Time-dependent wormhole solutions in conformal Weyl gravity

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We present the exact time-dependent solutions on inhomogeneous spherically symmetric space-time in the conformal invariant Weyl gravity. For this purpose, the subclass of the Lemaître-Tolman metric which is supported by an anisotropic fluid is used. For the first time, the exact solutions of the dynamical equations are obtained for two special cases. One of the exact solutions is a de Sitter space-time and other solution is a class of time-dependent wormhole geometries which can be supported by exotic matter similar to the general relativistic solutions.

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I. INTRODUCTION

The first solution for wormholes was studied by Flamm in the context of general relativity (GR) [1] in 1930. Later in 1933 Weyl [2] introduced a wormhole as a tunnel-like structure lying in the same universe or linking two remotely separated regions. In mid-1935, Einstein and Rosen constructed an unstable wormhole known as the Einstein-Rosen bridge [3]. Morris and Thorne proposed a solution to Einstein's field equation by imposing inhomogeneous and static spherical symmetry on space-time. Their solutions (called traversable wormholes) were topological objects with a throat connecting two asymptotically flat regions [4]. Traversable wormholes have no horizons or curvature singularities but they made gravitational forces that are assumed to be bearable by travelers. The most important characteristic of these wormholes is the throat, where the exotic matter should exist to prevent them from collapsing. Indeed such space-time requires a stress-energy tensor that violates the null energy condition [5].

From the theoretical point of view in quantum field theory, the possibility of exotic energy has been accepted in the Casimir effect [6]. Interesting evidence of the experimental effect is an attractive force between two parallel metallic plates in a vacuum that is generated by exotic matter [7]. Wormholes with negative energy densities in quantum gravity have been studied using the path integral in [8,9]. Moreover, Hochberg and Kephart have indicated that the wormhole geometry with negative energy might be produced by the gravitational squeezing of the vacuum [10]. Another field in which we deal with exotic matter is cosmology. Based on [11], the exotic matter with the property of $w < -1/3$ and the equation of state $p = w\rho$ is responsible for accelerating the Universe. Phantom energy with $w < -1$ has some other properties such as negative temperature and energy and Big Rip whose energy density evolves with expanding Universe [12]. Sato *et al.* [13] have investigated the possibility of dynamical wormhole formation in the inflationary era. Other aspects of evolving wormholes of the Planck length scale have been considered by Friedman [14] and Roman [15].

Although wormholes are explained by Einstein's gravity, there are still traversable wormhole solutions in alternative theories of gravity. Wormhole solutions have been studied in Brans-Dicke theory [16–19], $f(R)$ [20–27], $f(R,T)$ gravity [28], massive gravity [29], scalar-tensor theory [30,31], third order Lovelock [32] and Kaluza–Klein gravity [33]. Also static wormholes in the presence of a cosmological constant and Born-Infeld theory have been reported in [34] and [35,36], respectively. In Ref. [37] authors have studied the wormholes in $f(R,T)$ modified gravity theory by using an exponential shape function. In Einstein-Gauss-Bonnet (EGB) gravity, vacuum wormhole solutions have been obtained [38–41]. Also the higher-dimensional wormholes have been of interest in the last years [42,43]. Investigation of classical wormholes based on conformal Weyl gravity has been done by Varieschi and Ault [44]. Some works on the subject already exists in the literature (see for example [45,46]). Time-dependent wormhole solutions on inhomogeneous and spherically symmetric space-time in the presence of matter source with radial and transverse stresses have been obtained in [47]. Also by considering an inhomogeneous brane embedded in 5-dimensional constant curvature bulk, time-dependent wormhole solutions as exact solutions on the brane have been found in [48]. For the study of the evolving Lorentzian wormholes and the null energy condition (NEC) and weak energy condition (WEC) see [49–56]. Also evolving wormholes in $f(R)$ gravity theory, Einstein-Cartan gravity, EGB gravity, Lovelock gravity and Rastall theory were investigated in Refs. [57–61], respectively.

