac{189 a_{2,\mathrm{ST}}}{4} + 4b_{1,\mathrm{ST}} + \frac{1681\pi^2}{128} - \frac{2939}{3}\right) + \frac{33}{16}a_{2,\mathrm{ST}}^2 b_{1,\mathrm{ST}} + \frac{41}{32}a_{2,\mathrm{ST}} b_{1,\mathrm{ST}}^2 - \frac{287 a_{2,\mathrm{ST}} b_{1,\mathrm{ST}}}{8}\\
&\quad - \frac{41 a_{2,\mathrm{ST}} b_{2,\mathrm{ST}}}{8} - \frac{1}{16}11a_{2,\mathrm{ST}}^3 + \frac{461 a_{2,\mathrm{ST}}^2}{8} - \frac{3995 a_{2,\mathrm{ST}}}{8} - \frac{41 a_{3,\mathrm{ST}} b_{1,\mathrm{ST}}}{8} - \frac{369 a_{3,\mathrm{ST}}}{4} - \frac{41 a_{4,\mathrm{ST}}^{\mathrm{loc}}}{4} - \frac{19 b_{1,\mathrm{ST}} b_{2,\mathrm{ST}}}{8}\\
&\quad + \frac{19 b_{1,\mathrm{ST}}^3}{32} + \frac{505 b_{1,\mathrm{ST}}}{8} - \frac{41 b_{1,\mathrm{ST}}^2}{16} + \frac{79 b_{2,\mathrm{ST}}}{4} + \frac{19 b_{3,\mathrm{ST}}^{\mathrm{loc}}}{4} + 60\nu^2 - \frac{21 q_{3,\mathrm{ST}}^{\mathrm{loc}}}{4} + \frac{3381}{4}\bigg]
\end{aligned}
$$

$$+\frac{1}{\alpha}\left[2a_{2,\mathrm{ST}}a_{3,\mathrm{ST}}+\nu\left(\frac{55a_{2,\mathrm{ST}}}{4}+\frac{7b_{1,\mathrm{ST}}}{2}+\frac{533\pi^2}{128}-\frac{1153}{3}\right)+\frac{13}{32}a_{2,\mathrm{ST}}b_{1,\mathrm{ST}}^2+\frac{1}{2}a_{2,\mathrm{ST}}^2b_{1,\mathrm{ST}}-\frac{95a_{2,\mathrm{ST}}b_{1,\mathrm{ST}}}{8}\right.$$
$$-\frac{13a_{2,\mathrm{ST}}b_{2,\mathrm{ST}}}{8}+\frac{29a_{2,\mathrm{ST}}^2}{2}-\frac{1279a_{2,\mathrm{ST}}}{8}-\frac{13a_{3,\mathrm{ST}}b_{1,\mathrm{ST}}}{8}-\frac{121a_{3,\mathrm{ST}}}{4}-\frac{13a_{4,\mathrm{ST}}^{\mathrm{loc}}}{4}-\frac{9b_{1,\mathrm{ST}}b_{2,\mathrm{ST}}}{8}+\frac{9b_{1,\mathrm{ST}}^3}{32}-\frac{13b_{1,\mathrm{ST}}^2}{16}$$
$$\left.+\frac{179b_{1,\mathrm{ST}}}{8}+\frac{31b_{2,\mathrm{ST}}}{4}+\frac{9b_{3,\mathrm{ST}}^{\mathrm{loc}}}{4}+\frac{135\nu^2}{4}-\frac{27q_{3,\mathrm{ST}}^{\mathrm{loc}}}{8}+\frac{1221}{4}\right]+\frac{1}{\alpha^3}\left[-\frac{a_{2,\mathrm{ST}}b_{1,\mathrm{ST}}}{2}-6a_{2,\mathrm{ST}}-a_{3,\mathrm{ST}}-\frac{b_{1,\mathrm{ST}}b_{2,\mathrm{ST}}}{6}\right.$$
$$\left.\left.+\nu\left(b_{1,\mathrm{ST}}-\frac{69}{2}\right)+\frac{b_{1,\mathrm{ST}}^3}{24}+\frac{4b_{1,\mathrm{ST}}}{3}+\frac{2b_{2,\mathrm{ST}}}{3}+\frac{b_{3,\mathrm{ST}}^{\mathrm{loc}}}{3}+\frac{55\nu^2}{8}-\frac{13q_{3,\mathrm{ST}}^{\mathrm{loc}}}{16}+\frac{64}{3}\right]\right\}. \tag{3.18}$$

Here for simplicity we do not substitute the values of the ST corrections $a_{i,\mathrm{ST}}$, $b_{i,\mathrm{ST}}$, and $q_{i,\mathrm{ST}}$. The explicit expressions of the corrections have been derived in Refs. [22–26].

Finally, the last contribution, I_χ, to the 3PN scattering angle given in Eq. (3.17) is defined as the following integral:

$$I_\chi=-\frac{ja_{4,\mathrm{ST}}^{\log}}{2}\,\mathrm{Pf}\int_0^{u_{(\max)}}\frac{u^4\ln(u)}{(2\bar{E}+2u-j^2u^2)^{3/2}}du, \tag{3.19}$$

where Pf denotes Hadamard's partie finie regularization of the divergent integral [44]. The above integral cannot be solved explicitly using the standard techniques of integration; therefore, we simplify the integral by using suitable integration by parts as

$$I_\chi=a_{4,\log}^{\mathrm{ST}}\frac{(15\alpha^4+18\alpha^2+3)}{16j^6\alpha^2(\alpha^2+1)}B(\alpha)+a_{4,\mathrm{ST}}^{\log}\frac{(15\alpha^2+13)}{16(\alpha^2+1)\alpha j^6}+\mathcal{I}_\chi, \tag{3.20}$$

where the last term is now a *convergent* integral defined as

$$\mathcal{I}_\chi=\frac{2ja_{4,\mathrm{ST}}^{\log}}{(1+2j^2\bar{E})}\int_0^{u_{(\max)}}\frac{u^3(uj^2-1)\ln(u)}{\sqrt{2\bar{E}+2u-j^2u^2}}du. \tag{3.21}$$

