Variations on the Goroff-Sagnotti operator

Enrique Álvarez,* Jesús Anero,† and Eduardo Velasco-Aja‡
Departamento de Física Teórica and Instituto de Física Teórica, IFT-UAM/CSIC, Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain

(Received 4 September 2023; accepted 10 October 2023; published 21 November 2023)

The Goroff-Sagnotti operator, corresponding to the contraction of three Weyl tensors, is the first counterterm of general relativity (GR) nonvanishing on shell. We study the classical effects of including this operator in the effective gravitational Lagrangian. The results obtained for the Goroff-Sagnotti operator are proved to hold for some higher-curvature operators that generalize it. We find solutions to those operators' equations of motion (EM); in particular, we find the general condition for the spherically symmetric case and provide several example solutions. Concerning the EM for GR supplemented with the Goroff-Sagnotti operator, we study spherically symmetric perturbative corrections to the GR solution. In less symmetric instances, we only study the subset of solutions that solve the EM separately for GR and the Goroff-Sagnotti operator.

DOI: 10.1103/PhysRevD.108.104043

I. INTRODUCTION

Within the context of effective field theories (EFTs) applied to general relativity (GR), the Einstein-Hilbert Lagrangian[1] is only the lowest-energy approximation[2] of a Lagrangian involving higher powers of curvature (maybe all of them). Increasing powers of the energy scale suppress these higher-curvature terms $\Lambda_{\rm eff}$, at which the perturbative EFT description breaks down, and the input from all the higher-curvature operators is required.

From this vantage point, a timely question is to what extent a solution to GR's equations of motion (EM) is stable, in the sense of [1,2] under perturbations corresponding to these higher-dimensional operators in the gravitational Lagrangian.

Our main purpose is to examine the effect of changing the EM due to the presence, in the gravitational Lagrangian, of higher-curvature terms, specifically quadratic and cubic in curvature. A general classification of such operators was given in [3]. We present in Table I those operators corresponding to quadratic and cubic operators.

One can start by considering the quadratic operators in the first row of Table I. Since we will be concerned with the structural stability of solutions to the EM, we can neglect the $\Box R$ operator, as this boundary term does not change the EM, provided an adequate Gibbons-Hawking-York surface counterterm is added to the action. Furthermore, since we will only consider $n=4$-dimensional spacetimes, we can further use the fact that the Euler characteristic for a four-dimensional manifold corresponds to

$$\mathcal{X}_4 = \frac{1}{32\pi^2}\int d^4x\sqrt{-g}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2), \quad (1)$$

where $g \equiv \det g_{\mu\nu}$, to realize that only two out of the three remaining operators are independent. Therefore, we can choose as a basis for quadratic operators $\{R^2, R_{\mu\nu}R^{\mu\nu}\}$.

The resulting action,

$$S = -\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R - \kappa^2(\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu})), \quad (2)$$

has the following EM:

$$\begin{aligned} -\frac{1}{2\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = {} & \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) - 2\beta R\left(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}\right) \\ & - 2\alpha R^{\rho\sigma}\left(R_{\mu\rho\nu\sigma} - \frac{1}{4}R_{\rho\sigma}g_{\mu\nu}\right) \\ & - (\alpha + 2\beta)(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R \\ & - \alpha\Box\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right). \end{aligned} \quad (3)$$

From inspection of Eq. (3), one can see that all the solutions to GR in vacuum $R_{\mu\nu} = 0$ are solutions to Eq. (3) as well.[3]

*enrique.alvarez@uam.es
†jesusanero@gmail.com
‡eduardo.velasco@uam.es

[1] That gives rise to the equations of motion of GR.

[2] If one ignores the elephant in the room that the cosmological constant represents.

[3] Note this is not a symmetry-dependent statement, GR solutions are structurally stable under the addition of quadratic operators.

TABLE I. Quadratic and cubic curvature operators. The operator $R^{\alpha\beta}R_{\rho\sigma\lambda\alpha}R^{\rho\sigma\lambda}{}_{\beta}$ is independent if the space dimensionality is $n \geq 5$. The operators $R^{\alpha\beta\gamma\delta}R_{\alpha\mu\gamma\nu}R_{\beta}{}^{\mu}{}_{\delta}{}^{\nu}$, $R^{\alpha\beta\gamma\delta}R_{\gamma\delta\mu\nu}R^{\mu\nu}{}_{\alpha\beta}$ are independent if the space dimensionality is $n \geq 6$.

Curvature order	Operators
Quadratic	$R^2, \Box R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$
Cubic	$\Box^2 R, R\Box R, R_{\mu\nu}\nabla^\mu\nabla^\nu R, R_{\mu\nu}\Box R^{\mu\nu}, R_{\mu\alpha\nu\beta}\nabla^\alpha\nabla^\beta R^{\mu\nu}, \nabla^\mu R\nabla_\mu R, \nabla^\alpha R^{\mu\nu}\nabla_\alpha R_{\mu\nu}$, $\nabla^\alpha R^{\mu\nu}\nabla_\mu R_{\nu\alpha}, \nabla^\alpha R^{\mu\nu\rho\sigma}\nabla_\alpha R_{\mu\nu\rho\sigma}, R^3, RR_{\mu\nu}R^{\mu\nu}, R^{\mu\nu}R_\mu{}^\alpha R_{\alpha\nu}, R^{\mu\nu}R^{\rho\sigma}R_{\nu\sigma\mu\rho}$, $RR^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}, R^{\alpha\beta\gamma\delta}R_{\alpha\mu\gamma\nu}R_\beta{}^\mu{}_\delta{}^\nu, R^{\alpha\beta\gamma\delta}R_{\gamma\delta\mu\nu}R^{\mu\nu}{}_{\alpha\beta}, R^{\alpha\beta}R_{\rho\sigma\lambda\alpha}R^{\rho\sigma\lambda}{}_\beta$

The main goal of this work is to study operator deformations that can make spacetimes potentially depart from their GR solution status. However, Eq. (3) shows that quadratic operators will not affect structural stability. Thus, as discussed below, we will omit the presence of quadratic operators in our stability analysis.[4]

On the other hand, dealing with the entire set of cubic operators in Table I is generally complicated. There have been attempts in specific simplifying scenarios, for example, regarding perturbative solutions around a GR spacetime with a high degree of symmetry [4,5] or a particularly well-behaved subset of operators, e.g., [6,7].

Our approach here is similar to that in [1,4], where only one out of all such cubic operators is considered; we will generalize the statements for nonspherically symmetric spacetimes and obtain some further generalizations. In particular, we will consider the Goroff-Sagnotti (GS) operator,

$$\mathcal{O}_{\rm GS} \equiv W_{\mu\nu\alpha\beta}W^{\alpha\beta\rho\sigma}W_{\rho\sigma}{}^{\mu\nu}, \tag{4}$$

where $W_{\alpha\beta\gamma\delta}$ is the Weyl tensor,[5]

$$W_{\alpha\beta\gamma\delta} \equiv R_{\alpha\beta\gamma\delta} - \frac{1}{2}\{g_{\alpha\gamma}R_{\beta\delta} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\delta\alpha} + g_{\beta\delta}R_{\gamma\alpha}\} + \frac{1}{6}R\{g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta}\}. \tag{5}$$

The main property of Weyl's tensor is that it is inert[6] under the so-called Weyl rescalings (a gauge transformation) of the spacetime metric

$$g_{\mu\nu} \to \Omega(x)^2 g_{\mu\nu}, \qquad W^\mu{}_{\nu\rho\sigma} \to W^\mu{}_{\nu\rho\sigma}. \tag{6}$$

Restricting to a single operator means that there are potential effects that our analysis will not capture.[7] However, since such corrections are unlikely[8] to cancel the effects of the operator considered, we believe that any departure from the GR solutions here displayed would, most likely, only be accentuated by the presence of more general operators.

The reason for considering $\mathcal{O}_{\rm GS}$ is twofold: First, it is the first nonvanishing, on-shell counterterm that appears when renormalizing gravity. The one-loop counterterm [8] to GR,

$$L_\infty^{(1)} = \frac{1}{n-4}\frac{1}{(4\pi)^2}\int d^n x\sqrt{-g}\left(\frac{1}{60}R^2 + \frac{7}{10}R^2_{\mu\nu}\right), \tag{7}$$

vanishes on shell.

It is then essential to consider also the first nonvanishing on-shell counterterm, which appears at the two-loop order, the Goroff-Sagnotti (GS) operator [9], and is proportional to the trace of the cube of Weyl's tensor,

$$L_\infty^{(2)} = \frac{1}{n-4}\frac{209}{2880}\frac{1}{(4\pi)^4}\int d^n x\sqrt{-g}W_{\mu\nu\alpha\beta}W^{\alpha\beta\rho\sigma}W_{\rho\sigma}{}^{\mu\nu}. \tag{8}$$

Note that for $n \leq 5$, see Table I, there is only one independent such contraction [3]. It is interesting to remark that, as first pointed out in [1,10], for the static and spherically symmetric case, the EM corresponding to $\mathcal{O}_{\rm GS}$ excludes the general relativity solutions. The main goal of this work is to generalize the claims for spherically symmetric spacetimes and study to which extent less symmetric solutions are incompatible with the GR ones under the GS counterterm perturbation.