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Despite the incredible successes of GR theory, there are basic problems in astronomy and cosmology, such as dark energy and dark matter. An alternative approach to describe the cosmic structure of the Universe without considering dark matter is modifying the theory of gravity. Amongst the many modifications of GR, conformal invariant Weyl gravity is proposed in 1918 by Weyl [62,63] and developed by Bach [64]. Although finding exact solution of this fourth-order conformal Weyl gravity is a formidable endeavor, the exact vacuum solution to conformal Weyl gravity and its implications have been studied by Mannheim and Kazanas (MK metric) [65,66]. This exact static spherically symmetric vacuum solution is given by the following metric:

$$ds^2 = -B(r)^2 dt^2 + \frac{1}{B(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where

$$B(r) = 1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2,$$

and β, γ and k are integration constants. This exterior solution includes three new extra terms to the standard Schwarzschild metric which can explain the observed galactic rotation curves without introducing dark matter [67,68]. Cylindrically symmetric solutions in conformal gravity were presented in Refs. [69–71] which are a generalization of the Melvin solution and cosmic strings of the Abelian Higgs model. Dynamical cylindrical symmetric solutions in conformal Weyl gravity have been investigated in [72]. The purpose of this paper is to find the dynamical spherically symmetric solutions in the framework of the conformal Weyl gravity.

The exact solutions to the Reissner-Nordstrom, Kerr and Kerr-Newmann space-times have been studied in [73]. An interesting application of fourth-order conformal Weyl gravity is an analysis of the traversable wormhole solutions in this theory. The wormhole solutions in the theory of GR are supported by exotic matter which violates main energy conditions [74]; so an interesting challenge in wormhole physics is the demand to find a realistic matter that will support these exotic space-times. The computation of light bending angle by a spherically symmetric object using MK metric has been studied in detail [75–77]. For an asymptotically nonflat geometry such as MK metric by using Rindler-Ishak method, the total light deflection angle to second order has been calculated in [78] and [79,80]. Correct light deflection in Weyl conformal gravity has been obtained in [81]. In [82] authors have investigated the perihelion shift of planetary motion in conformal Weyl gravity. For astrophysical tests in conformal Weyl gravity see [83,84].

The structure of paper is as follows. In Sec. II, after a brief review of Weyl gravity, the Szekeres-Szafron metric is introduced. Then we obtain the field equations for this inhomogeneous space-time in the framework of conformal Weyl gravity in Sec. II C. Finally we solve these equations to find two physical and important solutions in Sec. III. The paper ends with concluding remarks in Sec. IV.

II. FIELD EQUATIONS IN CONFORMAL WEYL GRAVITY

A. Weyl action

Conformal Weyl gravity is based on the following action [85]:

$$\begin{aligned} I_w &= -\alpha \int d^4x\sqrt{-g}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa} \\ &= -2\alpha \int d^4x\sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right), \end{aligned} \tag{1}$$

where $g \equiv \det(g_{\mu\nu})$, α is the coupling constant and

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - g_{\lambda[\nu}R_{\kappa]\mu} + g_{\mu[\nu}R_{\kappa]\lambda} + \frac{1}{2}Rg_{\lambda[\nu}g_{\kappa]\mu} \tag{2}$$

is the Weyl tensor [2].

By varying the action (1) with respect to the $g_{\mu\nu}$, we obtain the following field equations:

$$2\alpha W_{\mu\nu} = \frac{1}{2}T_{\mu\nu}, \tag{3}$$

or

$$2\alpha\left[-\frac{1}{3}W^{(1)}_{\mu\nu} + W^{(2)}_{\mu\nu}\right] = \frac{1}{2}T_{\mu\nu}, \tag{4}$$

where

$$W_{\mu\nu} \equiv W^{(2)}_{\mu\nu} - \frac{1}{3}W^{(1)}_{\mu\nu}, \tag{5}$$

here $W^{(1)}_{\mu\nu}$ and $W^{(2)}_{\mu\nu}$ are defined as

$$W^{(1)}_{\mu\nu} = 2g_{\mu\nu}R^{;\beta}{}_{;\beta} - 2R_{;\mu\nu} - 2RR_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R^2, \tag{6}$$

and

$$\begin{aligned} W^{(2)}_{\mu\nu} = {}& \frac{1}{2}g_{\mu\nu}R^{;\alpha}{}_{;\alpha} + R_{\mu\nu}{}^{;\beta}{}_{;\beta} - R^{\beta}_{\mu;\nu\beta} - R^{\beta}_{\nu;\mu\beta} \\ & - 2R_{\mu\beta}R^{\beta}{}_{\nu} + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}, \end{aligned} \tag{7}$$

respectively.

In Eq. (3) $W_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric, traceless and covariantly conserved. We will use these properties in the next sections. Also the energy-momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta(g^{\mu\nu})}, \tag{8}$$

where L_m is the matter Lagrangian density.

