From static to Vaidya solutions in scalar tensor theories

Mokhtar Hassaine, Ulises Hernandez-Vera, and Franco Lara-Munoz
Instituto de Matemática, Universidad de Talca, Casilla 747, Talca, Chile

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We consider some classes of Horndeski theories in four dimensions for which a certain combination of the Einstein equations within a spherical ansatz splits into two distinct branches. Recently, for these theories, some integrability and compatibility conditions have been established which have made it possible to obtain black hole solutions depending on a single integration constant identified as the mass. Here, we will show that these compatibility conditions can be generalized to accommodate a time dependence by promoting the constant mass to an arbitrary function of the retarded (advanced) time. As a direct consequence, we prove that all the static black hole solutions can be naturally promoted to nonstatic Vaidya-like solutions. We extend this study in arbitrary higher dimensions where the pure gravity part is now described by the Lovelock theory and, where the scalar field action enjoyed the conformal invariance. For these theories, the splitting in two branches is also effective, and we show that their known static black hole solutions can as well be promoted to Vaidya-like solutions.

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I. INTRODUCTION

One of the important generalizations of the Schwarzschild solution with the aim of describing the exterior of a star which is either emitting or absorbing null dusts is undoubtedly the so-called Vaidya solution [1]. This "radiating" solution can be conveniently written in the Eddington-Finkelstein coordinates by promoting the Schwarzschild constant mass M to a function of the retarded (advanced) time

$$ds^2 = -\left(1 - \frac{2M(u)}{r}\right)du^2 + 2\epsilon du dr + r^2 d\Omega_2^2,$$

where $\epsilon = \pm 1$ describes incoming (resp. outgoing) radiation shells. This in turn implies that the exterior field of the radiating star behaves as a pure radiation field with an energy-momentum $T_{\mu\nu}$ that has component only along the retarded (advanced) time,

$$T_{\mu\nu} = -\frac{2\dot{M}(u)}{r^2} k_\mu k_\nu, \tag{1}$$

with $\mathbf{k} = \partial_r$, and where the dot stands for the derivative with respect to u. In contrast with the Schwarzschild metric, the Vaidya spacetime has no timelike Killing vector field, and hence it is a nonstatic metric. One of the interesting aspect of this solution is that it provided us with one of the oldest counterexamples to the cosmic censorship conjecture. Indeed, in the original paper of Papapetrou [2], it was shown that this solution could give rise to the formation of naked singularities. In addition, since its discovery, the Vaidya solution has been intensively studied and also generalized in presence of source. In particular its electrical extension [3] has given rise to numerous researches whether from a thermodynamic point of view [4] or about the possible violation of the weak energy condition [5]. Extension of Vaidya solutions in the case of Lovelock gravity and their thermodynamics features were also discussed in [6].

In the present article, we are interested in searching for Vaidya type solutions within the framework of scalar tensor theories. An example of such solution was exhibited in the case of a particular Horndeski theory in Ref. [7]. As is now well known, the Horndeski theory refers to the most general scalar tensor theory leading to second-order equations for the metric and the scalar field [8]. In the last decade this theory has been "rediscovered," and a considerable progress has been made in studying black holes in Horndeski theory. Recently, in Ref. [9], the authors have selected some subclasses of Horndeski theories by requiring that a certain combination of the Einstein equations within a spherical ansatz splits into two distinct branches. This splitting allows in some cases to determine the form of the scalar field without knowing the metric function explicitly. In this case, some integrability and compatibility conditions have been established which permit to derive interesting black hole solutions whose asymptotic behaviors resemble to those of Schwarzschild-(A)dS. It is important for our work to mention that these solutions depend on a single integration constant identified with the mass of the black hole.

The resulting action permitting such solutions, is devoid of any apparent symmetry (such as the shift symmetry), but surprisingly the scalar field action[1] can be seen as the sum of an action yielding a conformally invariant scalar field equation (with subscribes 4) with a another piece that is conformally invariant in five dimensions (with subscribes 5),

$$\begin{aligned} S = \int d^4x\sqrt{-g}[&R - 2\Lambda - 2\lambda_4 e^{4\phi} - 2\lambda_5 e^{5\phi} \\ &- \beta_4 e^{2\phi}(R + 6(\nabla\phi)^2) - \beta_5 e^{3\phi}(R + 12(\nabla\phi)^2) \\ &- \alpha_4(\phi\mathcal{G} - 4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 4\Box\phi(\nabla\phi)^2 - 2(\nabla\phi)^4) \\ &- \alpha_5 e^{\phi}(\mathcal{G} - 8G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 12\Box\phi(\nabla\phi)^2 - 12(\nabla\phi)^4)]. \end{aligned} \tag{2}$$

Note that for the vanishing parameters with subscribes 5, this action reduces to the one first considerer in [10], and for which two classes of black hole solutions have been found.

