Hidden conformal symmetries for black holes in modified gravity

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We determine hidden conformal symmetries behind the evolution equations of black hole perturbations in a vector-tensor theory of gravity. Such hidden symmetries are valid everywhere in the exterior region of a spherically symmetric, asymptotically flat black hole geometry. They allow us to factorize second order operators controlling the black hole perturbations into a product of two commuting first order operators. As a consequence, we are able to analytically determine the most general time-dependent solutions for the black hole perturbation equations. We focus on solutions belonging to a highest weight representation of a conformal symmetry, showing that they correspond to quasibound states with an ingoing behavior into the black hole horizon, and exponential decay at spatial infinity. Their time dependence is characterized by purely imaginary frequencies, with imaginary parts separated by integer numbers, as the overtones of quasinormal modes in general relativity.

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I. INTRODUCTION

The study of emergent conformal symmetries is a topic of active research for black holes in general relativity. Conformal symmetries are important for understanding black hole entropy in terms of microstate counting for BTZ [1] and extremal [2] black holes, as well as for configurations equipped with anti–de Sitter (AdS) symmetries (see e.g. [3–6]). Given the need of a deeper understanding of black hole entropy to address the black hole information problem, exploring the emergence of conformal symmetries in black hole geometries may prove particularly insightful.

Conformal symmetries emerge in the near horizon region of Schwarzschild black holes. They are associated with diffeomorphism invariance of geometrical quantities in the proximity of the black hole horizon [7,8], or with the properties of the near horizon optical metric [9]. However, they can also appear as *hidden symmetries* [10], without apparent relation to any geometric properties of the black hole space-time (see also [11]). This is reminiscent of the situation for Kerr [12], whereby such hidden symmetries and their Virasoro central charge extensions have been related with the black hole entropy. The interesting results of said hidden symmetries in [10] have been further explored in the literature: see (for example) [13–17]. Recently, they have played a key role in [18] where they have been used to uncover a black hole *Love symmetry*, able to explain the vanishing of Love numbers for four-dimensional black holes in terms of underlying symmetry structures. Further work has demonstrated that conformal symmetries emerge not only near the horizon, but also in the proximity of the photon ring region [19–28] of a black hole, explaining some of its features.

It is interesting to explore if these results pertain to general relativity, or instead whether hidden conformal symmetries can be found in alternative theories of gravity (see e.g. [29] for a comprehensive review). In this work we focus on a specific case of a vector-tensor theory of gravity [30,31], in which vector fields are nonminimally coupled with curvature. The dynamics of parity-odd fluctuations around a class of spherically symmetric, asymptotically flat geometries appear to be particularly simple. We concentrate on a background solution with a geometry corresponding to a Schwarzschild black hole [32,33]. The evolution equations for vector and metric fluctuations are characterized by hidden conformal symmetries, associated with $SL(2,R)$ algebras, and their extensions to centerless Virasoro symmetries. This is somehow unexpected, since our geometrical configurations do not enjoy AdS asymptotics, nor correspond to extremal black hole geometries. Interestingly, our symmetries apply on the *entire* exterior geometry of the configuration as opposed to only being realized in the near horizon limit.

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The evolution equations for the vector-field and metric perturbations can be factorized into a product of two commuting first order operators. This property allows us to analytically determine the most general time-dependent solutions for the system of equations. In fact, we associate this factorizability property with emergent conformal symmetries behind the structure of the equations controlling the field fluctuations, and directly link the two structures.

We proceed to study the highest weight representation for one of the $SL(2,R)$ algebras, and analytically determine the expression for the highest weight time-dependent solutions, and their descendants. As pointed out more generally in [11], the highest weight representations are known to have common properties with quasinormal modes of black holes in general relativity. Indeed, we find that elements belonging to the highest weight multiplets have frequencies separated by integer numbers, resembling the behavior of overtones of black hole quasinormal modes. We note, however, that although our solutions have the time-dependent profile of ingoing modes into the black hole horizon, they do not describe outgoing modes asymptotically far from it, as quasinormal modes do. Instead, they decay exponentially with the radial distance from the black hole horizon, behaving as quasibound states.

The details of the system and the dynamics of fluctuations around a spherically symmetric Schwarzschild solution are discussed in Secs. II and III, where we analyze in detail the emergent conformal symmetries. Their physical implications are discussed in Sec. IV. Our conclusions and further considerations may be found in Sec. V.

II. SYSTEM UNDER CONSIDERATION

We consider Einstein-Maxwell gravity, including a nonminimal coupling between a vector field A_μ and the Einstein tensor $G_{\mu\nu}$. The Lagrangian density is [30,31]

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}G_{\mu\nu}A^\mu A^\nu. \tag{2.1}$$

This system leads to second order equations of motion, hence it is free of Ostrogradsky instabilities. The choice of the nonminimal coupling between the vector and the curvature—the factor $1/4$ in front of the $G_{\mu\nu}A^\mu A^\nu$ combination—leads to particularly simple black hole and spherically symmetric solutions, see e.g. [32,33] (see also [34–40] for further developments on this topic), and we focus on this specific choice in what follows. Notice that the theory breaks a $U(1)$ Abelian symmetry through the direct coupling of the vector field to gravity. We consider the spherically symmetric Ansatz

$$\begin{aligned} ds^2 &= \bar{g}_{\mu\nu}dx^\mu dx^\nu \\ &= -\bar{A}(r)dt^2 + \frac{dr^2}{\bar{B}(r)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \end{aligned} \tag{2.2}$$

for the metric, and an electric-type ansatz

$$\bar{V}_\mu dx^\mu = \bar{\alpha}_0(r)dt + \bar{\Pi}(r)dr, \tag{2.3}$$

for the vector field. From now on, a bar denotes background quantities that depend on the radial coordinate only. Since the Abelian gauge symmetry is broken, we cannot gauge away the radial component of the vector profile.