B. Space-time geometry

The cosmological principle states at any cosmic time the Universe appears homogeneous and isotropic at very large scales i.e., invariant under translation and rotation around each comoving observer. It gives precise meaning that there are no special points and directions in the Universe. The kind of geometry consistent with cosmological principle is the Friedmann-Lemaître-Robertson-Walker (FLRW) line element [86]. While isotropy has been confirmed by cosmic microwave background radiation; realistic observations from the structure formation such as filaments, sheets, dark haloes and so on, indicate that homogeneity has been challenged in the different epoch of the Universe. Although, it is accepted that the Universe is currently passing through an accelerated phase of expansion, on a small scale, we are faced with self-gravitating solutions (i.e., spherically symmetric solutions based on GR) regardless of the expansion. The gravitational bending of light in the Schwarzschild space-time and the direct observation of black hole inside of some galaxies such as black hole at the center of M87 reported by the event horizon telescope [87,88], and the observation of gravitational waves [89] indicate the validity of GR on small scales.

By contrast, Swiss cheese models considered an exterior expanding pressureless FLRW universe with nonzero cosmological constant Λ, while there is a point mass of the center [90–92]. A more realistic description is provided by Lemaître and Tolman that is the first model which allows the study of inhomogeneous cosmology. They proposed the model whose space-time is filled by perfect fluid with dust equation of state [93,94]. The crucial point in these two models is that they constructed the geometries that is a spatially homogeneous spherically symmetric background and replaced it with an inhomogeneous distribution in small scales. In other words, inhomogeneous cosmological models are those which do not satisfy the cosmological principle, but they provide the limit FLRW space-time. Since then, there have been many papers that investigate on inhomogeneous models. Hellaby and Lake studied on geometrical aspects of inhomogeneous cosmology [95,96]. Shear and rotation-free inhomogeneous model have been studied by several authors e.g., Stravrinos *et al.* [97] and Clarkson [98]. Inhomogeneous model build up by rotating fluid may be found in Patel and Pandya [99]. The Szekeres-Szafron model [100–105] is among such models that allow us to study the spherically symmetric and inhomogeneous space-times which merge smoothly to the cosmological background.

In comoving coordinate, the form of Szekeres-Szafron metric is

$$ds^2 = -dt^2 + R(t)^2[(1+a(r))dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2], \tag{9}$$

here $R(t)$ is the cosmic scale factor and $a(r)$ is an unknown function of the radial coordinates of r. Note that we use our metric signature $(-,+,+,+)$. We have the Robertson-Walker metric as a special case

$$1 + a(r) = \frac{1}{1-kr^2}, \tag{10}$$

where k is the spatial curvature index which take the values: -1, 0, 1; corresponding to the open, flat and closed cases, respectively.

C. Field equations

Now by inserting the metric (9) in the field equations (3), we obtain

$$W^t_t = \frac{f(r)}{R(t)^4}, \tag{11}$$

$$W^r_r = \frac{g(r)}{R(t)^4}, \tag{12}$$

$$W^\theta_\theta = W^\varphi_\varphi = \frac{h(r)r^2}{R(t)^4}, \tag{13}$$

where $f(r)$, $g(r)$ and $h(r)$ are defined as

$$\begin{aligned} f(r) = \frac{1}{12(a+1)^5 r^4}[(4a^2a''' - 26aa'a'' \\ + 28a'^3 - 26a'a'' + 4a''' + 8aa''')r^3 \\ + (4a^2a'' - 7aa'^2 + 8aa'' - 7a'^2 \\ + 4a'')r^2 - (8a^2a' + 16aa' + 8a')r \\ + 20a^4 + 36a^3 + 28a^2 + 8a], \end{aligned} \tag{14}$$

$$\begin{aligned} g(r) = \frac{1}{12(a+1)^4 r^4}[(-4aa'' + 7a'^2 - 4a'')r^2 + 4a^4 \\ + 16a^3 + 20a^2 + 8a], \end{aligned} \tag{15}$$

$$\begin{aligned} h(r) = -\frac{1}{12(a+1)^5 r^6}[(2a^2a''' - 13aa'a'' \\ + 14a'^3 + 4aa''' - 13a'a'' + 2a''')r^3 \\ + (-4a^2a' - 8aa' - 4a')r + 4a^5 \\ + 20a^4 + 36a^3 + 28a^2 + 8a], \end{aligned} \tag{16}$$

where the prime denotes the derivative with respect to the radial coordinate r.