The integral of Eq. (3.21) cannot be expressed in terms of the elementary functions. After a suitable change of variables,

$$u=\frac{\sqrt{2\bar{E}}}{j}x';\qquad \epsilon\equiv 2\alpha=\frac{2}{p_\infty j}, \tag{3.22}$$

Eq. (3.21) yields

$$\mathcal{I}_\chi=\frac{16a_{4,\mathrm{ST}}^{\log}}{\epsilon^2j^6(4+\epsilon^2)}\,\mathrm{Pf}\int_0^{x'_{(\max)}}\frac{(2x'-\epsilon)x'^3}{\sqrt{1-x'^2+2x'\epsilon}}\ln\left(\frac{2x'}{\epsilon j^2}\right)dx', \tag{3.23}$$

where the upper limit $x'_{(\max)}=\epsilon/2+\sqrt{1+\epsilon^2/4}=1+\mathcal{O}(\epsilon)$. This integral can be computed in a small ϵ expansion (i.e., a large-j expansion) at fixed p_∞ using the approach introduced in Ref. [42] to compute the finite part of the real integral. This is obtained by the following: (i) using $x'_{(\max)}=1$ as the upper limit of the integral; (ii) PN expanding only the integrand; and (iii) taking the Hadamard partie finie of the divergent integrals. We then follow the steps of Ref. [31] to compute the j expansion of the integral. Here, we display the first three contributions to the integral in j expansion,

$$\mathcal{I}_\chi=\frac{a_{4,\mathrm{ST}}^{\log}\bar{E}}{j^4}\mathcal{I}_4+\frac{a_{4,\mathrm{ST}}^{\log}\bar{E}^{1/2}}{j^5}\mathcal{I}_5+\frac{a_{4,\mathrm{ST}}^{\log}}{j^6}\mathcal{I}_6+\mathcal{O}\left(\frac{\bar{E}^{-1/2}}{j^7}\right), \tag{3.24}$$

where

$$\mathcal{I}_4=\frac{\pi}{16}\left[7+6\ln\left(\frac{\bar{E}}{2j^2}\right)\right],$$
$$\mathcal{I}_5=\sqrt{2}\left[-2+8\ln(2)+2\ln\left(\frac{\bar{E}}{2j^2}\right)\right],$$
$$\mathcal{I}_6=\frac{\pi}{32}\left[77+30\ln\left(\frac{\bar{E}}{2j^2}\right)\right]. \tag{3.25}$$

The higher order contributions can be computed following the same approach.

B. Final expression of the local part of the 3PN scattering angle in a large-j expansion

The result presented in Eq. (3.17) of the scattering angle at the 3PN order is unexpanded in j except the integral I_χ of Eq. (3.19). To compute this integral, we expressed it into a simpler integral of Eq. (3.20). Then at the end of the last subsection, we computed a large-j expansion of this remaining part, $\mathcal{I}_\chi$, of the integral I_χ.

Let us now present the large-j expansion $\chi^{(3\mathrm{PN})}_{\mathrm{loc}}/2$ given by Eq. (3.17). Note that the auxiliary variable α entering the exactly known part of the scattering angle at 3PN, $\chi^{(3\mathrm{PN})}_{\mathrm{loc}}/2$, is dependent on j [see Eq. (3.12)]. As the contributions to $\mathcal{I}_\chi$ and hence I_χ given by Eq. (3.20) start at the $1/j^4$ order in their large-j expansion, we here present the large-j expansion of $\chi^{(3\mathrm{PN})}_{\mathrm{loc}}/2$ up to $1/j^4$,

$$\frac{\chi_{3\mathrm{PN}}}{2} = \frac{p_\infty^4}{j^2}\left[-\frac{3}{32}\pi q^{\mathrm{loc}}_{3,\mathrm{ST}} + \frac{9}{16}\pi\nu^2 - \frac{3}{4}\pi\nu\right] + \frac{p_\infty^3}{j^3}\left[-\frac{a_{2,\mathrm{ST}}b_{1,\mathrm{ST}}}{2} - 6a_{2,\mathrm{ST}} - a_{3,\mathrm{ST}} - \frac{b_{1,\mathrm{ST}}b_{2,\mathrm{ST}}}{6} + \nu b_{1,\mathrm{ST}} + \frac{b^3_{1,\mathrm{ST}}}{24} + \frac{4b_{1,\mathrm{ST}}}{3} + \frac{2b_{2,\mathrm{ST}}}{3} + \frac{b^{\mathrm{loc}}_{3,\mathrm{ST}}}{3} + 8\nu^2 - 36\nu - q^{\mathrm{loc}}_{3,\mathrm{ST}} + \frac{64}{3}\right] + \frac{p_\infty^2}{j^4}\left[\frac{3}{64}\pi a_{2,\mathrm{ST}}b^2_{1,\mathrm{ST}} - \frac{21}{16}\pi a_{2,\mathrm{ST}}b_{1,\mathrm{ST}} - \frac{3}{16}\pi a_{2,\mathrm{ST}}b_{2,\mathrm{ST}} + \frac{9}{8}\pi\nu a_{2,\mathrm{ST}} + \frac{3\pi a^2_{2,\mathrm{ST}}}{4} - \frac{273\pi a_{2,\mathrm{ST}}}{16} - \frac{3}{16}\pi a_{3,\mathrm{ST}}b_{1,\mathrm{ST}} - \frac{27\pi a_{3,\mathrm{ST}}}{8} - \frac{3\pi a^{\mathrm{loc}}_{4,\mathrm{ST}}}{8} + \frac{3\pi a^{\log}_{4,\mathrm{ST}}}{32} - \frac{3}{16}\pi b_{1,\mathrm{ST}}b_{2,\mathrm{ST}} + \frac{3}{4}\pi\nu b_{1,\mathrm{ST}} + \frac{3\pi b^3_{1,\mathrm{ST}}}{64} - \frac{3\pi b^2_{1,\mathrm{ST}}}{32} + \frac{45\pi b_{1,\mathrm{ST}}}{16} + \frac{9\pi b_{2,\mathrm{ST}}}{8} + \frac{3\pi b^{\mathrm{loc}}_{3,\mathrm{ST}}}{8} + \frac{45\pi\nu^2}{8} + \frac{123\pi^3\nu}{256} - \frac{109\pi\nu}{2} - \frac{9\pi q^{\mathrm{loc}}_{3,\mathrm{ST}}}{16} + \frac{315\pi}{8} + \frac{\pi a^{\log}_{4,\mathrm{ST}}}{32}\left\{7 + 6\ln\left(\frac{\bar{E}}{2j^2}\right)\right\}\right] + \mathcal{O}\left(\frac{1}{j^5}\right). \tag{3.26}$$