In addition to this physical motivation, the simplicity of the EM for $\mathcal{O}_{\rm GS}$ will allow us to extract certain general properties of the deformation, see Sec. II A 2. Ideally, we would want to get exact solutions to the EM. However, those are few and far between. This can be seen by observing the trace equations for the static and spherically symmetric case in Appendix D. Even in this simple case, they are nonlinear fourth-order differential equations.

Nonetheless, given the structure of $\mathcal{O}_{\rm GS}$, we provide a necessary condition for a spacetime to be a solution to the EM for the $\mathcal{O}_{\rm GS}$ operator. The full EM can be checked in

[4]We stress that they play a role when studying how solutions depart from GR solutions, but leave that analysis for future work.

[5]In general spacetime dimension the form of the Well tensor will change and the Weyl classification discussed in Appendix C will not be valid.

[6]Bear in mind the position of the indices in Eq. (6).

[7]Unless some fine-tuning in the coefficients of all the other third-order curvature operators would justify the choice here made. We are not aware of any reason for this to be the case.

[8]Given the different nature of the cubic operators in Table I.

Appendix A. This is independent of the symmetry of the spacetime considered, but some symmetry assumptions can strengthen the condition to a necessary and sufficient condition. The main tool for obtaining these results combines the trace of the EM and the Petrov classification, see Appendix C.

In proving the results for the GS operator, we generalize the results to the family of operators defined like

$$\mathcal{O}_p \equiv \mathrm{tr} W^p \equiv W_{\mu_1\nu_1\rho_1\sigma_1}\ldots W_{\mu_p\nu_p\rho_p\sigma_p} I^{\vec{\mu}\vec{\nu}\vec{\rho}\vec{\sigma}}, \tag{9}$$

where $I^{\vec{\mu}\vec{\nu}\vec{\rho}\vec{\sigma}}$ is the product of $2p$ inverse metric tensors, for example,

$$I^{\vec{\mu}\vec{\nu}\vec{\rho}\vec{\sigma}} \equiv g^{\mu_1\mu_2}\ldots g^{\mu_{p-1}\mu_p} g^{\nu_1\nu_2}\ldots g^{\nu_{p-1}\nu_p} g^{\rho_1\rho_2}\ldots g^{\rho_{p-1}\rho_p} g^{\sigma_1\sigma_2}\ldots g^{\sigma_{p-1}\sigma_p}. \tag{10}$$

Although more complicated contractions are possible, they are all similar in character.

After this, we discuss solutions to the whole system, including the Einstein-Hilbert operator, providing details regarding static and spherically symmetric perturbative solutions in ω.

To summarize, the starting point would be the Lagrangian

$$L = \sqrt{-g}\left\{-\frac{1}{2\kappa^2}R + aR_{\mu\nu}^2 + bR^2 + \omega\kappa^2 W_{\mu\nu\alpha\beta}W^{\alpha\beta\rho\sigma}W_{\rho\sigma}{}^{\mu\nu}\right\}, \tag{11}$$

with dimensionless coupling constant a, b, ω, and $\kappa^2 \equiv 8\pi G$.

The coefficients of the quadratic operators R^2 and $R_{\mu\nu}^2$ in (11) can and will be put equal to zero ($a = b = 0$) to study the structural stability of GR solutions.[9] Hence, we will study

$$L = \sqrt{-g}\left\{-\frac{1}{2\kappa^2}R + \omega\kappa^2 W_{\mu\nu\alpha\beta}W^{\alpha\beta\rho\sigma}W_{\rho\sigma}{}^{\mu\nu}\right\}. \tag{12}$$

The outline for this paper is the following. In Sec. II A, we consider the solutions to the EM of $\mathcal{O}_{\mathrm{GS}}$, starting in Sec. II A 1 from the spherically symmetric scenario, where the symmetry allows us to make stronger statements. After this, in Sec. II A 2, we consider less symmetric spacetimes and study how the results of the prior section are changed. Both in Secs. II A 1 and II A 2, the results are obtained from proofs for the family of operators $\mathcal{O}_p$ in Eq. (9), from which setting $p = 3$ the results for $\mathcal{O}_{\mathrm{GS}}$ are obtained.[10] This is followed by some results concerning perturbative solutions in ω of the EM in Sec. III. Finally, in Sec. IV, we close with a summary of this work's results, conclusions, and implications.

II. ANALYTIC SOLUTIONS TO $\mathcal{O}_{\mathrm{GS}}$

Given the difficulties that a direct approach to solving the EM presents, we will start by considering the possibility of solving the EM separately for $\mathcal{O}_{\mathrm{GS}}$. In Sec. II A, we discuss the solution to the EM for the $\mathcal{O}_{\mathrm{GS}}$ operator by analyzing the family of $\mathcal{O}_p$ operators defined in Eq. (9). After this, in Sec. II B, we comment on the solutions to the EM of Eq. (12).

A. The $\mathcal{O}_{\mathrm{GS}}$ EM

In this section, we only consider the EM for the $\mathcal{O}_{\mathrm{GS}}$ operator. For this, we first study spherically symmetric spacetimes in Sec. II A 1 and then turn to more general spacetimes in Sec. II A 2.

1. Spherically symmetric spacetimes

Let us start with an observation that is the root of many of the following results. Direct computation of the Weyl tensor for spherically symmetric spacetimes with line element

$$ds^2 = B(r,t)dt^2 - A(r,t)dr^2 - r^2 d\Omega_2^2 \tag{13}$$

yields

$$W^{\mu\nu}{}_{\rho\sigma} = \frac{G}{12A^2B^2r^2}C^{\mu\nu}{}_{\rho\sigma}. \tag{14}$$

In Eq. (14), we have defined

$$\begin{aligned} G(A, A', B, B', \cdots) \equiv\; & A(r^2(B'^2 - \dot{A}\dot{B}) \\ & + 2rB(r\ddot{A} + B' - rB'') - 4B^2) \\ & + rB(A'(rB' - 2B) - r\dot{A}^2) + 4A^2B^2, \end{aligned} \tag{15}$$

where $\dot{f} = \partial_t f(r,t)$, $f' = \partial_r f(r,t)$, and $C^{\mu\nu}{}_{\rho\sigma}$ is a metric-independent tensor,

[9] Even if these operators do not render GR solutions unstable, they would be important to determine how the spacetime is deformed in the presence of the $\mathcal{O}_{\mathrm{GS}}$ operator. Thus, to illustrate the nature of the deformation in Sec. III, we will only consider the GS operator.

[10] The reason for presenting the more general proofs is the completeness of the results, even if such operators are higher order in curvature and, as such, Λ_{eff} suppressed.

$$
\begin{aligned}
&C^{01}{}_{\alpha\beta} = -C^{10}{}_{\alpha\beta} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad C^{02}{}_{\alpha\beta} = -C^{20}{}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
&C^{03}{}_{\alpha\beta} = -C^{30}{}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \qquad C^{12}{}_{\alpha\beta} = -C^{21}{}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
&C^{13}{}_{\alpha\beta} = -C^{31}{}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix}, \qquad C^{23}{}_{\alpha\beta} = -C^{32}{}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{16}
$$

This was first noted in [10], where it is written in terms of projectors. Here, the result is showcased in detail. These observation led, in [1], to a proof of the fact that all spherically symmetric solutions to $\mathcal{O}_{\mathrm{GS}}$ have to be Weyl flat; $W^{\alpha}{}_{\beta\gamma\delta} = 0$.

Here we provide an alternative proof valid for the family of $\mathcal{O}_p$ operators that will prove useful for the upcoming results in Sec. II A 2.

Claim. For a spherically symmetric action consisting of p contracted Weyl tensors,

$$
\begin{aligned}
S_{\mathrm{Weyl}^p} &\equiv \int d^4x\sqrt{-g}\mathcal{O}_p \\
&= \int d^4x\sqrt{-g}W_{\mu_1\nu_1\rho_1\sigma_1}\ldots W_{\mu_p\nu_p\rho_p\sigma_p}I^{\vec{\mu}\,\vec{\nu}\,\vec{\rho}\,\vec{\sigma}},
\end{aligned} \tag{17}
$$

the condition

$$G[A, A', \dot{A}, \ddot{A}, B, B', \dot{B}, B'', r] = 0 \tag{18}$$

contains all nonsingular solutions to the EM. Equation (18) is equivalent to Weyl flatness.

Proof. The general action in Eq. (17) is defined for $n = 4$ dimensions. We have defined the tensor

$$I^{\vec{\mu}\,\vec{\nu}\,\vec{\rho}\,\vec{\sigma}} \equiv g^{\mu_{i_1}\nu_{j_1}}\ldots g^{\rho_{k_p}\sigma_{l_p}} \quad (2p \text{ metrics}), \tag{19}$$

where $(i_1\ldots i_p)$, $(j_1\ldots j_p)$, $(k_1\ldots k_p)$, $(l_1\ldots l_p)$ are permutations of $(1\ldots p)$, such that it represents all possible scalars containing p Weyl tensors.