The energy-momentum tensor required to support such a space-time is in the form,

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho, P_r, P_t, P_t), \tag{17}$$

where $\rho(r,t)$ is the energy density and $P_r(r,t)$, $P_t(r,t)$ are the radial and transverse pressures, respectively. Use of Eqs. (11)–(13) and (17) and substituting into Eq. (3) lead to the following equations:

$$\rho(r,t) = -4\alpha \frac{f(r)}{R(t)^4}, \tag{18}$$

$$P_r(r,t) = 4\alpha \frac{g(r)}{R(t)^4}, \tag{19}$$

$$P_t(r,t) = 4\alpha \frac{h(r) r^2}{R(t)^4}. \tag{20}$$

To calculate $a(r)$ and $R(t)$ in Weyl gravity, we use two properties of Weyl's tensor, Bianchi and trace identities as

$$\nabla_{\mu} W^{\mu\nu} = 0, \tag{21}$$

$$W^{\mu}_{\mu} = 0, \tag{22}$$

use of Eq. (21) for $\nu = t$ leads to

$$\frac{1}{R(t)} \frac{dR(t)}{dt} \left[\frac{f(r) + g(r) + 2h(r) r^2}{R(t)^4}\right] = 0, \tag{23}$$

and for $\nu = r$ from Eq. (21), we have

$$g'(r) + \frac{2g(r)}{r} - 2rh(r) = 0. \tag{24}$$

Also, we use the trace identity (22) to obtain

$$\frac{f(r) + g(r) + 2h(r) r^2}{R(t)^4} = 0, \tag{25}$$

or

$$-\rho + P_r + 2P_t = 0. \tag{26}$$

As we know there are two unknown functions $R(t)$ and $a(r)$ to obtain the metric, as well as ρ, P_t and P_r, are unknown and functions of r and t. In the standard GR [47] and the brane-world model [48] in order to obtain the inhomogeneous exact solutions, authors have chosen the generalized equation of state as follows:

$$\rho + \alpha P_r + 2\beta P_t = 0, \tag{27}$$

where α and β are constant parameters. But, we note that in the conformal Weyl gravity the energy-momentum tensor components are constrained through the trace identity (26), which means $\alpha = \beta = -1$. Thus in contrast to standard GR, we cannot use Eq. (27) to obtain a new equation to find $a(r)$ and $R(t)$.

Weyl equations (18)–(20) together with Eqs. (23)–(25) make a set of equations which can be solved. In the next section by imposing constrain between the radial and transverse pressures, we obtain exact solutions for $R(t)$ and $a(r)$.

III. EXACT SOLUTIONS IN WEYL GRAVITY

In this section we are going to obtain inhomogeneous exact solutions in the framework of Weyl gravity. As we know the equation of state has an important role in the study of the geometry of space-time. For example $\omega = -1$ correspond with the vacuum energy or cosmological constant and $-1 < \omega < -1/3$ are mentioned for the quintessence matter and used as a candidate for explaining the accelerated expansion of the Universe. Phantom field as an exotic matter with equation of state parameter $\omega < -1$ also accelerate the expansion of the Universe.

A. Case I: Isotropic fluid

First we focus on the cosmic scale factor $R(t)$. By comparing two equations (23) and (25), we conclude that

$$\frac{\dot{R}(t)}{R(t)} \neq 0. \tag{28}$$

The above equation shows that there are different choices to get the scale factor. So the scale factor can be an arbitrary function of time. Inflating Lorentzian wormholes in the framework of GR was investigated by Roman [15] which explores the possibility that inflation provides a natural mechanism for the enlargement wormholes from microscopic size to macroscopic. For having an exponential inflation we consider the simplest choice. By choosing $\frac{\dot{R}(t)}{R(t)} =$ constant, we have the following solution for $R(t)$:

$$R(t) = R_0 e^{H_0 t}, \tag{29}$$

where H_0 is the constant of integration.

Now, we consider an isotropic fluid

$$P_r(r,t) = P_t(r,t), \tag{30}$$

which gives

$$g(r) = h(r) r^2. \tag{31}$$

Substituting Eq. (31) into Eq. (24) we obtain

$$rg'(r) = 0, \tag{32}$$

which have the following solution:

$$g(r) = c_1, \tag{33}$$

where c_1 is an integration constant.

By substituting $g(r)$ from Eq. (33) into Eq. (15), we obtain

$$12c_1(a+1)^4r^4 = 4a^4 - 4aa''r^2 + 7a'^2r^2 + 16a^3 - 4a''r^2 + 20a^2 + 8a. \tag{34}$$

The above equation has not an exact solation. For the case $c_1 = 0$ we find the following exact solution as

$$a(r) = \frac{c_2r^2}{1 - c_2r^2}, \tag{35}$$

where c_2 is a constant of integration. Now from Eqs. (29) and (35), the line element (9) takes the form

$$ds^2 = -dt^2 + R_0^2e^{2H_0t}\left[\frac{dr^2}{1 - c_2r^2} + r^2d\Omega^2\right], \tag{36}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. For this case from Eq. (18)–(20) we find $\rho = P_r = P_t = 0$, which is the simplest case which satisfying the trace equation (26).