The corresponding Einstein and vector field equations admit two branches of solutions (we refer the reader to [32] for details on the system of equations). One branch contains vanishing vector radial component $\bar{\Pi}(r) = 0$, and is continuously connected with the Reissner-Nordström configuration. No known exact solutions exist in this branch. The equations of motion for the other branch—on which we focus our attention—are satisfied by choosing profiles for $\bar{B}(r)$, $\bar{\alpha}_0(r)$, and $\bar{\Pi}(r)$ obeying the following relations:

$$\bar{B}(r) = \frac{\bar{A}(r)}{\bar{A}(r) + r\bar{A}'(r)}, \tag{2.4}$$

$$\left[\frac{d}{dr}\left(\frac{r\bar{\alpha}_0(r)}{2}\right)\right]^2 = \frac{d}{dr}(r\bar{A}(r)), \tag{2.5}$$

$$\bar{\Pi}(r) = \sqrt{\frac{\bar{\alpha}_0^2(r) - 4\bar{A}(r)}{\bar{A}(r)\bar{B}(r)}}. \tag{2.6}$$

Hence the configuration is determined by a choice of the arbitrary function $\bar{A}(r)$, compatible with the boundary conditions one wishes to impose. For example, choosing $\bar{A}(r) = 1–2M/r$, one finds an asymptotically flat, spherically symmetric solution corresponding to a Schwarzschild geometry

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\Omega^2, \tag{2.7}$$

$$\bar{\alpha}_0(r) = 2 + \frac{2Q}{r}, \tag{2.8}$$

$$\bar{\Pi}(r) = \frac{2\sqrt{Q^2 + 2(M+Q)r}}{r - 2M}, \tag{2.9}$$

characterized by two arbitrary constants M, Q. Notice that—differently from the Reissner-Nordström black hole—the background geometry does not depend on Q (whilst, as we will learn, the dynamics of fluctuations depends on this quantity). This geometrical configuration is equipped with a Schwarzschild horizon located at radius $R_S = 2M$.

However a solution with a Schwarzschild horizon is not the unique asymptotically flat configuration solving Eqs. (2.4)–(2.6). There is the possibility to smoothly connect the exterior Schwarzschild geometry (2.7) to a regular interior configuration with no horizon, dubbed "ultracompact vector star" in [33]. We briefly discuss this configuration since, in an appropriate limit, its symmetry properties resemble what we will find next for the dynamics of the fluctuations. The metric and vector-field components for the interior solution—recall our metric ansatz (2.2)—are

$$\bar{A}(r) = \sigma^2 + \frac{2\sigma(1-\sigma)}{1+\gamma}\left(\frac{r}{\bar{R}}\right)^{\gamma} + \frac{(1-\sigma)^2}{1+2\gamma}\left(\frac{r}{\bar{R}}\right)^{2\gamma}, \quad (2.10)$$

$$\bar{B}(r) = \frac{\bar{A}}{[\sigma+(1-\sigma)(r/\bar{R})^{\gamma}]^2}, \quad (2.11)$$

and

$$\bar{\alpha}_0(r) = 2\left(Q + \frac{\gamma(1-\sigma)}{1+\gamma}\right)\frac{\bar{R}}{r} + 2\sigma + \frac{2(1-\sigma)}{1+\gamma}\left(\frac{r}{\bar{R}}\right)^{\gamma}, \quad (2.12)$$

while the (long) expression for $\bar{\Pi}(r)$ can be found by plugging the previous equations into equation (2.6). This solution is characterized by the additional dimensionless constants $0 \le \sigma \le 1$ and $\gamma \ge 0$, as well as a radius $\bar{R}$ corresponding to the boundary of the vector star (the Q is the same as in the exterior solution above). This configuration solves the equations in the Appendix under conditions (2.4)–(2.6), which characterize the branch of solutions we are interested in. It connects smoothly with the exterior configuration, with no need to consider contributions from the extra surface energy momentum tensor, if its parameters satisfy the relation

$$\bar{R} = \frac{(1+\gamma)(1+2\gamma)}{2\gamma(1-\sigma)(1+\gamma+\sigma\gamma)} R_S \ge R_S. \quad (2.13)$$

The solutions described by Eq. (2.7) (in the exterior) and Eq. (2.10) (in the interior) corresponds to an asymptotically flat vector star, with compactness

$$\mathcal{C} \equiv \frac{M}{\bar{R}} = \frac{\gamma(1-\sigma)(1+\gamma+\sigma\gamma)}{(1+\gamma)(1+2\gamma)} \le \frac{1}{2} \quad (2.14)$$

smaller than a Schwarzschild black hole. The black hole compactness limit $\mathcal{C} = 1/2$ is reached for $\sigma \to 0$ and $\gamma \to \infty$. These objects can avoid Buchdahl theorem and be as compact as black holes, thanks to their internal anisotropic stress [33].

When $\sigma \to 0$, the configuration of Eqs. (2.10) and (2.11) becomes singular and develops self-similar properties, with a scaling symmetry resembling a singular isothermal sphere [33]. This feature indicates that scaling symmetries and conformal transformations can play an important role in the characterization of this system. In fact, a rich pattern of conformal symmetries emerge when studying the dynamics of parity-odd fluctuations, as we are going to discuss in the following.

III. PARITY-ODD FLUCTUATIONS AND CONFORMAL SYMMETRIES

We now consider the dynamics of parity odd fluctuations around a spherically symmetric background configuration satisfying the system of Eqs. (2.4)–(2.6). For definiteness, we focus on the exterior Schwarzschild geometry described in Eqs. (2.7)–(2.9). We demonstrate that the evolution equations for the fluctuations enjoy a large set of symmetries [including conformal $SL(2,R)$ symmetries], which allows us to analytically characterize their time-dependent solutions, and their corresponding properties.