First, we will generalize the integrability conditions given in [9] for a spherical metric (written in Eddington-Finkelstein) whose base manifold has constant curvature $\kappa = \pm 1$ or $\kappa = 0$, namely

$$ds^2 = -f(r)du^2 - 2du\,dr + \frac{r^2 d\theta^2}{1 - \kappa\theta^2} + r^2\theta^2 dy^2. \tag{3}$$

The topological black holes generalizing the solutions found in [9] will be exhibited. A particular attention is devoted to the planar case $\kappa = 0$ where we show that the integrability conditions impose that the couplings $\lambda_4 = \lambda_5 = \beta_4 = \beta_5 = 0$. In this planar case, we will also see that the scalar field has a free parameter unconstrained by the field equations. The reason of this freedom will be clarified. In a second time, we show that demanding the metric function f to depend as well on the retarded (advanced) time u, the generalization of the integrability conditions naturally lead to Vaidya-like metric. More precisely, we establish that each topological black holes can be promoted to Vaidya-like metric by promoting the constant mass to a function of u. In the third part, we extend our analysis in higher dimensions where the action is now given by the Lovelock Lagrangian with a scalar field action that is conformally invariant. We first see that a certain combinations of the equations of motion within our ansatz of the form (3) splits in two branches as in the Einstein gravity case. Similarly, we establish that the Lovelock static black hole solutions of these theories found in [11,12], can as well be promoted to Vaidya-like solutions by allowing the mass to depend on the coordinate u.

[1] In what follows, by scalar field action, we mean the terms in the action that depend on the scalar field in opposition with those of pure gravity.

II. INTEGRABILITY CONDITIONS FOR TOPOLOGICAL BLACK HOLES

Let us denote by

$$E_{\mu\nu} := G_{\mu\nu} + \Lambda g_{\mu\nu} - T_{\mu\nu}, \tag{4}$$

the field equations arising from the variation of the action (2) with respect to the metric, and whose explicit expressions are reported in the Appendix. For an ansatz metric of the form (3) with a scalar field depending only on the radial coordinate r, the equation $E_{rr} = 0$ splits in two radically distinct branches given by

$$\begin{aligned} (\phi'^2 - \phi'')[&r^2(2\beta_4 + 3\beta_5 e^{\phi})e^{2\phi} \\ &+ 4\alpha_5(\kappa - f(1 + r\phi')(1 + 3r\phi'))e^{\phi} \\ &+ 4\alpha_4(\kappa - f(1 + r\phi')^2)] = 0. \end{aligned} \tag{5}$$

The first branch will refer to the equation $(\phi'^2 - \phi'') = 0$ for which the integration of the scalar field can be done without knowing *a priori* the expression of the metric function. For the second branch, as already mentioned in [9], the integration of the scalar field can only be achieved in the case $\alpha_5 = \beta_5 = \lambda_5 = 0$, and the factorization (5) reduces to

$$(\phi'^2 - \phi'')[2\beta_4 r^2 e^{2\phi} + 4\alpha_4(\kappa - f(1 + r\phi')^2)] = 0. \tag{6}$$

In what follows, we first consider the first branch for which we give the compatibility conditions and exhibit as well its topological black holes. In the second sub-section, we will establish that these static topological black hole solutions can be extended to Vaidya-like solutions by turning the constant mass to be an arbitrary function of the retarded time u. Although the second branch is somewhat different in that the scalar field depends explicitly on the constant M and the couplings with the subscribe 5 are taken to zero, we will see that a similar analysis will surprisingly lead to the same conclusions.

A. First branch of topological black hole solutions

Here, we focus on the solution given by the first branch of (5), namely $\phi'^2 - \phi'' = 0$. Although the general solution should include two integration constants, the full integration of the system will impose one of them to be zero and, hence we anticipate the following form for the scalar field solution

$$\phi(r) = \ln\left(\frac{\eta}{r}\right), \tag{7}$$

where η is *a priori* a free integration of constant. Plugging this ansatz into the remaining Einstein equations, E_{uu} and $E_{\theta\theta}$, one obtains that