The spatial part of metric (36) shows an exponentially expanding 3-sphere, and therefore describes a closed empty universe for $c_2 > 0$ and a open empty universe for $c_2 < 0$.

Also for the special case $c_2 = 0$ it corresponds to

$$ds^2 = -dt^2 + R_0^2e^{2H_0t}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{37}$$

which presents the de Sitter space-time.

Therefore as mentioned in Ref. [72] the conformal invariance imposes a sharp constraint on isotropic distributions of matter in the Universe; so that in an empty Friedmann-Robertson-Walker (FRW) universe, the scale factor can be an arbitrary function of time. The merit of this is that we do not need any exotic matter to explain the acceleration expansion of the Universe [72].

B. Case II: Anisotropic fluid

Now, we consider the case with the following relation between the energy density $\rho(r)$ and the radial pressure $P_r(r)$:

$$P_r(r,t) = \omega\rho(r,t), \tag{38}$$

where ω is the equation of state parameter.

Substituting Eq. (18) and (19) into Eq. (38), we have

$$g(r) = -\omega f(r), \tag{39}$$

by omitting $h(r)$ between Eq. (24) and Eq. (25), we have

$$rg'(r) + 3g(r) = -f(r), \tag{40}$$

by combining Eqs. (39) and (40), we have

$$rg'(r) = \frac{(1-3\omega)}{\omega}g(r), \tag{41}$$

with the following solution:

$$g(r) = c_1r^{\frac{(1-3\omega)}{\omega}}, \tag{42}$$

where c_1 is an integration constant. By substituting $g(r)$ from Eq. (42) into Eq. (15), we obtain

$$12c_1r^{\frac{(1-3\omega)}{\omega}}(a+1)^4r^4 = 4a^4 - 4aa''r^2 + 7a'^2r^2 + 16a^3 - 4a''r^2 + 20a^2 + 8a. \tag{43}$$

In general the above equation could not be solved unless we set $\omega = -1$. Unfortunately, even in this case, the equation does not have an explicit form of the exact solution. However, in the Appendix, we present a solution containing an integration term with three constants of integration c_1, c_2 and c_3. For different values of these constants there are many different solutions, however, some of them do not have the physical meaning. Now, in what follows we consider the case of $c_1 = \frac{1}{3}$ which leads to a solution satisfying all of the wormhole conditions.

1. The case of $c_1 = \frac{1}{3}$

As is clear from Eq. (A1), by choosing $c_1 = \frac{1}{3}$ the integrand takes a simple form and thus one can easily find the following exact solution for $a(r)$ function. Also, by choosing $c_1 = \frac{1}{3}$ in Eq. (43) we have

$$4aa''r^2 + 4a''r^2 - 7a'^2r^2 + 8a + 4a^2 + 4 = 0. \tag{44}$$

It can be shown that this equation has the following exact solution:

$$a(r) = -1 + \frac{1}{(\frac{3}{8})^{\frac{4}{3}}(c_2r^{\frac{3}{2}} - c_3r^{-\frac{1}{2}})^{\frac{4}{3}}}, \tag{45}$$

where c_2 and c_3 are integration constants. The line element (9) takes the form

$$ds^2 = -dt^2 + R_0^2 e^{\sqrt{\frac{\Delta}{3}}t}\left[\frac{dr^2}{(\frac{3}{8})^{\frac{4}{3}}(c_2 r^{\frac{3}{2}} - c_3 r^{-\frac{1}{2}})^{\frac{4}{3}}} + r^2 d\Omega^2\right], \quad (46)$$

where $H_0 \equiv \sqrt{\frac{\Delta}{3}}$.

The time-dependent wormholes have been introduced by Roman with the following line element [15]:

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \quad (47)$$

where $R(t)$ and $b(r)$ are the scale factor and the shape function of wormhole, respectively [106]. The minimum value of r is a throat radius of wormhole $r = r_0$, so the radial coordinate change in the interval $r_0 \leq r \leq \infty$. Since the shape function $b(r)$ is responsible to define the shape of the wormhole, hence for a wormhole solution it should satisfy the certain conditions: (i) The radius of the wormhole throat corresponds with the point where $b(r_0) = r_0$, (ii) the flaring-out condition implies that $b'(r) < 1$ and (iii) for $r > r_0$ the throat condition implies that $\frac{b(r)}{r} < 1$ (for more study the reader is referred to [107–110]).