Fluctuations around our background configuration are parametrized in terms of small quantities $h_{\mu\nu}$ and a_μ, as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (3.1)$$

$$A_\mu = \bar{A}_\mu + a_\mu. \quad (3.2)$$

In Regge-Wheeler gauge, the metric fluctuations are controlled by two nonvanishing components h_0 and h_1, which depend on time and on the radial direction:

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & -h_0(t,r)(\partial_\varphi/\sin\theta) & h_0(t,r)\sin\theta\partial_\theta \\ 0 & 0 & -h_1(t,r)(\partial_\varphi/\sin\theta) & h_1(t,r)\sin\theta\partial_\theta \\ -h_0(t,r)(\partial_\varphi/\sin\theta) & -h_1(t,r)(\partial_\varphi/\sin\theta) & 0 & 0 \\ h_0(t,r)\sin\theta\partial_\theta & h_1(t,r)\sin\theta\partial_\theta & 0 & 0 \end{pmatrix} Y(\theta,\varphi), \quad (3.3)$$

with $Y(\theta,\varphi)$ denoting spherical harmonics. There is a single parity-odd component for the perturbations a_μ of the vector field, which we denote with $\beta(t,r)$:

$$a_\mu = \left\{0, 0, \frac{r\bar{\alpha}_0^2(r)}{4}\beta(t,r)\sin\theta^{-1}\partial_\varphi, -\frac{r\bar{\alpha}_0^2(r)}{4}\beta(t,r)\sin\theta\partial_\theta\right\} Y(\theta,\varphi). \quad (3.4)$$

For convenience, we multiply the parity-odd vector fluctuation $\beta(t,r)$ by the background quantity $r\bar{\alpha}_0^2(r)/4$ in order to simplify the corresponding evolution equations. It is straightforward to determine the quadratic action governing the dynamics of fluctuations $h_0(t,r)$, $h_1(t,r)$, and $\beta(t,r)$ around a configuration satisfying Eqs. (2.4)–(2.6). This task has been carried out[1] in [41]. We write the corresponding equations of motion in the Appendix. Upon using conditions (2.4)–(2.6), formulas simplify dramatically. We can algebraically solve for $h_1(t,r)$, and express this quantity as a linear combination of $h_0(t,r)$ and $\beta(t,r)$ in the exterior geometry of Eqs. (2.7)–(2.9):

$$h_1 = \frac{r}{r+Q}\frac{\sqrt{Q^2+2(M+Q)r}}{r-2M}h_0 - \frac{\sqrt{Q^2+2(M+Q)r}}{2}\beta - \frac{r^2}{2(\ell+2)(\ell-1)}\partial_t\beta + \frac{r^2(r+Q)}{2(\ell+2)(\ell-1)}\partial_t\partial_r\beta - \frac{r^3\sqrt{Q^2+2(M+Q)r}}{2(r-2M)(\ell+2)(\ell-1)}\partial_t^2\beta. \tag{3.5}$$

The quantity ℓ is the multipole number, an integer $\ell \geq 2$. Hence, only two degrees of freedom are dynamical in this system: we take them to be $\beta(t,r)$ and $h_0(t,r)$. Upon substituting condition (3.5) into the remaining equations, the dynamics of the vector fluctuations $\beta(t,r)$ are decoupled from the metric perturbation $h_0(t,r)$. We study the interesting symmetry properties of the decoupled vector equation in Sec. III A, and then analyze the metric fluctuation dynamics in Sec. III B. The physical implications of our findings are studied in Sec. IV.

A. Vector-field perturbations and associated symmetries

After inserting Eq. (3.5) into the remaining evolution equations, the vector fluctuation $\beta(t,r)$ decouples from the metric fluctuation $h_0(t,r)$. The evolution equation reads

$$\mathcal{E}_\ell[\beta] = 0. \tag{3.6}$$

$\mathcal{E}_\ell[\ldots]$ is a linear operator defined in the exterior black hole region, $r \geq 2M$:

$$\mathcal{E}_\ell[\beta(t,r)] = 2\sqrt{\bar{\Delta}(r)}\partial_r\beta + \bar{\Delta}(r)\partial_r^2\beta(t,r) - 2\bar{\Sigma}(r)\partial_t\partial_r\beta + \frac{\bar{\Sigma}^2(r)\partial_t^2\beta}{\bar{\Delta}(r)} - \bar{\Sigma}'(r)\partial_t\beta - \ell(\ell+1)\beta, \tag{3.7}$$

where the integer $\ell \geq 2$ denotes the multipole number. The definition of the radial-dependent functions $\bar{\Delta}$ and $\bar{\Sigma}$ are given in terms of the background quantities

$$\bar{\Delta}(r) = \frac{r^2\bar{\alpha}_0^2}{4(\bar{A}+r\bar{A}')}, \tag{3.8}$$

$$= (r+Q)^2, \tag{3.9}$$

and

$$\bar{\Sigma}(r) = \frac{r^2\bar{\alpha}_0}{4\bar{A}}\sqrt{\frac{\bar{\alpha}_0^2-4\bar{A}}{\bar{A}+r\bar{A}'}}, \tag{3.10}$$

$$= \frac{r(r+Q)}{r-2M}\sqrt{Q^2+r(2M+2Q)}. \tag{3.11}$$

In passing from the first to the second line in each of the previous equations we are specializing to the Schwarzschild-type "exterior" solution of Eqs. (2.7)–(2.9).

The vector Eq. (3.7) has intriguing properties. Although it involves second derivatives along the temporal and radial coordinates, it can be factorized as a product of two commuting first-order operators D^+ and D^-, as

$$\mathcal{E}_\ell[\beta(t,r)] = (D^-D^+)[\beta(t,r)] = (D^+D^-)[\beta(t,r)] \tag{3.12}$$

with

$$D^\pm[\beta(t,r)] = \frac{\bar{\Sigma}}{\sqrt{\bar{\Delta}}}\partial_t\beta - \sqrt{\bar{\Delta}}\partial_r\beta - \sigma_\pm\beta. \tag{3.13}$$

The constants $\sigma_\pm$ in Eq. (3.13) depend on the multipole number as

$$\sigma_+ = 1+\ell, \qquad \sigma_- = -\ell. \tag{3.14}$$

Crucially, the factorizability feature (3.12) is valid in the entire exterior region of the black hole, and not only in the proximity of the horizon. Such a property can be expected for the evolution of fluctuations in the context of extremal black holes [for example, see Ref. [42], their Eq. (2.30), setting the extremal condition $r_S = 2r_Q$]. However it is quite unexpected in our case, where we deal with a Schwarzschild background. The property (3.12) of the evolution Eq. (3.6) implies that, for any given multipole ℓ, the general solution $\beta(t,r)$ of the second order equation is a linear combination of the solutions of the two first order equations

