Revisiting the *I*-Love-*Q* relations for superfluid neutron stars

Eneko Aranguren,[1,*] José A. Font,[2,3,†] Nicolas Sanchis-Gual,[2,‡] and Raül Vera[1,§]

[1]*Fisika Saila, University of the Basque Country UPV/EHU, 48080 Bilbao, Basque Country*
[2]*Departamento de Astronomía y Astrofísica, Universitat de València, Dr. Moliner 50, 46100, Burjassot (Valencia), Spain*
[3]*Observatori Astronòmic, Universitat de València, Catedrático José Beltrán 2, 46980, Paterna, Spain*

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We study the tidal problem and the resulting I-Love-Q approximate universal relations for rotating superfluid neutron stars in the Hartle-Thorne formalism. Superfluid stars are described in this work by means of a two-fluid model consisting of superfluid neutrons and all other charged constituents. We employ a stationary and axisymmetric perturbation scheme to second order around a static and spherically symmetric background. Recently, we used this scheme to study isolated rotating superfluid stars. In this paper it is applied to analyze the axially symmetric sector of the tidal problem in a binary system. We show that a consistent use of perturbative matching theory amends the original two-fluid formalism for the tidal problem to account for the possible nonzero value of the energy density at the boundary of the star. This is exemplified by building numerically different stellar models spanning three equations of state. Significant departures from universality are found when the correct matching relations are not taken into account. We also present an augmented set of universal relations for superfluid neutron stars which includes the contribution to the total mass of the star at second order, δM. Therefore, our results complete the set of universal relations for rotating superfluid stars, generalizing our previous findings in the perfect fluid case.

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I. INTRODUCTION

The analysis of the inspiral gravitational-wave signal emitted during a binary neutron star (BNS) coalescence provides information on the internal structure of neutron stars and on the supranuclear equation of state (EOS). In a BNS system, the tidal field of the companion induces a mass-quadrupole moment and accelerates the coalescence. The ratio of the induced quadrupole moment to the external tidal field is proportional to the tidal Love number of the star, k_2, or to the tidal deformability $\lambda_2 = (2/3)k_2[(c^2/G)(R/M)]^5$, where R and M refer to the radius and mass of the star. The strength of tidal interactions increases rapidly during the final tens of gravitational-wave inspiral cycles before merger, making their effects potentially measurable [1–5]. This has been put into practice in the analysis of GW170817 and GW190425, the first two (and, so far, only) BNS systems detected by the LIGO-Virgo-KAGRA Collaboration [6–9]. The tidal deformability of these systems was measured using EOS-insensitive relations between the moment of inertia I, the tidal deformability λ_2 (or the Love number k_2) and the spin-induced quadrupole moment Q, known as I-Love-Q relations [10,11]. In the case of GW170817, the observational constraints on the tidal deformation of the binary components allowed to rule out some of the stiffest supranuclear EOS models.

The most basic theoretical treatment of the tidal problem in a binary system [12,13] fits in the Hartle-Thorne scheme (HT hereafter) [14,15], a pioneer proposal that provides a perturbative framework in general relativity to describe the equilibrium configuration of a compact and isolated perfect fluid body around a static and spherically symmetric configuration, up to second order. Within the HT scheme the tidal problem can be solved in the regime of stationary and axial perturbations (see [16] and references therein). For this problem, the I-Love-Q relations found in [10] were first seen to split into two categories, one valid for ordinary neutron stars and another one for quark stars, the latter characterized by the presence of a nonvanishing energy density at the boundary of the star. Shortly after, [17] amended the results of [10] by considering a term used by [1] [see Eq. (15) in the latter reference] to account for a possible nonzero value of the energy density at the stellar boundary when computing the Love number. This, in turn, justified the result obtained by [16] for the limiting case of homogeneous stars. The correction reported by [17] provided universal I-Love-Q relations regardless of the EOS type.

*eneko.aranguren@ehu.eus
†j.antonio.font@uv.es
‡nicolas.sanchis@uv.es
§raul.vera@ehu.eus

The proof that Eq. (15) in [1] is indeed the correct expression for the Love number was reported in [18] building on the amendment to the original HT scheme provided in [19]. We recall that the original HT scheme implicitly assumes that *all* functions describing the perturbations are continuous everywhere, in particular at the boundary of the star. Apart from providing the needed results to put the HT scheme on firm grounds, the main point of the amendment was to prove the inconsistency of this assumption because, although most of the interior and exterior parts of the functions must indeed share the same value at the boundary, some of the functions do present a jump that is proportional to the value of the energy density there.

The rigorous support and partial correction to the original HT scheme reported in [19] (see also [20,21]) is obtained by producing an initial framework resorting to perturbation theory in purely geometric terms. On top of that, the equations for the matter content at the stellar interior are to be imposed. Reference [19] focused on perfect fluid stars (with barotropic EOS) finding that the discontinuity of one perturbation function due to the nonvanishing of the energy density at the boundary affects the computation of the contribution to the mass of the star at second order, δM (for a given fixed central pressure). This was first used in [22] to revisit the seminal work of [23] on homogeneous rotating stars, and the significant correction to the total mass was underlined. Second, the correction in the computation of the mass was used in [18] to show that I-Love-Q EOS-insensitive relations also apply to δM, thus extending the universality to a family of four parameters, I-Love-$Q-\delta M$.

The original HT model was also adapted in [24] to describe slowly rotating, *superfluid* neutron stars, building on a two-fluid formalism introduced by [25,26]. This adaptation, however, inherited the incorrect (implicit) assumptions from the original HT scheme regarding the continuity of the perturbation functions at the stellar surface. This has been recently fixed in [27] where we have used the geometrical perturbation scheme of [19] (see also [21]) to amend the two-fluid formalism of isolated rotating superfluid stars.

Despite the fact that the results in [18,19] provide the perturbation formalism for the tidal problem with a geometrical justification to correctly compute the tidal number leading to the universality of the I-Love-Q relations, those works have been overlooked by several subsequent studies. In particular, the two-fluid model has been also used by [28] to study the I-Love-Q relations for superfluid neutron stars imposing the continuity of all functions (and some derivatives) without justification to compute the tidal deformability. We note that, in principle, one cannot resort to any known explicit result or correct expression for the Love number, since those apply to the perfect fluid case.

The aim of the present paper is to explore the tidal problem and the approximate universal relations for superfluid neutron stars, revisiting the results of [28] using the corrected HT scheme we started developing in our previous work [27]. As in [24,27–29] we describe superfluid stars by a simple two-fluid model which accounts for superfluid neutrons and all other constituents. Using a toy-model EOS for which the number densities of the two constituents do not vanish at the boundary of the star, we showed in [27] that the corrections to the HT formalism do impact the structure of rotating superfluid neutron stars in a significant way. In this paper we demonstrate that the study of the tidal problem for superfluid stars is also affected by the same continuity issues. Therefore, although we check that the EOSs used in [28,29] do not present those issues due to the vanishing of the relevant physical quantities at the boundary,[1] the correction of the HT formalism we report here needs to be considered for general-purpose (i.e. EOS insensitive) computations of the tidal problem in a binary system.

The structure of this paper is as follows: In Sec. II we briefly recall the two-fluid formalism and the construction of the global interior/exterior configuration. In Secs. III and IV we briefly describe the perturbation scheme for a two-fluid model and we develop the background configuration of superfluid neutron stars. Thus, these sections lay the groundwork for the notation that will be employed later on. Next, Secs. V A and V B describe the first and second order problems, respectively. Once the general setup has been constructed, Sec. VI addresses the tidal problem and the Love numbers obtained therein. Correspondingly, Sec. VII presents our results to test the universality of our I-Love-Q-δM relations, for a variety of physically motivated EOS, as well as a toy model. Our conclusions are summarized in Sec. VIII. Unless otherwise stated (see, for example, Appendix A), we will be using units such that $G = c = 1$.

II. TWO-FLUID MODEL

In the two-fluid formalism, as originally developed in [25,30] (see also [24,26]), the flow of neutrons and protons are described, respectively, by two vectors

$$n^\alpha = n u^\alpha, \qquad p^\alpha = p v^\alpha,$$

where u^α and v^α are two unit timelike vectors and n and p are the neutron and proton number densities. The coupling of the neutrons and protons is described by the quantity $x^2 := -p_\alpha n^\alpha$. The EOS of the whole system is provided by specifying a master function

$$\Lambda = \Lambda(n^2, p^2, x^2),$$

that depends on three arguments.

[1]Unfortunately, we find no explicit mention on the behavior of the fluid quantities at the boundary in those works.

In terms of the auxiliary functions

$$\mathcal{A} \coloneqq -\frac{\partial\Lambda(n^2, p^2, x^2)}{\partial x^2}, \qquad \mathcal{B} \coloneqq -2\frac{\partial\Lambda(n^2, p^2, x^2)}{\partial n^2}, \qquad \mathcal{C} \coloneqq -2\frac{\partial\Lambda(n^2, p^2, x^2)}{\partial p^2},$$

the 1-forms,

$$\mu_\alpha \coloneqq \mathcal{B}n_\alpha + \mathcal{A}p_\alpha, \qquad \chi_\alpha \coloneqq \mathcal{C}p_\alpha + \mathcal{A}n_\alpha,$$

are the dynamically and thermodynamically conjugates to n^α and p^α, respectively. The energy-momentum tensor of the fluid is then given by

$$T^\alpha{}_\beta = \Psi\delta^\alpha_\beta + p^\alpha\chi_\beta + n^\alpha\mu_\beta, \tag{1}$$

where

$$\Psi \coloneqq \Lambda - n^\alpha\mu_\alpha - p^\alpha\chi_\alpha \tag{2}$$

acts as a generalized pressure.

The equations of motion are given by the conservation equations

$$\nabla_\alpha n^\alpha = 0, \qquad \nabla_\alpha p^\alpha = 0, \tag{3}$$

plus the Euler equations

$$n^\alpha(\nabla_\alpha\mu_\beta - \nabla_\beta\mu_\alpha) = 0, \qquad p^\alpha(\nabla_\alpha\chi_\beta - \nabla_\beta\chi_\alpha) = 0. \tag{4}$$

Equations (3) and (4) imply $\nabla^\alpha T_{\alpha\beta} = 0$.

The two problems at hand will be framed in a stationary and axially symmetric setting (describing the perturbations) over a static and spherically symmetric background configuration. As customary, we use spherical coordinates $\{t, r, \theta, \phi\}$ arranged so that the timelike and axial (spacelike) Killing vector fields of the whole setting read ∂_t and ∂_ϕ, respectively. Thus, the functions describing the stationary and axisymmetric spacetime geometry g_{STAX} and the fluids only depend on r and θ.

Moreover, if the fluids are assumed to rotate around the axis so that there are no convective motions, and the rotation is rigid, then

$$u \propto (\partial_t + \tilde{\Omega}_n\partial_\phi), \qquad v \propto (\partial_t + \tilde{\Omega}_p\partial_\phi), \tag{5}$$

for some constants $\tilde{\Omega}_n$ and $\tilde{\Omega}_p$, which represent the angular velocities of neutrons and protons, respectively. In this case (3) are automatically satisfied and (4) are equivalent to

$$\mu_c = -g_{\mathrm{STAX}}(\partial_t + \tilde{\Omega}_n\partial_\phi, \mu), \qquad \chi_c = -g_{\mathrm{STAX}}(\partial_t + \tilde{\Omega}_p\partial_\phi, \chi), \tag{6}$$

for some constants μ_c and χ_c. We use $g_{\mathrm{STAX}}(\cdot,\cdot)$ for the scalar product in the index-free notation.