$$E_{uu}^{\rm static}=\left[\frac{\alpha_4 f^2}{r}-\left(r+\frac{\beta_5\eta^3}{2r^2}+\frac{2\kappa\alpha_5\eta}{r^2}+\frac{2\alpha_4\kappa}{r}\right)f+\frac{1}{r^2}\left(\frac{\lambda_5\eta^5}{2}+\frac{\beta_5\eta^3\kappa}{2}\right)+\frac{1}{r}(\lambda_4\eta^4+\beta_4\kappa\eta^2)-\frac{r^3\Lambda}{3}+\kappa r\right]',$$

$$E_{\theta\theta}^{\rm static}=\left[\frac{\alpha_4}{r}f^2-\left(r-\frac{\beta_5\eta^3}{r^2}-\frac{\beta_4\eta^2}{r}\right)f-\frac{\lambda_5\eta^5}{3r^2}+\frac{1}{r}(-\lambda_4\eta^4)-\frac{r^3\Lambda}{3}\right]''. \tag{8}$$

Integrating both equations we obtain two quadratic equations for the metric f,

$$\frac{\alpha_4 f^2}{r}-\left(r+\frac{\beta_5\eta^3}{2r^2}+\frac{2\kappa\alpha_5\eta}{r^2}+\frac{2\alpha_4\kappa}{r}\right)f+\frac{1}{r^2}\left(\frac{\lambda_5\eta^5}{2}+\frac{\beta_5\eta^3\kappa}{2}\right)+\frac{1}{r}(\lambda_4\eta^4+\beta_4\kappa\eta^2)-\frac{r^3\Lambda}{3}+\kappa r+C_1=0, \tag{9a}$$

$$\frac{\alpha_4}{r}f^2-\left(r-\frac{\beta_5\eta^3}{r^2}-\frac{\beta_4\eta^2}{r}\right)f-\frac{\lambda_5\eta^5}{3r^2}-\frac{\lambda_4\eta^4}{r}-\frac{r^3\Lambda}{3}+C_3 r+C_2=0, \tag{9b}$$

where C_1, C_2 and C_3 are *a priori* three different integration constants. In order for these equations to be compatibles, the coupling constants of the system are forced to be tied as follows

$$\beta_5=-\frac{4}{3\eta^2}\alpha_5\kappa,\qquad \beta_4=-\frac{2\alpha_4}{\eta^2}\kappa,$$
$$\lambda_5=-\frac{3}{5\eta^2}\beta_5\kappa,\qquad \lambda_4=-\frac{1}{2\eta^2}\beta_4\kappa, \tag{10}$$

while the constants of integration must be fixed as

$$C_3=\kappa,\qquad C_1=C_2=-2M, \tag{11}$$

and, where M is a truly integration constant. Note that in the planar case $\kappa=0$, the couplings β_4, λ_4, β_5 and λ_5 must vanish (10), and this case will be treated separately below. Otherwise for $\kappa=\pm1$, the constant η of the scalar field becomes fixed in term of the coupling constants of the theory, and defining

$$\mathcal{E}^{\rm static}=\frac{\alpha_4 f^2}{r}-\left(r+\frac{4\alpha_5\eta\kappa}{3r^2}+\frac{2\alpha_4\kappa}{r}\right)f-\frac{4\eta}{15r^2}\alpha_5\kappa^2-\frac{\alpha_4\kappa^2}{r}-\frac{r^3\Lambda}{3}+\kappa r, \tag{12}$$

the two quadratic equations for the metric function (9) at the "point" defined by the compatibility conditions (10) and (11) become a single relation

$$\mathcal{E}^{\rm static}-2M=0,$$

whose general metric solution can be parametrized in terms of $\alpha_4\neq0$, α_5 and η as

$$f(r)=\kappa+\frac{2\alpha_5\eta\kappa}{3r\alpha_4}+\frac{r^2}{2\alpha_4}\left(1\pm\left[\left(1+\frac{4\alpha_5\eta\kappa}{3r^3}\right)^2+4\alpha_4\left(\frac{\Lambda}{3}+\frac{2M}{r^3}+\frac{2\alpha_4\kappa^2}{r^4}+\frac{8\alpha_5\eta\kappa^2}{5r^5}\right)\right]^{\frac{1}{2}}\right), \tag{13}$$

On the other hand, for $\alpha_4=0$, the equation defining f becomes linear and in this case, the topological black hole metric function is given by

$$f(r)=\frac{1}{1+\frac{4\alpha_5\eta\kappa}{3r^3}}\left[\kappa-\frac{\Lambda r^2}{3}-\frac{2M}{r}-\frac{4\alpha_5\eta\kappa^2}{15r^3}\right]. \tag{14}$$

Note that both solutions (13) and (14) correspond to those reported in [9] for $\kappa=1$.