[1]The work [41] shows that the dynamics of parity odd fluctuations around spherically symmetric solutions for the theory (2.1) are generically plagued by instabilities in the near-horizon region. However, the specific background configuration (2.7)–(2.9) we are considering avoids their arguments. In fact—in the notation of [41]—it leads a vanishing quantity C_1 and C_6 as defined in Appendix C of [41]. But C_1 is assumed to be nonvanishing in [41], since it appears in the denominator of many equations, upon solving constraint equations. Hence, the instability arguments as developed in [41] do not apply in the present instance.

$$D^+[\beta(t,r)] = 0, \tag{3.15}$$

$$D^-[\beta(t,r)] = 0. \tag{3.16}$$

The exact time-dependent solutions of Eqs. (3.15) and (3.16) are not difficult to determine. The complete solution then reads

$$\beta(t,r) = c_+ \frac{F_+(t+r_\star)}{(r+Q)^{\sigma_+}} + c_- \frac{F_-(t+r_\star)}{(r+Q)^{\sigma_-}}, \tag{3.17}$$

where $c_\pm$ are two arbitrary constants, $\sigma_\pm$ appear as defined in Eq. (3.14), and $F_\pm$ are two arbitrary functions of the combination

$$t + r_\star \equiv t + \int^r \frac{\bar{\Sigma}(\tilde{r})}{\bar{\Delta}(\tilde{r})} d\tilde{r}. \tag{3.18}$$

The "tortoise" radial coordinate $r_\star$ is given by the radial integral in Eq. (3.18): we will study its properties in Sec. IV. Further specifications of physically interesting solutions depend both on the boundary conditions we are interested in, and on the structure of the configurations we wish to study.

When focusing on static fluctuations, first-order operators similar to our definition in Eq. (3.13) have been introduced in [42,43] as "ladder" operators in the context of Schwarzschild solutions of general relativity (see also [44,45] for related proposals). The ladder operators studied in [43] do not satisfy a commutation relation like our (3.12). Nevertheless they allow one to generate solutions of different multipole numbers starting from a given solution at multipole level ℓ. Such features have proven to be helpful for understanding the vanishing of Love numbers of black hole solutions.

We do not pursue this line of investigation. Instead, we analyze the full time-dependent solutions for the equations of motion, as given in Eq. (3.17). We ask whether the particularly simple structure of our solutions (3.17) can be associated with some underlying symmetry, possibly also explaining the factorizability property (3.12) of the equations of motion. We answer affirmatively, and we determine underlying conformal symmetries behind the vector evolution equation. Interestingly, these conformal symmetries are not just a near-horizon property of the configuration: they extend to the entire exterior black hole geometry.

1. A first set of symmetries

We introduce the first-order operators L_p involving solely first derivatives

$$L_p = -\frac{e^{\frac{p}{4M}(t+r_\star)}}{\sqrt{\bar{\Delta}}} [(4M\sqrt{\bar{\Delta}} + p\bar{\Sigma})\partial_t - p\bar{\Delta}\partial_r], \tag{3.19}$$

where p is an arbitrary integer and the tortoise radial variable $r_\star$ is defined in Eq. (3.18). The quantities $\bar{\Delta}$, $\bar{\Sigma}$ are given in Eqs. (3.9) and (3.10). The operators L_p satisfy a centerless Virasoro algebra

$$[L_p, L_s] = (p-s)L_{p+s}, \tag{3.20}$$

for every integer p, s. Moreover, for each p, the L_p operators commute with the operator $\mathcal{E}_\ell$ of (3.6) controlling the vector equations of motion:

$$[\mathcal{E}_\ell, L_p] = 0, \tag{3.21}$$

indicating that they can be used for generating solutions from existing ones. Hence they represent a symmetry for the system. The choice of operators (3.19) is inspired by the structure of operators studied in [10], adapted to the particular form of equations of motion in our context. In fact, while a similar algebraic structure has been found in the near-horizon region of Schwarzschild black hole in GR [10], in our case the commutation relations are valid everywhere outside the horizon.

In fact, we can consider an $SL(2,R)$ conformal subalgebra constituted by the three operators L_{+1}, L_0, L_{-1}, by focusing on $|p| \le 1$. The operator $\mathcal{E}_\ell$ is associated with the quadratic Casimir of the $SL(2,R)$ algebra:

$$\mathcal{E}_\ell[\beta(t,r)] + \ell(\ell+1)\beta(t,r) = \left(L_0^2 - \frac{1}{2}\{L_1, L_{-1}\}\right)[\beta(t,r)]. \tag{3.22}$$

Subsequently, solutions of the equations of motion can be related, starting from a highest weight representation of the $SL(2,R)$ algebra [10,11,18]. Such a representation is defined through the conditions

$$L_1[\beta^{(0)}(t,r)] = 0, \tag{3.23}$$

$$L_0[\beta^{(0)}(t,r)] = \sigma_\pm \beta^{(0)}(t,r). \tag{3.24}$$

Both the choices $\sigma_\pm$ are allowed in principle, depending on the physical structures one wishes to analyze, and also depending on whether one studies (as we do here) a highest weight or a lowest weight representation [18]. It is not difficult to convince ourselves, using the commutation relations (3.20), that a function $\beta^{(0)}(t,r)$ satisfying the previous conditions also satisfies the equations of motion $\mathcal{E}_\ell[\beta^{(0)}(t,r)] = 0$. Moreover, starting from $\beta^{(0)}(t,r)$, we can build its descendants as

$$\beta^{(n)}(t,r) \equiv (L_{-1})^n[\beta^{(0)}(t,r)], \tag{3.25}$$

which also satisfy the equations of motion, and belong to the conformal multiplet associated with the highest weight solution $\beta^{(0)}$.

The general solution of conditions (3.23) and (3.24) contains a plane-wave structure (but with a purely imaginary frequency). It results

$$\beta^{(0)}(t,r) = c_+ \frac{e^{-\frac{\sigma_+(t+r_\star)}{4M}}}{(r+Q)^{\sigma_+}} + c_- \frac{e^{-\frac{\sigma_-(t+r_\star)}{4M}}}{(r+Q)^{\sigma_-}}, \qquad (3.26)$$

for two arbitrary constants $c_\pm$. The solution (3.26) has the correct structure we determined in (3.17) for solving the vector equations of motion. As anticipated above, the exponential structure (3.26) indicates that the highest weight solution (3.26) behaves as a plane wave with a purely imaginary frequency ω such that $4iM\omega = \sigma_\pm$ for each of the two contributions in Eq. (3.26) [recall that the $\sigma_\pm$ of Eq. (3.14) are integers]. We will return in Sec. IV to study its properties and the properties of its descendants.