For example, the GS operator corresponds to

$$I^{\vec{\mu}\,\vec{\nu}\,\vec{\rho}\,\vec{\sigma}} = g^{\mu_1\rho_3}g^{\nu_1\sigma_3}g^{\rho_1\mu_2}g^{\sigma_1\nu_2}g^{\rho_2\mu_3}g^{\sigma_2\nu_3}. \tag{20}$$

The EM can be formally expressed as

$$\frac{1}{2}g_{\alpha\beta}G^p C^p_{\mu\nu\rho\sigma} + pG^{p-1}\frac{\delta G}{\delta g^{\alpha\beta}}C^p_{\mu\nu\rho\sigma} + G^p\frac{\delta C^p_{\mu\nu\rho\sigma}}{\delta g^{\alpha\beta}} = 0, \tag{21}$$

where

$$C^p_{\mu\nu\rho\sigma} = C_{\mu_1\nu_1\rho_1\sigma_1}\ldots C_{\mu_p\nu_p\rho_p\sigma_p}I^{\vec{\mu}\,\vec{\nu}\,\vec{\rho}\,\vec{\sigma}}. \tag{22}$$

This means that all solutions to $G[A, A', \dot{A}, \ddot{A}, B, B', \dot{B}, B'', r] = 0$ are also solutions of the EM ("sufficient condition"). They depend on an arbitrary function since a unique differential equation determines $A(r,t)$ and $B(r,t)$.

Also, since the action is conformal in dimension $n = 2p$, the trace of the EM is proportional[11] to

$$g^{\alpha\beta}\frac{\delta S_{\mathrm{Weyl}^p}}{\delta g^{\alpha\beta}} \propto (2p-4)W_{\mu_1\nu_1\rho_1\sigma_1}\ldots W_{\mu_p\nu_p\rho_p\sigma_p}I^{\vec{\mu}\,\vec{\nu}\,\vec{\rho}\,\vec{\sigma}}. \tag{23}$$

We provide an alternative proof of this fact in Sec. II A 2. Nonetheless, this is forced by conformal invariance and dimensional arguments. Indeed, Deser-Schwimmer's [11] conjecture, proved by Alexakis [12], states that the most general form of the integrand of a conformal (Weyl) invariant is of the form

$$P \equiv W + \nabla_\mu j^\mu + c\mathrm{Pf}(R_{\mu\nu\rho\sigma}), \tag{24}$$

where c is a constant, W is a conformal invariant (i.e., $\sqrt{-g}W^2$ in $n = 4$ dimensions), and the Pfaffian is proportional to the integrand of Euler's characteristic of a compact manifold M through

$$\int_M \mathrm{Pf}(R_{\mu\nu\rho\sigma})d(\mathrm{vol}) = \frac{2^n\pi^{\frac{n}{2}}(n/2-1)!}{2(n-1)!}\chi(M). \tag{25}$$

Combining Eq. (23) with the observation in Eq. (14), we can see that, for the spherically symmetric case, the

[11]It turns out that it is not only proportional but equal. See Sec. II A 2 below.

condition $G = 0$ is also a necessary condition for the spherically symmetric solutions to the EM for $\mathcal{O}_p$. ■

The fact that all Weyl flat solutions satisfy $G = 0$ (barring divergences in the metric tensor) means that, for all powers[12] $p \neq 2$, we capture all solutions ("necessary condition") of the EM by simply solving for this condition.

This is no longer true in the conformal dimension ($p = 2$); see Appendix B. This is why Schwarzschild spacetime is a viable solution for four-dimensional conformal gravity [13].

Finally, we note that, taking into account the proof that conformal gravity satisfies Birkhoff's theorem [14], i.e., any spherically symmetric solution is static, and the fact that the set of solutions to conformal gravity contains the solutions for any action of the form (17) we conclude that any spherically symmetric solution to an action made of by p contractions of the Weyl tensor will be static. Therefore, Birkhoff's theorem, in the sense of spherically symmetric solutions having to be spherical,[13] holds for this larger set of actions.

For illustrative purposes, we now apply the result we have just proven to obtain some solutions to the complete EM in (A2) for the static and spherically symmetric case. Since Eq. (18) is a unique condition for the two functions $A(r)$, $B(r)$, to find solutions, we make an ansatz for $B(r)$ (seed function) and integrate the first-order ordinary differential equation for $A(r)$.

(i) First, we would like to compare the classical result [13]; for that, we further restrict the freedom of the pair $A(r)$, $B(r)$[14] and set

$$B(r) = \frac{1}{A(r)}. \tag{26}$$

Plugging this in (18), one can integrate the resulting equation, obtaining

$$A(r) = \frac{1}{c_2 r^2 + c_1 r + 1}, \tag{27}$$

which corresponds to the solution first reported in [10], without the $\frac{1}{r}$ term of the solution in [13]. This further shows that there is no Schwarzschild solution, as first pointed out by Deser and Tekin [10]. Compare this with the conformal gravity solution (B5).

To obtain distinct solutions, we will consider independent $A(r)$, $B(r)$.

(1) We can obtain asymptotically flat solutions with Schwarzschild-like behavior at large r, using the seed

$$B(r) = 1 - \frac{\mathcal{M} r^n}{r^{n+1} + \lambda}, \quad \text{with} \quad \mathcal{M}, \lambda > 0. \tag{28}$$

These solutions satisfy that

$$\text{for } r \gg \sqrt[n+1]{\lambda}, \qquad B(r) \simeq 1 - \frac{\mathcal{M}}{r}. \tag{29}$$

Some particular choices of n and corresponding $A(r)$ are

(a) $n = 0$,

$$B(r) = 1 - \frac{\mathcal{M}}{r + \lambda}, \tag{30}$$

$$A(r) = \frac{(2r^2 + r(4\lambda - 3\mathcal{M}) + 2\lambda(\lambda - \mathcal{M}))^2}{(\lambda + r)^2(\lambda + r - \mathcal{M})(c_1 r^2(\lambda + r) + 4(\lambda + r - \mathcal{M}))}. \tag{31}$$

(b) $n = 1$,

$$B(r) = 1 - \frac{\mathcal{M} r}{r^2 + \lambda}, \tag{32}$$

$$A(r) = \frac{(2\lambda^2 + r^3(2r - 3\mathcal{M}) + \lambda r(4r - \mathcal{M}))^2}{(\lambda + r^2)^2(\lambda + r^2 - r\mathcal{M})(r(r(c_1(\lambda + r^2) + 4) - 4\mathcal{M}) + 4\lambda)}. \tag{33}$$

The unique condition (18) allows for a large number of solutions. The examples explored here are such that they recover asymptotic flatness if $c_1 = 0$. These examples are obtained using Eq. (18) and later checked to satisfy the EM in Appendix D.

[12] Note then this proof also contains the GS operator, $\mathcal{O}_{\text{GS}}$.

[13] They are clearly nonunique.

[14] This oversimplification works here even though the gauge only depends on one function instead of two [10].

2. Nonspherically symmetric spacetime: The trace equations

The proof in Sec. II A 1 is based on the variation of the action corresponding to the integral of an arbitrary $\mathcal{O}_p$ operator. Spherical symmetry is only crucial to drawing the conclusion that

$$\mathcal{O}_p = 0 \overset{\text{sph symm}}{\Longrightarrow} W^{\alpha}{}_{\beta\gamma\delta} = 0. \tag{34}$$

On the other hand, we stress that the need for $\mathcal{O}_p$ to vanish is not symmetry dependent.

Given the action

$$S_p \equiv \int d^n x \sqrt{-g}\mathcal{O}_p, \tag{35}$$

we have set the dimensionality to n to keep track of where the dimensionality plays a role in this derivation. We will later set $n = 4$ below. Upon a linearized Weyl rescaling in an arbitrary dimension, say, n,

$$\delta_W g_{\mu\nu} = \omega(x) g_{\mu\nu}, \tag{36}$$

and because it is invariant precisely when $n = 2p$ the action in Eq. (35) satisfies

$$\delta_W S_p = \int d^n x \delta_W (\sqrt{-g}\mathcal{O}_p) = \int d^n x (n-2p)\omega(\sqrt{-g}\mathcal{O}_p) = \int d^n x \frac{\delta S_p}{\delta g^{\mu\nu}} \omega g^{\mu\nu}. \tag{37}$$

It follows that

$$g^{\mu\nu} \frac{\delta S_p}{\delta g^{\mu\nu}} = (n-2p)(\sqrt{-g}\mathcal{O}_p). \tag{38}$$

This is a rederivation of the relation in Eq. (23), accounting for all the proportionality our previous discussion missed.

The result in Eq. (38) generally holds for S_p actions; we now try to provide a way of assessing whether this identity is satisfied for different spacetimes by studying the properties of the Weyl tensor. The natural way of doing so is through Petrov's classification, which we briefly introduce in Appendix C for the reader's convenience.