A. Global configuration: Vacuum exterior

The global model of the star consists of two spacetimes $(\mathcal{M}^+, g^+_{\mathrm{STAX}})$ and $(\mathcal{M}^-, g^-_{\mathrm{STAX}})$ with timelike boundaries Σ^+ and Σ^- which are pointwise identified $\Sigma \equiv \Sigma^+ = \Sigma^-$, to produce a joined spacetime $(\mathcal{M}, g_{\mathrm{STAX}})$ with $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$, and g_{STAX} is $g^\pm_{\mathrm{STAX}}$ on each region $\mathcal{M}^\pm$ accordingly. The identification is required to be isometric, so that Σ has an induced metric h. This requires the well-known first matching (or junction) conditions, $h \coloneqq h^+ = h^-$, where $h^\pm$ are the induced metrics of Σ as embedded on $(\mathcal{M}^\pm, g^\pm_{\mathrm{STAX}})$, respectively. Then, g_{STAX} can be extended continuously on $\mathcal{M}$. To avoid a distributional Riemann tensor on $(\mathcal{M}, g_{\mathrm{STAX}})$, which is equivalent to avoid energy surface layers at the boundary of the star in general relativity, we must demand that the second fundamental forms (extrinsic curvatures) $\kappa^\pm$ of Σ as embedded on $(\mathcal{M}^\pm, g^\pm_{\mathrm{STAX}})$ agree. To sum up, the full matching conditions require then that $h^+ = h^-$ and $\kappa^+ = \kappa^-$ hold on Σ.

We take the $+$ part to describe the interior of the star, thus solving the two-fluid model problem, and the $-$ part to describe the vacuum exterior. The (history of the) surface of the star is provided by Σ. For the two problems we are interested in, we assume that both spacetimes are stationary and axisymmetric, so that the boundaries inherit the two symmetries [31].

The interior and exterior problems are then solved imposing "regularity" at the origin, and whatever conditions we want to impose on the exterior, in addition to the relations on Σ provided by the matching conditions. In particular, the matching conditions also determine, in principle, the surface of the star. As shown in [27] for our stationary and axisymmetric setting, the matching conditions imply the continuity of Ψ across Σ. Therefore

$$\Psi(r, \theta) = 0$$

determines the surface of the star $r = r(\theta)$ implicitly. This condition is not sufficient, but it is the only one involving only the interior side, as shown in [32] (see also [33]) for the general stationary and axisymmetric setting. The rest of the matching conditions provide the matching hypersurface from the other side Σ^- and relations between the boundary data for the interior and exterior problems.

In the following we will denote by $[f]$, for any function f, the difference of f evaluated at both sides of the hypersurface Σ, i.e., $[f](p) \coloneqq f^+|_{p^+\in\Sigma^+} - f^-|_{p^-\in\Sigma^-}$, where $p(\in\Sigma) = p^+ = p^-$ after the identification. Moreover, if $[f] = 0$ we will simply use f when evaluated on either Σ^+ or Σ^-. All expressions in square brackets will denote their difference.

III. PERTURBATION SCHEME

In this paper we focus on two problems, namely, the deformation due to the rotation of an isolated star, and the axially symmetric sector of the (even-parity) tidal problem caused by a companion star. The two issues are dealt with as two perturbative stationary and axisymmetric problems over a static and spherically symmetric configuration.

For the perturbative problems we use perturbation theory in metric theories of gravity, which is, in effect, a gauge-field theory of symmetric tensors on a given background configuration at each order. In particular, our work is based on the concept of perturbation scheme, which includes the notion of classes of gauges, that inherits some of the symmetries of the background. We refer the reader to [20] for the detailed definitions and the gauge fixing procedures involved in axially symmetric and axistationary second order perturbations around spherical backgrounds. To deal with the matching of the exterior and interior regions we use the theory of perturbed matchings, based on perturbations of hypersurfaces to second order [34], particularized to the case of stationary and axially symmetric perturbations in [21] (see also [19]).

A. Perturbation theory in rigidly rotating two-fluid stars

The perturbative problem of the isolated rotating star modeled by a two-fluid has been already dealt with in our previous Ref. [27], revisiting and amending the approach and results in [24]. However, for completeness we include here an outline of the whole procedure because the tidal problem shares most part of the setting. Thus, we follow the stationary and axisymmetric perturbative scheme to second order around a static and spherically symmetric background $(\mathcal{M}, g)$ as described in [20] (see also [19,21]) based on an abstract perturbation parameter ε. In short, we have a family of stationary and axisymmetric spacetimes $(\mathcal{M}_\varepsilon, \tilde{g}_\varepsilon)$, where $(\mathcal{M}_0, \tilde{g}_0) = (\mathcal{M}, g)$ is our static and spherically symmetric background, together with a class of point identifications $\Gamma_\varepsilon\colon \mathcal{M} \to \mathcal{M}_\varepsilon$ (spacetime gauges), where Γ_0 is the identity. This class of gauges is, so far, only restricted to inherit the stationarity and axial symmetry generated by ∂_t and ∂_ϕ in the background as defined in [20].

On each $(\mathcal{M}_\varepsilon, \tilde{g}_\varepsilon)$ we have defined the two-fluid model quantities, and the equations they satisfy, that depend on ε. The metrics $\tilde{g}_\varepsilon$ as well as all the fluid quantities, and the corresponding equations, are pulled back using Γ_ε^* onto $(\mathcal{M}, g)$.

In particular, the procedure defines a family of metrics $g_\varepsilon = \Gamma_\varepsilon^*(\tilde{g}_\varepsilon)$ on $\mathcal{M}$. The first order K_1 and second order K_2 perturbation tensors are defined as the first and second order derivatives of g_ε with respect to ε, evaluated at $\varepsilon = 0$. As a result, the ε family of metrics can be written as the usual

$$g_\varepsilon = g + \varepsilon K_1 + \frac{1}{2}\varepsilon^2 K_2 + O(\varepsilon^3).$$

If we take, as explained above, spherical coordinates $\{t, r, \theta, \phi\}$ on $(\mathcal{M}, g)$, then the inheriting of the symmetries by the class of gauges Γ_ε means that ∂_t and ∂_ϕ are Killings of the whole family g_ε. Therefore, just like g, the perturbation tensors K_1 and K_2 do not depend on t nor ϕ. Now, suitable gauge-fixing procedures can be used to simplify further the forms of K_1 and K_2.

Similarly, for every two-fluid model quantity we have a corresponding ε family of quantities defined on $\mathcal{M}$, and thence background, first and second order corresponding quantities. Explicitly, the number density of neutrons and protons are decomposed to second order as (we follow the notation from [24])

$$n_\varepsilon(r,\theta) = n_0(r)(1 + \varepsilon^2\eta(r,\theta)) + O(\varepsilon^3), \tag{7}$$

$$p_\varepsilon(r,\theta) = p_0(r)(1 + \varepsilon^2\Phi(r,\theta)) + O(\varepsilon^3). \tag{8}$$

The fact that there is no contribution at first order is a consequence of the forms K_1 and K_2 take in the stationary and axisymmetric perturbative setting over a static and spherical background configuration. A rigorous account on this matters is made in [20,21] for the perfect fluid case. For the purposes of this work we will assume the usual forms of the perturbation tensors and this decomposition for the two-fluid quantities, which is consistent, from the beginning.

As in the perfect fluid case, where the same equation of state is assumed for the whole (background and perturbations) configuration, here one demands $\Lambda_\varepsilon(n_\varepsilon^2, p_\varepsilon^2, x_\varepsilon^2) = \Lambda(n_\varepsilon^2, p_\varepsilon^2, x_\varepsilon^2)$. In the following we use the notation $\Lambda_\varepsilon \coloneqq \Lambda(n_\varepsilon^2, p_\varepsilon^2, x_\varepsilon^2)$, so that $\Lambda_0 = \Lambda(n_0^2, p_0^2, x_0^2)$. We will also use $\Lambda_0(r) \coloneqq \Lambda(n_0^2(r), p_0^2(r), x_0^2(r))$ and equivalently for $\Psi_0(r)$. The flows u_ε and v_ε have the form of (5) with some $\tilde{\Omega}_{n\varepsilon}$ and $\tilde{\Omega}_{p\varepsilon}$ (only dependency on ε). Since the background is static we have $\tilde{\Omega}_{n0} = \tilde{\Omega}_{p0} = 0$. On the other hand, it is (implicitly) assumed that after a redefinition of the perturbation parameter to absorb the second order contributions in $\tilde{\Omega}$s, we have[2]

$$\tilde{\Omega}_{n\varepsilon} = \varepsilon\Omega_n + O(\varepsilon^3),$$
$$\tilde{\Omega}_{p\varepsilon} = \varepsilon\Omega_p + O(\varepsilon^3)$$

for some pair of constants Ω_n and Ω_p. The full form of the flows u_ε and v_ε as well as the set of ε families,

[2]The fact that the redefinition of the perturbation parameter ε to absorb the second order contribution to Ω is consistent with the problem to second order should be proven after all the problem has been set. To our knowledge this has been only (rigorously) proven in the rigidly rotating perfect fluid case, in [20,21].

$\{x_\varepsilon, \mu_\varepsilon^\alpha, \chi_\varepsilon^\alpha, \Psi_\varepsilon, \mu_{c\varepsilon}, \chi_{c\varepsilon}\}$, are then found using the expressions from Sec. II, taking into account that

$$\mathcal{A}_\varepsilon = -\frac{\partial\Lambda(n_\varepsilon^2, p_\varepsilon^2, x_\varepsilon^2)}{\partial x_\varepsilon^2}, \qquad \mathcal{B}_\varepsilon = -2\frac{\partial\Lambda(n_\varepsilon^2, p_\varepsilon^2, x_\varepsilon^2)}{\partial n_\varepsilon^2},$$
$$\mathcal{C}_\varepsilon = -2\frac{\partial\Lambda(n_\varepsilon^2, p_\varepsilon^2, x_\varepsilon^2)}{\partial p_\varepsilon^2}, \tag{9}$$

plus the tensors K_1 and K_2. From those quantities we construct ${T_\varepsilon}^\alpha{}_\beta$ using (1) accordingly. The expressions for the rotating perturbation case (to second order) are given in full in [27].

Since the Einstein field equations hold on each $(\mathcal{M}_\varepsilon, \tilde{g}_\varepsilon)$, the corresponding pullbacks onto $(\mathcal{M}, g)$ must also hold, and therefore

$$Ein(g_\varepsilon)^\alpha{}_\beta = \varkappa {T_\varepsilon}^\alpha{}_\beta, \tag{10}$$

must be satisfied for all ε, where $\varkappa = 8\pi G/c^4$ and $Ein(g_\varepsilon)$ is the Einstein tensor computed from g_ε. The background equations are (10) evaluated at $\varepsilon = 0$, while the first and second order Einstein equations correspond to the first and second order derivatives with respect to ε evaluated at $\varepsilon = 0$ respectively.

Similarly, the Euler equations (6) apply for $g_{\mathrm{STAX}} = g_\varepsilon$ and all the quantities substituted by their ε counterparts on the right-hand side. To use the notation of [24,27], the ε families of constants $\mu_{c\varepsilon}$ and $\chi_{c\varepsilon}$ are explicitly written as

$$\mu_{c\varepsilon} = \mu_\infty(1 + \varepsilon^2\gamma_n) + O(\varepsilon^3),$$
$$\chi_{c\varepsilon} = \chi_\infty(1 + \varepsilon^2\gamma_p) + O(\varepsilon^3), \tag{11}$$

which define the four constants $\mu_\infty(= \mu_{c0})$, $\chi_\infty(= \chi_{c0})$, γ_n and γ_p.

We finish this section with a brief comment on the perturbation parameters. Let us first stress that, apart from the boundary data needed to solve the background configuration, the exact model only contains two free parameters. These correspond to the rotating parameters $\tilde{\Omega}_n$ and $\tilde{\Omega}_p$. In the perturbative approach we have instead three, namely Ω_n, Ω_p and ε. The introduction of a spurious parameter is a consequence of the scalability property of perturbation theory. Computationally one chooses freely one of the three parameters, say $\Omega_p = 1$. Then, for a desired value of the relative rotation rate $\Delta \coloneqq \Omega_n/\Omega_p$ fixes Ω_n accordingly. After solving the problems, one finds the convenient measurable physical quantities and uses the scalability property to fix the model to the data needed.