IV. NONLOCAL CONTRIBUTIONS TO THE SCATTERING ANGLE

In this section, we compute the leading order (LO) nonlocal contributions to the scattering angle using the order-reduction approach of Ref. [38] for bound orbits. This approach has recently been used to derive the nonlocal contributions to the EOB metric potentials for bound orbits in ST theories [25,26]. Here, we will use this approach for hyperboliclike orbits in ST theories following Refs. [31,45].

As the tail contribution to the Hamiltonian starts at 3PN order in ST theory, one can compute the LO contribution to the scattering angle χ_{nonloc} by considering the Hamiltonian

$$H = H_{\mathrm{N}} + H^{\mathrm{tail}}, \tag{4.1}$$

where H_{N} is the Newtonian-order Hamiltonian and H^{tail} is the LO tail contribution [18], as only the Newtonian order radial momentum is required for computing the 3PN LO tail contribution to the scattering angle. The Newtonian order contribution to the Hamiltonian in dimensionless variables given in Eq. (2.14) reads

$$\frac{H_{\mathrm{N}}}{\mu} = \frac{1}{2}\left(\hat{p}_r^2 + \frac{j^2}{\hat{r}^2}\right) - \frac{1}{\hat{r}}. \tag{4.2}$$

For simplicity we work in $M = 1$ units such that $\mu = \nu$ in this section.

From Sec. III, let us recall the formula of the scattering angle derived using the general Hamilton-Jacobi approach,

$$\chi(\bar{E}, j) = -\frac{\partial}{\partial j}\int \hat{p}_r(\bar{E}, j, \hat{r})d\hat{r}, \tag{4.3}$$

where the radial momentum function, $\hat{p}_r(\bar{E}, j, r)$, is first computed by solving for $\hat{p}_r^2(\bar{E}, j, r)$ the energy conservation law,

$$\bar{E} = \frac{H(\hat{r}, \hat{p}_r, j)}{\nu} = \frac{1}{2}\left(\hat{p}_r^2 + \frac{j^2}{\hat{r}^2}\right) - \frac{1}{\hat{r}} + \frac{H^{\mathrm{tail}}}{\nu}. \tag{4.4}$$

At LO in tail, the solution of the equation in $\hat{p}_r$ is

$$\hat{p}_r = \hat{p}_r^0 - \frac{1}{\hat{p}_r^0}\frac{H^{\mathrm{tail}}(\hat{r}, \hat{p}_r^0, j)}{\nu}, \tag{4.5}$$

where $\hat{p}_r^0$ is the Newtonian contribution [see Eq. (3.9)]. Inserting the solution in Eq. (4.3), we obtain

$$\chi(\bar{E}, j) = -\frac{\partial}{\partial j}\int \hat{p}_r^0(\bar{E}, j, \hat{r})dr + \frac{\partial}{\partial j}\int \frac{dr}{\hat{p}_r^0}\frac{H^{\mathrm{tail}}(\hat{r}, \hat{p}_r, j)}{\nu}, \tag{4.6}$$

where the first term is the Newtonian order contribution to the scattering angle derived in Eq. (3.14). Then the nonlocal contribution to the scattering angle reads

$$\chi_{\mathrm{nonloc}} = \frac{1}{\nu}\frac{\partial}{\partial j}W^{\mathrm{tail}}(\bar{E}, j), \tag{4.7}$$

where

$$W^{\mathrm{tail}} = \int \frac{d\hat{r}}{\hat{p}_r^0}H^{\mathrm{tail}}(\hat{r}, \hat{p}_r, j) = \left[\!\left[\int dt H^{\mathrm{tail}}\right]\!\right]. \tag{4.8}$$

In the second expression we have used the property that the time localized Hamiltonian is simply obtained by using the solutions of the Newtonian-level Hamilton's equation for the phase space variables (see Ref. [31] for the hyperbolic case and Ref. [38] for the elliptic case), and that $d\hat{r}/\hat{p}_r^0 = dt$ is along the Newtonian Hamiltonian flow.