Let us represent, following [15–17],[15] the ten independent components of Weyl's tensor as a six-dimensional block matrix,

$$W^{\mu\nu}{}_{\rho\sigma} = \begin{pmatrix} E & -B \\ B & E \end{pmatrix}, \tag{39}$$

where B and E are three-dimensional, traceless, symmetric, and real matrices. We refer the reader to Appendix C for details on this. From Eq. (39),[16]

$$W^2 = N\begin{pmatrix} E^2 - B^2 & -(EB+BE) \\ BE+EB & E^2 - B^2 \end{pmatrix} \to \mathrm{tr}W^2 = N\mathrm{tr}(E^2 - B^2). \tag{40}$$

Similarly,

$$W^3 = N\begin{pmatrix} E^3 - EB^2 - B^2E - BEB & B^3 - BE^2 - E^2B - BE^2 - EBE \\ BE^2 - B^3 + EBE + E^2B & E^3 - EB^2 - B^2E - BEB \end{pmatrix}, \tag{41}$$

where now, using the cyclic property of the trace,

$$\mathcal{O}_{\mathrm{GS}} = \mathrm{tr}W^3 = N\mathrm{tr}(E^3 - 3EB^2). \tag{42}$$

Given the fact that Schwarzschild spacetime (as, in fact, all static, spherically symmetric spacetimes) has purely electric Weyl tensors, it can be easily shown that, for them,

$$\mathcal{O}_{\mathrm{GS}} = 0 \Longrightarrow W_{\mu\nu\rho\sigma} = 0. \tag{43}$$

For purely magnetic spacetimes, $\mathcal{O}_{\mathrm{GS}} = 0$ is given. Unfortunately, there are not many known purely magnetic exact solutions. A theorem stated in [18] claims that there are no purely magnetic type D vacuum solutions of Einstein's equations.

Starting with $\mathcal{O}_4$, the cyclic property of the trace is not enough to get a simple result. To be more specific, we need some explicit results [16,19] on Petrov's classification of spacetimes. They are frequently expressed in terms of the properties of the matrix

$$Q \equiv E + iB. \tag{44}$$

The algebraically general spacetime is dubbed Petrov type I. For these spacetimes, the general criterion is

$$(Q - \lambda_1 Id)(Q - \lambda_2 Id)(Q - \lambda_3 Id) = 0 \tag{45}$$

[15]In fact, Matte's work [15] precedes Petrov's by one year, although he did not achieve a full classification of Weyl's tensor.

[16]Here, N takes care of index undercounting, and its exact value is immaterial for our purposes.

(where $\sum \lambda_i = 0$ and $\lambda_i \equiv e_i + ib_i$). The electric and magnetic Weyl tensors obey

$$E = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}, \tag{46}$$

with $\sum e_i = \sum b_i = 0$. It is plain that, in this case, $[E, B] = 0$ so the trace is given by

$$\mathcal{O}_{\mathrm{GS}} = 2N \sum_i e_i (e_i^2 - 3b_i^2). \tag{47}$$

Type D is a subtype corresponding to $\lambda_1 = \lambda_2 = -2\lambda_3$ in Eq. (45). This particular Petrov type is important for us because all spherically symmetric solutions [19] are either type D or type O (meaning that the spacetime is conformally flat).

The general result for the all-important types I and D is obtained from the above discussion,

$$\mathcal{O}_{2m} = 2 \sum_{k=0}^{m} (-1)^k \binom{2m}{2k} \mathrm{tr}(E^{2(m-k)} B^{2k}), \tag{48}$$

$$\mathcal{O}_{2m+1} = 2 \sum_{k=0}^{m} (-1)^k \binom{2m+1}{2k} \mathrm{tr}(E^{2(m-k)+1} B^{2k}). \tag{49}$$

This gives the conditions for the vanishing trace in terms of products of the electric and magnetic eigenvalues $\{e_i, b_i\}$.

Let us apply these results to a couple of example spacetimes. These depart from spherical symmetry and thus from the results in Sec. II A 1.

(a) First, we consider the spacetime of pp-waves with line element [19]

$$ds^2 = -2du^2 H(x, y, u) + 2du dv - (dx^2 + dy^2). \tag{50}$$

Given the fact that pp-waves belong to the class of vanishing scalar invariant [20] spacetimes (included in Kundt's class), we only have to impose Einstein's equations in vacuum, which means that $H(x, y, u)$ has to satisfy

$$R_{\mu\nu} = 0 \Leftrightarrow \partial_x^2 H(x, y, u) + \partial_y^2 H(x, y, u) = 0. \tag{51}$$

The above-introduced analysis yields electric and magnetic components of the Weyl tensor corresponding to

$$E = \frac{1}{2H(x, y, u) + 2} \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & \partial_x^2 H(x, y, u) & \partial_x \partial_y H(x, y, u) \\ 0 & \partial_x \partial_y H(x, y, u) & -\partial_x^2 H(x, y, u) \end{pmatrix} \tag{52}$$

and

$$B = \frac{1}{2H(x, y, u) + 2} \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\partial_x \partial_y H(x, y, u) & \partial_x^2 H(x, y, u) \\ 0 & \partial_x^2 H(x, y, u) & \partial_x \partial_y H(x, y, u) \end{pmatrix}. \tag{53}$$

In Eqs. (52) and (53), Eq. (51) has been used. The direct calculation then gives

$$Q^2 = 0. \tag{54}$$

Therefore, we see that pp-waves are everywhere in Petrov type N. One then has that the trace condition is satisfied for them, i.e.,

$$\mathcal{O}_{\mathrm{GS}} \propto \mathrm{tr} Q^3 = 0. \tag{55}$$

Thus, pp-waves comprise an example of a spacetime satisfying the necessary condition to be a solution to the EM. Here, direct computation lets us see that this spacetime satisfies the Goroff-Sagnotti EM, Eq. (A2).

In [1], it was proven that no spherically symmetric spacetime could simultaneously solve Einstein's and Goroff-Sagnotti EM. It is interesting to remark that pp-waves are the only example of a solution to both EM we have identified to this date.

(b) As a second nonspherically symmetric nor static example, we consider Kerr-Newman spacetime, whose line element in Boyer-Lindquist coordinates [21] reads

$$ds^2 = \frac{\Delta}{\rho^2}(dt - a \sin^2\theta d\phi)^2 - \frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)d\phi - a dt]^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2, \tag{56}$$

where

$$\Delta \equiv r^2 - r r_s + a^2 + r_q^2 \quad \text{and} \quad \rho \equiv a^2 \cos^2(\theta) + r^2. \tag{57}$$

As usual, a is related to the angular momentum, r_s is the Schwarzschild radius related to the mass, and r_q is related to the electric charge[17] of the solution.

In this case, the electric and magnetic matrices are

[17] We set $r_q = 0$ such that the line element in Eq. (56) is a solution to the vacuum Einstein's EM.

$$E = e(r,\theta)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad \text{with} \quad e(r,\theta) = \frac{4r_s r(3a^2 - 2r^2 + 3a^2\cos(2\theta))}{(a^2 + 2r^2 + a^2\cos(2\theta))^3}, \tag{58}$$

$$B = b(r,\theta)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad \text{with} \quad b(r,\theta) = \frac{4r_s a\cos(\theta)(a^2 - 6r^2 + a^2\cos(2\theta))}{(a^2 + 2r^2 + a^2\cos(2\theta))^3}. \tag{59}$$

This corresponds to a Petrov type D spacetime; following the above discussion,

$$\mathcal{O}_{\mathrm{GS}} = \frac{3}{4}e(r,\theta)(e(r,\theta)^2 - 3b(r,\theta)^2). \tag{60}$$

The trace in Eq. (60) is nonvanishing[18] for arbitrary r values. We conclude that an uncharged Kerr spacetime with mass cannot solve the EM for the Goroff-Sagnotti counterterm. We find this remarkable. Schwarzschild metric and its most important stationary extension, namely, Kerr spacetime (widely believed to be the general end point of gravitational collapse) is incompatible with the Goroff-Sagnotti Lagrangian deformation.

B. General remarks on the complete EM

It is physically evident that the effect of all those operators is suppressed by powers of the energy scale Λ_{eff} (two powers for the Goroff-Sagnotti operator). This means that the perturbative corrections to an exact solution of the Einstein-Hilbert Lagrangian (like the simplest Schwarzschild spacetime one) and even quantum corrections are not expected to be relevant at low energies. The nature of perturbations for spherically symmetric spacetimes is considered in Sec. III.

Nevertheless, when an exact solution can be found in a nonlinear theory like GR, one always learns new things, some of them very important physically, like the presence of a horizon, which is a global concept. It would then be desirable to find exact solutions of the Einstein-Hilbert Lagrangian coupled to those higher-dimensional operators. Unfortunately, we have not been able to identify any solution in which the two operators in the Lagrangian interact in a nontrivial way. The already mentioned solutions are in the class of "vanishing scalar invariants" like pp-waves, in which (when Ricci flatness is imposed) the EM of the two operators at play vanish separately.