B. Perturbed matching

Let us be given a static and spherically symmetric background global configuration $(\mathcal{M}, g)$, composed by $(\mathcal{M}^+, g^+, \Sigma^+)$ and $(\mathcal{M}^-, g^-, \Sigma^-)$ with identified boundaries $\Sigma \coloneqq \Sigma^+ = \Sigma^-$, and such that the matching conditions $h^+ = h^-$ and $\kappa^+ = \kappa^-$ hold on Σ. Assume now that the global configuration setting described in Sec. II A applies to a ε family of spacetimes $(\mathcal{M}_\varepsilon, \tilde{g}_\varepsilon)$ such that $(\mathcal{M}, g) = (\mathcal{M}_0, \tilde{g}_0)$. That is, we take $(\mathcal{M}_\varepsilon, \tilde{g}_\varepsilon)$, for each ε around 0, to be composed by two spacetimes with boundary $(\mathcal{M}_\varepsilon^+, \tilde{g}_\varepsilon^+, \tilde{\Sigma}_\varepsilon^+)$ and $(\mathcal{M}_\varepsilon^-, \tilde{g}_\varepsilon^-, \tilde{\Sigma}_\varepsilon^-)$ so that $\tilde{\Sigma}_\varepsilon \coloneqq \tilde{\Sigma}_\varepsilon^+ = \tilde{\Sigma}_\varepsilon^-$ after some identification of points, and $\mathcal{M}_\varepsilon^+ \cap \mathcal{M}_\varepsilon^- = \tilde{\Sigma}_\varepsilon$. The matching conditions $\tilde{h}_\varepsilon^+ = \tilde{h}_\varepsilon^-$, $\tilde{\kappa}_\varepsilon^+ = \tilde{\kappa}_\varepsilon^-$ are satisfied on each $\tilde{\Sigma}_\varepsilon$ by construction, and $h_0^\pm = h^\pm$ and $\kappa_0^\pm = \kappa^\pm$.

Prior to prescribing that identification of points between the boundaries at each ε, we must also prescribe the identification of points amongst each of the two ε families of boundaries $\tilde{\Sigma}_\varepsilon^+$ and $\tilde{\Sigma}_\varepsilon^-$. After the identification of points, and thus the construction of $\tilde{\Sigma}_\varepsilon$, we are only left with a prescription of the identification of points along the ε family of hypersurfaces $\tilde{\Sigma}_\varepsilon$, namely $\Upsilon_\varepsilon\colon \Sigma \to \tilde{\Sigma}_\varepsilon$. This gives rise to the so-called hypersurface gauge [34] (see also [35] for a different approach to first order). As the families of metrics $\tilde{g}_\varepsilon^\pm$ are pulled back onto $\mathcal{M}$ using the spacetime gauges at each side to obtain the families of metrics $g_\varepsilon^\pm = \Gamma_\varepsilon^{\pm *}(\tilde{g}_\varepsilon^\pm)$ at each side of $\mathcal{M}$, the matching conditions are pulled back using Υ onto Σ to obtain the relations

$$h_\varepsilon^+ = h_\varepsilon^-, \qquad \kappa_\varepsilon^+ = \kappa_\varepsilon^-, \tag{12}$$

where $h_\varepsilon^\pm \coloneqq \Upsilon_\varepsilon^*(\tilde{h}_\varepsilon^\pm)$ and $\kappa_\varepsilon^\pm \coloneqq \Upsilon_\varepsilon^*(\tilde{\kappa}_\varepsilon^\pm)$.

To understand how the perturbation of the hypersurface is described in this setting at each side, we simply have to define the family of hypersurfaces Σ_ε^+ on $\mathcal{M}^+$ by $\Sigma_\varepsilon^+ = \Gamma_\varepsilon^{-1}(\tilde{\Sigma}_\varepsilon^+)$, with $\Sigma_0^+ = \Sigma^+$ by construction, and the same for the $-$ side. Let us focus on the $+$ side, the $-$ side will be analogous. The family of hypersurfaces Σ_ε^+ describes how $\Sigma(= \Sigma^+)$ changes as a set of points in $\mathcal{M}^+$. On the other hand, if we take $p \in \Sigma$, the family of maps $\gamma_\varepsilon^+(p) \coloneqq \Gamma_\varepsilon^{+-1}(\Upsilon_\varepsilon^+(p))$ generates a curve on $\mathcal{M}^+$ starting at p and moving across each Σ_ε^+. The vector field Z_1 defined at every point p in Σ^+ as the velocity of the curve γ_ε, and the acceleration Z_2 at p can be decomposed as $Z_1^+ = Q_1^+\mathfrak{n}^+ + T_1^+$ and $Z_2^+ = Q_2^+\mathfrak{n}^+ + T_2^+$, for some functions $Q_{1/2}^+$ and tangential vectors $T_{1/2}^+$ to Σ^+, where $\mathfrak{n}^+$ is the unit normal to Σ^+. The information of the deformation of Σ, as a set of points, is thus encoded in the functions Q_1 and Q_2 at first and second order, respectively. The vectors $T_{1/2}^+$ determine how the points are identified within the family Σ_ε and therefore depend on both spacetime and hypersurface gauges. On the other hand, since the hypersurface gauge does not modify the matching hypersurfaces as sets of points and only affects how they are identified pointwise, Q_1^+ does not depend on the hypersurface gauge. However, at second order both gauges get involved in the quantity Q_2^+. This whole construction (dropping the $\pm$ indicators) is depicted in Fig. 1.

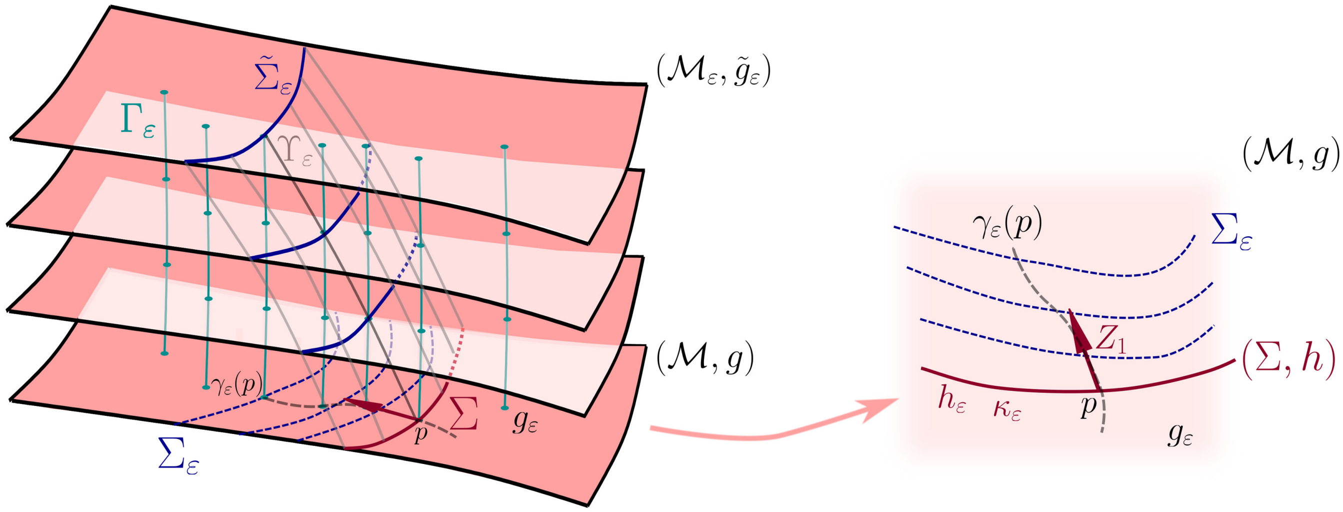

FIG. 1. Diagram to describe the setup of the perturbation theory for spacetimes and hypersurfaces as described in the main text. This picture applies to both the + and − families. The region that is to be matched, say $\mathcal{M}^+$ with boundary Σ^+, lies either on the left- or on the right-hand side of Σ^+. Observe that we have depicted only the members of the family $\mathcal{M}_\varepsilon$ for $\varepsilon \geq 0$, with $\mathcal{M} = \mathcal{M}_0$, but we could continue for negative values of ε. This is irrelevant because the perturbation procedure only involves the derivatives of the various ε-family objects evaluated at $\varepsilon = 0$, and the limit taken from positive ε equals the limit taken from negative values by construction. As shown, Z_1 is the tangent vector of the curve defined by $\gamma_\varepsilon(p)$ at p, while Z_2 corresponds to the acceleration of that curve at p, and it is not depicted here.

The first and second order matching conditions are the first and second order derivatives of the Eq. (12) with respect to ε on $\varepsilon = 0$. The explicit expressions at each side $\pm$ in terms of the perturbation tensors K_1 and K_2, plus $Q_{1/2}$ and $T_{1/2}$ were found in [34]. The particularization to stationary and axisymmetric perturbations around a spherical static background assuming axisymmetric surface deformations was presented in [19], while for arbitrarily deformed surfaces the job was done in [21]. Let us stress that this set of perturbed matching conditions arises by demanding that the Riemann tensor does not present a delta distribution, so it is thus *purely geometric and therefore independent of the field equations.* In [27] we used those results to write down the perturbed matching conditions for the two-fluid model at the boundary of the star, and used them to solve the isolated rotating star global problem. We will recall the relevant results below, and use them to obtain the perturbed matching for the tidal problem for two-fluid stars.

IV. BACKGROUND

As explained above, the background configuration is a globally static and spherically symmetric spacetime composed of the interior and exterior regions of the star. The geometric configuration is shared by both the isolated rotating star and the tidal problems. It therefore corresponds to the background configuration constructed in [24,26,27]. We briefly review the construction of the configuration in this section to fix some notation.

We consider two static spherically symmetric spacetimes with boundary $(\mathcal{M}^+, g^+, \Sigma^+)$ and $(\mathcal{M}^-, g^-, \Sigma^-)$ describing the interior and exterior of the star. In spherical coordinates $\{t_+, r_+, \theta_+, \phi_+\}$ and $\{t_-, r_-, \theta_-, \phi_-\}$ for the corresponding region, we take

$$g^\pm = -e^{\nu^\pm(r_\pm)}\mathrm{d}t_\pm^2 + e^{\lambda^\pm(r_\pm)}\mathrm{d}r_\pm^2 + r_\pm^2(\mathrm{d}\theta_\pm^2 + \sin^2\theta_\pm \mathrm{d}\phi_\pm^2),$$

for some pair of functions on each region, $\lambda^\pm$ and $\nu^\pm$. The boundaries, assumed to be timelike and taken to preserve the spherical and static symmetry [31], are given by $\Sigma^\pm \coloneqq \{r_\pm = R_\pm\}$, for some positive numbers $R_\pm > 0$. The gluing of Σ^+ and Σ^- is specified, without loss of generality, by $\theta_+ = \theta_-$, $\phi_+ = \phi_-$ and $t_+ = t_-$ on the boundaries, that we will denote as ϑ, φ and τ, respectively, as coordinates on Σ.

The interior of the neutron star is described in the background configuration by the two-fluid model introduced in Sec. II as described in Sec. III. We are thus given a master function $\Lambda \coloneqq \Lambda(n_0^2, p_0^2, x_0^2)$ as a function of three arguments. From that we can compute (9) with $\varepsilon = 0$ and construct the quantities

$$\mathcal{A}_0^0 \coloneqq \mathcal{A}_0 + 2\frac{\partial \mathcal{B}_0}{\partial p_0^2} n_0 p_0 + 2\frac{\partial \mathcal{A}_0}{\partial n_0^2} n_0^2 + 2\frac{\partial \mathcal{A}_0}{\partial p_0^2} p_0^2 + \frac{\partial \mathcal{A}_0}{\partial x_0^2} n_0 p_0,$$

$$\mathcal{B}_0^0 \coloneqq \mathcal{B}_0 + 2\frac{\partial \mathcal{B}_0}{\partial n_0^2} n_0^2 + 4\frac{\partial \mathcal{A}_0}{\partial n_0^2} n_0 p_0 + \frac{\partial \mathcal{A}_0}{\partial x_0^2} p_0^2,$$

$$\mathcal{C}_0^0 \coloneqq \mathcal{C}_0 + 2\frac{\partial \mathcal{C}_0}{\partial p_0^2} p_0^2 + 4\frac{\partial \mathcal{A}_0}{\partial p_0^2} n_0 p_0 + \frac{\partial \mathcal{A}_0}{\partial x_0^2} n_0^2$$

that encode second derivatives.