Let us now go back in more details to the planar case $\kappa=0$. As said before, in this case the couplings $\beta_4=\lambda_4=\beta_5=\lambda_5=0$, and consequently the field equations reduce to

$$G_{\mu\nu}+\Lambda g_{\mu\nu}=\alpha_4\mathcal{H}^{(4)}_{\mu\nu}+\alpha_5 e^{\phi}\mathcal{H}^{(5)}_{\mu\nu},$$
$$\alpha_4\mathcal{H}^{(4)}+\alpha_5 e^{\phi}\mathcal{H}^{(5)}=0, \tag{15}$$

where the different expressions of the tensors can be found in the Appendix. It is a matter of check to see that the following metric function and scalar field

$$f(r)=\frac{r^2}{2\alpha_4}\left(1\pm\sqrt{1+4\alpha_4\left(\frac{\Lambda}{3}+\frac{2M}{r^3}\right)}\right),$$
$$\phi(r)=\ln\left(\frac{\tilde\eta}{r}\right), \tag{16}$$

satisfy the Eqs. (15) in the planar case $\kappa=0$. Various comments can be made concerning this solution. First, one can see that although the coupling $\alpha_5\neq0$, it does not appear in the expression of the metric neither in that of the scalar field. Also, in contrast with the cases $\kappa=\pm1$, the constant $\tilde\eta$ is a truly integration constant unconstrained by the field equations. These two features of the solution (16) can be explained by the fact that the configuration (16), in addition to satisfying the equations of motion (15), makes also that the stress tensor $\mathcal{H}^{(5)}_{\mu\nu}$ and the scalar quantity $\mathcal{H}^{(5)}$ to identically vanish on-shell. Consequently, this explains the absence of the coupling α_5 in the parametrization of the solution. Concerning the constant $\tilde\eta$, it is clear that, since the α_4-part of the action is invariant under the shift constant of the scalar field $\phi \to \phi + \mathrm{cst}$, and since the α_5-part of the equations vanish on-shell, our scalar field solution will always be defined up to a constant.

B. The Vaidya-like extension of the first branch of solutions

We now show that all the previous solutions can be extended to Vaidya type solutions thanks to a generalization of the previous comparability relations. In order to achieve this task, we allow the metric function f to depend as well on the retarded time but we restrict the form of the scalar field as in the static case[2]

$$ds^2 = -f(u,r)du^2 - 2dudr + \frac{r^2 d\theta^2}{1-\kappa\theta^2} + r^2\theta^2 dy^2,$$
$$\phi(r) = \ln\left(\frac{\eta}{r}\right), \tag{17}$$

where η will be fixed by the compatibility conditions as in the static case. As a consequence of this choice, the Einstein equation $E_{rr} = 0$ is automatically satisfied since the first part of the factorization (5) is unchanged. Now, evaluating the expression of the scalar field into the remaining independent Einstein equations at the special tuning point where the topological black holes exist (10), one gets

$$E_{uu} = \frac{1}{r^2}(f(u,r)\partial_r - \partial_u)\mathcal{E}^{\mathrm{static}}(f(u,r)), \tag{18}$$

$$E_{\theta\theta} = \frac{r}{2(\kappa\theta^2 - 1)}\partial_{rr}\mathcal{E}^{\mathrm{static}}(f(u,r)), \tag{19}$$

where $\mathcal{E}^{\mathrm{static}}(f(u,r))$ refers to the expression defined in (12) but now evaluated at $f = f(u,r)$. It is clear that this system is incompatible unless $\mathcal{E}^{\mathrm{static}}(f(u,r))$ is a constant, and in this case one would end up with the static solution previously derived. Nevertheless, let us explore other possibilities, and for this we first focus on the equation $E_{\theta\theta} = 0$. Its general solution is given by $\mathcal{E}^{\mathrm{static}}(f(u,r)) = C_1(u)r + 2M(u)$ where $C_1(u)$ and $M(u)$ are two arbitrary functions of u. Injecting this expression into the equation E_{uu}, one gets

$$E_{uu} = \frac{f(u,r)C_1(u)}{r^2} - \frac{\dot{C}_1(u)}{r} - \frac{2\dot{M}(u)}{r^2}. \tag{20}$$