2. A second set of symmetries

Interestingly, besides the conformal symmetries associated with operators L_p of Eq. (3.19), the vector fluctuations also enjoy *extra* symmetries, that shed light on the factorizability property of the vector Eq. (3.12).

We introduce a new set of operators

$$P_n^{(\sigma)} = -e^{\frac{n}{4M}(t+r_\star)}[4M\partial_t + n\sigma], \qquad (3.27)$$

for an integer n. The quantity σ in the previous definition is in principle arbitrary. These operators satisfy a Witt-like algebra

$$[P_m^{(\sigma_1)}, P_n^{(\sigma_2)}] = (m-n)P_{m+n}^{(\sigma_3)} + m^2(\sigma_1-\sigma_2)\delta_{m+n,0}, \qquad (3.28)$$

with a contribution resembling a central extension. The quantity σ_3 is related to $\sigma_{1,2}$ by

$$(m^2-n^2)\sigma_3 = (m^2\sigma_1 - n^2\sigma_2)(1-\delta_{m^2,n^2}). \qquad (3.29)$$

When $\sigma_1 = \sigma_2$, one finds the usual Witt algebra. The operators $P_m^{(\sigma)}$ commute with the equations of motion

$$[\mathcal{E}_\ell, P_n^{(\sigma)}] = 0, \qquad (3.30)$$

indicating they are also symmetries of the system of equations. As for the commutation relations we met for the L's operators, also the commutation relations for the P operators of Eq. (3.30) are valid everywhere in the exterior black hole geometry. As far as we are aware, we are the first to study the properties of these operators in a black hole space-time in modified gravity.

The associated quadratic Casimir gives

$$\ell(\ell+1)\beta(t,r) = P_0^2[\beta(t,r)] - \frac{1}{2}\{P_1, P_{-1}\}[\beta(t,r)]. \qquad (3.31)$$

The quantities $P_n^{(\sigma)}$ have also elegant commutation relations with the operators L_m defined in (3.19): we find

$$(m+n)[L_m, P_n^{(\sigma)}] = m^2 L_{m+n} - n^2 P_{m+n}^{(\sigma)}. \qquad (3.32)$$

For $m+n=0$, the previous relation gives a trivial identity. To deal with this case, we introduce another operator ($m \neq 0$)

$$\mathcal{D}^{(\sigma)} \equiv \frac{2}{m}L_0 - \frac{1}{m^2}[L_m, P_{-m}^{(\sigma)}], \qquad (3.33)$$

$$= \frac{\bar{\Sigma}}{\sqrt{\bar{\Delta}}}\partial_t\beta - \sqrt{\bar{\Delta}}\partial_r\beta - \sigma\beta, \qquad (3.34)$$

which "closes the algebra" and commutes with all the remaining operators:

$$0 = [\mathcal{D}^{(\sigma_1)}, \mathcal{E}_\ell] = [\mathcal{D}^{(\sigma_1)}, P_n^\sigma] = [\mathcal{D}^{(\sigma_1)}, L_n] = [\mathcal{D}^{(\sigma_1)}, \mathcal{D}^{(\sigma_2)}]. \qquad (3.35)$$

Interestingly, comparing Eqs. (3.13) and (3.34), we realize that for $\sigma = \sigma_\pm$ the operators $\mathcal{D}^{(\sigma_\pm)}$ coincide with the operators $\mathcal{D}^\pm$ of Eq. (3.13). We then find an algebraic origin of the factorizable operators $\mathcal{D}^\pm$ as related with a combination of the two $SL(2,R)$ (or more generally, centerless Virasoro) algebras involving the operators L_m and P_n. It would be interesting to find other examples of similar rich algebraic structures for other black holes in general relativity and beyond. We now demonstrate very similar algebraic structures for metric fluctuations, before turning to some physical implications of our findings in Sec. IV.

B. Metric perturbations

Interestingly, under particular hypotheses, symmetries identical to the ones we determined in the vector sector of Sec. III A also apply to the metric sector. Whilst, as we learned, the evolution equation for vector perturbations decouple from the remaining propagating degrees of freedom, metric fluctuations are sourced by vector fluctuations. To study the system more simply, we choose appropriate boundary conditions to set the independent vector fluctuations to zero, $\beta(t,r) = 0$. We can then study the dynamics of the metric perturbations only. We learned in Eq. (3.5) that the quantity $h_1(t,r)$ is algebraically related with $h_0(t,r)$. Hence we focus on the dynamics of h_0: it is convenient to rescale it as

$$h_0(t,r) = r^2\gamma(t,r). \qquad (3.36)$$

The metric perturbation γ satisfies the evolution equation

$$\mathcal{G}_\ell[\gamma(t,r)] = 0, \qquad (3.37)$$

in the exterior geometry, with

$$\mathcal{G}_\ell[\gamma(t,r)] = \frac{\bar{\Sigma}(r)^2\partial_t^2\gamma}{\bar{\Delta}(r)} - \left(\frac{2\bar{\Sigma}(r)}{\sqrt{\bar{\Delta}(r)}} + \bar{\Sigma}'(r)\right)\partial_t\gamma - 2\bar{\Sigma}(r)\partial_t\partial_r\gamma + 4\sqrt{\bar{\Delta}(r)}\partial_r\gamma + \bar{\Delta}(r)\partial_r^2\gamma - (\ell-1)(\ell+2)\gamma. \tag{3.38}$$

The functions $\bar{\Delta}$ and $\bar{\Sigma}$ are defined in the exterior region of the black hole in Eqs. (3.9) and (3.11). Similarly to the vector case, the metric evolution Eq. (3.38) may also be factorized into a product of two first-order operators, with an identical structure.