Comparison of metric (46) with (47) leads to the following shape function:

$$b(r) = r - \left(\frac{3}{8}\right)^{\frac{4}{3}}(c_2 r^{\frac{9}{4}} - c_3 r^{\frac{1}{4}})^{\frac{4}{3}}, \quad (48)$$

and from condition (i) the throat radius is

$$r_0 = \left(\frac{c_3}{c_2}\right)^{\frac{1}{2}}, \quad (49)$$

which is real only if ($c_2 > 0$, $c_3 > 0$). One can find c_3 in terms of r_0, c_2; so we rewrite the shape function as

$$b(r) = r - \left(\frac{3}{8}\right)^{\frac{4}{3}}(c_2 r^{\frac{9}{4}} - c_2 r_0^2 r^{\frac{1}{4}})^{\frac{4}{3}}, \quad (50)$$

in this case. In Fig. 1 we have plotted the shape function with various conditions. The figure shows all necessary conditions are satisfied by the given shape function.

Quasicosmological traversable wormhole solutions in the context of $f(R)$ gravity have been studied in Ref. [20]. In contrast to the GR one can find the asymptotically spherical, flat and hyperbolic wormhole solutions in modified gravity theories. We have plotted the behavior of function $(1 - \frac{b(r)}{r})$ in Fig. 2. It shows that the wormhole solutions in Weyl gravity at large r match the hyperbolic FRW universe and so the asymptotically flatness condition is violated.

As we mentioned before the traversable wormholes violate the main energy conditions such as NEC, WEC, strong energy condition and dominated energy condition (DEC) for the stress-energy tensor and so they invoke the existence of exotic matter i.e., matter with negative energy density places at or near the wormhole throat. However, in higher-dimensional theories, $f(R)$ gravity theories and modified gravity theories with higher order curvature terms, the wormhole solutions may satisfy some energy conditions [20–27,111].

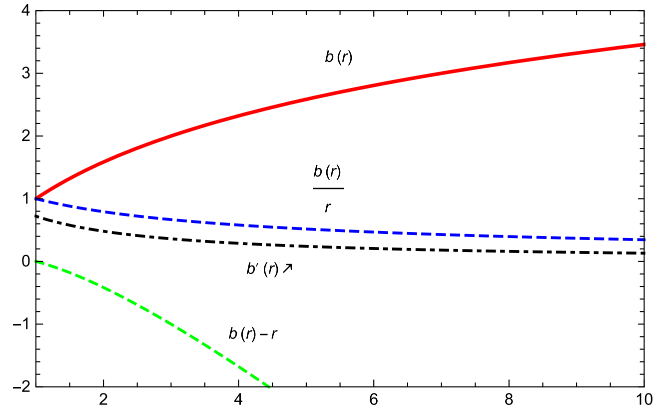

FIG. 1. Shape function $b(r)$, throat condition $\frac{b(r)}{r} < 1$, flaring-out condition $b'(r) < 1$ for throat radius $r_0 = 1$, $c_2 = 0.1$ and $c_3 = 0.1$.

By substituting Eq. (45) into Eqs. (14)–(16), we have

$$f(r) = +\frac{1}{3r^4}, \quad (51)$$

$$g(r) = +\frac{1}{3r^4}, \quad (52)$$

$$h(r) = -\frac{1}{3r^4}. \quad (53)$$

Now we can obtain the energy density, the radial and transverse pressure by substituting Eqs. (51)–(53) and (29) into Eqs. (18)–(20) as follows:

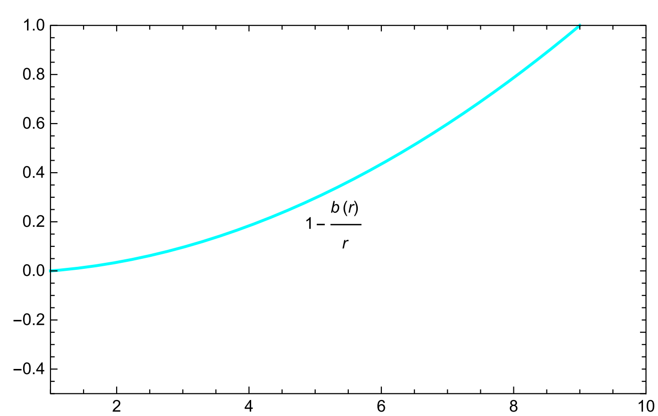

FIG. 2. The behavior of $1 - \frac{b(r)}{r}$ as a function of r.