By definition we first have that $x_0^2 = n_0 p_0$. The interior problem is then composed of a system of four differential equations for the set $\{\lambda^+, \nu^+, n_0, p_0\}$. We refer to Sec. IV A in [27] for a full account and explicit expressions of the background interior problem. It is convenient to define the mass function in the interior of the star as $M^+(r_+) = r_+(1 - e^{-\lambda^+(r_+)})/2$. The exterior solution is given by $e^{\nu^-(r_-)} = e^{-\lambda^-(r_-)} = 1\text{–}2M/r_-$; i.e. it is the Schwarzschild geometry of mass M.

The matching conditions are $R \coloneqq R_+ = R_-$, together with $[\lambda] = [\nu] = [\nu'] = 0$, where the prime denotes the derivative with respect to the argument. Given the field equations, after imposing $R_+ = R_-$, the set of two matching conditions $\{[\lambda] = 0, [\nu'] = 0\}$ are equivalent to $\Psi_0(R) = 0$ and $M = M^+(R)$. In particular we then have $e^{\nu(R)} = e^{-\lambda(R)} = 1\text{–}2M/R$ and

$$\nu'(R) = 2e^{\lambda(R)}\frac{M}{R^2} = \frac{1}{R}\frac{2M}{R - 2M}. \tag{13}$$

In addition, the background field equations can be used to obtain [27]

$$[\lambda'] = -\varkappa R e^{\lambda(R)}\Lambda_0(R), \tag{14}$$

$$[\nu''] = -\varkappa\left(1 + \frac{R\nu'(R)}{2}\right)e^{\lambda(R)}\Lambda_0(R). \tag{15}$$

For some specific forms of Λ, the global background problem can be solved, i.e. the solution exists and is unique, given central values $n_0(0)$ and $p_0(0)$ within some ranges, at least numerically.

Once the interior problem is solved for some given values of n_0 and p_0 at the origin, $\Psi_0(R) = 0$ fixes the value of R and $M = M^+(R)$ determines M. We will assume from now on that $R > 2M$. The condition $[\nu] = 0$ is just used to set the value at the origin $\nu^+(0)$. Observe that $n_0(R)$, $p_0(R)$ and thus $\Lambda_0(R)$ take their values from the interior problem, are not constrained by the matching whatsoever and do not necessarily vanish.

Later we will make use of the background functions $\mu_0 \coloneqq n_0\mathcal{B}_0 + p_0\mathcal{A}_0$ and $\chi_0 \coloneqq p_0\mathcal{C}_0 + n_0\mathcal{A}_0$.

V. ISOLATED ROTATING STAR

The complete analysis for rotating stars is reported in our previous article [27]. The reader is addressed to this reference for details on the full sets of equations and a complete description of the computational procedure to solve the global problem at each order. Here, for the sake of completeness, we provide a succinct summary of the approach, using the same notation as in [27].

A. First order problem

We assume there exists a class of gauges for which the first order perturbation tensor at both sides has the form (we drop the $\pm$ indexes)

$$K_1 = -2r^2\omega(r)\sin^2\theta\mathrm{d}t\mathrm{d}\phi, \tag{16}$$

for some function $\omega(r)$ of the radial coordinate only and bounded at the origin.[3] The field equation in the interior for ω_+ is given by Eq. (49) in [27], while the equation in the exterior, for ω_-, is the same with a vanishing right-hand side.

Within the class of gauges that keeps K_1 with the form of (16), we have two gauge freedoms to set, one at each region $\pm$, that amount to the addition of a constant to $\omega_\pm$ correspondingly [19,21]. The gauge at the exterior can be fixed so that ω_- vanishes at infinity. With that choice the solution is given by $\omega_-(r) = 2J/r^3$, for some constant J, which accounts eventually for the total angular momentum.

Finally, the gauge in the interior can be fixed so that the first order matching conditions read [19,21,27]

$$[\omega] = [\omega'] = 0,$$

while the deformation quantities $Q_1^\pm(\tau, \vartheta, \varphi)$ satisfy $[Q_1] = 0$, $Q_1[\lambda'] = 0$, $Q_1[\nu''] = 0$.

The angular momentum of the individual fluids, defined in [25], are given explicitly by [24] (we drop the + subindex)

$$\begin{aligned} J_n &= -\frac{8\pi}{3}\int_0^R \mathrm{d}r\, r^4 e^{(\lambda-\nu)/2} \\ &\quad \times (\mu_0 n_0(\omega_+ - \Omega_n) + \mathcal{A}_0 n_0 p_0(\Omega_n - \Omega_p)), \\ J_p &= -\frac{8\pi}{3}\int_0^R \mathrm{d}r\, r^4 e^{(\lambda-\nu)/2} \\ &\quad \times (\chi_0 p_0(\omega_+ - \Omega_p) + \mathcal{A}_0 n_0 p_0(\Omega_n - \Omega_p)). \end{aligned}$$

The total angular momentum is recovered with $J = J_n + J_p$. Similarly, the moments of inertia of the individual fluids are given by $I_n = J_n/\Omega_n$ and $I_p = J_p/\Omega_p$, and the total moment of inertia is given by $I = I_n + I_p$.

B. Second order problem

At second order we assume that there exists a class of gauges in which the second order perturbation tensor at both sides (dropping the $\pm$ indexes) is given by the usual form

[3]We take this as an assumption. Although it has been extensively argued in the literature that this is an eventual consequence of the global problem, to our knowledge, a full proof of the analogous problem in the perfect fluid case has only been produced recently in [20,21].

$$K_2 = (-4e^{\nu(r)}h(r,\theta) + 2r^2\omega^2(r)\sin^2\theta)\mathrm{d}t^2 + 4e^{\lambda(r)}v(r,\theta)\mathrm{d}r^2 + 4r^2k(r,\theta)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{17}$$

with

$$h(r,\theta) = h_0(r) + h_2(r)P_2(\cos\theta),$$
$$v(r,\theta) = v_0(r) + v_2(r)P_2(\cos\theta),$$
$$k(r,\theta) = k_2(r)P_2(\cos\theta), \tag{18}$$

where $P_2(\cos\theta)$ is the Legendre polynomial $P_\ell(\cos\theta)$ with $\ell = 2$, and such that all functions are bounded at the origin. The fact that there is no $k_0(r)$ term fixes partially the class of gauges in the perturbation scheme. The gauge freedom that keeps the form (17) (see Proposition 6.11 in [20]) together with (18) is given by the second order gauge vector $V_2 \propto t\partial_t$ (plus any Killing vector of the background metric g).

As for the matter content, the contribution at second order of the number density of neutrons and protons is assumed to be of the form $\eta(r,\theta) = \eta_0(r) + \eta_2(r)P_2(\cos\theta)$ and $\Phi(r,\theta) = \Phi_0(r) + \Phi_2(r)P_2(\cos\theta)$, respectively. As explained in [27] in more length, the fact that there appear no $\ell > 2$ terms in the expansions of these quantities is justified in [24] using the arguments in the literature for the perfect fluid problem and assuming equatorial symmetry.

For convenience, we substitute the set $\{\eta_\ell(r), \Phi_\ell(r)\}$ by some auxiliary functions $\{\mathcal{P}_{\ell n}(r), \mathcal{P}_{\ell p}(r)\}$ [defined by Eq. (62) in [27]] that are more easily recognizable as "pressure"-like functions when compared to the perfect fluid case.

1. Second order matching

Let us consider K_2^+ and K_2^- of the form (17) with no conditions on $h(r,\theta)$, $v(r,\theta)$ and $k(r,\theta)$, and assume that the background and first order matching conditions are satisfied (no field equations used). The second order matching conditions are satisfied if and only if there exists a pair of functions $\Xi^\pm(\tau,\vartheta,\varphi)$ on Σ and free constants c_0, c_1, H_0 and H_1 such that [Eqs. (5.69)–(5.75) in [21], see also Proposition 2 [19]]

$$Q_1[\omega''] = 0, \tag{19a}$$

$$[\Xi] = Re^{\lambda(R)/2}(2c_0 + (2c_1 + H_1)\cos\vartheta), \tag{19b}$$

$$[k] = c_0 + c_1\cos\vartheta, \tag{19c}$$

$$[h] = \frac{1}{2}(H_0 + R\nu'(R)c_0) + \frac{1}{4}R\nu'(R)(H_1 + 2c_1)\cos\vartheta, \tag{19d}$$

$$[v - 2k - rk_{,r}] = \left(H_1 - \frac{1}{2}e^{\lambda(R)}(2c_1 + H_1)\right)\cos\vartheta + \frac{1}{2}\left[\Xi e^{-\lambda/2}\left(\frac{\lambda'}{2} - \frac{1}{r}\right)\right] - \frac{1}{4}e^{-\lambda(R)}Q_1^2[\lambda''], \tag{19e}$$

$$[h_{,r}] - \frac{R\nu'(R)}{2}[k_{,r}] - \nu'(R)\left(1 - \frac{R\nu'(R)}{2}\right)[k] = \frac{\nu'(R)}{2}\left\{\left(1 - \frac{R\nu'(R)}{2}\right)H_1 - \frac{1}{2}e^{\lambda(R)}(2c_1 + H_1)\right\}\cos\vartheta + \frac{1}{4}\left[\Xi e^{-\lambda/2}\left(\nu'' + \nu'^2 - \frac{\nu'}{r}\right)\right] - \frac{1}{4}e^{-\lambda(R)}Q_1^2[\nu'''] \tag{19f}$$

are satisfied. The function Ξ^- provides the second order deformation, as seen from the exterior, since the hypersurface gauge can be partially chosen so that $Q_2^- = \Xi^-$ [19,21] (see also [27]). We have included the full set of second order matching conditions because we will use them for the tidal problem below.

Now, returning to the rotating isolated star model, let us assume the functions h, v, and k satisfy (18). Then, we necessarily have $c_0 = c_1 = H_1 = 0$, cf. (19c) and (19d), and therefore (19b) yields $[\Xi] = 0$, so only one function Ξ (out of $\Xi^\pm$) appears in the matching. Now, using the decompositions

$$(Q_1)^2(\tau,\vartheta,\varphi) = \sum_{\ell=0}^{2}\mathcal{Q}_\ell(\tau,\varphi)P_\ell(\cos\vartheta) + \mathcal{Q}_\perp(\tau,\vartheta,\varphi),$$
$$\Xi(\tau,\vartheta,\varphi) = \sum_{\ell=0}^{2}\Xi_\ell(\tau,\varphi)P_\ell(\cos\theta) + \Xi_\perp(\tau,\vartheta,\varphi), \tag{20}$$

where we denote by $f_\perp$ the part of f orthogonal to $\ell = 0$, 1, 2, and given that the background and first order matching conditions hold, the set of equations in (19) is equivalent to the set $Q_1[\omega''] = 0$ plus

$$[h_0] = \frac{1}{2}H_0, \tag{21a}$$

$$[v_0] = \frac{1}{4}e^{-\lambda(R)/2}\Xi_0[\lambda'] - \frac{1}{4}e^{-\lambda(R)}\mathcal{Q}_0[\lambda''], \tag{21b}$$

$$[h_0'] = \frac{1}{4}e^{-\lambda(R)/2}\Xi_0[\nu''] - \frac{1}{4}e^{-\lambda(R)}\mathcal{Q}_0[\nu'''], \tag{21c}$$

$$[k_2] = 0, \qquad [h_2] = 0, \tag{22a}$$

$$[v_2] - R[k_2'] = \frac{1}{4}e^{-\lambda(R)/2}\Xi_2[\lambda'] - \frac{1}{4}e^{-\lambda(R)}\mathcal{Q}_2[\lambda''], \tag{22b}$$

$$[h_2'] - \frac{R}{2}\nu'(R)[k_2'] = \frac{1}{4}e^{-\lambda(R)/2}\Xi_2[\nu''] - \frac{1}{4}e^{-\lambda(R)}\mathcal{Q}_2[\nu'''], \tag{22c}$$

and

$$[\lambda']\Xi_1 = [\lambda']\Xi_\perp = 0, \qquad [\nu'']\Xi_1 = [\nu'']\Xi_\perp = 0, \tag{23a}$$

$$[\lambda'']\mathcal{Q}_1 = [\lambda'']\mathcal{Q}_\perp = 0, \quad [\nu''']\mathcal{Q}_1 = [\nu''']\mathcal{Q}_\perp = 0. \tag{23b}$$

These last equations for Ξ_1, $\mathcal{Q}_1$, $\Xi_\perp$ and $\mathcal{Q}_\perp$ are not matching conditions as such, since their purpose is to determine those quantities involved in the deformation (in the class of gauges we are working on). Observe that $\Xi_1 = \mathcal{Q}_1 = \Xi_\perp = \mathcal{Q}_\perp = 0$ satisfy the relations.