The LO tail contribution to the Hamiltonian in ST theories in the Jordan-frame conventions reads [18]

$$H_{\rm LO}^{\rm tail} = -\frac{2}{3}(3+2\omega_0)\mathrm{Pf}_{2s_1/c}\int_{-\infty}^{\infty}\frac{d\tau}{|\tau|}I_{s,i}^{(2)}(t)I_{s,i}^{(2)}(t+\tau), \quad (4.9)$$

where Pf is the Hadamard partie finie function with the Hadamard partie finie scale s_1 and $I_{s,i}^{(2)}$ is the second time derivative of the scalar dipole moment, $I_{s,i}$. In the center-of-mass (COM) frame, the scalar dipole moment in DEF conventions is

$$I_{s,i} = \nu\alpha_0(\alpha_B - \alpha_A)\hat{x}^i, \quad (4.10)$$

where α_0 is the asymptotic value of Eq. (2.2) at $\varphi = \varphi_0$, and α_A and α_B are given in Table I. Here, $\hat{x}^i(=(Z_A - Z_B)^i)/(G_{AB}M)$ is the dimensionless relative separation vector and $Z_{A,B}$ indicate the positions of the two bodies.

Thus the LO potential $W^{\rm tail}$ is

$$W^{\rm tail} = \int dt H_{\rm LO}^{\rm tail}, \quad (4.11)$$

where $H_{\rm LO}^{\rm tail}$ is given in Eq. (4.9).

In Refs. [25,26], using Kepler's equations for elliptic orbits it is shown that the scalar dipole moment is a periodic function in the action-angle variables, and hence can be decomposed into Fourier series. Here, we are considering hyperbolic motions; therefore, the Cartesian coordinates are parametrized as

$$x(t) = -a(\cosh\bar{u} - e), \quad (4.12)$$

$$y(t) = -a\sqrt{e^2-1}\sinh\bar{u}, \quad (4.13)$$

and the hyperbolic Kepler equation is

$$\bar{n}t = e\sinh\bar{u} - \bar{u}, \quad (4.14)$$

where e is the eccentricity, a is the semimajor axis, $\bar{u}$ is the eccentric anomaly, and

$$\bar{n} = \frac{1}{\bar{a}^{3/2}}; \qquad \bar{a} = -a = \frac{1}{2\bar{E}}. \quad (4.15)$$

Similar to Fourier series expansion of the scalar dipole moment for elliptic orbits in Refs. [24,26], the scalar dipole moment for hyperbolic motions can also be decomposed into Fourier series, i.e.,

$$I_{\rm s,i}(t) = \int\frac{d\omega}{2\pi}\tilde{I}_{s,i}(\omega)\mathrm{e}^{-i\omega t},$$
$$\tilde{I}_{s,i}(\omega) = \int dt I_{s,i}(t)\mathrm{e}^{i\omega t}, \quad (4.16)$$

where $\tilde{I}_{s,i}(\omega)$ is the Fourier transform of the scalar dipole moment.

Inserting the Fourier transformation of the scalar dipole moment [and $\tau = G_{AB}M(t'-t)$][3] in Eq. (4.11) and converting to DEF conventions yields

$$\begin{aligned} W_{\rm tail} &= -\frac{1}{(1+\alpha_A\alpha_B)^2}\frac{2}{3\alpha_0^2}\int dt\,\mathrm{Pf}_{2\hat{s}_1/c}\int\frac{dt'}{|t-t'|}\int\frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\omega^2\omega'^2\tilde{I}_{s,i}(\omega)\tilde{I}_{s,i}(\omega')\mathrm{e}^{-i\omega t}\mathrm{e}^{-i\omega' t'} \\ &= -\frac{2}{3\alpha_0^2(1+\alpha_A\alpha_B)^2}\mathrm{Pf}_{2\hat{s}_1/c}\int\frac{dt'}{|t-t'|}\int\frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\omega^2\omega'^2\tilde{I}_{s,i}(\omega)\tilde{I}_{s,i}(\omega')\mathrm{e}^{-i\omega'\hat{\tau}}2\pi\delta(\omega+\omega') \\ &= -\frac{2}{3\alpha_0^2(1+\alpha_A\alpha_B)^2}\mathrm{Pf}_{2\hat{s}_1/c}\int\frac{dt'}{|t-t'|}\int\frac{d\omega}{2\pi}\omega^4\tilde{I}_{s,i}(\omega)\tilde{I}_{s,i}(-\omega)\mathrm{e}^{i\omega\hat{\tau}} \\ &= -\frac{2}{3\alpha_0^2(1+\alpha_A\alpha_B)^2}\int\frac{d\omega}{2\pi}\omega^4\tilde{I}_{s,i}(\omega)\tilde{I}_{s,i}(-\omega)\mathrm{Pf}_{2\hat{s}_1/c}\int\frac{d\hat{\tau}}{|\hat{\tau}|}\mathrm{e}^{i\omega\hat{\tau}}. \end{aligned} \quad (4.17)$$

The partie finie integral of the last term in the above equation is (see Ref. [46])

$$\mathrm{Pf}_{2\hat{s}_1/c}\int_{-\infty}^{\infty}\frac{d\tau}{|\tau|}\mathrm{e}^{i\omega\tau} = -2\ln\left(\frac{2|\omega|\hat{s}_1\mathrm{e}^{\gamma_{\rm Euler}}}{c}\right), \quad (4.18)$$

where $\gamma_{\rm Euler}$ is Euler's Gamma. Inserting Eq. (4.18) into Eq. (4.17) gives the Fourier-domain formula for the potential

$$W^{\rm tail} = \frac{8}{3\alpha_0^2(1+\alpha_A\alpha_B)^2}\int_0^{\infty}\frac{d\omega}{2\pi}\omega^4\tilde{I}_{s,i}(\omega)\tilde{I}_{s,i}^*(\omega) \times\ln\left(\frac{2|\omega|\hat{s}_1\mathrm{e}^{\gamma_{\rm Euler}}}{c}\right), \quad (4.19)$$

where $\tilde{I}_{s,i}(-\omega) = \tilde{I}_{s,i}^*(\omega)$.