A straightforward computation shows that the compatibility of the equation $E_{uu} = 0$ together with the expression (12) would yield the previous static solution, namely $C_1(u) = 0$ and $M(u) = M = \mathrm{cst}$. However, one can opt for the option that this configuration behaves as a pure Vaidya-like radiation field. Indeed, this can occur by choosing $C_1(u) = 0$ and leaving free the dependence of the function $M(u)$. Indeed, in this case, the full Einstein equations become

$$E_{\mu\nu} \coloneqq G_{\mu\nu} + \Lambda g_{\mu\nu} - T_{\mu\nu} = -\frac{2\dot{M}(u)}{r^2}\delta^u_\mu\delta^u_\nu, \tag{21}$$

for an ansatz of the form (17) with a metric function given by

$$f(u,r) = \kappa + \frac{2\alpha_5\eta\kappa}{3r\alpha_4} + \frac{r^2}{2\alpha_4}\left(1 \pm \left[\left(1 + \frac{4\alpha_5\eta\kappa}{3r^3}\right)^2 + 4\alpha_4\left(\frac{\Lambda}{3} + \frac{2M(u)}{r^3} + \frac{2\alpha_4\kappa^2}{r^4} + \frac{8\alpha_5\eta\kappa^2}{5r^5}\right)\right]^{\frac{1}{2}}\right). \tag{22}$$

In sum, we have shown that the topological black hole $\kappa = \pm 1$ defined at the special point (10) can naturally be promoted to a Vaidya-like solution (20) by promoting the constant mass of the metric function to an arbitrary function of the retarded time (21).

Along the same lines, the $\kappa = 0$ black hole solution (14) with the couplings $\alpha_4 = \beta_4 = \lambda_4 = 0$ can as well rendered to satisfy the same Vaidya conditions (20) for a metric function, and a scalar field given by

$$f(u,r) = \frac{1}{1 + \frac{4\alpha_5\eta\kappa}{3r^3}}\left[\kappa - \frac{\Lambda r^2}{3} - \frac{2M(u)}{r} - \frac{4\alpha_5\eta\kappa^2}{15r^3}\right],$$
$$\phi(r) = \ln\left(\frac{\eta}{r}\right).$$

We also mention that for $M = M(u)$, the Vaidya extension of the static solution with a planar base manifold (16) also satisfy the stealth equations $\mathcal{H}^{(5)}_{\mu\nu} = 0 = \mathcal{H}^{(5)}$ as in the static case.

C. Second branch of solutions

We now consider the second branch of equation (5). Unfortunately, as shown in Ref. [9], this branch can only be solved analytically for $\alpha_5 = \beta_5 = \lambda_5 = 0$, and in this case, the solution for $\kappa = 1$ was given in [10]. Its topological generalization can be conveniently parametrized as follows

$$f(r) = \kappa + \frac{r^2}{2\alpha_4}\left[1 \pm \sqrt{1 + 4\alpha_4\left(\frac{2M}{r^3} + \frac{\Lambda}{3}\right)}\right],$$
$$\phi(r) = \ln\left(\frac{\sqrt{-\frac{2\kappa\alpha_4}{\beta_4}} + (1-\kappa^2)\sqrt{\frac{2\kappa\alpha_4}{\beta_4}}}{r\sinh\left[\sqrt{\kappa}\left(c_1 \pm \int^r \frac{dr}{r\sqrt{f(r)}}\right)\right]}\right), \tag{23}$$

and the solution holds at the special fine tuning $\lambda_4 = 3\beta_4^2/4\alpha_4$. Note that for this second branch, the emergence of an unconstrained constant of integration c_1, a sort of hair. For this second branch, it is clear that

[2] One could also have considered a scalar field of the form $\phi = \phi(u,r)$, and this will considerably complicate the problem, but for our purpose it is enough to consider $\phi = \phi(r)$.

the scalar field depends on the mass parameter M through the expression of the metric function f. Moreover, in contrast with the previous case, if one would naively turn on the time dependence of the metric function and on the scalar by promoting the mass M to a function $M(u)$, the Einstein field equation $E_{rr}=0$ would not be satisfied. This is mainly because for a time-dependent ansatz for the scalar field, $\phi(u,r)$, the static equation (5) for $\alpha_5=\beta_5=\lambda_5=0$ becomes factorized as

$$[(\partial_r\phi)^2-\partial_{rr}\phi]\times[r^2\beta e^{2\phi}+2\alpha_4 r(1+r)(f\partial_r\phi-\partial_u\phi)+\alpha_4(f-1)]=0, \tag{24}$$

and, because of the presence of the term $\partial_u\phi$, the second branch will not be satisfied by just turning the constant M to a function $M(u)$. Despite this inconvenient, it can be shown that the Einstein's equations for a nonstatic ansatz of the form (17) with a metric function

$$f(u,r)=\kappa+\frac{r^2}{2\alpha_4}\left[1\pm\sqrt{1+4\alpha_4\left(\frac{2M(u)}{r^3}+\frac{\Lambda}{3}\right)}\right]$$

are such that

$$E_{\mu\nu}\coloneqq G_{\mu\nu}+\Lambda g_{\mu\nu}-T_{\mu\nu}=-\frac{2\dot{M}(u)}{r^2}\delta^u_\mu\delta^u_\nu, \tag{25}$$

provided that the scalar field satisfied the equation of the second branch (23), namely

$$r^2\beta e^{2\phi}+2\alpha_4 r(1+r)(f\partial_r\phi-\partial_u\phi)+\alpha_4(f-1)=0. \tag{26}$$

Note that the spherical case $\kappa=1$ was already reported along the same lines in [7].