The above analysis of the matching has not taken into account the field equations at any order (not even the background). If the background field equations are used, Eqs. (21) and (22) take the form of Eqs. (79)–(82) in [27].[4] Moreover, (23) reduce to $\Lambda_0(R)\Xi_1 = \Lambda_0(R)\Xi_\perp = 0$ and $\Lambda_0'(R)\mathcal{Q}_1 = \Lambda_0'(R)\mathcal{Q}_\perp = 0$. In any case, if the first order equation for ω [Eq. (49) in [27]] is also used, then $Q_1[\omega''] = 0$ holds automatically.

The global problem, that is, the interior and exterior problems with common boundary data provided by the matching conditions, can be split onto the $\ell = 0$ and $\ell = 2$ sectors. We review the problems as presented in [27] next.

2. $\ell = 0$

The $\ell = 0$ interior problem for the set of functions $\{h_0^+, v_0^+, \mathcal{P}_{0n}, \mathcal{P}_{0p}\}$ comprises Eqs. (65), (67)–(68) in [27]. The exterior solution is given by Eq. (72) in [27] after fixing the gauge at the exterior so that h_0^- vanishes at infinity (using the freedom driven by $V_2^- \propto t\partial_t$ appropriately). This fixes the spacetime gauge at the exterior completely.

Regarding the matching, let us first note that using $V_2^+ = H_0 t\partial_t$ at the interior we can set $H_0 = 0$ in (21a). This fixes the spacetime gauge at the interior completely.

As detailed in [27], next we must consider the difference of the field equations at both sides on Σ. The difference of Eq. (67) in [27] does not provide useful information (just gives $[v_0']$), but the difference of Eq. (68) in [27] provides, after using the matching up to first order, a relation between $[h_0']$, $[v_0]$ and $\mathcal{P}_0(R) \coloneqq n_0\mathcal{P}_{0n}(R) + p_0\mathcal{P}_{0p}(R)$. Now, the system composed by that relation and the two Eqs. (21b) and (21c) is shown to be equivalent to one equation for a combination of Ξ_0 and $\mathcal{Q}_0$, plus an equation for $[v_0]$ [see (25) below], in terms of $\mathcal{P}_0(R)$, and the original relation from the field equations.

To sum up, the $\ell = 0$ sector of the second order perturbative problems match if and only if the two equations

$$[h_0] = 0, \tag{24}$$

$$[v_0] = \varkappa\frac{R}{\nu'(R)}e^{\lambda(R)}\left\{\frac{1}{3}R^2 e^{\lambda(R)} n_0(R)p_0(R)\mathcal{A}_0(R)(\Omega_n - \Omega_p)^2 - \mathcal{P}_0(R)\right\}, \tag{25}$$

hold. Moreover, the matching produces an equation for a combination of Ξ_0 and $\mathcal{Q}_0$ [Eq. (91) in [27]] which we do not include here for brevity. The field equations produce then a value for $[h_0']$ that is consistent with the geometrical matching condition (21c).

The exterior solution is $h_0^-(r) = -v_0^-(r)$ with

$$v_0^-(r) = \frac{\delta M}{r - 2M} - \frac{J^2}{r^3(r - 2M)}, \tag{26}$$

for some constant δM. Thus, the $\ell = 0$ exterior solution only involves δM, which turns out to be the contribution to the mass at second order. Indeed, the ADM mass of the family of geometries given by g_ε at $r \to \infty$, given some central values $n_0(0)$ and $p_0(0)$, is

$$M_T = M + \varepsilon^2 \delta M.$$

Now, using the identity $v_0^-(R) \equiv v_0^+(R) - [v_0]$ with (25) and (26) we obtain

$$\delta M = \frac{J^2}{R^3} + (R - 2M)v_0^+(R) - \varkappa\frac{R(R-2M)}{\nu'(R)}e^{\lambda(R)} \times\left\{\frac{1}{3}R^2 e^{\lambda(R)} n_0(R)p_0(R)\mathcal{A}_0(R)(\Omega_n - \Omega_p)^2 - \mathcal{P}_0(R)\right\}. \tag{27}$$

This expression of δM corrects the expression (60) in [24], that does not contain the last term.

C. $\ell = 2$

The $\ell = 2$ problem in the interior region involves the set $\{h_2, v_2, k_2, \mathcal{P}_{2n}, \mathcal{P}_{2p}\}$ that satisfy the five relations in Eqs. (66), (69)–(71) in [27].

The general exterior solution is given explicitly by Eqs. (73) and (74) in [27] for $h_2^-(r)$, $k_2^-(r)$, and

$$v_2^-(r) = -h_2^-(r) + \frac{6J^2}{r^3}, \tag{28}$$

in terms of a free parameter C that is to be fixed.[5] On the other hand, the pole structure at the origin implies that the general interior solution for $\{h_2^+(r), k_2^+(r), v_2^+(r)\}$ depends

[4] Equation (82) in [27] contains a typo: the second Ξ_2 should read $\mathcal{Q}_2$.

[5] Expression (28) corrects a typo in the last term in Eq. (75) of [27], and also (76) of [19], where last term should have a global minus. This has no other consequences whatsoever.

on a free parameter, denoted by A in [27], that multiplies the homogeneous part of the solution.

As for the difference on Σ of the field equations, the set of three equations given by (22a) plus the difference of [Eqs. (69)–(71) in [27]] is equivalent to the set $\{$(22a)–(22c)$\}$ plus another equation that determines a combination of Ξ_2 and $\mathcal{Q}_2$, explicitly given by Eq. (93) in [27].

In short, the $\ell = 2$ sector of the second order perturbation problems match if and only if

$$[h_2] = 0, \qquad [k_2] = 0, \tag{29}$$

and then the matching produces an equation for a combination of Ξ_2 and $\mathcal{Q}_2$ [Eq. (93) in [27]]. The field equations produce then values for $[v_2]$, $[h_2']$ and $[k_2']$ consistent with the geometrical equations (22b) and (22c).

Once the interior problem is integrated in terms of the inhomogenous and homogenous part of the general solution, the parameters A (from the interior) and C (form the exterior) are fixed using the two relations (29).

The value of C of the exterior solution is related with the quadrupole moment Q by

$$Q = \frac{8}{5} C M^3 + \frac{J^2}{M}. \tag{30}$$

VI. THE TIDAL PROBLEM

We summarize the problem for the linearized analysis of perturbations for a compact body immersed in a quadrupolar tidal field [12,16]. Given a static and spherically symmetric background, the even-parity first order perturbation tensor in the Regge-Wheeler gauge is given by [36] (we drop the $\pm$ indexes)

$$K_1^T = \sum_{\ell,m} \{ e^{\nu(r)} H_{0\ell m}(r) \mathrm{d}t^2 + e^{\lambda(r)} H_{2\ell m}(r) \mathrm{d}r^2 + r^2 K_{\ell m}(r)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\} Y_{\ell m}(\theta,\phi), \tag{31}$$

where $Y_{\ell m}(\theta,\phi)$ are the spherical harmonics. The equations for each mode $\{\ell, m\}$ decouple, and for $\{\ell \geq 2, m = 0\}$ we have that (we drop the $m = 0$ label)

$$H_{2\ell} = H_{0\ell}, \tag{32}$$

$$K_\ell' = H_{0\ell}' + \nu' H_{0\ell}, \tag{33}$$

$$r^2 \nu' H_{0\ell}' = e^\lambda(\ell(\ell+1) - 2) K_\ell + (r(\lambda' + \nu') - (r\nu')^2 - e^\lambda \ell(\ell+1) + 2) H_{0\ell}. \tag{34}$$

This system is usually written as a single second order ODE for $H_{0\ell}$, and then K_ℓ is obtained algebraically from (34).

For the interior problem the system (33) and (34) is integrated for each pair $\{H_{0\ell}^+, K_{0\ell}^+\}$ from a regular origin (see e.g. [37]). In the exterior vacuum problem, for which $e^{\nu^-} = e^{-\lambda^-} = 1$–$2M/r_-$, the second order equation for $H_{0\ell}$ is the general Legendre equation (with $m = 2$). The general solution is thus given by

$$H_{0\ell}^-(r_-) = a_{\ell P} \hat{P}_\ell^2\left(\frac{r_-}{M} - 1\right) + a_{\ell Q} \hat{Q}_\ell^2\left(\frac{r_-}{M} - 1\right), \tag{35}$$

for some constants $a_{\ell P}$, $a_{\ell Q}$ (to keep the notation of [16]), with

$$\hat{P}_\ell^2(x) \coloneqq \left(\frac{2^\ell}{\sqrt{\pi}} \frac{\Gamma(\ell + 1/2)}{\Gamma(\ell - 1)}\right)^{-1} P_\ell^2(x),$$

$$\hat{Q}_\ell^2(x) \coloneqq \left(\frac{\sqrt{\pi}}{2^{\ell+1}} \frac{\Gamma(\ell + 3)}{\Gamma(\ell + 3/2)}\right)^{-1} Q_\ell^2(x),$$

where $P_\ell^2(x)$ and $Q_\ell^2(x)$ denote the associated Legendre functions of the first and second kind $P_\ell^n(x)$ and $Q_\ell^n(x)$ with $n = 2$, respectively. The task now is to obtain the necessary and sufficient set of matching conditions for the interior and exterior problem for the sector $\{\ell \geq 2, m = 0\}$. That constitutes a stationary and axially symmetric perturbation problem over a static and spherically symmetric background. Specifically, the problem for a first order perturbation tensor of the form K_1^T for $\{\ell \geq 2, m = 0\}$ at each side is equivalent to a second order perturbation problem provided by (16) and (17) with the trivial choice $\omega = \Omega_n = \Omega_p = 0$, and the substitutions

$$h(r,\theta) \to \bar{h}(r,\theta) \coloneqq -\frac{1}{4} \sum_{\ell \geq 2} H_{0\ell}(r,\theta), \tag{36a}$$

$$v(r,\theta) \to \bar{v}(r,\theta) \coloneqq \frac{1}{4} \sum_{\ell \geq 2} H_{2\ell}(r,\theta), \tag{36b}$$

$$k(r,\theta) \to \bar{k}(r,\theta) \coloneqq \frac{1}{4} \sum_{\ell \geq 2} K_\ell(r,\theta). \tag{36c}$$

Since the first order perturbation is vanishing, the second order problem effectively becomes first order. Observe that using (36), Eqs. (69)–(71) in [27] with $\omega = f_\omega = 0$ translate to Eqs. (32)–(34) for $H_{2\ell}$, K_ℓ and $H_{0\ell}$ with $\ell = 2$, respectively.