III. SOME REMARKS ON PERTURBATIVE SOLUTIONS

It is well known that, while the Einstein-Hilbert action admits the Schwarzschild metric as a solution, this is not true anymore when the GS operator is included, as first shown by Deser and Tekin [10]. This section aims to ascertain where an initial Schwarzschild spacetime is driven in the presence of the GS counterterm.[19]

In this section, we are interested in analyzing the static spherically symmetric perturbations to Schwarzschild spacetime when the Einstein-Hilbert action is deformed with the GS term, that is,

$$S = \int d^4x\sqrt{-g}\left\{-\frac{1}{2\kappa^2}R + \omega\kappa^2 W^3\right\}. \tag{61}$$

In Sec. III A below, we analyze the case of a polar perturbation of the type

$$ds^2 = \left[1 - \frac{r_s}{r} + B(r)\right]dt^2 - \frac{1}{\left[1 - \frac{r_s}{r} + A(r)\right]}dr^2 - r^2 d\Omega^2. \tag{62}$$

In this formalism, we will consider the first-order corrections in $\omega\kappa^4$ to the EM for a spherically symmetric spacetime, displayed in Appendix D, that determine the small corrections to g_{tt} given by $B(r)$ and discuss the changes this brings into the thermodynamic quantities of the original spacetime.

A. Perturbations of the horizon

Since the perturbation in (62) does not change the static and spherically symmetric nature of the spacetime, we will have that every event horizon will correspond to the Killing horizon for a certain timelike vector $\mathcal{K}$ [22]. For the particular form of the metric we are considering,

$$\mathcal{K} \equiv \partial_t \tag{63}$$

is a Killing vector. Then the Lie derivative of the metric with respect to the vector field $\mathcal{K}$ satisfies

[18] Unless $r_s = 0$.

[19] It is important to acknowledge that the results under the $\mathcal{O}_{\mathrm{GS}}$ deformation should, on a complete calculation, be computed including the quadratic operators which, even if they do not affect the structural stability, are important to see how the Schwarzschild solution would change.

$$\mathcal{L}_{\mathcal{K}}g = \nabla_\alpha \mathcal{K}_\beta + \nabla_\beta \mathcal{K}_\alpha = 0. \tag{64}$$

The Killing horizon of $\mathcal{K}$ will correspond to a null hypersurface in which the norm of $\mathcal{K}$ vanishes, i.e.,

$$\mathcal{K}^2 = 1 - \frac{r_s}{r} + B(r) = 0. \tag{65}$$

To see how the perturbation $B(r)$ introduced by the GS counterterm can affect the horizon structure, we consider the first-order EM for $B(r)$,

$$r^8 A(r) + (r - r_s)(r^8 B'(r) + 12\kappa^2 r_s^2(4r_s - 3r)) - r^7 r_s B(r) = 0, \tag{66}$$

$$r^8 A'(r) + r^7 A(r) + 12\kappa^2 r_s^2(16 r_s - 15 r) = 0. \tag{67}$$

Solving Eqs. (66) and (67) gives

$$A(r) = \frac{c_1}{r} + \frac{32\kappa^2 r_s^3}{r^7} - \frac{36\kappa^2 r_s^2}{r^6}, \tag{68}$$

$$B(r) = -\frac{c_2 r_s}{r} + \frac{c_1}{r} + c_2 + \frac{8\kappa^2 r_s^3}{r^7} - \frac{12\kappa^2 r_s^2}{r^6}. \tag{69}$$

Setting c_2, $c_1 = 0$ to ensure asymptotic flatness and the right spacetime when $\omega \to 0$, respectively, gives

$$B(r) = \frac{8\kappa^2 r_s^3}{r^7} - \frac{12\kappa^2 r_s^2}{r^6}. \tag{70}$$

With Eq. (70), we will first consider small displacements of the horizon and study how these are reflected in the horizon temperature and the associated Bekenstein-Hawking entropy. Here we will consider the Killing horizon corresponding to the timelike Killing vector of the metric. If we consider $r = r_0$ to be the value at which

$$g_{tt}(r_0)|_{\omega=0} = 0, \tag{71}$$

then $r_0 = r_s$. If we consider that when $\omega \neq 0$ the new Killing horizon is at

$$r_* = r_s + \omega\kappa^2\rho + \cdots, \tag{72}$$

we have that

$$\rho = \frac{4\kappa^2}{r_s^3}, \tag{73}$$

and then the position of the horizons is given by

$$r_* \equiv r_s + \kappa^4\omega\left(\frac{4}{r_s^3}\right). \tag{74}$$

In this particular case, we can check that this value of r_* makes the first order in g^{rr} vanish.

B. Thermodynamics

For arbitrarily small perturbations, the only effect of the GS counterterm is to shift the location of the horizon. Suppose we assume no further change occurs in the spacetime's causal structure. In that case, we can, in principle, calculate the first-order changes that the temperature and entropy associated with the event horizon suffer under the GS counterterm. Changes in the thermodynamical variables coming from interaction with classical matter (dirty black holes) were studied in [23], while an analysis for general third-order curvature perturbations was presented in [5]. Here we follow the latter to discuss the changes to first order in the perturbative parameter $\omega\kappa^2$.

We note that there are some instances in higher-derivative theories of gravity in which the changes in the first black hole thermodynamical law [24] can be exactly calculated [6]. However, these results depend crucially on the fact that $A(r) = 1/B(r)$, which allows expressing the series near the horizon in terms of the surface gravity. We will restrain our analysis to a qualitative study of the lowest order and leave the more technical details for elsewhere.

In order to study the change in temperature, we expand the near-horizon geometry. This leads to the known Rindler-like behavior near the horizon. Now, we analytically continue the geometry to complex coordinates. Then, we use the Euclidean time periodicity.

To avoid conical defects at $r = r_\star$, the Euclidean time variable has to have a period whose inverse is related to the temperature. To first order, this corresponds to

$$T_H = \frac{\hbar}{4\pi k_B r_s}\left(1 + \frac{4\omega\kappa^4}{r_s^4}\right), \tag{75}$$

which is clearly different from the classical result for the spherically symmetric gravitational collapse [25].

In order to study the change in the entropy, we note that from the asymptotic expansion in Sec. III A, the behavior for an observer at infinity has corrections of $\mathcal{O}(1/r)^6$ or higher. Therefore, the Arnowitt-Deser-Misner mass measured by an observer far from the horizon is unchanged. Therefore, the first law in this approximation reads

$$\frac{dS}{dM} = \frac{1}{T} = \frac{4\pi k_B r_s}{\hbar}\left(1 - \frac{4\omega\kappa^4}{r_s^4}\right). \tag{76}$$

This naive calculation shows the first-order change in the thermodynamical quantities associated with the horizon. A more careful consideration should incorporate quadratic operators into the picture. However, we have left them out of the question for simplicity.

IV. SUMMARY AND CONCLUSION

The leitmotif of the present paper is that we believe that it is physically important to take into account irrelevant (in the renormalization group sense) operators in the gravitational action, which are present in a naive calculation of the radiative corrections to the Einstein-Hilbert action.

With that in mind, we have examined in some detail the effect of the GS operator,

$$\mathcal{O}_{\rm GS} \equiv W_{\mu\nu\alpha\beta} W^{\alpha\beta\rho\sigma} W_{\rho\sigma}{}^{\mu\nu}. \tag{77}$$

We found that many arguments could be extended to Lagrangians built out of arbitrary powers of Weyl's tensor. We generalized, in particular, Deser and Ryzhov's arguments [10] to the nonstatic case and found the general solution of the EM with spherical symmetry in the nonconformal situation. Furthermore, we have studied how the necessary and sufficient condition is changed when spherical symmetry is relaxed.

In detail, the operators we have studied are built of p contractions of the Weyl tensor, symbolically,

$$\int d^n x \sqrt{-g}\mathcal{O}_p. \tag{78}$$

These are conformally invariant in dimension $n = 2p$. Using the so-called trace equation and the Petrov classification of Weyl's tensor, we have found a necessary condition for the solutions of the EM in the general case.

In the more simple instance of spherically symmetric spacetimes, which motivated this study originally [1], we can find the general solution of the EM. This is because, in this case, the trace condition becomes a necessary and sufficient condition for a spacetime to solve for the complete EM.

Examples for the less trivial pp-waves and Kerr-spacetime are explicitly worked out. pp-waves are found to be a solution to the EM of GR with the $p = 3$ operator included.

Concerning the physically relevant EM (including the Einstein-Hilbert contribution), we were able to prove that there are no spherically symmetric Ricci flat solutions. Some computations regarding perturbative solutions are provided for spherically symmetric and static spacetimes, but these are not substitutes for analytic solutions. Although no complete solution to the EM has been identified for the spherically symmetric case, we have shown that, when the Einstein-Hilbert Lagrangian is deformed with the W^3 operator, the resulting spacetime is expected to change its asymptotic behavior. A perturbative analysis indicates that the horizon gets modified and so do the thermodynamic properties of spacetime.