III. EXTENSION TO HIGHER DIMENSIONS

As already mentioned, the scalar field equation of the action (2) for $\alpha_5=\beta_5=\lambda_5=0$, enjoys the conformal invariance although the scalar field action is not conformally invariant [10]. Recently, this kind of symmetry present at the level of the equation and not at the level of the action has been dubbed non-Noetherian conformal symmetry [13]. In Ref. [7], the authors presented a procedure to obtain this non-Noetherian conformally invariant action from a Noetherian conformally action in higher dimensions. This latter action, whose pure gravity action included the Lovelock theory, was first considered in [14,11], and is nothing but the most general theory of gravity conformally coupled to a scalar field that yields second-order field equations for the scalar field and the metric. In order to be self-contained, we will present the action with the useful notations as introduced in Refs. [14,11].

Let us first define the following tensor

$$S_{\mu\nu}{}^{\gamma\delta}=\Phi^2R_{\mu\nu}{}^{\gamma\delta}-4\Phi\delta^{[\gamma}_{[\mu}\nabla_{\nu]}\nabla^{\delta]}\Phi+8\delta^{[\gamma}_{[\mu}\nabla_{\nu]}\phi\nabla^{\delta]}\Phi-2\delta^{[\gamma}_{[\mu}\delta^{\delta]}_{\nu]}\nabla_\rho\Phi\nabla^\rho\Phi.$$

where now the scalar field Φ is related to the previous one ϕ by $\Phi=e^\phi$, and the action under consideration is given by

$$S=\int d^Dx\sqrt{-g}\left\{\sum_{k=0}^{[\frac{D-1}{2}]}\frac{1}{2^k}\delta^{(k)}\left(a_kR^{(k)}+b_k\Phi^{D-4k}S^{(k)}\right)\right\}, \tag{27}$$

where the a_k and b_k are coupling constants. The Kronecker tensor $\delta^{(k)}$ is defined by

$$\delta^{(k)}=k!\delta^{\mu_1}_{[\alpha_1}\delta^{\nu_1}_{\beta_1}\ldots\delta^{\mu_k}_{\alpha_k}\delta^{\nu_k}_{\beta_k]},$$

while the expressions of $R^{(k)}$ and $S^{(k)}$ read

$$R^{(k)}=\prod_{r=1}^{k}R_{\mu_r\nu_r}{}^{\alpha_r\beta_r},\qquad S^{(k)}=\prod_{r=1}^{k}S_{\mu_r\nu_r}{}^{\alpha_r\beta_r}.$$

As in four dimensions, the Einstein equation $E_{rr}=0$ for a static ansatz of the form (3) with a radial scalar field can be factorized in a similar form of (5). On the other hand, static black hole solutions similar to those described previously were constructed in Refs. [11,12]. In [12], it was shown that solutions of the action (26) can be projected through a limiting process to the four-dimensional solutions of the action (2) with $\alpha_5=\beta_5=\lambda_5=0$. It is therefore natural to ask whether the static high-dimensional solutions can be converted into Vaidya-type solutions which would project into those previously derived. This is indeed the case, and these Vaidya-like solutions for the first branch read

$$ds^2=-f(u,r)du^2-2dudr+r^2d\Omega^2_{D-2,\kappa},\qquad \Phi(r)=\frac{\eta}{r}, \tag{28}$$

where the metric function $f(u,r)$ satisfies a polynomial equation of order $[\frac{D-1}{2}]$ that reads

$$\sum_{k=0}^{[\frac{D-1}{2}]}\frac{a_k(D-1)!}{(D-2k-1)!}\left(\frac{\kappa-f(u,r)}{r^2}\right)^k=\frac{M(u)(D-1)(D-2)}{r^{D-1}}-\frac{q(D-1)(D-2)}{r^D}. \tag{29}$$