[3]Here, the dimensionless variable $\hat{\tau} = \frac{\tau}{G_{AB}}$ and $t = \frac{T}{G_{AB}M}$.

To compute the explicit expression of Eq. (4.19) of potential $W^{\rm tail}$ in terms of $\bar{E}$ and j, we insert the Fourier transform of the scalar dipole moment. For this, we evaluate the Fourier transforms of (x, y) for hyperbolic orbits, i.e.,

$$\tilde{x}(\omega) = \int dt\, e^{i\omega t} x(t),$$
$$\tilde{y}(\omega) = \int dt\, e^{i\omega t} y(t). \tag{4.20}$$

After inserting Eqs. (4.12)–(4.14) into Eq. (4.20) and using the definition of Hankel functions of first kind $H^{(1)}_{p'}$ [see Eq. (9.1.25) of [47]],

$$\int_{-\infty}^{\infty} e^{q' \sinh\xi - p'\xi} = i\pi H^{(1)}_{p'}(q'), \tag{4.21}$$

we find the Fourier transform of (x, y). The computation gives

$$\tilde{x}(\omega) = \frac{\pi a}{\omega}\left(\frac{p'_\omega}{q'_\omega} H^{(1)}_{p'_\omega}(q'_\omega) - H^{(1)}_{p'_\omega+1}(q'_\omega)\right), \tag{4.22}$$

$$\tilde{y}(\omega) = -\frac{\pi a}{\omega e}\sqrt{e^2-1}\, H^{(1)}_{p}(q), \tag{4.23}$$

where for our case

$$q' = q'_\omega = ie\frac{\omega}{\bar{n}}; \qquad p' = p'_\omega = \frac{q'_\omega}{e}. \tag{4.24}$$

We then consider the Fourier transform of (x, y) in the large-j limit which is equivalent to the large-e limit as $e = \sqrt{1 + 2\bar{E}j^2}$. The large-$e$ limit of Eqs. (4.22) and (4.23) yields

$$\tilde{x}(\omega) = -\frac{\pi a}{\omega} H^{(1)}_1(ig_\omega), \tag{4.25}$$

$$\tilde{y}(\omega) = -\frac{\pi a}{\omega} H^{(1)}_0(ig_\omega), \tag{4.26}$$

where $q'_\omega = ig_\omega$ and $g_\omega = e\frac{\omega}{\bar{n}}$. The Hankel functions evaluated at purely imaginary arguments are related to modified Bessel functions K_ν as [see Eq. (9.6.4) of [47]]

$$K_0(x') = i\frac{\pi}{2} H^{(1)}_0(ix'); \quad K_1(x') = -\frac{\pi}{2} H^{(1)}_1(ix'). \tag{4.27}$$

Finally, inserting Eqs. (4.25)–(4.27) into Eq. (4.19) and then taking the j derivative of potential $W^{\rm tail}$, the explicit expression of the scattering angle in the large-e limit yields

$$\chi^{\rm tail}_{\rm 3PN} = -\frac{2\pi\nu}{3}\frac{p_\infty^2}{j^4}\left[2\delta_+ + \frac{\bar{\gamma}_{AB}(\bar{\gamma}_{AB}+2)}{2}\right] \times\left\{7 + 3\ln\left(\frac{p_\infty^2 \hat{s}_1}{4j}\right)\right\}, \tag{4.28}$$

where we recall that $\hat{s}_1 = s_1/(G_{AB}M)$ is the dimensionless regularization scale defining the near zone–far zone separation, $\delta_+ = \frac{\delta_A+\delta_B}{2}$, and $\bar{\gamma}_{AB}$ is defined in Table I.

V. SUMMING THE LOCAL AND NONLOCAL CONTRIBUTIONS TO $\chi_{\rm 3PN}$ IN A LARGE-j EXPANSION

In Sec. III, we first computed the local scattering angle up to 3PN order, and then in Sec. IV we separately computed the nonlocal contributions at the 3PN order. The results at 1PN and 2PN levels were given in fully explicit and exact forms. However, the results at the 3PN order were obtained in the large-j expansion for both the local contribution (due to the logarithmic term $\mathcal{I}_\chi$) and the nonlocal contribution. On combining the two separate 3PN order contributions to the scattering angle at 3PN, we find