$$\mathcal{G}_\ell[\gamma(t,r)] = (D^-D^+)[\gamma(t,r)] = (D^+D^-)[\gamma(t,r)], \tag{3.39}$$

with

$$D^\pm[\gamma(t,r)] = \frac{\bar{\Sigma}}{\sqrt{\bar{\Delta}}}\partial_t\gamma - \sqrt{\bar{\Delta}}\partial_r\gamma - \rho_\pm\gamma. \tag{3.40}$$

However this time the constants $\rho_\pm$ read

$$\rho_+ = 2+\ell, \qquad \rho_- = 1-\ell. \tag{3.41}$$

Hence the general solution to Eq. (3.37) can be expressed as a linear combination of solutions to equations $D^-[\gamma(t,r)] = 0$ and $D^+[\gamma(t,r)] = 0$. We find the general solution to be

$$\gamma(t,r) = d_+\frac{S_+(t+r_\star)}{(r+Q)^{\rho_+}} + d_-\frac{S_-(t+r_\star)}{(r+Q)^{\rho_-}}, \tag{3.42}$$

with $d_\pm$ arbitrary constants, and $S_\pm$ arbitrary functions of their argument.

Supplementing this, we find an underlying algebraic structure which lies behind the factorizability property of Eq. (3.39). It is actually *the very same structure* that applied to the vector case. In fact, the very same operators L_n, P_m^ρ commute with the operator $\mathcal{G}_\ell$ introduced in Eq. (3.38), which controls the evolution equation for metric perturbations. Consequently, the commutation relations remain the same as the ones already studied in Sec. III A. Combining these operators as in Eq. (3.34), we can form the combinations ($m \neq 0$)

$$\mathcal{D}^{(\rho_\pm)} \equiv \left(\frac{2}{m}L_0 - \frac{1}{m^2}[L_m, P_{-m}^{(\rho_\pm)}]\right)[\gamma(t,r)], \tag{3.43}$$

$$= \frac{\bar{\Sigma}}{\sqrt{\bar{\Delta}}}\partial_t\beta - \sqrt{\bar{\Delta}}\partial_r\beta - \rho_\pm\beta, \tag{3.44}$$

which precisely coincide with the factorizing operators (3.40). Equivalent to the vector fluctuations, in the case of the metric perturbations the structure of the evolution equations can be related with $SL(2,R)$ symmetries of the system.

IV. SOME CONSEQUENCES OF THE CONFORMAL SYMMETRIES

The rich structure of symmetries associated with the evolution equations for vector and metric fluctuations offers a deeper understanding of the factorizability properties of the system, Eqs. (3.12) and (3.39), which lead to exact time-dependent solutions for the system. Solutions can be organized in $SL(2,R)$ multiplets, and have properties in common with quasinormal modes of black hole perturbations in general relativity. There are, however, some important differences. In this section, we seek to investigate these differences further.

We begin by focusing on the vector perturbations of Sec. III A, since the metric perturbations studied in Sec. III B behave very similarly. We consider solutions belonging to the highest weight representation of the $SL(2,R)$ algebra associated with the operators L_p (with $p = 0, \pm 1$). We start with the time-dependent solution corresponding to the highest weight (primary) vector. It is written in Eq. (3.26). We reiterate it here substituting the values given in Eq. (3.14) for the quantities $\sigma_\pm$:

$$\beta^{(0)}(t,r) = c_+\frac{e^{-\frac{(1+\ell)(t+r_\star)}{4M}}}{(r+Q)^{(1+\ell)}} + c_- e^{\frac{\ell(t+r_\star)}{4M}}(r+Q)^\ell. \tag{4.1}$$

The solution depends on two free parameters, c_+ and c_-; recall that the multipole numbers satisfy $\ell \geq 2$. But the contribution to Eq. (4.1) proportional to c_- exponentially grows in time. Due to this, it is not a physically interesting solution for describing a small perturbation: hence we set $c_- = 0$. We are left with a perturbation exponentially decaying in time, as expected for a black hole quasinormal mode (see e.g. [46] for a comprehensive review). However, the radial dependence of the solution is somehow peculiar. Recall the definition of tortoise coordinate $r_\star$ of Eq. (3.18):

$$r_\star = \int^r \frac{\bar{\Sigma}(\tilde{r})}{\bar{\Delta}(\tilde{r})}d\tilde{r} = \int^r \frac{\tilde{r}}{\tilde{r}+Q}\frac{\sqrt{Q^2+2(M+Q)\tilde{r}}}{\tilde{r}-2M}d\tilde{r}. \tag{4.2}$$

In order for the integrand to be well defined for all $r > 2M$, we impose the condition $(M+Q) \geq 0$.

For the case $Q = -M$, the integral is particularly simple and gives

$$r_\star = 2M\ln\left(\frac{r-2M}{\sqrt{r-M}}\right). \tag{4.3}$$

Hence we learn that $r_\star \sim 2M\ln(r-2M) \to -\infty$ nearby the black hole horizon $r \to 2M$, while $r_\star \sim 2M\ln(r) \to +\infty$ at large distances $r \to +\infty$ from the black hole. This is the expected behavior for the tortoise coordinate. When expressed in terms of the original (t,r) coordinates, the resulting exact solution for the highest weight vector is

$$\beta^{(0)}(t,r)=\frac{c_+ e^{-\frac{(1+\ell)t}{4M}}}{(r-2M)^{\frac{1+\ell}{2}}(r-M)^{\frac{3+3\ell}{4}}},\quad Q=-M. \tag{4.4}$$

For $Q+M>0$ the integral can still be solved analytically, but the resulting formula is more complicated. It scales as $r_\star\sim\ln(r-2M)$ near the black hole horizon, and as $r_\star\sim r^{1/2}$ at plus infinity; again a behavior compatible with what we should expect from a tortoise coordinate. When expressed in terms of the original (t,r) coordinates, the resulting exact solution for the highest weight vector is

$$\beta^{(0)}(t,r)=\frac{c_+ e^{-\frac{(1+\ell)t}{4M}}}{(r+Q)^{1+\ell}}\left[\frac{2M+Q+\sqrt{Q^2+2(M+Q)r}}{2M+Q-\sqrt{Q^2+2(M+Q)r}}\right]^{\frac{1+\ell}{2}} \times e^{-\frac{1+\ell}{2M}\sqrt{Q^2+2(M+Q)r}}\, e^{\frac{(1+\ell)Q^2}{2M\sqrt{Q(2M+Q)}}\operatorname{arccot}\left(\sqrt{\frac{2MQ+Q^2}{Q^2+2(M+Q)r}}\right)}. \tag{4.5}$$