Here, the constant η is defined in term of the coupling constants of the Lagrangian through the relation

$$\sum_{k=1}^{[\frac{D-1}{2}]}k\frac{b_k}{(D-2k-1)!}\kappa^{k-1}\eta^{2-2k}=0, \tag{30}$$

and the constant q is given by

$$q = -\frac{b_0^{(i)}}{(D-2)}\eta^D - \sum_{k=1}^{[\frac{D-1}{2}]} \frac{b_k (D-3)!\kappa^k}{(D-2k-2)!}\eta^{D-2k}. \qquad (31)$$

It is a matter of check to see that injecting the solutions given by (27)–(30) into the field equations of the action (26) will give rise the Vaidya conditions

$$\mathcal{G}_{\mu\nu} - T_{\mu\nu} = -\frac{(D-2)\dot{M}(u)}{r^{D-2}}\delta_\mu^u \delta_\nu^u, \qquad (32)$$

where $\mathcal{G}_{\mu\nu}$ is the Lovelock tensor. As anticipated, the metric solution of the polynomial equation (29) by means of the limiting process $D \to 4$ described in [12] will yield to (21) with the couplings $\alpha_5 = \beta_5 = \lambda_5 = 0$.

IV. CONCLUSIONS

In general, it is rather difficult to find time-dependent spherically symmetric solutions to Einstein's equations in the presence of a some matter source. There are, however, a few examples, such as stealth configurations, see e.g. [15–17] and [7], or even the example of time-dependent spherically symmetric solution that describes the gravitational collapse to a scalar black hole in three dimensions, see [18,19]. Here, we have considered some classes of scalar tensor theories such that a certain combination of the Einstein equations can be factorized out as (5) within a spherical ansatz of the form (3). For these theories, we have shown that from static black hole configurations, and by extending their mass parameter to a function of the retarded time, one can end up with Vaidya-like configurations satisfying

$$\mathcal{G}_{\mu\nu} - T_{\mu\nu} = -\frac{(D-2)\dot{M}(u)}{r^{D-2}}\delta_\mu^u \delta_\nu^u.$$

In general, it is a nontrivial task to find matter that may source the Vaidya geometries, that is some source that compensates the right-hand side of the previous equation by means of its energy-momentum tensor. For example, as shown in [20], it would be impossible for a massless scalar field minimally coupled to Einstein gravity, and this even if the scalar field with lightlike gradient behaves like a pure radiation field.

On the other hand, we are convinced that promoting the mass constant to an arbitrary function of the retarded time will not always yield to Vaidya-like configurations. In fact, the possibility of generating such Vaidya-type solutions from static solutions in our case is essentially due to the factorization of the equation as given in (6) together with the fact that the scalar field solution of the first branch does not depend on M. In order to reinforce our intuition, we can consider the examples of the static black hole solution of a conformally scalar field known as the BBMB solution [21,22] or its self-interacting extension [23]. These both theories are particular cases of those considered here since they correspond to the action (26) with $\lambda_5 = \beta_5 = \alpha_4 = \alpha_5 = \lambda_4 = 0$ (and in the self-interacting case $\lambda_4 \neq 0$). Nevertheless, the main differences are due to the fact since $\alpha_4 = 0$, the factorization (6) yields only to the first branch, and, in this case, the static scalar field solution depends explicitly on the mass constant M. It is then a matter of check to see that even by promoting the constant M to an arbitrary function of time $M = M(u)$ the full equations will yield inconsistencies unless $M = \text{cst}$.

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APPENDIX: FIELD EQUATIONS

The field equations obtained by varying the action (2) with respect to the metric read

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \qquad (A1)$$

where

$$T_{\mu\nu} = \beta_4 e^{2\phi}\mathcal{A}_{\mu\nu}^{(4)} + \alpha_4 \mathcal{H}_{\mu\nu}^{(4)} - \lambda_4 e^{4\phi} g_{\mu\nu} + \beta_5 e^{3\phi}\mathcal{A}_{\mu\nu}^{(5)} + \alpha_5 e^{\phi}\mathcal{H}_{\mu\nu}^{(5)} - \lambda_5 e^{5\phi} g_{\mu\nu}$$

where the terms $\mathcal{A}_{\mu\nu}^{(4)}$, $\mathcal{A}_{\mu\nu}^{(5)}$, $\mathcal{H}_{\mu\nu}^{(4)}$ and $\mathcal{H}_{\mu\nu}^{(5)}$ are those associated with the energy-momentum tensor of the scalar field, and are given by