$$\rho(t,r) = -\frac{4\alpha}{3R_0^4}\frac{1}{r^4}e^{-\sqrt{\frac{16\Lambda}{3}}t}, \tag{54}$$

$$P_r(t,r) = +\frac{4\alpha}{3R_0^4}\frac{1}{r^4}e^{-\sqrt{\frac{16\Lambda}{3}}t}, \tag{55}$$

$$P_t(t,r) = -\frac{4\alpha}{3R_0^4}\frac{1}{r^4}e^{-\sqrt{\frac{16\Lambda}{3}}t}, \tag{56}$$

where $H_0 \equiv \sqrt{\frac{\Lambda}{3}}$.

Now, let us check whether the matter is exotic or not by calculating some energy conditions namely

$$\mathrm{WEC}:\ \rho \geq 0\rho + P_r \geq 0. \tag{57}$$

$$\mathrm{NEC}:\ \rho + P_r \geq 0\rho + P_t \geq 0. \tag{58}$$

$$\mathrm{DEC}:\ \rho - |P_r \geq 0\rho - |P_t \geq 0. \tag{59}$$

In Figs. 3–5 we have plotted variation of the energy density $\rho(r,t)$, $\rho(r,t) + P_r(r,t)$ and $\rho(r,t) - |P_t(r,t)|$ for

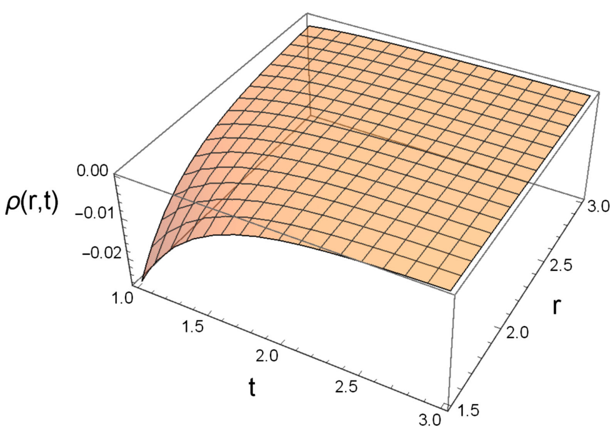

FIG. 3. The variation of WECs $[\rho(r,t)]$ for $R_0 = \alpha = \Lambda = 1$ and the throat radius $r_0 = 1$.

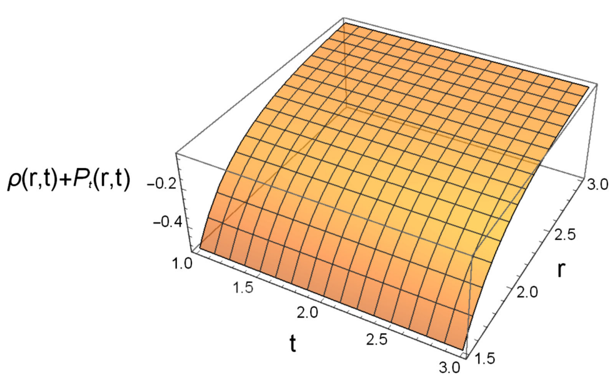

FIG. 4. The variation of NECs $[\rho(r,t) + P_t(r,t)]$ for $R_0 = \alpha = 1$ and $\Lambda = 10^{-35}$ and the throat radius $r_0 = 1$.

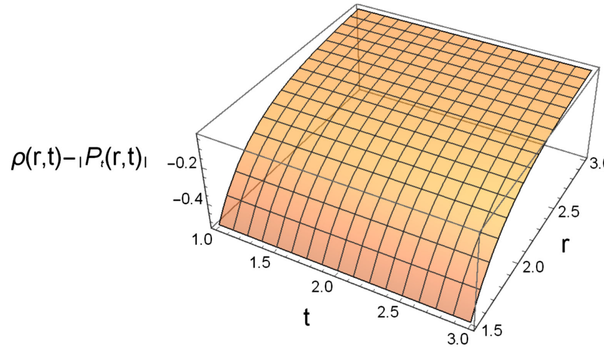

FIG. 5. The variation of DECs $[\rho(r,t) - |P_t(r,t)|]$ for $R_0 = \alpha = 1$ and $\Lambda = 10^{-35}$ and the throat radius $r_0 = 1$.