$$\begin{aligned}\frac{\chi(\bar{E},j)^{(3{\rm PN})}}{2} =& \frac{p_\infty^4}{j^2}\pi\left\{\nu\left[-\frac{3}{4}-\frac{15}{64}\bar{\gamma}_{\rm AB}^2-\frac{13}{16}\bar{\gamma}_{\rm AB}-\frac{1}{16}\langle\bar{\beta}\rangle+\frac{1}{16}\langle\delta\rangle\right]+\nu^2\left[\frac{3}{8}\bar{\gamma}_{\rm AB}-\frac{3}{16}\langle\bar{\beta}\rangle+\frac{9}{16}\right]\right\}\\ &+\frac{p_\infty^3}{j^3}\left\{\frac{64}{3}-\frac{4}{3}[\langle\delta\rangle(\bar{\gamma}_{\rm AB}+2)-2\bar{\gamma}_{\rm AB}(\bar{\gamma}_{\rm AB}^2+6\bar{\gamma}_{\rm AB}+12)]+\frac{\nu}{6}[-216+4\langle\bar{\beta}\rangle(8\bar{\gamma}_{\rm AB}-3)+8\bar{\gamma}_{\rm AB}\delta_+-6\bar{\gamma}_{\rm AB}^3\right.\\ &-93\bar{\gamma}_{\rm AB}^2-262\bar{\gamma}_{\rm AB}-4X_{\rm AB}\bar{\gamma}_{\rm AB}\beta_-+8\langle\delta\rangle-57X_{\rm AB}\beta_-+2X_{\rm AB}\epsilon_-+36\beta_++4\delta_+-2\zeta-2\epsilon_+]+\frac{\nu^2}{3}[\bar{\gamma}_{\rm AB}^2+22\bar{\gamma}_{\rm AB}\\ &\left.-6\langle\bar{\beta}\rangle-18\beta_++4\delta_+-6\zeta+2\epsilon_++24]\right\}\\ &+\frac{p_\infty^2}{j^4}\pi\left\{\frac{315}{8}+\frac{1}{32}[4\langle\delta\rangle(-20\bar{\gamma}_{\rm AB}+12\beta_+^3+12\beta_-^2\beta_+-35)+\langle\bar{\beta}\rangle(-187\bar{\gamma}_{\rm AB}^2-556\bar{\gamma}_{\rm AB}+12\langle\delta\rangle-412)+16\bar{\gamma}_{\rm AB}\langle\epsilon\rangle\right.\\ &+236\bar{\gamma}_{\rm AB}^3+1229\bar{\gamma}_{\rm AB}^2+2148\bar{\gamma}_{\rm AB}+48\langle\bar{\beta}\rangle^2-8\langle\kappa\rangle+24\langle\epsilon\rangle+48\beta_-(\beta_-^2+\beta_+^2)(18\beta_+-\delta_+)X_{\rm AB}-48\beta_-\delta_-(\beta_-^2+\beta_+^2)]\end{aligned}$$

$$+\nu\Bigg[\frac{123\pi^2}{256}-\frac{109}{2}+\bar{\gamma}_{\rm AB}\left(\frac{49\langle\bar{\beta}\rangle}{4}-\frac{3\langle\delta\rangle}{16}-\frac{3\beta_- X_{\rm AB}}{4}+\frac{39\beta_+}{4}+\frac{\delta_+}{2}-\frac{21\pi^2\delta_+}{256}+\frac{15\zeta}{4}-\epsilon_+ +\frac{225\pi^2}{512}-\frac{3353}{48}\right)$$
$$+\left(\frac{\langle\bar{\beta}\rangle}{4}+\frac{15\pi^2}{256}-\frac{2425}{96}\right)\bar{\gamma}_{\rm AB}^2+\left(-\frac{71}{64}-\frac{21\pi^2}{1024}\right)\bar{\gamma}_{\rm AB}^3+\left(\delta_+ +\frac{7}{4}-\frac{3}{4}\langle\bar{\beta}\rangle\right)\langle\bar{\beta}\rangle+\left(\frac{9\beta_- X_{\rm AB}}{2}+\frac{81}{4}\right)\beta_+$$
$$+\frac{9}{4}(\beta_-^2+\beta_+^2)-\left(\frac{17}{12}+\frac{21\pi^2}{128}\right)\delta_+ +\frac{9}{8}\langle\delta\rangle+\left(-\frac{45\beta_-}{4}-\frac{\delta_-}{8}+\frac{3\epsilon_-}{8}\right)X_{\rm AB}+\beta_-\delta_- +3\zeta+\frac{\kappa_+}{2}-\frac{15\epsilon_+}{8}+\frac{\langle\kappa\rangle}{4}-\frac{3}{8}\langle\epsilon\rangle$$
$$+\left(2\delta_+ +\frac{\bar{\gamma}_{\rm AB}(\bar{\gamma}_{\rm AB}+2)}{2}\right)\ln\left(\frac{2}{p_\infty}\right)-\frac{3\bar{\gamma}_{\rm AB}(11(\bar{\gamma}_{\rm AB}+2)^2-4\langle\delta\rangle)}{32\alpha_{\rm AB}(\bar{\gamma}_{\rm AB}+2)}+\frac{12(\beta_-^2-\beta_+^2)\langle\bar{\beta}\rangle}{\bar{\gamma}_{\rm AB}^2}$$
$$+\frac{\beta_+(3\langle\epsilon\rangle+4\delta_- X_{\rm AB}-8\delta_+)}{2\bar{\gamma}_{\rm AB}}+\frac{\beta_-(X_{\rm AB}(4\delta_+ +3\epsilon_+)-8\delta_- -3\epsilon_-)}{2\bar{\gamma}_{\rm AB}}\Bigg]$$
$$-\frac{3}{8}\nu^2[-\bar{\gamma}_{\rm AB}^2-16\bar{\gamma}_{\rm AB}+3-4\beta_-^2+18\beta_+ -4\delta_+ +6\zeta-2\epsilon_+ -15]\Bigg\}, \tag{5.1}$$

where we use the notations of Refs. [24,25] with $X_{\rm A,B}\equiv m^0_{A,B}/M$ and

$$X_{AB}\equiv X_A - X_B, \tag{5.2}$$

$$\langle\bar{\beta}\rangle\equiv -X_{AB}\beta_- + \beta_+, \tag{5.3}$$

$$\langle\bar{\kappa}\rangle\equiv -X_{AB}\kappa_- + \kappa_+, \tag{5.4}$$

$$\langle\delta\rangle\equiv X_{AB}\delta_- + \delta_+, \tag{5.5}$$

$$\langle\epsilon\rangle\equiv -X_{AB}\epsilon_- + \epsilon_+. \tag{5.6}$$

Here, the subscript "±" denotes the symmetric and antisymmetric parts of the ST parameters, e.g., $z_\pm = (z_A \pm z_B)/2$. As the scattering angle is gauge invariant, the arbitrary scale $\hat{s}_1$ has been canceled between the two contributions as expected.