The matching conditions for K_1^{T+} and K_1^{T-} thus correspond to (19) with $\omega = f_\omega = \mathcal{Q} = 0$ and the substitutions in (36). Direct inspection shows that the equations decouple in terms of ℓ. Once each $\Xi^\pm$ is expanded in Legendre polynomials, see (20), Eq. (19b) implies that $[\Xi_\ell] = 0$ for $\ell \geq 2$, so we only have one Ξ_ℓ for each $\ell \geq 2$. Then, the rest of the matching conditions for $\ell \geq 2$ read

$$[K_\ell] = 0, \qquad [H_{0\ell}] = 0, \tag{37}$$

$$[H_{2\ell}] - R[K'_\ell] = e^{-\lambda(R)/2}\Xi_\ell[\lambda'], \tag{38}$$

$$[H'_{0\ell}] + \frac{R}{2}\nu'(R)[K'_\ell] = -e^{-\lambda(R)/2}\Xi_\ell[\nu'']. \tag{39}$$

Now, taking into account the background field equations, so that (13)–(15) hold, the set of two Eqs. (38) and (39), for each $\ell \geq 2$, is equivalent to the set

$$[H_{2\ell}] - R[K'_\ell] = -\varkappa R e^{\lambda(R)/2}\Xi_\ell \Lambda_0(R), \tag{40}$$

$$[H'_{0\ell}] - [K'_\ell] = -\frac{1}{R}\left(1 + \frac{M}{R}e^{\lambda(R)}\right)[H_{2\ell}]. \tag{41}$$

On the other hand, we must use the information provided by the first order field equations. Taking the differences of (32)–(34) on Σ, and using the background matching conditions and (37), together with (14), we obtain

$$[H_{2\ell}] = [H_{0\ell}], \qquad [K'_\ell] = [H'_{0\ell}],$$
$$[H'_{0\ell}] = -\varkappa\frac{R}{2M}H_{0\ell}(R)\Lambda_0(R),$$

respectively. The combination of these three equations with (37) and (40) is equivalent to the set of (six) equations on Σ given by

$$[K_\ell] = 0, \qquad [H_{0\ell}] = 0, \qquad [H_{2\ell}] = 0, \tag{42}$$

$$[H'_{0\ell}] = [K'_\ell] = -\varkappa\frac{R^2}{2M}H_{0\ell}(R)\Lambda_0(R), \tag{43}$$

and

$$\left(\frac{R^2}{2M}H_{0\ell} + e^{\lambda(R)/2}\Xi_\ell\right)\Lambda_0(R) = 0, \tag{44}$$

while (41) then holds identically.

It is important to note that if the three equations in (42) hold and the field equations are imposed, then the two relations in (43) are automatically satisfied. As a result, and to sum up, *the interior and exterior problems for each ℓ match if and only if* (42) *hold, and then the deformation satisfies* (44). The field equations imply the fulfillment of the rest of the matching conditions.

A. Computation of the Love numbers

Given the exterior solution is provided by (35), the tidal Love numbers are defined by

$$k_\ell \coloneqq \frac{1}{2}\left(\frac{M}{R}\right)^{2\ell+1} a_\ell,$$

where $a_\ell \coloneqq a_{\ell Q}/a_{\ell P}$, for each mode ℓ. Therefore

$$a_\ell = -\frac{\partial_{r_-}\hat{P}^2_\ell - (y^-_\ell/R)\hat{P}^2_\ell}{\partial_{r_-}\hat{Q}^2_\ell - (y^-_\ell/R)\hat{Q}^2_\ell}\bigg|_{r_-=R},$$

with $y_\ell \coloneqq rH'_{0\ell}/H_{0\ell}$ defined at both $\pm$ sides.

Integrating the interior problem provides $y^+_\ell(R)$, and then $y^-_\ell(R)$ is obtained using the identity $y^-_\ell(R) = -[y_\ell] + y^+_\ell(R)$. Equations (42) and (43) establish

$$[y_\ell] = -\frac{\varkappa R^3}{2M}\Lambda_0(R). \tag{45}$$

As a result,

$$a_\ell = -\frac{\partial_{r_+}\hat{P}^2_\ell - (y^+_\ell/R)\hat{P}^2_\ell - (\varkappa R^2\Lambda_0(R)/2M)\hat{P}^2_\ell}{\partial_{r_+}\hat{Q}^2_\ell - (y^+_\ell/R)\hat{Q}^2_\ell - (\varkappa R^2\Lambda_0(R)/2M)\hat{Q}^2_\ell}\bigg|_{r_+=R}. \tag{46}$$

The error encountered in previous literature (e.g. [28,29]) concerns the implicit assumption that y_ℓ are continuous across the surface of the star, which is inconsistent with the perturbative procedure if $\Lambda_0(R)$ does not vanish. Reverting the argument, in those works it was not proven (not even stated) that $\Lambda_0(R) = 0$ if and only if the functions y_ℓ are continuous.

In the numerical analysis we discuss next we will only care about the $\ell = 2$ term, and the tidal problem will be accounted for with the alternative quantity $\lambda_2 = a_2/3$, known as the tidal deformability. The Love number k_2 obtained from (46) reads

$$\begin{aligned} k_2 = &\frac{8}{5}C_o^5(1-2C_o)^2(2C_o(\mathcal{Y}-1)-\mathcal{Y}+2) \\ &\times\{2C_o(4(\mathcal{Y}+1)C_o^4+(6\mathcal{Y}-4)C_o^3+(26-22\mathcal{Y})C_o^2 \\ &+3(5\mathcal{Y}-8)C_o-3\mathcal{Y}+6) \\ &+3(1-2C_o)^2(2C_o(\mathcal{Y}-1)-\mathcal{Y}+2)\log((1-2C_o))\}^{-1}, \end{aligned} \tag{47}$$

where $C_o \coloneqq M/R$, with

$$\mathcal{Y} \coloneqq y^-_{\ell=2}(R) = y^+_{\ell=2}(R) + \frac{\varkappa R^3}{2M}\Lambda_0(R). \tag{48}$$

The form of expression (47) coincides (replacing $\mathcal{Y}$ by "y") with that in the literature, e.g. [13], Eq. (25) in [29] and Eq. (24) in [28]. This is so because "y" there corresponds to the exterior $y^-_{\ell=2}(R)$, and the exterior problem is the same. However, in [28,29] "y" is given a single value on the boundary, implicitly assuming *a priori* that, in the usual wording, $y_{\ell=2}$ is continuous. Because of (45), the correct result requires (48). The final values for k_2 found in [28,29]

TABLE I. Set of parameters for the NL3 [38] and GM1 [39] relativistic mean field models obtained from [28].

Model	c_σ^2	c_ω^2	c_ρ^2	b	c
NL3	15.739	10.530	5.324	0.002055	−0.002650
GM1	11.785	7.148	4.410	0.002948	−0.001071

turn out to be valid because, as we have checked, the EOS forces Λ_0 to approach zero at the boundary in the models presented in those works.

Expression (48) can be compared with Eq. (15) in [1], which was put forward following the previous discussion in [16] for homogeneous stars, and finally proved for perfect fluids in general in [18]. Let us stress, however, that the relation between the discontinuities of the physical quantities and the jumps of the relevant metric functions is far from obvious, *a priori*. Note that whereas the discontinuities affecting y_ℓ and δM in the perfect fluid case are both proportional to the value of the energy density at the boundary [see Eqs. (13) and (24) in [18]], in the two-fluid model that is not the case. The last term in (27) is not proportional to $\Lambda_0(R)$.

VII. *I*-LOVE-*Q*-*δM* RELATIONS

We turn next to apply the theoretical developments of the previous sections to different two-fluid models, spanning three different EOS. Our aim is to investigate if, and by how much, the corrections reported here impact the universality of the *I*-Love-*Q*-*δM* relations, extending our previous work in the perfect fluid with barotropic EOS case [18]. For all of our models we impose chemical equilibrium ($\mu_0 = \chi_0$).

A. Models

1. Two-fluid polytropic model

This model is the one presented in [24], and describes a star where the individual constituents (the superfluid neutrons and all other components) do not interact, i.e. there is no entrainment. The corresponding master function is given by

$$\Lambda(n_0^2, p_0^2) = -m_n n_0 - \sigma_n n_0^{\beta_n} - m_n p_0 - \sigma_p p_0^{\beta_p}, \quad (49)$$

where m_n is the mass of the neutron and

$$\sigma_n = 0.2 m_n, \qquad \beta_n = 2.3, \qquad \sigma_p = 2 m_n, \qquad \beta_p = 1.95.$$

We choose units such that $m_n = c = G = 1$ and the number densities of both fluids (n_0 and p_0) are given in fm^{-3} [see Eqs. (A1)–(A3) in Appendix A].

2. Toy model

This EOS is the toy model we suggested in [27], for which the master function reads

$$\Lambda(n_0^2, p_0^2, x_0^2) = -(2n_0 + p_0 + x_0^2) m_n, \quad (50)$$

where the same units as before are chosen. Hence, the same conversion factors to SI units apply [Eqs. (A1)–(A3) in Appendix A]. This equation does account for entrainment between the two fluids, and the energy density at the boundary of the star $\Lambda_0(R)$ turns out to be nonvanishing.

3. Relativistic mean field model

We also consider the model employed in [28] in their study of the *I*-Love-*Q* relations for superfluid neutron stars. Due to the complexity of the master function and the derived relations, we refer to Appendix B for the explicit expressions and the techniques to deal with them within a numerical approach. As shown by Eq. (B1), the master function for this EOS depends on a number of parameters. In our computations we consider the two sets of parameters listed in Table I for the NL3 [38] and GM1 [39] relativistic mean field models. The conversion factors to SI units are shown in Eqs. (A4)–(A6) in Appendix A.

B. Results

In what follows we show the different *I*-Love-*Q*-*δM* relations we obtain for each of our four models. The results are displayed in Figs. 2 and 3 in terms of the usual dimensionless quantities

$$\bar{I} \coloneqq \frac{I}{M^3},$$
$$\bar{Q} \coloneqq \frac{QM}{J^2},$$
$$\overline{\delta M} \coloneqq \delta M \frac{M^3}{J^2}.$$

Note that M refers to the mass of the spherical background configuration. Although the approximate universal relations involve only the above dimensionless quantities, we follow the customary practice in the literature and refer to those relations as the *I*-Love-*Q*-*δM* universal relations (i.e. not explicitly including the overline in the quantities).

Each of the symbols in Figs. 2 and 3 corresponds to a particular stellar model, which has been computed numerically using a modification of the code employed in [18]. The tidal deformability λ_2 (which is directly related to the Love number k_2) and the contribution to the mass at second order, δM, are the only quantities that depend on the value of the energy density at the boundary of the star.

Out of the four models we consider here, the toy model is the only one which does present a nonvanishing value of $\Lambda_0(R)$. For that reason, in the plots in Figs. 2 and 3 where