The only spacetime we have found to solve the complete EM are pp-waves. This could have potential implications in gravitational waves within the context of effective field theories, but more work is required to make any precise statement.

The results regarding structural stability would not change very much if we had included quadratic terms in the action. The reason is that quadratic theories admit Schwarzschild spacetime as a solution, so they do not upset the stability of the initial solution. They become relevant when studying where the solution is driven when operators that hinder structural stability, such as $\mathcal{O}_{\rm GS}$, are included in the action. In the same spirit, the introduction of a cosmological constant can be done easily, and it does not affect the result in any essential way, provided the perturbations are done to the corresponding de Sitter/anti–de Sitter-Schwarzschild background.

ACKNOWLEDGMENTS

We have enjoyed fruitful discussions with Roberto Emparan and Jose Senovilla. We acknowledge partial financial support by the Spanish MINECO through the Centro de Excelencia Severo Ochoa Program under Grant No. CEX2020-001007-S funded by MCIN/AEI/10.13039/501100011033. All authors acknowledge the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie Grant Agreement No. 860881-HIDDeN and also by Grant No. PID2019-108892RB-I00 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF—A way of making Europe." This project has received funds/support from the European Union's Horizon Europe program under Marie Sklodowska-Curie Actions-Staff Exchanges (SE) Grant Agreement No. 101086085-ASYMMETRY. E. A. and E. V. A. acknowledge partial financial support by the Spanish Research Agency (Agencia Estatal de Investigación) through the Grant No. PID2022-137127NB-I00 funded by MCIN/AEI/10.13039/501100011033/ FEDER, UE.

APPENDIX A: EQUATIONS OF MOTION FOR $\mathcal{O}_{\rm GS}$

Here we present, for completeness, the EM for the $\mathcal{O}_{\rm GS}$ operator. Again, we will denote the spacetime dimensions as n to illustrate the dependence on dimensionality. However, the reader must bear in mind that, for the Petrov classification and the decomposition into electric and magnetic parts of the Weyl tensor to work, one must have $n = 4$. Therefore, the n in the expressions below should only be considered as a case study of the general claims made in Sec. II A 2 regarding the trace of $\mathcal{O}_p$ operators being conformally invariant in $n = 2p$ dimensions.

The action for $\mathcal{O}_{\rm GS}$ is

$$S_{\rm GS} \equiv \int d^4 x \sqrt{-g} W_{\mu\nu\alpha\beta} W^{\alpha\beta\rho\sigma} W_{\rho\sigma}{}^{\mu\nu}. \tag{A1}$$

Making explicit the dimensionality, $n = 4$, they read

$$V_{\mu\nu} = \frac{1}{2} g_{\mu\nu}\mathcal{O}_{\rm GS} + \frac{6}{(n-2)}\left[\frac{1}{n-1}(RW_\mu{}^{\alpha\beta\lambda}W_{\nu\alpha\beta\lambda} - R_{\mu\nu}W^2) - R^{\alpha\beta}W_{\mu\alpha\lambda\tau}W^{\lambda\tau}{}_{\nu\beta}\right.$$
$$- 2\nabla^\tau\nabla_\mu(W_\nu{}^{\alpha\beta\lambda}W_{\tau\alpha\beta\lambda}) + g_{\mu\nu}\nabla^\rho\nabla^\sigma(W_\rho{}^{\alpha\beta\lambda}W_{\sigma\alpha\beta\lambda}) - (n-2)\nabla^\rho\nabla^\sigma(W_{\mu\rho\alpha\beta}W^{\alpha\beta}{}_{\nu\sigma})$$
$$\left. + \left(\frac{(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)}{n-1}W^2 + \Box(W_\mu{}^{\alpha\beta\lambda}W_{\nu\alpha\beta\lambda})\right) + R_\mu{}^\tau W_\nu{}^{\alpha\beta\lambda}W_{\tau\alpha\beta\lambda}\right] - 3W_\mu{}^{\alpha\beta\lambda}W_{\nu\alpha\rho\sigma}W^{\rho\sigma}{}_{\beta\lambda}. \tag{A2}$$

The trace of Eq. (A2) corresponds to

$$V_{\mu\nu}g^{\mu\nu} = \frac{n-6}{2}W_{\mu\nu\alpha\beta}W^{\alpha\beta\rho\sigma}W_{\rho\sigma}{}^{\mu\nu}. \tag{A3}$$

We see that in any dimension different from the conformal one ($n = 6$), a *necessary* condition for the EM to be satisfied is that

$$\mathcal{O}_{\rm GS} = W_{\mu\nu\alpha\beta}W^{\alpha\beta\rho\sigma}W_{\rho\sigma}{}^{\mu\nu} = 0. \tag{A4}$$

APPENDIX B: CONFORMAL GRAVITY

In this appendix, we study conformal gravity (CG), corresponding to the action

$$S_{\rm CG} = \int d^4x\sqrt{-g}W_{\mu\nu\alpha\beta}W^{\mu\nu\alpha\beta} = \int d^4x\sqrt{-g}\mathcal{O}_{\rm CG}. \tag{B1}$$

The purpose of this section is to shed some light on the reasons why conformal invariance for the action in Eq. (B1) implies that the results proven in Secs. II A 1 and II A 2 fail to hold for this action.

The action (B1) defined by the trace of two Weyl tensors is conformally (gauge) invariant in $n = 4$ dimensions (and it is the only operator on the family with this property). The equations of motion correspond to the vanishing of the traceless Bach tensor

$$H_{\alpha\beta} \equiv \frac{\delta S}{\delta g^{\alpha\beta}} \equiv B_{\alpha\beta} \equiv K^{\mu\nu}W_{\mu\alpha\beta\nu} + \nabla^\lambda(\nabla_\lambda K_{\alpha\beta} - \nabla_a K_{\beta\lambda}), \tag{B2}$$

where Schouten's tensor $K_{\mu\nu}$ is defined as

$$K_{\alpha\beta} \equiv \frac{1}{n-2}\left(R_{\alpha\beta} - \frac{1}{2(n-1)}Rg_{\alpha\beta}\right). \tag{B3}$$

Physically, the most important property of Bach's tensor is that it corresponds to a primary operator of dimension 2; that is, under a Weyl rescaling (6), it transforms as

$$B_{\mu\nu} \to \Omega^{-2}B_{\mu\nu}. \tag{B4}$$

It was shown in [13] that conformal gravity admits a solution,

$$ds^2 = \left(c_0 + \frac{c_{-1}}{r} + c_1 r + c_2 r^2\right)dr^2 - \frac{dt^2}{\left(c_0 + \frac{c_{-1}}{r} + c_1 r + c_2 r^2\right)} - r^2 d\Omega_2^2, \tag{B5}$$

while Deser and Tekin show [10] that the GS counterterm does not allow for the c_{-1} term. This is again because, for the conformal dimension $p = 2$, the trace identically vanishes, and therefore $G = 0$ will capture some, but not all, spherically symmetric and static solutions to

$$H_{\alpha\beta} = 0. \tag{B6}$$

This means that the set of all spherically symmetric solutions corresponding to (18) belongs to the larger set of solutions of conformal gravity.

As an example of the difference between $\mathcal{O}_{\rm GS}$ and the family of operators $\mathcal{O}_p$, note that, for static, spherically symmetric spacetimes,

$$\mathcal{O}_4 \equiv W^{\alpha\beta}{}_{\gamma\delta}W^{\gamma\delta}{}_{\kappa\zeta}W^{\kappa\zeta}{}_{\lambda\sigma}W^{\lambda\sigma}{}_{\alpha\beta}, \tag{B7}$$

is proportional to the Lagrangian of CG in Eq. (B1). Using the notation of Sec. II A 2, it so happens that

$$\sqrt{-g}\mathcal{O}_2 = \frac{\sqrt{r^4AB\sin^2(\theta)}}{12r^4A^4B^4}G^2 \tag{B8}$$

and

$$\sqrt{-g}\mathcal{O}_4 = \frac{\sqrt{r^4AB\sin^2(\theta)}}{576r^8A^8B^8}G^4. \tag{B9}$$

This simple relationship between both Lagrangians does not carry over to the corresponding solutions. As shown in [1], the solutions for CG contain those of $\mathcal{O}_4$ but not conversely. Of course, for CG, the trace equation does not yield any restriction on the conformal case $n = 4$. On the other hand, the trace condition applies for the EM corresponding to the operator $\mathcal{O}_4$. Therefore, the only way in which a metric can be a solution to both CG and quartic gravity is to satisfy, in addition,

$$W^2 = W^{\alpha\beta}{}_{\gamma\delta}W^{\gamma\delta}{}_{\alpha\beta} = 0 \Rightarrow G = 0 \Rightarrow W^\alpha{}_{\beta\gamma\delta} = 0, \tag{B10}$$

i.e., it has to be conformally flat. In fact, for any $p \neq 2$, static, spherically symmetric solutions are determined by Eq. (B10).