VI. CONCLUSIONS

Building upon the results of [22–26] for the corrections in the EOB metric coefficients (A, B, Q_e) for massless scalar-tensor theory for the conservative part of the dynamics, we determined the scattering angle for hyperboliclike orbits up to the 3PN order for both the local-in-time and the nonlocal-in-time parts of the dynamics. First, we compute the scattering angle for the local part of the dynamics by the following: (i) deriving the radial momentum as a function of u, orbital angular momentum, and energy by iteratively solving the EOB energy conservation law; (ii) calculating the scattering angle using the standard techniques of Ref. [42] for solving divergent integrals arising in the PN expansion of the radial momentum except the integral I_χ at the 3PN order; and (iii) computing the integral I_χ by using the appropriate integration by parts and expanding in large-j the remaining integral after a change of variables [31]. We then computed the total contribution to the 3PN order scattering angle in the large-j expansion.

Then, we computed the nonlocal-in-time contribution by using the approach introduced in Ref. [38] for GR of order reducing (time localization) the Hamiltonian in small-eccentricity case for hyperboliclike encounters [31,45]. Finally, we substituted the ST corrections of the metric potentials (A, B, Q_e) and sum both the local and the nonlocal contributions in the large-j expansion at 3PN order. As the first test of our results, we checked that the scattering angle coincides with the scattering angle of GR (see Ref. [31] for GR results) in the GR limit as expected.

Another test we perform on our results is to study the binary black hole in the limit in ST theories assuming that the sensitivity for the stationary black hole holds for a binary system. The sensitivity parameter (s_A) for the stationary black hole in the Jordan-frame conventions is exactly $s_A = \frac{1}{2}$ (see Refs. [48,49]), which in our conventions[4] implies that the parameter $\alpha_A = 0$, and hence all other ST parameters entering the scattering angle, vanishes (see Table I). This shows that our results are indistinguishable with the results of GR under this assumption.

This paper must be seen as a first step to compute the gauge-invariant scattering angle within the PN expansion for massless scalar-tensor theories. In future work we will address radiation reaction contributions to scattering.

[4]The Jordan-frame sensitivity (s_A) is related to the parameter α_A as

$$s_A = \frac{1}{2} - \frac{\alpha_A}{2\alpha_0}. \tag{6.1}$$

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[1] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Observation of gravitational waves from a binary black hole merger, Phys. Rev. Lett. **116**, 061102 (2016).
[2] D. Reitze *et al.*, Cosmic explorer: The U.S. contribution to gravitational-wave astronomy beyond LIGO, Bull. Am. Astron. Soc. **51**, 035 (2019), arXiv:1907.04833.
[3] M. Punturo, M. Abernathy, F. Acernese, B. Allen, N. Andersson *et al.*, The Einstein telescope: A third-generation gravitational wave observatory, Classical Quantum Gravity **27**, 194002 (2010).
[4] P. Amaro-Seoane, Detecting intermediate-mass ratio inspirals from the ground and space, Phys. Rev. D **98**, 063018 (2018).
[5] M. Maggiore *et al.*, Science case for the Einstein telescope, J. Cosmol. Astropart. Phys. 03 (2020) 050.
[6] M. Evans *et al.*, A horizon study for cosmic explorer: Science, observatories, and community, arXiv:2109.09882.
[7] T. Damour and G. Esposito-Farese, Tensor multiscalar theories of gravitation, Classical Quantum Gravity **9**, 2093 (1992).
[8] T. Damour and G. Esposito-Farese, Nonperturbative strong field effects in tensor—scalar theories of gravitation, Phys. Rev. Lett. **70**, 2220 (1993).
[9] T. Damour and G. Esposito-Farese, Testing gravity to second post-Newtonian order: A field theory approach, Phys. Rev. D **53**, 5541 (1996).
[10] P. C. C. Freire, N. Wex, G. Esposito-Farese, J. P. W. Verbiest, M. Bailes, B. A. Jacoby, M. Kramer, I. H. Stairs, J. Antoniadis, and G. H. Janssen, The relativistic pulsar-white dwarf binary PSR J1738 + 0333 II. The most stringent test of scalar-tensor gravity, Mon. Not. R. Astron. Soc. **423**, 3328 (2012).
[11] M. Khalil, R. F. P. Mendes, N. Ortiz, and J. Steinhoff, Effective-action model for dynamical scalarization beyond the adiabatic approximation, arXiv:2206.13233.
[12] T. Gautam, P. C. C. Freire, A. Batrakov, M. Kramer, C. C. Miao, E. Parent, and W. W. Zhu, Relativistic effects in a mildly recycled pulsar binary: PSR J1952 + 2630, arXiv:2210.03464.
[13] A. De Felice and S. Tsujikawa, f(R) theories, Living Rev. Relativity **13**, 3 (2010).
[14] R. N. Lang, Compact binary systems in scalar-tensor gravity. II. Tensor gravitational waves to second post-Newtonian order, Phys. Rev. D **89**, 084014 (2014).
[15] R. N. Lang, Compact binary systems in scalar-tensor gravity. III. Scalar waves and energy flux, Phys. Rev. D **91**, 084027 (2015).
[16] S. Mirshekari and C. M. Will, Compact binary systems in scalar-tensor gravity: Equations of motion to 2.5 post-Newtonian order, Phys. Rev. D **87**, 084070 (2013).
[17] L. Bernard, Dynamics of compact binary systems in scalar-tensor theories: Equations of motion to the third post-Newtonian order, Phys. Rev. D **98**, 044004 (2018).
[18] L. Bernard, Dynamics of compact binary systems in scalar-tensor theories: II. Center-of-mass and conserved quantities to 3PN order, Phys. Rev. D **99**, 044047 (2019).
[19] L. Bernard, Dipolar tidal effects in scalar-tensor theories, Phys. Rev. D **101**, 021501 (2020); **107**, 069901(E) (2023).
[20] L. Bernard, L. Blanchet, and D. Trestini, Gravitational waves in scalar-tensor theory to one-and-a-half post-Newtonian order, J. Cosmol. Astropart. Phys. 08 (2022) 008.
[21] O. Schön and D. D. Doneva, Tensor-multiscalar gravity: Equations of motion to 2.5 post-Newtonian order, Phys. Rev. D **105**, 064034 (2022).
[22] F.-L. Julié and N. Deruelle, Two-body problem in scalar-tensor theories as a deformation of general relativity: An effective-one-body approach, Phys. Rev. D **95**, 124054 (2017).
[23] F.-L. Julié, Reducing the two-body problem in scalar-tensor theories to the motion of a test particle: A scalar-tensor effective-one-body approach, Phys. Rev. D **97**, 024047 (2018).
[24] T. Jain, P. Rettegno, M. Agathos, A. Nagar, and L. Turco, Effective-one-body Hamiltonian in scalar-tensor gravity at third post-Newtonian order, Phys. Rev. D **107**, 084017 (2023).
[25] T. Jain, Nonlocal-in-time effective one body Hamiltonian in scalar-tensor gravity at third post-Newtonian order, Phys. Rev. D **107**, 084018 (2023).
[26] F.-L. Julié, V. Baibhav, E. Berti, and A. Buonanno, Third post-Newtonian effective-one-body Hamiltonian in scalar-tensor and Einstein-scalar-Gauss-Bonnet gravity, Phys. Rev. D **107**, 104044 (2023).
[27] R. Gamba, S. Akçay, S. Bernuzzi, and J. Williams, Effective-one-body waveforms for precessing coalescing compact binaries with post-Newtonian twist, Phys. Rev. D **106**, 024020 (2022).
[28] R. Gamba and S. Bernuzzi, Resonant tides in binary neutron star mergers: Analytical-numerical relativity study, Phys. Rev. D **107**, 044014 (2023).