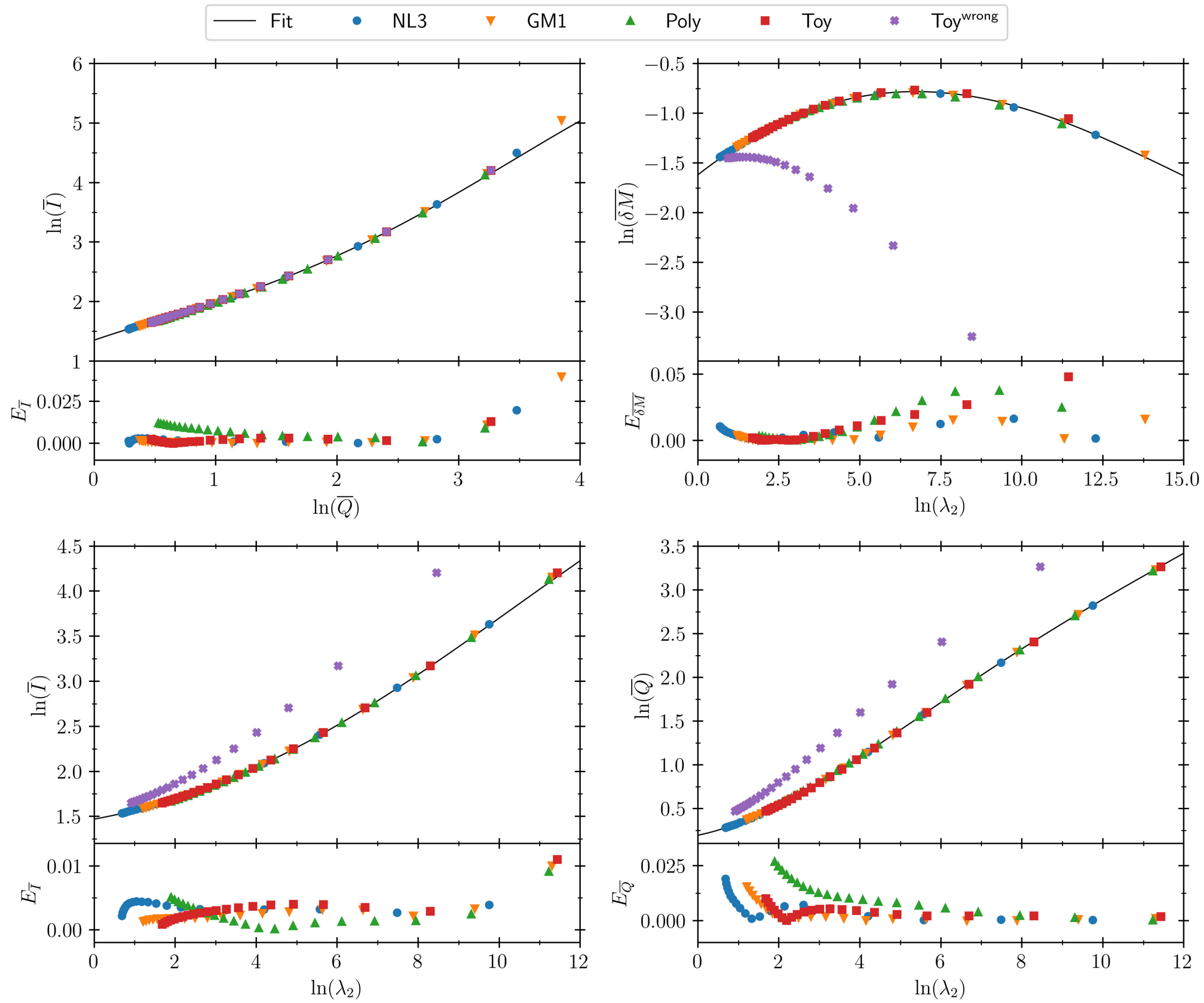

FIG. 2. Relations between $\bar{I}-\bar{Q}$ (top left), $\overline{\delta M}-\lambda_2$ (top right), $\bar{I}-\lambda_2$ (bottom left) and $\bar{Q}-\lambda_2$ (bottom right) for the three models with $\Delta = 1$. The purple dots represent the values obtained by taking $[v_0] = 0$ in the toy model, i.e. the incorrect values that would have been predicted with the original HT model. This will only have an effect on the relations involving $\overline{\delta M}$ and λ_2. The lower panels in each plot represent the relative errors between the individual plots and the fitting curves, $E_X = |(\ln X - \ln X^{\mathrm{fit}})/\ln X^{\mathrm{fit}}|$. The errors of Toy$^{\mathrm{wrong}}$ are not displayed.

either λ_2 or $\overline{\delta M}$ appear, we include an additional set of points labeled "Toy$^{\mathrm{wrong}}$." Those show the results obtained for the same toy model but without considering the correction to the original perturbative frameworks in which it was (implicitly) assumed all metric functions to be continuous.

In Fig. 2 we show the different *I*-Love-*Q*-δM relations for the four EOS with $\Delta = 1$. The top part of each panel shows the actual correlations between pairs of parameters while the bottom part displays the corresponding relative errors. In all cases the relations found between pairs of parameters follow approximate universal relations (when using the correct matching conditions). These relations can be accurately fitted with polynomial curves, using the logarithm of the parameters as variables. The individual fitting formulae can be summarized with the expression

$$\ln(y_i) = a_i + b_i \ln(x_i) + c_i \ln(x_i)^2 + d_i \ln(x_i)^3 + e_i \ln(x_i)^4, \tag{51}$$

where the values of the coefficients are given in Table II. Those fits are displayed with solid lines in the top panels of each plot in Fig. 2. Our results show that an augmented set of universal relations for the tidal problem in binary systems of superfluid neutron stars, involving the four

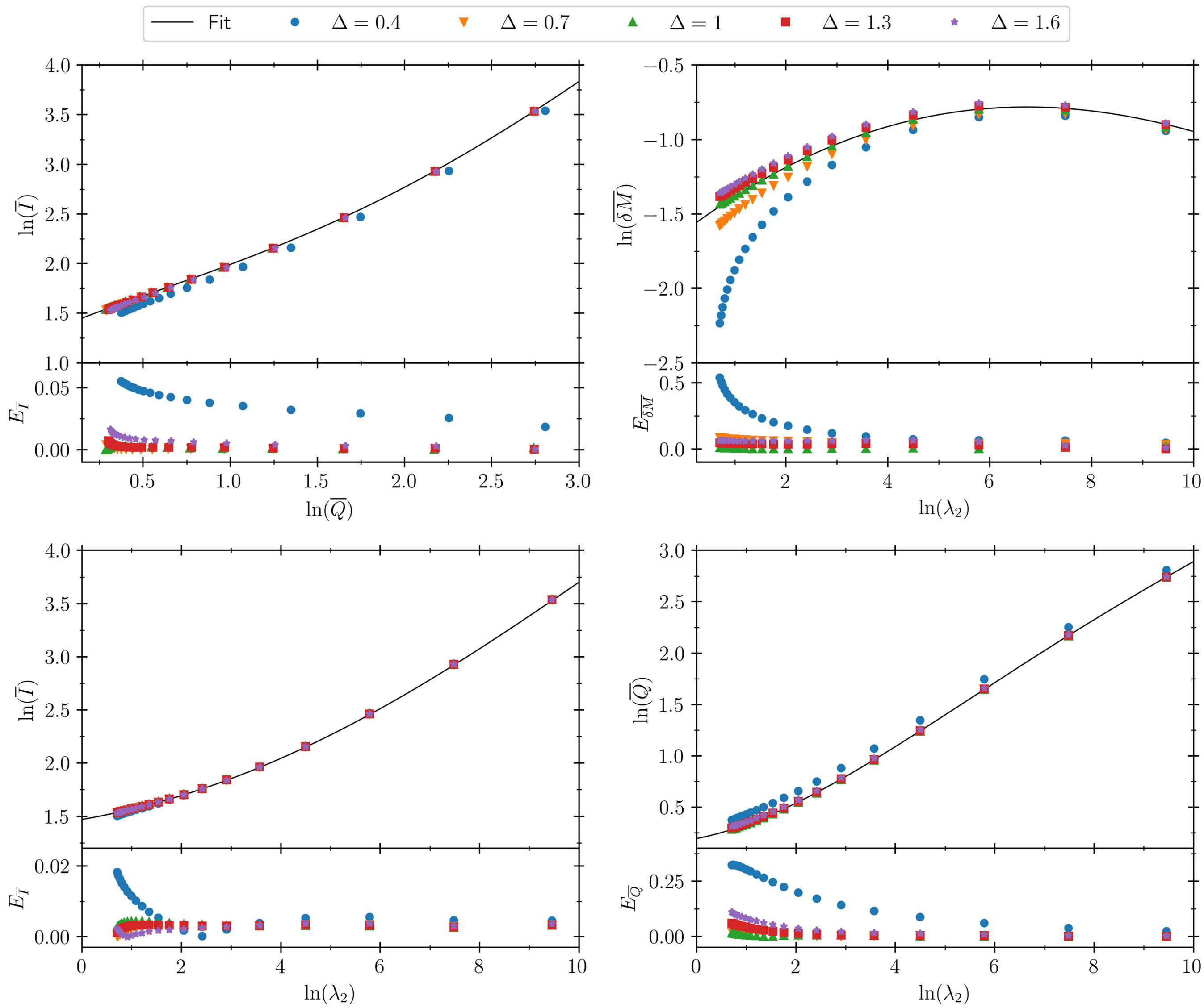

FIG. 3. Relations between $\bar{I} - \bar{Q}$ (top left), $\overline{\delta M} - \lambda_2$ (top right), $\bar{I} - \lambda_2$ (bottom left) and $\bar{Q} - \lambda_2$ (bottom right) for the NL3 model with different values of Δ.

perturbation parameters $\bar{I}$, λ_2, $\bar{Q}$ and also $\overline{\delta M}$, exists. This result reproduces the previous findings for the tidal problem in the perfect fluid with barotropic EOS case [18].

The deviation from universality stands out when not taking into account the correction to the formalism (see the purple symbols in the $\overline{\delta M} - \lambda_2$, $\bar{I} - \lambda_2$ and $\bar{Q} - \lambda_2$ plots). The correct formalism yields fully universal relations for all the EOS considered, irrespective of the existence of jumps of the energy density, as shown by the red symbols.

In Fig. 3 we illustrate the relations for the specific case of the NL3 model, for different values of Δ. In this case, universality is lost when the two fluids do not corotate (i.e. $\Delta \neq 1$). The smallest departures from universality are found for the $\bar{I} - \lambda_2$ pair (bottom-left panel in Fig. 3), with the maximum relative error at the 2% level. These results were already found in [28], except for the analysis of the second order contribution to the mass $\overline{\delta M}$. In our study, the inclusion of $\overline{\delta M}$ into the set of quantities to analyze shows that the $\overline{\delta M} - \lambda_2$ curve (top-right panel in Fig. 3) is significantly more sensitive to the variation of the relative rotation rate between the two fluids Δ than the rest of the

TABLE II. Parameters for the fitting curves of Eq. (51). The first three rows have been obtained from [11], whereas the last row is from [18] after applying a suitable logarithm base conversion.

y_i	x_i	a_i	b_i	c_i	d_i	e_i
$\bar{I}$	λ_2	1.47	0.0817	0.0149	2.87×10^{-4}	-3.64×10^{-5}
$\bar{I}$	$\bar{Q}$	1.35	0.697	−0.143	9.94×10^{-2}	-1.24×10^{-2}
$\bar{Q}$	λ_2	0.194	0.0936	0.0474	-4.21×10^{-3}	1.23×10^{-4}
$\overline{\delta M}$	λ_2	−1.619	0.255	−0.0195	-1.08×10^{-4}	1.81×10^{-5}

relations. Relative errors as large as 50% are found for $\Delta = 0.4$.

VIII. CONCLUSIONS

In this work we have studied the tidal problem and the resulting I-Love-Q approximate universal relations for rotating superfluid neutron stars in the Hartle-Thorne formalism. To do so we have adapted the stationary and axisymmetric perturbation scheme for global stellar models developed in [27] to the first order tidal problem in binary systems. Our approach is based on the geometrical formalism developed in [19] and fully generalized in [20,21]. The outcome provides the expected correction to the computation of the Love numbers caused by a nonvanishing energy density at the interior side of the stellar boundary. Such correction is analogous to that of the perfect fluid case found for homogeneous stars in [16] and proven in full generality in [18].

The analytic formalism has been applied to different two-fluid stellar models built numerically, spanning three different EOS. On the one hand, in particular, we have checked that the relevant physical quantities produced by the EOSs used in [28,29] tend to zero at the boundary, thus providing firm grounds to those results. Further, we have shown how the contribution to the mass at second order δM also satisfies universal relations with I, Love and Q for all EOS when the two fluids corotate ($\Delta = 1$). This result is in agreement with the perfect fluid case [18]. The universal I-Love-Q relations are known to fail when $\Delta \neq 1$, as shown by [28]. We have also found that in the numerical stellar models analyzed in this work, the departure from universality in the relations involving δM are significantly more sensitive than the rest.

The results presented in this paper, thus, complete the set of universal relations for rotating superfluid stars, generalizing our previous findings in the perfect fluid case. Using the extended set of universal relations reported in this work in order to improve observational constraints on the supranuclear EOS of neutron stars is an effort worth pursuing next. Our findings in this direction will be reported elsewhere [40].

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APPENDIX A: UNITS

For the two-fluid polytropic and toy model EOSs, the conversion factors from the given code units (CU) to the SI units were derived in [27]. They read

$$r^{\mathrm{SI}} = r^{\mathrm{CU}} \times c\sqrt{\frac{\mathrm{fm}^3}{G m_n}}, \tag{A1}$$

$$t^{\mathrm{SI}} = t^{\mathrm{CU}} \times \sqrt{\frac{\mathrm{fm}^3}{G m_n}}, \tag{A2}$$

$$m^{\mathrm{SI}} = m^{\mathrm{CU}} \times c^3\sqrt{\frac{\mathrm{fm}^3}{G^3 m_n}}, \tag{A3}$$

where G, c and m_n recover their SI values.