For both cases, when $Q+M\geq 0$ the resulting solution for the primary vector has a structure

$$\beta^{(0)}(t,r)\propto\frac{e^{-\frac{(1+\ell)(t+r_\star)}{4M}}}{(r+Q)^{(1+\ell)}}. \tag{4.6}$$

It has the correct ingoing behavior $e^{-i\omega(t+r_\star)}$ at the black hole horizon, $r_\star\to-\infty$; however, it does *not* exhibit the correct outgoing behavior $e^{-i\omega(t-r_\star)}$ at $r_\star\to+\infty$, as expected for a quasinormal mode. The solution is better classified as a quasibound state, since there is energy dissipation inside the black hole horizon; yet these modes decay exponentially at radial infinity. This behavior is in common with perturbations of massive fields [47–49]. In our case, the nonminimal coupling of the vector with gravity induces contributions to the energy momentum tensor of the vector fluctuations, that appear to mimic the effects of the vector mass, and cause the aforementioned decay.

Let us now examine the behavior of the frequencies of the solutions involved. The structure of the solution (4.6) indicates that the highest weight primary vector has a purely imaginary frequency:

$$\omega_0=-i\frac{(1+\ell)}{4M}, \tag{4.7}$$

with no real part. Starting from the primary vector solution of Eq. (4.6), we can generate its descendants $\beta^{(n)}$ by repeated application of the operators L_{-1}, as explained around equation (3.25). The corresponding frequencies result

$$M\omega_n=M\omega_0-\frac{\mathrm{i}n}{4},\quad n=0,1,2,3\ldots \tag{4.8}$$

Hence the elements of the highest weight representation have frequencies shifted by an integer n with respect to the frequency of the primary vector.

For large values of n, this formula greatly resembles the behavior of frequency overtones of Schwarzschild black hole perturbations in general relativity, which follows the law [50,51]

$$\text{GR, large } n:\ M\omega_n=\frac{\ln 3}{8\pi}-\frac{i}{4}\left(n+\frac{1}{2}\right)+\mathcal{O}[(n+1)^{-1/2}]. \tag{4.9}$$

The relations between overtones and elements of a highest representation were already noticed in the context of general relativity [11], exploiting near-horizon conformal symmetries in proximity of the horizon. In our modified gravity setup, the result extends in the entire exterior geometry, since our conformal symmetries are defined in all exterior space. Comparing Eqs. (4.8) and (4.9), we notice that in our modified gravity framework the frequencies are purely imaginary and lack a real part. Moreover, as explained above, we can not talk of quasinormal modes in our system, but instead of quasibound states.

We have largely omitted discussion of the static solution, however we make a brief note here as a basis for further work. If one takes the zero-frequency limit of the time-dependent solutions considered in Sec. III, then one may directly solve the equations of motion in the static limit. With this procedure, the vector equation appears to become exactly that of the equation controlling the dynamics of perturbations around an extremal Reissner-Nordström configuration derived in [42]

$$\bar\Delta(r)\partial_r^2\beta(r)+2\sqrt{\bar\Delta(r)}\partial_r\beta(r)-\ell(\ell+1)\beta(r)=0, \tag{4.10}$$

where solutions may be obtained by repeated applications of the L_{-1} operator to the highest weight vector (or, conversely, applying the L_1 operator to the lowest weight vector) [18]. These solutions read

$$\beta(r)=\frac{d_{(+)}}{(r+Q)^{(1+\ell)}}+d_{(-)}(r+Q)^{\ell}, \tag{4.11}$$

for two constants $d_{(\pm)}$. When $(Q+2M)>0$, both the contributions proportional to $d_{(+)}$ and $d_{(-)}$ are regular at the black hole horizon: subsequently, the associated Love numbers do not necessarily vanish. This highlights an immediate difference when compared to general relativity, where typically only one among the two solutions of the second-order static equation is physically acceptable. If instead $(Q+2M)\leq 0$, then the solution proportional to

$d_{(+)}$ is not regular at the horizon, and should be discarded. Some care must be taken when considering the metric perturbations, as h_1 is algebraically connected to h_0 via Eq. (3.5), it would appear as if these perturbations immediately diverge on approach to the horizon. This divergence, however, may be removed (as usual) by moving into the tortoise coordinate, r_*, suggesting to us that this is purely a coordinate singularity.

The exterior background configuration that formed the primary focus of this work can be smoothly joined into a regular interior, and consequently static solutions of vector and metric fluctuations may also be smoothly connected with fluctuations in the interior [33] by the imposition of particular constraints on the relevant parameters. Due to our conformal symmetries appearing to be a global phenomena, it would be interesting to develop our conformal symmetry structures into the interior solutions.

V. SUMMARY AND OUTLOOK

We studied the evolution of parity-odd, time-dependent field fluctuations around an asymptotically flat solution of a vector-tensor theory of gravity. The geometry corresponds to a Schwarzschild black hole. However the dynamics of parity-odd fluctuations—both in the vector and in the metric sectors—resemble the behavior of perturbations around extremal black holes in general relativity. Their evolution equations, controlled by operators containing second order derivatives along time and space, can be factorized into a product of two commuting first order operators. This property allows us to analytically determine the most general time-dependent solutions for the system of equations.

We shown that the aforementioned factorizability property is associated with the existence of a large set of symmetries behind the system of equations. We identified two sets of operators belonging to $SL(2,R)$ algebras, which represent conformal symmetries for the system. Once combined, they produce the aforementioned first-order operators that generate the equations of motion. Interestingly, while similar conformal $SL(2,R)$ symmetries have been found in proximity of the horizon of a Schwarzschild configuration [10], our symmetries apply to the entire exterior geometry of the black hole.