$$\begin{aligned}
\mathcal{A}_{\mu\nu}^{(4)} &= G_{\mu\nu} + 2\nabla_\mu\phi\nabla_\nu\phi - 2\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}(2\Box\phi + (\nabla\phi)^2), \\
\mathcal{A}_{\mu\nu}^{(5)} &= G_{\mu\nu} + 3\nabla_\mu\phi\nabla_\nu\phi - 3\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}(3\Box\phi + (\nabla\phi)^2), \\
\mathcal{H}_{\mu\nu}^{(4)} &= -2G_{\mu\nu}(\nabla\phi)^2 + 4P_{\mu\alpha\nu\beta}(\nabla^\alpha\nabla^\beta\phi - \nabla^\alpha\phi\nabla^\beta\phi) \\
&\quad + 4(\nabla_\alpha\phi\nabla_\mu\phi - \nabla_\alpha\nabla_\mu\phi)(\nabla^\alpha\phi\nabla_\nu\phi - \nabla^\alpha\nabla_\nu\phi) \\
&\quad + 4(\nabla_\mu\phi\nabla_\nu\phi - \nabla_\nu\nabla_\mu\phi)\Box\phi + g_{\mu\nu}(-2(\Box\phi)^2 + (\nabla\phi)^4) \\
&\quad + 2g_{\mu\nu}\nabla_\beta\nabla_\alpha\phi(\nabla^\beta\nabla^\alpha\phi - 2\nabla^\alpha\phi\nabla^\beta\phi), \\
\mathcal{H}_{\mu\nu}^{(5)} &= -4G_{\mu\nu}(\nabla\phi)^2 + 4P_{\mu\alpha\nu\beta}(\nabla^\alpha\nabla^\beta\phi - \nabla^\alpha\phi\nabla^\beta\phi) \\
&\quad + 8(\nabla_\mu\phi\nabla_\nu\phi - \nabla_\nu\nabla_\mu\phi)\Box\phi - 4g_{\mu\nu}\Box\phi(\Box\phi + (\nabla\phi)^2) \\
&\quad + 4g_{\mu\nu}\nabla_\alpha\nabla_\beta\phi(\nabla^\alpha\nabla^\beta\phi - 2\nabla^\alpha\phi\nabla^\beta\phi) \\
&\quad + 8(\nabla_\mu\phi\nabla_\nu\nabla_\alpha\phi\nabla^\alpha\phi + \nabla_\nu\phi\nabla_\mu\nabla_\alpha\phi\nabla^\alpha\phi) \\
&\quad - 8\nabla_\mu\nabla_\alpha\phi\nabla_\nu\nabla^\alpha\phi + 4(\nabla\phi)^2(\nabla_\mu\nabla_\nu\phi - 3\nabla_\mu\phi\nabla_\nu\phi).
\end{aligned} \qquad (A2)$$

Here, the tensor $P_{\mu\alpha\nu\beta}$ is defined as follows

$$P_{\mu\alpha\nu\beta} = R_{\mu\alpha\nu\beta} + g_{\mu\beta}R_{\alpha\nu} + g_{\alpha\nu}R_{\mu\beta} - g_{\mu\nu}R_{\alpha\beta} - g_{\alpha\beta}R_{\mu\nu} + \frac{1}{2}(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu})R.$$

The variation of the action (2) with respect to the scalar field yields

$$0 = \beta_4 e^{2\phi}\mathcal{A}^{(4)} + \alpha_4 e^{\phi}\mathcal{H}^{(4)} + 8\lambda_4 e^{4\phi} + \beta_5 e^{3\phi}\mathcal{A}^{(5)} + \alpha_5 e^{\phi}\mathcal{H}^{(5)} + 10\lambda_5 e^{5\phi}, \tag{A3}$$

where $\mathcal{A}^{(4)}$, $\mathcal{A}^{(5)}$, $\mathcal{H}^{(4)}$ and $\mathcal{H}^{(5)}$ are is given by

$$\mathcal{A}^{(4)} = 2(R - 6\Box\phi - 6(\nabla\phi)^2),$$
$$\mathcal{A}^{(5)} = 3(R - 8\Box\phi - 12(\nabla\phi)^2),$$

and where

$$\mathcal{H}^{(4)} = \mathcal{G} + 8(G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - R^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \Box\phi(\nabla\phi)^2) + 8(2\nabla^\mu\phi\nabla_\mu\nabla_\nu\phi\nabla^\nu - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi + \phi(\Box\phi)^2),$$
$$\mathcal{H}^{(5)} = \mathcal{G} + 16(G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - R^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 3\Box\phi(\nabla\phi)^2) + 24(((\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi + (\nabla\phi)^4)) - 4R(\nabla\phi)^2 + 48\nabla^\mu\phi\nabla_\mu\nabla_\nu\phi\nabla^\nu\phi.$$

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