$r_0 = 1$. As can be seen from Eqs. (54)–(56) for $c_1 = \frac{1}{3}$ the WEC, NEC and DEC is violated throughout the space-time and so the matter is exotic for this case.

As mentioned before a fundamental ingredient of static traversable wormhole solutions in GR is the NEC violation. However, for time-dependent wormhole solutions in GR the NEC and the WEC can be avoided for a specific interval of time and in certain regions at the throat [47–51]. Nevertheless, in some alternative gravity theories such as $f(R)$ gravity, EGB theory, Lovelock and Rastall gravity the energy conditions can be satisfied depending on the parameters of theory and so it is not necessarily an exotic matter to build wormholes. For these alternative gravity theories, similar to GR, the time-dependent spherically symmetric wormhole solutions have been extensively studied in the literature. For time-dependent wormhole solutions in $f(R)$ gravity theory the energy conditions are satisfied for the specific values of the model constants [57]. However, this is not the case for time-dependent wormhole solutions analyzed in this work.

In conformal Weyl gravity as a fourth-order gravitational theory both the static and time-dependent wormhole solutions differ from their similar cases in GR. For the static worm hole solution in Weyl gravity for example in the simple case $b(r) = r_0$ in contrast to GR the radial pressure is positive at the throat and the energy density is negative whereas similar to GR the NEC is violated throughout the space-time [45]. However, for the case of the time-dependent wormhole geometry in this work, we have verified that the NEC and WEC are violated, as shown in the analysis above for the specific case $c_1 = \frac{1}{3}$. Finally, we mention that the restriction for choosing the constants results from mathematical/technical reason not the physical one.

IV. CONCLUSION

There are two methods for formulating wormhole solutions. One method involves joining two asymptotically flat space-times via boundary conditions, while the other method involves smoothly merging the wormhole metrics with a cosmological background. In this paper, we employ the latter method and present a spherically inhomogeneous structure that smoothly joins with a cosmological background within the context of conformal Weyl gravity. Our ansatz metric belongs to the category of the Szekeres-Szafron metric, with two unknown functions, $a(r)$ and $R(t)$. Based on reasonable constraints on the energy-momentum tensor of an anisotropic space-time, we obtain the Weyl equations. These equations, together with resulting equations from the Bianchi and trace identities, i.e., $-\rho + P_r + 2P_t = 0$ make a set of equations, which have no exact solution in the general case. Considering two special cases, isotropic and anisotropic fluid, leads us to categories of equations based on the amount of c_1 as an integration constant. We obtain the de Sitter space-time as an exact solution which corresponds to $c_1 = 0$, $\rho = P_r = P_t = 0$. Another exact solution has been obtained for special case $P_r = -P_t$, which corresponds to time-dependent wormhole for $c_1 = -\frac{1}{3}$, which can be supported by exotic matter which at large r match two hyperbolic FRW universes.

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APPENDIX

In this section we obtain the solution of Eq. (43) for $\omega = -1$ by using MAPLE software as follows:

$$a(r) = -1 + \frac{1}{r^2} \times \text{Root Of}\left[1 + 2c_3 r^2 + 2\int^{Z_-} \frac{df}{\sqrt{c_2 f^{\frac{7}{2}} - 12c_1 f^4 + 4f^4}} r^2\right], \qquad \text{(A1)}$$

where c_2 and c_3 are integration constants and function RootOf is a placeholder for representing all the roots of an equation in one variable. As one can see from Eq. (A1) for $c_1 = \frac{1}{3}$ the integrate can be solved easily and we find the following exact solution:

$$c_2 - \frac{c_3}{r^2} - \frac{8}{3}\frac{1}{r^{3/2}(1 + a(r))^{3/4}} = 0, \qquad \text{(A2)}$$

or

$$a(r) = -1 + \frac{1}{\left(\frac{3}{8}\right)^{\frac{4}{3}} (c_2 r^{\frac{3}{2}} - c_3 r^{-\frac{1}{2}})^{\frac{4}{3}}}. \qquad \text{(A3)}$$

Also, for the case of $c_2 = 0$, Eq. (A1) leads to the following solution:

$$a(r) = -1 + \frac{1}{1 - c_3 r^2 + c_4}, \qquad \text{(A4)}$$

where $c_4 = \sqrt{3c_1 + 1} - 1$. The above solution cannot describe the wormhole solution since it does not satisfy the wormhole conditions and is not physically suitable. We do not discuss this solution in this paper.

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