We then studied the highest weight representation for one of the $SL(2,R)$ algebras. We analytically determined the expression for the highest weight time-dependent solutions, and their descendants. The highest weight representations are known to have common properties with quasinormal modes of black holes in general relativity. In fact, we found that elements belonging to the highest weight multiplets have frequencies separated by integer numbers, resembling the behavior of overtones of black hole quasinormal modes. However, although our solutions have the time-dependent profile of modes ingoing into the black hole horizon, they do not describe outgoing modes asymptotically far from it, as quasinormal modes do. Instead, they decay exponentially with the radial distance from the black hole, behaving as quasibound states.

Our results leave many open questions for future work. We do not truly understand the origin of our conformal symmetries, since we do not have AdS asymptotic boundary conditions, nor do we consider configurations corresponding to an extremal black hole. It would be nice to relate our conformal symmetries with some hidden symmetry of our vector-tensor system, or, at least, with classes of its solutions within some specific ansatz. A hint of this possibility was discussed in Sec. II (building on [33]), where we pointed out that our system admits spherically symmetric solutions with scaling symmetries. Such solutions can be used to smooth out the black hole horizon, and describe ultracompact stars with a Schwarzschild exterior geometry. It may also be possible that extra hidden symmetries relate our configurations with an asymptotically AdS space in terms of subtracted geometries—as shown in [10] for Schwarzschild geometries within general relativity—or with some extremal or near-horizon black hole space-times, hence geometrically motivating our conformal symmetries as some form of isometries.

As stated above, among the general time-dependent solutions for vector and metric perturbations, the highest weight solutions are somehow special, and describe quasibound states around the black hole geometry. It would be interesting to relate such states with modes accounting for the black hole entropy, or the entropy of regularized compact stars discussed in Sec. II (see Ref. [33]). Regarding this important point, we have been able to extend $SL(2,R)$ symmetries to centerless Virasoro algebras. It would be interesting to investigate whether our symmetries can be formed in terms of center-full Virasoro symmetries, and find ways to include central charges. In fact, the latter can be associated with the entropy of our geometrical configurations, analogous to the case for the Kerr black hole [12].

In this work, we focused exclusively on time-dependent solutions. However, the system also admits static solutions, which can be determined by directly solving the static limit of the equations of motion, *or* in terms of descendant (or ascendant) elements of highest (lowest) weight representations [18] of the $SL(2,R)$ symmetries. By imposing appropriate boundary conditions, the static solutions can be related with the Love numbers of our configurations, a topic which received much attention in the recent literature in the context of emergent symmetries [18,43]. Some preliminary results on Love numbers for our configurations have been explored in [33], but certainly a more complete analysis is needed.

We hope to be able to answer some of these questions in the near future.

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APPENDIX: EVOLUTION EQUATIONS

We collect in this appendix the three evolution equations for the parity-odd fluctuations h_0, h_1, and β, obtained varying the Lagrangian (2.1) expanded at second order in perturbation around the exterior solution of Eqs. (2.7)–(2.9):

$$\begin{aligned}
0 &= 2r^4(Q+r)^2\partial_t^2\beta(t,r) - r^4\sqrt{Q^2+2(M+Q)r}\partial_t\partial_r h_0(t,r)\\
&\quad + 2r^2(2M-r)(2Q\partial_r h_0(t,r) + (r-Q)\partial_t h_1(t,r)) + 2r^3\sqrt{Q^2+2(M+Q)r}\partial_t h_0(t,r)\\
&\quad + r^3(r-2M)(Q+r)\partial_r^2 h_0(t,r) + r^4\sqrt{Q^2+2(M+Q)r}\partial_t^2 h_1(t,r)\\
&\quad + r^3(2M-r)(Q+r)\partial_t\partial_r h_1(t,r) - 2r^2(r-2M)^2(Q+r)^2\partial_r^2\beta(t,r)\\
&\quad + 4r(2M-r)(Q+r)(3MQ-(M+Q)r+r^2)\partial_r\beta(t,r)\\
&\quad - 2(2M-r)(6MQ^2+(\ell-1)(\ell+2)Q^2r+\ell(\ell+1)(2Qr^2+r^3)-2Mr^2)\beta(t,r)\\
&\quad - (\ell-1)(\ell+2)(2M-r)r\sqrt{Q^2+2Mr+2Qr}h_1(t,r)\\
&\quad - (4Mr(r-3Q) - r((\ell(\ell+1)-8)Q+\ell(\ell+1)r))h_0(t,r),
\end{aligned} \tag{A1}$$

$$\begin{aligned}
0 &= \frac{2(\ell-1)(\ell+2)\sqrt{Q^2+2(M+Q)r}}{(Q+r)^2(r-2M)r}h_1(t,r) - \frac{2(\ell-1)(\ell+2)(Q^2+2(M+Q)r)}{(Q+r)^3(r-2M)^2}h_0(t,r)\\
&\quad + \left(\frac{2}{(Q+r)^2} - \frac{(\ell-1)(\ell+2)}{(r-2M)r}\right)\beta(t,r) - \frac{2}{Q+r}\partial_r\beta(t,r) - \partial_r^2\beta(t,r) + \frac{r\sqrt{Q^2+2(M+Q)r}}{(r-2M)(Q+r)}\partial_t\partial_r\beta(t,r)\\
&\quad - \frac{(2MQ^3+6MQ(M+2Q)r+(14M^2+13MQ-3Q^2)r^2-5(M+Q)r^3)}{(r-2M)^2(Q+r)^2\sqrt{Q^2+2(M+Q)r}}\partial_t\beta(t,r),
\end{aligned} \tag{A2}$$

$$\begin{aligned}
0 &= \frac{r\sqrt{Q^2+2(M+Q)r}}{(r-2M)(Q+r)}\partial_t^2\beta(t,r) + \partial_t\partial_r\beta(t,r) - \frac{1}{Q+r}\partial_t\beta(t,r)\\
&\quad - \frac{(\ell-1)(\ell+2)\sqrt{Q^2+2(M+Q)r}}{r^2(Q+r)}\beta(t,r) - \frac{2(\ell-1)(\ell+2)}{r^2(Q+r)}h_1(t,r)\\
&\quad - \frac{2(\ell-1)(\ell+2)\sqrt{Q^2+2(M+Q)r)}}{r(2M-r)(Q+r)^2}h_0(t,r).
\end{aligned} \tag{A3}$$

As explained in the main text, we can use (A3) to algebraically solve for h_1 as a function of the remaining variables.

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