Similarly, for the mean field model EOS the conversion factors are given by

$$r^{\mathrm{SI}} = r^{\mathrm{CU}} \times c\sqrt{\frac{c}{G\hbar}}\mathrm{fm}^2, \tag{A4}$$

$$t^{\mathrm{SI}} = t^{\mathrm{CU}} \times \sqrt{\frac{c}{G\hbar}}\mathrm{fm}^2, \tag{A5}$$

$$m^{\mathrm{SI}} = m^{\mathrm{CU}} \times c^3\sqrt{\frac{c}{G^3\hbar}}\mathrm{fm}^2. \tag{A6}$$

Note that m is a general unit of mass, given in kg, not the nucleon mass mentioned in Appendix B.

APPENDIX B: MEAN FIELD MODEL EOS

The master function for this EOS is given by [44]

$$\Lambda_0 = -\frac{c_\omega^2}{18\pi^4}(k_n^3+k_p^3)^2 - \frac{c_\rho^2}{72\pi^4}(k_p^3-k_n^3)^2 - \frac{1}{4\pi^2}\left(k_n^3\sqrt{k_n^2+m_\star^2|_0} + k_p^3\sqrt{k_p^2+m_\star^2|_0}\right)$$
$$-\frac{1}{4c_\sigma^2}\{(2m-m_\star|_0)(m-m_\star|_0) + m_\star|_0(bmc_\sigma^2(m-m_\star|_0)^2 + cc_\sigma^2(m-m_\star|_0)^3)\}$$
$$-\frac{1}{3}bm(m-m_\star|_0)^3 - \frac{1}{4}c(m-m_\star|_0)^4 - \frac{1}{8\pi^2}\left\{k_p(2k_p^2+m_e^2)\sqrt{k_p^2+m_e^2} - m_e^4\ln\left(\frac{k_p+\sqrt{k_p^2+m_e^2}}{m_e}\right)\right\}, \quad \text{(B1)}$$

where $k_n = (3\pi^2 n_0)^{1/3}$, $k_p = (3\pi^2 p_0)^{1/3}$, m is the nucleon mass (the average of the neutron and proton masses), and the parameter $m_\star|_0$ is the Dirac effective mass, coming from the transcendental equation

$$m_\star|_0 = m - m_\star|_0\frac{c_\sigma^2}{2\pi^2}\left\{k_n\sqrt{k_n^2+m_\star^2|_0} + k_p\sqrt{k_p^2+m_\star^2|_0} + \frac{1}{2}m_\star^2|_0\ln\left(\frac{-k_n+\sqrt{k_n^2+m_\star^2|_0}}{k_n+\sqrt{k_n^2+m_\star^2|_0}}\right)\right.$$
$$\left.+\frac{1}{2}m_\star^2|_0\ln\left(\frac{-k_p+\sqrt{k_p^2+m_\star^2|_0}}{k_p+\sqrt{k_p^2+m_\star^2|_0}}\right)\right\} + bmc_\sigma^2(m-m_\star|_0)^2 + cc_\sigma^2(m-m_\star|_0)^3. \quad \text{(B2)}$$

For convenience, we may work instead with a differential equation for $m_\star|_0$ (this strategy was discussed in [45], although for a different expression of the EOS)

$$m'_\star|_0 = \left.\frac{\partial m_\star}{\partial k_n}\right|_0 k'_n + \left.\frac{\partial m_\star}{\partial k_p}\right|_0 k'_p,$$

where (again [44])

$$\left.\frac{\partial m_\star}{\partial k_n}\right|_0 = -\frac{c_\sigma^2}{\pi^2}\frac{m_\star|_0 k_n^2}{\sqrt{k_n^2+m_\star^2|_0}}\left\{\frac{3m-2m_\star|_0+3bmc_\sigma^2(m-m_\star|_0)^2+3cc_\sigma^2(m-m_\star|_0)^3}{m_\star|_0}\right.$$
$$\left.-\frac{c_\sigma^2}{\pi^2}\left(\frac{k_n^3}{\sqrt{k_n^2+m_\star^2|_0}}+\frac{k_p^3}{\sqrt{k_p^2+m_\star^2|_0}}\right) + 2bmc_\sigma^2(m-m_\star|_0) + 3cc_\sigma^2(m-m_\star|_0)^2\right\}^{-1},$$

$$\left.\frac{\partial m_\star}{\partial k_p}\right|_0 = -\frac{c_\sigma^2}{\pi^2}\frac{m_\star|_0 k_p^2}{\sqrt{k_p^2+m_\star^2|_0}}\left\{\frac{3m-2m_\star|_0+3bmc_\sigma^2(m-m_\star|_0)^2+3cc_\sigma^2(m-m_\star|_0)^3}{m_\star|_0}\right.$$
$$\left.-\frac{c_\sigma^2}{\pi^2}\left(\frac{k_n^3}{\sqrt{k_n^2+m_\star^2|_0}}+\frac{k_p^3}{\sqrt{k_p^2+m_\star^2|_0}}\right) + 2bmc_\sigma^2(m-m_\star|_0) + 3cc_\sigma^2(m-m_\star|_0)^2\right\}^{-1}.$$

The generalized pressure is given by

$$\Psi_0 = \Lambda_0 + \frac{1}{3\pi^2}(\mu_0 k_n^3 + \chi_0 k_p^3),$$

where the two auxiliary functions μ_0 and χ_0 explicitly read

$$\mu_0 = \frac{c_\omega^2}{3\pi^2}(k_n^3 + k_p^3) - \frac{c_\rho^2}{12\pi^2}(k_p^3 - k_n^3) + \sqrt{k_n^2 + m_\star^2|_0},$$
$$\chi_0 = \frac{c_\omega^2}{3\pi^2}(k_n^3 + k_p^3) + \frac{c_\rho^2}{12\pi^2}(k_p^3 - k_n^3) + \sqrt{k_p^2 + m_\star^2|_0} + \sqrt{k_p^2 + m_e^2}.$$

The functions accounting for the first and second order derivatives of Λ_0 are given by

$$\begin{aligned}
\mathcal{A}_0 = {} & c_\omega^2 - \frac{1}{4}c_\rho^2 + \frac{c_\omega^2}{5\mu_0^2}\left\{2k_p^2\frac{\sqrt{k_n^2 + m_\star^2|_0}}{\sqrt{k_p^2 + m_\star^2|_0}} + \frac{c_\omega^2}{3\pi^2}\left(\frac{k_n^2 k_p^3}{\sqrt{k_n^2 + m_\star^2|_0}} + \frac{k_p^2 k_n^3}{\sqrt{k_p^2 + m_\star^2|_0}}\right)\right\} \\
& + \frac{c_\rho^2}{20\mu_0^2}\left\{2k_p^2\frac{\sqrt{k_n^2 + m_\star^2|_0}}{\sqrt{k_p^2 + m_\star^2|_0}} + \frac{c_\rho^2}{12\pi^2}\left(\frac{k_n^2 k_p^3}{\sqrt{k_n^2 + m_\star^2|_0}} + \frac{k_p^2 k_n^3}{\sqrt{k_p^2 + m_\star^2|_0}}\right)\right\} \\
& - \frac{c_\rho^2 c_\omega^2}{30\mu_0^2\pi^2}\left(\frac{k_n^2 k_p^3}{\sqrt{k_n^2 + m_\star^2|_0}} - \frac{k_p^2 k_n^3}{\sqrt{k_p^2 + m_\star^2|_0}}\right) + \frac{3\pi^2 k_p^2}{5\mu_0^2 k_n^3}\frac{k_n^2 + m_\star^2|_0}{\sqrt{k_p^2 + m_\star^2|_0}}, \\
\mathcal{B}_0 = {} & \frac{3\pi^2\mu_0}{k_n^3} - c_\omega^2\frac{k_p^3}{k_n^3} + \frac{1}{4}c_\rho^2\frac{k_p^3}{k_n^3} - \frac{c_\omega^2 k_p^3}{5\mu_0^2 k_n^3}\left\{2k_p^2\frac{\sqrt{k_n^2 + m_\star^2|_0}}{\sqrt{k_p^2 + m_\star^2|_0}} + \frac{c_\omega^2}{3\pi^2}\left(\frac{k_n^2 k_p^3}{\sqrt{k_n^2 + m_\star^2|_0}} + \frac{k_p^2 k_n^3}{\sqrt{k_p^2 + m_\star^2|_0}}\right)\right\} \\
& - \frac{c_\rho^2 k_p^3}{20\mu_0^2 k_n^3}\left\{2k_p^2\frac{\sqrt{k_n^2 + m_\star^2|_0}}{\sqrt{k_p^2 + m_\star^2|_0}} + \frac{c_\rho^2}{12\pi^2}\left(\frac{k_n^2 k_p^3}{\sqrt{k_n^2 + m_\star^2|_0}} + \frac{k_p^2 k_n^3}{\sqrt{k_p^2 + m_\star^2|_0}}\right)\right\} \\
& + \frac{c_\rho^2 c_\omega^2 k_p^3}{30\pi^2\mu_0^2 k_n^3}\left(\frac{k_n^2 k_p^3}{\sqrt{k_n^2 + m_\star^2|_0}} - \frac{k_p^2 k_n^3}{\sqrt{k_p^2 + m_\star^2|_0}}\right) - \frac{3\pi^2 k_p^5}{5\mu_0^2 k_n^6}\frac{k_n^2 + m_\star^2|_0}{\sqrt{k_p^2 + m_\star^2|_0}}, \\
\mathcal{C}_0 = {} & \frac{3\pi^2\chi_0}{k_p^3} + \frac{1}{4}c_\rho^2\frac{k_n^3}{k_p^3} - c_\omega^2\frac{k_n^3}{k_p^3} - \frac{c_\omega^2 k_n^3}{5\mu_0^2 k_p^3}\left\{2k_p^2\frac{\sqrt{k_n^2 + m_\star^2|_0}}{\sqrt{k_p^2 + m_\star^2|_0}} + \frac{c_\omega^2}{3\pi^2}\left(\frac{k_n^2 k_p^3}{\sqrt{k_n^2 + m_\star^2|_0}} + \frac{k_p^2 k_n^3}{\sqrt{k_p^2 + m_\star^2|_0}}\right)\right\} \\
& - \frac{c_\rho^2 k_n^3}{20\mu_0^2 k_p^3}\left\{2k_p^2\frac{\sqrt{k_n^2 + m_\star^2|_0}}{\sqrt{k_p^2 + m_\star^2|_0}} + \frac{c_\rho^2}{12\pi^2}\left(\frac{k_n^2 k_p^3}{\sqrt{k_n^2 + m_\star^2|_0}} + \frac{k_p^2 k_n^3}{\sqrt{k_p^2 + m_\star^2|_0}}\right)\right\} \\
& + \frac{c_\rho^2 c_\omega^2 k_n^3}{30\pi^2\mu_0^2 k_p^3}\left(\frac{k_n^2 k_p^3}{\sqrt{k_n^2 + m_\star^2|_0}} - \frac{k_p^2 k_n^3}{\sqrt{k_p^2 + m_\star^2|_0}}\right) - \frac{3\pi^2}{5\mu_0^2 k_p}\frac{k_n^2 + m_\star^2|_0}{\sqrt{k_p^2 + m_\star^2|_0}},
\end{aligned}$$

together with

$$\mathcal{A}_0^0 = c_\omega^2 - \frac{c_\rho^2}{4} + \frac{\pi^2}{k_p^2}\frac{m_\star|_0 \frac{\partial m_\star}{\partial k_p}|_0}{\sqrt{k_n^2 + m_\star^2|_0}},$$

$$\mathcal{B}_0^0 = c_\omega^2 + \frac{c_\rho^2}{4} + \frac{\pi^2}{k_n^2}\frac{k_n + m_\star|_0 \frac{\partial m_\star}{\partial k_n}|_0}{\sqrt{k_n^2 + m_\star^2|_0}},$$

and

$$\mathcal{C}_0^0 = c_\omega^2 + \frac{c_\rho^2}{4} + \frac{\pi^2}{k_p^2}\frac{k_p + m_\star|_0 \frac{\partial m_\star}{\partial k_p}|_0}{\sqrt{k_p^2 + m_\star^2|_0}} + \frac{\pi^2}{k_p}\frac{1}{\sqrt{k_p^2 + m_e^2}}.$$

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