[29] S. Ossokine *et al.*, Multipolar effective-one-body waveforms for precessing binary black holes: Construction and validation, Phys. Rev. D **102**, 044055 (2020).

[30] T. Damour, Gravitational scattering, post-Minkowskian approximation and effective one-body theory, Phys. Rev. D **94**, 104015 (2016).

[31] D. Bini and T. Damour, Gravitational scattering of two black holes at the fourth post-Newtonian approximation, Phys. Rev. D **96**, 064021 (2017).

[32] D. M. Eardley, Observable effects of a scalar gravitational field in a binary pulsar, Astrophys. J. Lett. **196**, L59 (1975).

[33] A. Buonanno and T. Damour, Effective one-body approach to general relativistic two-body dynamics, Phys. Rev. D **59**, 084006 (1999).

[34] A. Buonanno and T. Damour, Transition from inspiral to plunge in binary black hole coalescences, Phys. Rev. D **62**, 064015 (2000).

[35] T. Damour, P. Jaranowski, and G. Schäfer, On the determination of the last stable orbit for circular general relativistic binaries at the third post-Newtonian approximation, Phys. Rev. D **62**, 084011 (2000).

[36] D. Bini, T. Damour, and A. Geralico, Sixth post-Newtonian local-in-time dynamics of binary systems, Phys. Rev. D **102**, 024061 (2020).

[37] D. Bini, T. Damour, and A. Geralico, Sixth post-Newtonian nonlocal-in-time dynamics of binary systems, Phys. Rev. D **102**, 084047 (2020).

[38] T. Damour, P. Jaranowski, and G. Schäfer, Fourth post-Newtonian effective one-body dynamics, Phys. Rev. D **91**, 084024 (2015).

[39] L. Landau and E. Lifshitz, *Mechanics: Volume 1*, Course of Theoretical Physics, (Elsevier Science, 1976).

[40] T. Damour, High-energy gravitational scattering and the general relativistic two-body problem, Phys. Rev. D **97**, 044038 (2018).

[41] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.108.104071 for the *Mathematica* file containing the explicit expressions of radial momentum up to 3PN order for scalar-tensor theory.

[42] T. Damour and G. Schäfer, Higher order relativistic periastron advances and binary pulsars, Nuovo Cimento Soc. Ital. Fis. B **101**, 127 (1988).

[43] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. (Elsevier/Academic Press, Amsterdam, 2007), pp. XLVIII+1171, translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).

[44] L. Blanchet and G. Faye, Hadamard regularization, J. Math. Phys. (N.Y.) **41**, 7675 (2000).

[45] D. Bini and A. Geralico, Frequency domain analysis of the gravitational wave energy loss in hyperbolic encounters, Phys. Rev. D **104**, 104019 (2021).

[46] T. Damour, P. Jaranowski, and G. Schäfer, Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems, Phys. Rev. D **89**, 064058 (2014).

[47] M. Abramowitz, *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables*, (Dover Publications, Inc., USA, 1974).

[48] S. Hawking and J. Hartle, Energy and angular momentum flow into a black hole, Commun. Math. Phys. **27**, 283 (1972).

[49] T. P. Sotiriou and V. Faraoni, Black holes in scalar-tensor gravity, Phys. Rev. Lett. **108**, 081103 (2012).