APPENDIX C: PETROV'S CLASSIFICATION

Here we present the basic ingredients of Petrov's classification. We follow the very pedagogic explanations of [19]. Equally good presentations of this topic are present, for example, in [21,26]. However, any advanced book in general relativity with a section on the topic should suffice.

The curvature tensor, expressed in an orthogonal tetrad $\{e_a^\mu\}$, R_{abcd}, can be uniquely decomposed into parts transforming under irreducible representations of the Lorentz group,

$$R_{abcd} = W_{abcd} + E_{abcd} + G_{abcd}, \tag{C1}$$

where

$$E_{abcd} \equiv \frac{1}{2}(g_{ac}S_{bd} + g_{bd}S_{ac} - g_{ad}S_{bc} - g_{bc}S_{ad}),$$
$$G_{abcd} \equiv \frac{1}{12}R(g_{ac}g_{bd} - g_{ad}g_{bc}),$$
$$S_{ab} \equiv R_{ab} - \frac{1}{4}Rg_{ab}. \tag{C2}$$

Equation (C1) defines the Weyl tensor equivalent to Eq. (5). Defining the dual vector,

$$\widetilde{W}_{abcd} = \frac{1}{2}\epsilon_{cdef}W_{ab}{}^{ef},$$

where ϵ_{abcd} is the Levi-Civita symbol. From this, we can define the complex Weyl tensor,

$$W^*{}_{abcd} = W_{abcd} + i\widetilde{W}_{abcd}.$$

By using a normalized timelike vector, we can express the content of W_{abcd} in terms of a two-index complex tensor Q,

$$-Q_{ac} \equiv W^*{}_{abcd}u^b u^d \equiv E_{ac} + iB_{ac}, \quad \text{where } u_c u^c = 1. \tag{C3}$$

This defines the electric E_{ab} and magnetic parts B_{ab} of the Weyl tensor. Even though Eq. (C3) might seem to depend on the choice of u^a, this is not the case, as the tensor $W^*{}_{abcd}$ can be retrieved from Q_{ab} by the expression

$$-\frac{1}{2}W^*{}_{abcd} = 4u_{[a}Q_{b][d}u_{c]} + g_{a[c}Q_{d]b} - g_{b[c}Q_{d]a} + i\epsilon_{abef}u^e u_{[c}Q_{d]}{}^f + i\epsilon_{cdef}u^e u_{[a}Q_{b]}{}^f, \tag{C4}$$

where the square brackets mean antisymmetrization in the enclosed indices.

This means one can describe the Weyl tensor using the matrix Q_{ab} or, alternatively, by its electric and magnetic components. One can see that Q_{ab} satisfies

$$(i)\ Q_{ab} = Q_{ba}, \quad (ii)\ Q^a{}_a = 0, \quad (iii)\ Q_{ab}u^b = 0. \tag{C5}$$

TABLE II. Petrov classification.

Petrov type		
I	[1 1 1]	$(Q-\lambda_1 Id)(Q-\lambda_2 Id)(Q-\lambda_3 Id)=0$
D	[(1 1) 1]	$(Q-\lambda_1 Id)(Q+\frac{\lambda_2}{2}Id)(Q-\lambda Id)=0$
II	[2 1]	$(Q-\lambda_1 Id)(Q+\frac{\lambda_2}{2}Id)^2(Q-\lambda Id)=0$
N	[(2 1)]	$Q^2=0$
III	[3]	$Q^3=0$
O		$Q=0$

Conditions (i)–(iii) in Eq. (C5) mean that Q_{ab} has five independent, complex-valued entries, which correspond to the ten independent components of the Weyl tensor in $n = 4$ dimensions.

The last piece necessary to discuss Petrov's classification is the eigenvalue equation,

$$Q_{ab}X^b = \lambda X_a, \tag{C6}$$

which leads to the characteristic equation

$$\det(Q - \lambda \mathrm{Id}) = 0, \tag{C7}$$

which determines the eigenvalues and their multiplicity. Spacetimes can be classified according to these eigenvalues and their multiplicity, see Table II. It is also common to present a diagram in which the more general spacetimes are related to the more restrictive ones; see Fig. 1.

These properties of the Weyl tensor and the Petrov classification imply that we can, as in the original works, see, e.g., [15–17], identify the Weyl tensor with two indices raised, $W^{\alpha\beta}{}_{\gamma\delta}$, as

$$W^{\mu\nu}{}_{\rho\sigma} = \begin{pmatrix} E & -B \\ B & E \end{pmatrix}, \tag{C8}$$

$$\begin{tikzcd}
\mathrm{I} \arrow[d] \arrow[dr] & & \\
\mathrm{II} \arrow[r] \arrow[d] \arrow[dr] & \mathrm{D} \arrow[d] \arrow[dr] & \\
\mathrm{III} \arrow[r] & \mathrm{N} \arrow[r] & \mathrm{O}
\end{tikzcd}$$

FIG. 1. Petrov classification diagram. Arrows point toward less general spacetimes contained in the spacetimes from which arrows start.

which will be extensively used to discuss the properties of the trace equation for $\mathcal{O}_p$ operators in Sec. II A 2. Similarly, the properties of the spacetimes in Table II are used in Sec. II A 2 to study the spacetimes in terms of the eigenvalues of the electric and magnetic tensors, i.e., $\{e_i, b_i\}$.

APPENDIX D: SOME DETAILS OF THE COMPUTATIONS

Let us write the full EM, omitting the explicit r dependence,

$$\begin{aligned} H_{tt} = \frac{1}{144 r^6 A^7 B^5}\Big\{ & 132 r^4 \omega B^4 A'^4 (rB' - 2B)^2 - 48 A^6 B^6 (3\gamma r^4 - 8\omega) \\ & - 8A^7 B^6 (-18\gamma r^4 + 4\omega) + 3r^2 A^2 B^2 \mathcal{F}_\omega(A, B, A', B', A'', B'', A^{(3)}, B^{(3)}) \\ & + A^4 \mathcal{G}_\omega(A, B, A', B', A'', B'', A^{(3)}, B^{(3)}, B^{(4)}) \\ & - 3rA^3 B \mathcal{H}_\omega(A, B, A', B', A'', B'', A^{(3)}, B^{(3)}, B^{(4)}) \\ & + r^3 \omega A B^3 A'^2 (2B - rB')\Big[2rB(63rA''B' + A'(213rB'' - 97B')) - 265 r^2 A' B'^2 \\ & + 4B^2(149A' - 63rA'')\Big] + 6A^5 B^2 \mathcal{K}_\omega(A, B, A', B', A'', B'', A^{(3)}, B^{(3)}, B^{(4)})\Big\}, \end{aligned} \tag{D1}$$

with

$$\begin{aligned} \mathcal{F}_\omega(A, B, A', B', \cdots) \equiv {} & 105 r^4 \omega A'^2 B'^4 - 2r^3 \omega B A' B'^2 (27rA''B' + A'(130rB'' + 23B')) \\ & + 8rB^3\Big[14\omega r A'^3 B' - 2r^2 \omega A''^2 B' + \omega A'^2 (r(31B'' - 14rB^{(3)}) - 14B') \\ & + r\omega A'((32A'' - 2rA^{(3)})B' - 23rA''B'')\Big] + 4r^2 \omega B^2\Big[r^2 A''^2 B'^2 \\ & + A'^2(28r^2 B''^2 - 60B'^2 + rB'(14rB^{(3)} + 43B'')) \\ & + rA'B'(23rA''B'' + (rA^{(3)} + 13A'')B')\Big] \\ & - 16B^4(-r^2\omega A''^2 + 14\omega r A'^3 - 20\omega A'^2 + r\omega A'(18A'' - rA^{(3)})), \end{aligned} \tag{D2}$$

$$\begin{aligned} \mathcal{K}_\omega(A, B, A', B', \cdots) \equiv {} & 8B^4(A'(3\gamma r^5 + 2r\omega) - 14\omega) \\ & + 49\omega r^4 B'^4 - 16\omega r^4 B^3 B^{(4)} - 4r^3 B B'^2 (29\omega r B'' - 5\omega B') \\ & + 4r^2 B^2\Big[9\omega r^2 B''^2 + 3\omega B'^2 + 2rB'(6\omega r B^{(3)} - 3\omega B'')\Big], \end{aligned} \tag{D3}$$

$$\begin{aligned} \mathcal{G}_\omega(A, B, A', B', \cdots) \equiv {} & -4r^3 B^3[\omega B'^3(40 - 87rA') + 26r^3 \omega B''^3 + 6r^2 \omega B' B''(14rB^{(3)} + B'') \\ & + 6r\omega B'^2(r(rB^{(4)} - 6B^{(3)}) - 36B'')] \\ & + 24r^2 B^4[-rB'(\omega B''(27rA' - 8) + 2r\omega(rB^{(4)} + 8B^{(3)})) \\ & - 2\omega B'^2(3r^2 A'' - 3rA' + 2) + 2r^2\omega((rB^{(3)})^2 - 5B''^2 \\ & + r(rB^{(4)} + 2B^{(3)})B'')] + 48r^2 B^5[A'(\omega B' + r(6\omega r B^{(3)} - \omega B'')) \\ & + r(2\omega r(2A''B'' + B^{(4)}) + B'(\omega r A^{(3)} - \omega A''))] \\ & - 32B^6[3\alpha r^5 A^{(3)} + 18\beta r^5 A^{(3)} + 3r^3 \omega A^{(3)} - 9r^2 \omega A'' + 24r\omega A' - 10\omega] \\ & + 121 r^6 \omega B'^6 - 6r^5 \omega B B'^4 (85rB'' - 29B') \\ & + 84 r^4 \omega B^2 B'^2 (7r^2 B''^2 - 4B'^2 + 2rB'(rB^{(3)} - 2B'')), \end{aligned} \tag{D4}$$

and finally,

$$\begin{aligned}\mathcal{H}_\omega(A,B,A',B',\cdots)\equiv{}& 28r^4\omega BB'^3(rA''B'-A'(B'-10rB''))-85r^5\omega A'B'^5\\&-4r^3\omega B^2B'[A'(55r^2B''^2-46B'^2+2rB'(10rB^{(3)}+B''))\\&+rB'(18rA''B''+(rA^{(3)}+A'')B')]+8rB^4[\omega rA'^2(19rB''-5B')-2r\omega(2(A''-rA^{(3)})B'\\&+r(rA^{(3)}B''+A''(2rB^{(3)}-5B'')))+A'(B'(13\omega r^2A''+4\omega)\\&-2r\omega(2B''+r(rB^{(4)}-8B^{(3)})))]+2r^2B^3[-57\omega rA'^2B'^2+4\omega A'(13B'^2\\&+2r^2B''(6rB^{(3)}+5B'')+B'(r^3B^{(4)}-46rB''))+4r\omega(4r^2A''B''^2-10A''B'^2+rB'(rA^{(3)}B''\\&+A''(2rB^{(3)}+5B'')))]+8B^5[29r\omega A'^2-4r\omega(rA^{(3)}-3A'')-2\omega A'(13r^2A''+14)].\end{aligned}\tag{D5}$$

The component H_{rr}, in turn, reads

$$\begin{aligned}H_{rr}={}&\frac{1}{144r^6A^5B^6}\{8(-18\gamma r^4+4\omega)A^6B^6+11\omega(rA'(rB'-2B)B)^3\\&+48(AB)^5((3\gamma r^4-4\omega)B+r(3\gamma r^4+2\omega)B')+3(rB)^2\omega AA'(rB'-2B)^2[7r^2A'B'^2\\&-2r(rB'A''+A'(6rB''-4B'))B-4(5A'-rA'')B^2]+6A^4B^6[48\omega B^2\\&-16\omega B(r^3B^{(3)}+2rB')-4r^2(\omega B'^2-4\omega r(B''+rB^{(3)})B'+r^2\omega B''^2)\\&-12\omega r^3B^{-2}B'^2(B'+rB'')B+7\omega r^4B^{-2}B'^4]+3rA^2B[8\omega B^5(7rA'^2-8A'+4rA'')\\&+2r^2(7\omega rA'^2B'^2+4\omega A'B^3(3B'^2+r(B''-2rB^{(3)})B'-2r^2B''^2)\\&+8r\omega A''(B'-rB'')B')-8rB^4[7\omega rB'A'^2-2\omega(4B'+r(rB^{(3)}-3B''))A'\\&-2r\omega A''(rB''-3B')]+4r^3\omega B'B^2(A'(2r^2B''^2-5B'^2+r(11B''+rB^{(3)})B')\\&+rB'A''(B'+rB''))-2r^4\omega B'^3(rB'A''+A'B(7B'+9rB''))B+7r^5\omega A'B'^5]\\&+A^3(11\omega B'^6r^6-42\omega BB'^4B''r^6+12\omega B^2B'^2(-8B'^2+r(5B''+rB^{(3)})B'+3r^2B''^2)r^4\\&+4(rB)^3(\omega(16+9rA')B'^3+48r\omega B''B'^2-6r^2\omega B''(5B''+rB^{(3)})B'+2r^3\omega B''^3)\\&+24B^4(2\omega B''(B''+rB^{(3)})r^2-2rB'(\omega(5+rA')B''+2r\omega B^{(3)})\\&-B'^2(\omega A''r^2+2\omega A'r-2\omega))r^2+48rB^5(2\omega A'B''+B^{(3)}r^2\\&+B'(-3r\omega A'+2\omega(A''r^2+1)))-32\omega B^6(3A''r^2-6A'r+4)),\end{aligned}\tag{D6}$$

where $A^{(n)}$ corresponds to the nth partial derivative.

From spherical symmetry and the Bianchi identities, these are the two independent components of the vacuum EM.

[1] E. Álvarez, J. Anero, and E. Velasco-Aja, The fate of horizons under quantum corrections, J. Cosmol. Astropart. Phys. 02 (2023) 029; E. Alvarez, J. Anero, and R. Santos-Garcia, Structural stability of spherical horizons, arXiv: 2006.02463.

[2] V. I. Arnold, *Equations, Differentielles Ordinaires* (MIR, Moscow, 1974); *Ordinary Differential Equations (Universitext)* (Springer, New York, 1992); *Chapitres Supplementaires sur la theorie des equations differentialles* (MIR, Moscow, 1978).

[3] S. A. Fulling, R. C. King, B. G. Wybourne, and C. J. Cummins, Normal forms for tensor polynomials. 1: The Riemann tensor, Classical Quantum Gravity **9**, 1151 (1992).

[4] A. Dobado and A. López, Some consequences of the effective low-energy Lagrangian for gravity, Phys. Lett. B **316**, 250 (1993).

[5] M. Lu and M. Wise, M. Black holes with a generalized gravitational action, Phys. Rev. D **47**, R3095 (1993).

[6] P. Bueno and P. Cano, Four-dimensional black holes in Einsteinian cubic gravity, Phys. Rev. D **94**, 124051 (2016).

[7] J. Moreno and Á. Murcia, On the classification of generalized quasitopological gravities, Phys. Rev. D **108**, 044016 (2023).

[8] G. 't Hooft and M. J. G. Veltman, One loop divergencies in the theory of gravitation, Ann. Inst. Henri Poincare, Sect. A **20**, 69 (1974).

[9] M. H. Goroff and A. Sagnotti, The ultraviolet behavior of Einstein gravity, Nucl. Phys. **B266**, 709 (1986).

[10] S. Deser and J. Franklin, Birkhoff for Lovelock redux, Classical Quantum Gravity **22**, L103 (2005); S. Deser and A. V. Ryzhov, Curvature invariants of static spherically symmetric geometries, Classical Quantum Gravity **22**, 3315 (2005); S. Deser and B. Tekin, Shortcuts to high symmetry solutions in gravitational theories, Classical Quantum Gravity **20**, 4877 (2003).

[11] S. Deser and A Schwimmer, Geometric classification of conformal anomalies in arbitrary dimensions, Phys. Lett. B **309**, 279 (1993).

[12] S. Alexakis, On the decomposition of global conformal invariants I, Ann. Math. **170**, 1241 (2009).

[13] P. D. Mannheim and D. Kazanas, Exact vacuum solution to conformal Weyl gravity and galactic rotation curves, Astrophys. J. **342**, 635 (1989).

[14] R. J. Riegert, Birkhoff's theorem in conformal gravity, Phys. Rev. Lett. **53**, 315 (1984).

[15] A. Matte, Sur de nouvelles solutions oscillatoires des equations de la gravitation, Can. J. Math. **5**, 1 (1953).

[16] FAE Pirani, Invariant formulation of gravitational radiation theory, Phys. Rev. **105**, 1089 (1957).

[17] J. A. Wheeler, Gravitational and electromagnetic wave flux compared and contrasted, Phys. Rev. D **16**, 3384 (1977).

[18] C. B. G. McIntosh, R. Arianrhod, S. T. Wade, and C. Hoenselaers, Electric and magnetic Weyl tensors: Classification and analysis, Classical Quantum Gravity **11**, 1555 (1994).

[19] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 2003), ISBN 978-0-521-46702-5, 978-0-511-05917-9.

[20] V. Pravda, A. Pravdova, A. Coley, and R. Milson, All space-times with vanishing curvature invariants, Classical Quantum Gravity **19**, 6213 (2002).

[21] Barrett O'Neill, *The Geometry of Kerr Black Holes* (Dover, New York, 2014).

[22] B. Carter, Killing horizons and orthogonal transitivity in space-time, J. Math. Phys. (N.Y.) **10**, 70 (1969).

[23] M. Visser, Dirty black holes: Thermodynamics and horizon structure, Phys. Rev. D **46**, 2445 (1992).

[24] J. Bekenstein, Black holes and entropy, Phys. Rev. D **7**, 2333 (1973).

[25] S. Hawking, Particle creation by black holes, Commun. Math. Phys. **43**, 199 (1975).

[26] R. Wald, *General Relativity* (Chicago University Press, Chicago, 1984).